

# LORENTZ-SHIMOYAKI AND BOYD THEOREMS FOR WEIGHTED LORENTZ SPACES

ELONA AGORA, JORGE ANTEZANA, MARÍA J. CARRO, AND JAVIER SORIA

ABSTRACT. We prove the Lorentz-Shimogaki and Boyd theorems for the spaces  $\Lambda_u^p(w)$ . As a consequence, we give the complete characterization of the strong boundedness of  $H$  on these spaces in terms of some geometric conditions on the weights  $u$  and  $w$ , whenever  $p > 1$ . For these values of  $p$ , we also give the complete solution of the weak-type boundedness of the Hardy-Littlewood operator on  $\Lambda_u^p(w)$ .

## 1. INTRODUCTION AND MOTIVATION

Given a rearrangement invariant (r.i.) Banach function space  $X$  on  $\mathbb{R}$ , the Lorentz-Shimogaki theorem ([13], [18] see also [5, p. