The Quantum Hall Effect

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Abstract: About the reason for the presence of plateaux at the Hall resistivity graphic as a function of the magnetic field in a Hall effect system, and a brief introduction to its fractional regime.

I. INTRODUCTION [1]

After the hundredth anniversary of Hall's original work, it was found that under certain conditions in an effectively two-dimensional system of electrons subdued to a strong magnetic field, the Hall conductivity takes quantised values

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar}\nu \tag{1}$$

where ν was integer valued with extraordinary precision. The conductivity is inversely proportional to the Hall resistivity that we will see below.

This quantisation indicated that the door to the quantum regime had been opened and it led to the quantum Hall effect. The first theorical approximation to understand this effect was based on the model of independent electrons in the presence of imperfections.

A few years later of this first discovery, it was also found that in some nearly ideal samples at low temperatures, ν could take very specific fractional values as well. This led to the fractional quantum Hall effect, where the Coulomb interacions between electrons, neglected in the integer quantum Hall effect investigations, take an important role. An interesing fact is that the charged particles roaming around these systems carry a fraction of the charge of the electron, as if the electron has split himself.

We will introduce some considerations about the Hall effect system before entering in the quantum regime.

II. FORMATION OF A TWO-DIMENSIONAL ELECTRON GAS [2]

Let us consider a crystalline material with the free electron model, which assumes that the electrons are free to move within the crystal, but they are externally confined by potential barriers. The behaviour of these electrons will be the same as in an electron gas with Fermi energy

$$E_F = \frac{\hbar^2}{2m_e}(3\pi\rho)^{2/3} \tag{2}$$

Where ρ is the electronic density. Introducing periodic potentials, it can be seen that the energy levels of the electrons are grouped in bands, separated by energy band gaps. The presence of these periodic potentials is justified by the translational periodicity of the crystal, and the corresponding eigenstates of the resulting Hamiltonian are the Bloch functions. An interesting case to see is the semiconductor, which has multiple partially-filled bands. However, his behaviour is mainly dominated by the highest almost-filled band and the lowest almost-empty band, which correspond to the valence band and the conductor band respectively.

We are interested in two-dimensional structures where the electrons are confined to a thin sheet in the xy-plane by potential barriers that create a quantum well. This can be physically implemented by joining two dissimilar semiconductors with different band gaps in a A-B-A structure, forming a system with two heterojunctions. The discontinuity in either the valence band and the conduction band can be used to form a potential well that acts like an inversion band.

A. The Effective Mass Approximation [2] and the g-factor [3]

The electrons at the partially-filled bands are affected by the presence of the periodic potential. This effect can be easily included by replacing the value of the free electron mass, m_e, with the effective mass, $m = m^* m_e$.

Moreover, if we introduce the concept of holes as the missing electrons of the valence band with the same properties of the electrons but with positive charge, we can see that there exist two different effective masses: one for the electron of the conduction band, m_e^*, and one for the holes of the valence band, m_h^*.

On the other hand, the gyromagnetic factor of the electron, g, presents a dependence on the energy gap of the semiconductor, E_g. In order to distinguish it from the g-factor of the free electron, we introduce the effective g-factor, g^*, which can take different values such as $g^* \approx 2$ for wide-gap semiconductors, or $g^* = -0.44$ for GaAs.

It is interesting to see how these effective values affect to the system. For the last case, GaAs, the dielectric constant is $\epsilon = 12.4$ and the effective mass is $m_e^* = 0.067$ Taking effective atomic units $\hbar = e^2/\epsilon = m = 1$, the

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length unit and the energy unit are

$$a_0^* = \frac{\epsilon}{m^*} a_0 \sim 9.794 \quad nm \tag{3}$$

$$E_0^* = \frac{m^*}{\epsilon^2} E_0 \sim 11.26 \quad meV \tag{4}$$

Where a_0 is the Bohr radius. The magnetic field when $l_B = a_0$ is $10^5 T$, in contrast with the field $B^* = 4.5T$ when $l_B = a_0^*$. Is for this reason that for such systems, the magnetic field can not be treated perturbatively.

III. CLASSICAL HALL EFFECT [4]

Consider a system where the electrons are restricted to move in the xy-plane, with an applied electric field $\boldsymbol{E}$ in the x-direction. Let us turn on a magnetic field $\boldsymbol{B} = B\hat{k}$ in the z-direction. Because of the Lorentz force, the electrons are pushed to one side of the conductor, producing a difference of potential between the surface boundaries parallel to the x-axis, usually called Hall Voltage V_H. This phenomenon was discovered by Edwin H. Hall in 1879 and it is known as the Hall effect. It is usually used to measure unknown magnetic fields with Hall probe devices.

It is possible to define the Hall transverse resistivity

$$\rho_{xy} = \frac{E_y}{j_x} = \frac{1}{\sigma_{xy}} \tag{5}$$

This resistivity is proportional to the magnetic field and inversely proportional to the electron density

$$\rho_{xy} = \frac{F_y}{e} \frac{1}{e\rho v_x} = \frac{B}{e\rho} \tag{6}$$

However, this linear dependence on the magnetic field is only seen for low values of B. At low temperatures, by increasing the value of B, some plateaux appear in the graphic of the Hall resistivity dependence on the magnetic field. This can be explained considering quantum effects. But first, let us introduce the Landau levels.

IV. LANDAU LEVELS [4]

If we restrict the electrons to low energies, we can neglect the spin of these. The reason is that we would need an energy $\Delta = 2\mu_B B$, with μ_B the Bohr's magneton, to flip the spin, but with the constraints of energy there is not enough energy to do so. With the electrons restricted to the xy-plane, we write $\vec{x} = (x, y)$ and the magnetic field is perpendicular to this plane, with $\nabla \times \vec{A} = B\hat{k}$. We will find the spectrum and the eigenfunctions of the Hamiltonian

$$H = \frac{1}{2m}(\vec{p} + e\vec{A})^2 = \frac{1}{2m}\vec{\pi}^2 \tag{7}$$

where $\vec{\pi} = \vec{p} + e\vec{A} = m\dot{\vec{x}}$ is the mechanical momentum and obeys $[\pi_x, \pi_y] = -ie\hbar B$. Introducing the raising and lowering operators:

$$a = \frac{1}{\sqrt{2e\hbar B}}(\pi_x - i\pi_y) \tag{8}$$

$$a^\dagger = \frac{1}{\sqrt{2e\hbar B}}(\pi_x + i\pi_y) \tag{9}$$

that obey $[a, a^\dagger] = 1$, we now can rewrite the Hamiltonian as

$$H = \hbar\omega_B(aa^\dagger + \frac{1}{2}) \tag{10}$$

where $\omega_B = \frac{eB}{m}$ is the cyclotron frequency. Proceding in the same way as in an harmonic oscillator, we first introduce a ground state $|0\rangle$ obeying $a|0\rangle = 0$ and build the rest of the states with the operators a and $a^\dagger$:

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{11}$$

$$a|n\rangle = \sqrt{n}|n-1\rangle \tag{12}$$

The energy of the state $|n\rangle$ is

$$E_n = \hbar\omega_B(n + \frac{1}{2}) \qquad n \in \mathbb{N} \tag{13}$$

The energy levels of an electron become equally spaced, with a gap between levels of $\Delta = \hbar\omega_B = \frac{e\hbar B}{m}$. In order to find the corresponding wave fuctions, we introduce the Landau gauge, $\vec{A} = xB\hat{j}$, and the Hamiltonian becomes

$$H = \frac{1}{2m}\left(p_x^2 + (p_y + eBx)^2\right)\Big) \tag{14}$$

Because of the translational invariance in the y direction, we can use the separation of variables $\Psi_k(x, y) = e^{iky} f_k(x)$. Replacing it in the corresponding Schrödinger equation we find the following wave functions, which depend on two quantum numbers, $n \in \mathbb{N}$ and $k \in \mathbb{R}$

$$\Psi_{n,k}(x, y) \sim e^{iky} H_n(x + kl_B^2) exp\left(-\frac{(x + kl_B^2)^2}{2l_B^2}\right) \tag{15}$$

where H_n are the Hermite polynomials. These wavefunctions look like strips, extended in the y-axis but exponentially localised around $x = -kl_B^2$ in the x-axis. The Landau levels are labelled by $n \in \mathbb{N}$ and are independent of the value k.

Note that if we take a rectangle region with lengths L_x and L_y, the degeneracy of the Landau levels is given by

$$N = \frac{L_x L_y}{2\pi l_B^2} = \frac{L_x L_y eB}{2\pi\hbar} = \frac{L_x L_y B}{\phi_0} \tag{16}$$

where $\phi = \frac{2\pi\hbar}{e}$ is the quantum of flux. This degeneracy is very large and it will be responsible for much of the physics of the fractional quantum Hall effect.

If we turn on an electric field $\boldsymbol{E}$, we need to add the term $-eEx$ to the Hamiltonian, and the energies are now given by

$$E_{n,k} = \hbar\omega_B(n+\frac{1}{2}) + eE\left(kl_B^2 - \frac{eE}{m\omega_B^2}\right) + \frac{m}{2}\frac{E^2}{B^2} \quad (17)$$

So the Landau levels have now a linear dependence on k, and the degeneracy in each level has lifted.

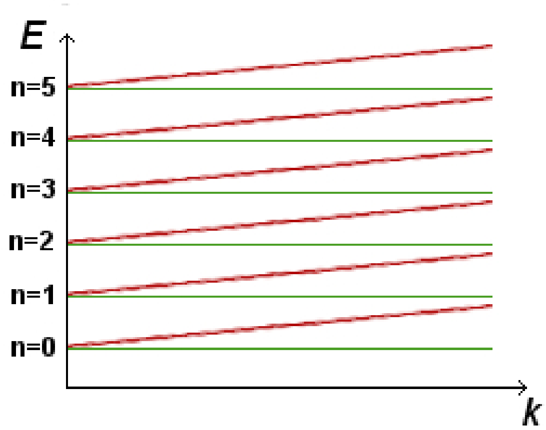

FIG. 1: Landau levels in absence (green) and in presence (red) of an electric field.

A. Introducing the concept of quantum dots [5]

We can think of the quantum dot as a 0-dimensional structure, created from a quantum well which has been etched first to leave wires and then etched again to form boxes. Let us consider the electrons motion restricted to a two-dimensional quantum dot, instead of the system defined above.We will neglect the Coulomb interaction between electrons and take into account the spin. If we define

$$\Omega^2 \equiv \omega_0^2 + \frac{1}{4}\omega_B^2 \quad (18)$$

that implements the confinement and the cyclotron frequency, we can write the Hamiltonian of the system as

$$H_0 = \sum_i \left\{ \frac{\vec{p}^2}{2m} + \frac{1}{2}\omega_B l_z + \frac{1}{2}m^*\Omega^2 r^2 + g^*\mu_B B\sigma_z \right\}_i \quad (19)$$

where $m = m^* m_e$ and $l_z = -i\hbar\delta/\delta\theta$ is the angular momentum relative to the z-axis. In this case, instead of taking the Landau gauge, we have used the symmetric gauge

$$\vec{A} = -\frac{1}{2}\vec{r}\times\vec{B} = -\frac{yB}{2}\hat{x} + \frac{xB}{2}\hat{y} \quad (20)$$

Since $[H_0, L_z] = [H_0, S_z] = 0$, we can write the wave functions as $\phi(\boldsymbol{r},\sigma) = e^{-il\theta}u(r)\chi_\sigma$. The solutions to the corresponding Schödinger equation were determined by Fock and Darwin. The energy levels are

$$E_{nl\sigma} = \hbar\Omega(2n+|l|+1) - \frac{1}{2}\hbar\omega_B l + \frac{1}{2}g^*\mu_B B\sigma \quad (21)$$

where $\sigma = \pm 1$, $n = 0, 1, 2...$ and $l = 0, \pm 1, \pm 2, ...$

- **The $\omega_0 \gg \omega_B$ limit case:** The energy becomes $E_{nl\sigma} = \hbar\omega_0(N+1)$ with $N \equiv 2n + |l| = 0, 1, 2, ...$ For a given N, n can take values from zero to the integer part of $N/2$ and $|l|$ can take values up to N with the same parity of N. These correspond to the levels of a two-dimensional harmonic oscillator The single-particle levels are arranged in straight lines as a function of $|l|$.

- **The $\omega_B \gg \omega_0$ limit case:** The energy becomes $E_{nl\sigma} = \hbar\omega_B(M+\frac{1}{2}) + \frac{1}{2}g^*\mu_B B\sigma$ with $M \equiv n + 1/2 + (|l|-l)/2$ the labels of the Landau levels. For each of the two spins, the first Landau level $M = 0$ corresponds to $n = 0$ and nonnegative values of l. For the second level $M = 1$, we have two options: $n = 1$ and nonegative values of l or $n = 0$ and $l = -1$, and so on.

So one can see the evolution of the harmonic oscillator spectrum for weak magnetic fields towards the Landau spectrum as we increase B.

V. INTEGER QUANTUM HALL EFFECT [4], [6]

In 1980 Klaus von Klitzing realized the first experiment exploring the quantum region of the Hall effect, with the collaboration of Dorda and Pepper. The material used in this experiment was a SiMOSFET, which consist in a metal-insulator-semiconductor heterojunction with the electrons trapped in the inversion band between the insulator and the semiconductor. He won the Nobel prize in 1985 for the discovery of the integer quantum Hall effect.

At low temperatures and values for the magnetic field above $1T$, one can observe that the Hall resistivity ρ_{xy} presents some plateaux for different ranges of magnetic field, jumping suddenly from one plateau to another. On each of these plateaux, the Hall resistivity takes the values

$$\rho_{xy} = \frac{2\pi\hbar}{e^2}\frac{1}{\nu} \qquad \nu \in \mathbb{Z} \quad (22)$$

The centre of each plateau occurs when the magnetic field takes the value

$$B = \frac{2\pi\hbar\rho}{\nu e} = \frac{\rho}{\nu}\phi_0 \quad (23)$$

where ρ is the electron density. Note that the integer ν that labels the plateaux corresponds to the integer $n+1$ that labels the Landau levels.

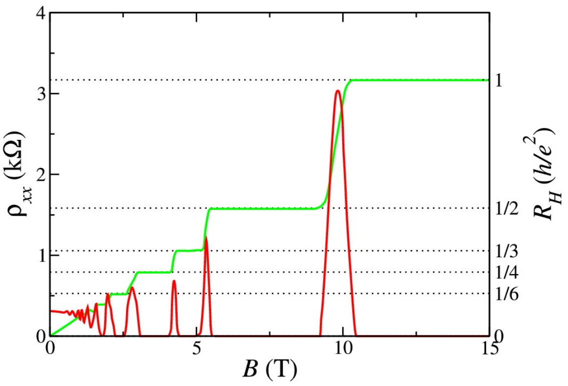

FIG. 2: The Integer Quantum Hall Effect. The green curve corresponds to the Hall resistivity ρ_{xy} and the red one to the longitudinal resistivity ρ_{xx}. Source: [6]

On the other hand, the longitudinal resisitivity ρ_{xx} is minimum while the Hall resisitivity sits on a plateau and it spikes when ρ_{xy} jumps.

We can explain the existence of these plateaux considering the quantum motion of the electrons. When ν Landau levels are filled, the many-electron wavefunctions generate very stable states against changes in the magnetic field. Then the Hall resistivity remains constant and gives room to the ν^{th} plateau meanwhile the longitudinal resistivity becomes small due to the large energy gap for exciting the electrons. This gap is caused by the presence of the closed shells that are formed when the magnetic field takes the values (23).

A. The Maximum Density Droplet (MDD) state [5]

We assume that the value of B is high enough and only the first Landau level $M = 0$ is occupied. Then $n = \frac{1}{2}(l - |l|)$ and (21) becomes

$$E_l = \frac{1}{2}\hbar\omega_B + \hbar\frac{\omega_0^2}{\omega_B}(l+1) + \frac{1}{2}g^*\mu_B B \tag{24}$$

The energy eigenstates of (19) are $E = \sum_l^{occ} E_l = C + \hbar\frac{\omega_0^2}{\omega_B}L$ where L is the eigenvalue with opposite sign of the total angular momentum operator. The ground state will be the one with the lowest value of L, L_0.

The Coulomb interaction would change all the scheme. If B is high enough to disregard the Coulomb effects with other Landau levels and we only take into account the first Landau level, the true ground state is of the form of the Slater determinant

$$\Psi_0 = \frac{1}{\sqrt{N!}}\begin{vmatrix} \phi_0(\vec{r_1}) & \dots & \phi_{N-1}(\vec{r_1}) \\ \vdots & \vdots & \vdots \\ \phi_0(\vec{r_N}) & \dots & \phi_{N-1}(\vec{r_N}) \end{vmatrix} \tag{25}$$

which is called the maximum density droplet state.

In the limit of infinite N, we can define the filling factor of Landau levels as

$$\nu = \rho\frac{2\pi\hbar}{eB} = \rho\frac{\phi_0}{B} \tag{26}$$

Its value is 1 when all the electrons are in the first Landau level. Decreasing B, the gap between the Landau levels will decrease as well and then some electrons will be placed on the second Landau level, giving $\nu = 2$ until it is completely filled, and so on.

B. The Presence of Disorder [4]

The samples used in the experiments usually contain impurities that can be modelled by adding some random potential to the Hamiltonian. The presence of this disorder will split the degenerate eigenstates, so it will change the energy spectrum of the Landau levels. Furthermore, the disorder turns many of the quantum states from extended to localised. Only the extended states can transport charge so they will be the only ones who contribute to the conductivity.

Suppose that all the extended states in a given Landau level are been filled. If we fix n and decrease B, the Fermi energy will increase so we will begin to populate the localised states, which do not contribute to the conductivity. This leads to the kind of plateaux observed, with constant conductivity -so with constant resistivity- over a range of B. So the presence of disorder explains the plateaux.

VI. FRACTIONAL QUANTUM HALL EFFECT [4], [5]

If the magnetic field is high enough, the filling factor becomes smaller than 1 and then we enter in the fractional regime, for which the ground state is no longer the MDD state. In 1982, some plateaux at non-integer filling factors were seen for the first time. For example, at the ground state there were plateaux at $\nu = \frac{1}{3}, \frac{2}{3}, \frac{1}{5}, \frac{2}{5}, \frac{3}{7}, \ldots$. In this fractional regime, the interactions between electrons plays an important role.

Using the Slater determinant in the Vandermonde way as reference, Robert B. Laughlin suggested the following wave function for the ground state at $\nu = 1/m$

$$\Psi_m(z_1, ..., z_N) = \prod_{i<j=1}^{N}(z_i - z_j)^m exp\left(-\sum_{i=1}^{N}\frac{|z_i|^2}{4}\right) \tag{27}$$

where $z_i = \frac{x_i - iy_i}{l_B}$. It can be seen numerically that this wave function has greater than 99% overlap with the true ground state, at least for small numbers of particles N. In these states the energy per electron is lower than in any other proposed state. Writing $|\Psi|^2 = e^{-\beta\Phi}$ with

$\beta = 1/m$ playing the role of an auxiliary dimensionless temperature, we find

$$\Phi(z_i) = -2m^2 \sum_{i<j=1}^{N} ln(z_i - z_j) + \frac{1}{2} m \sum_{i=1}^{N} |q_i|^2 \tag{28}$$

This is the potential energy for a two-dimensional plasma of charged particles, each one with charge $q = \sqrt{2m}$ and m and odd integer in order to preserve the Fermi statisctics. The density of the state at $\nu = 1/m$ is given by

$$\rho = \frac{1}{2\pi m} \tag{29}$$

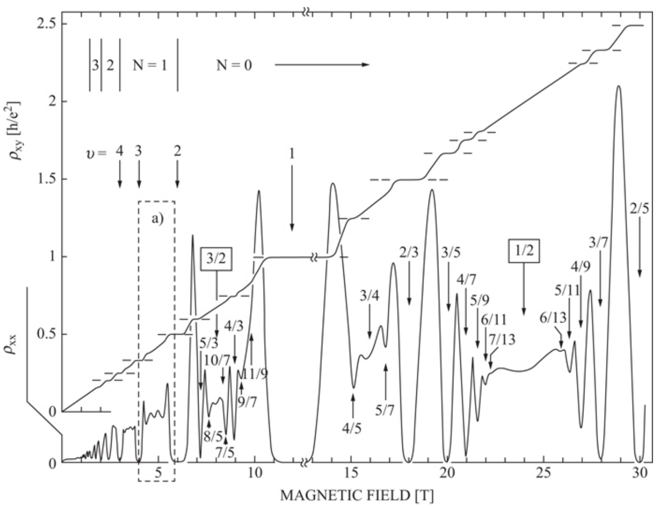

FIG. 3: The Fractional Quantum Hall Effect. Source: [4]

The Laughlin state should be thought of as an enterelly new phase of matter, but the classical analogy would be a liquid of states.

There is a competing solid phase in which the electrons form a two-dimensional triangular lattice, known as Wigner crystal. For low densities of electrons, this crystal has a lower energy than the Laughlin state.

VII. CONCLUSIONS

- One of the particularities of the integer quantum Hall effect is that it is best seen in a low purity material, with a rather high level of disorder and where very few electronic states extend across the whole system.

- The studies of the quantum Hall effect reveal surprising results that are of particular interest to electrical metrology and condensed matter physics disciplines [1]. The independence of the plateaux from the material and from the geometry of the device makes the quantum Hall effect a perfect candidate to be used as an absolute resistance standard that only depends on fundamental constants.

- Increasing the magnetic field and taking into account the Coulomb interaction, our integer quantum Hall effect becomes a more complex phenomenon, the fractional quantum Hall effect. This needs the introduction of some new conceps such as the quasi-particles and the quasi-holes, which have a fraction of the electron charge, or such as anyons, a type of quasi-particle with properties much less restricted than fermions and bosons. Furthermore, a thorough study of this effect would require of some topology, but that goes beyond the aim of this paper.

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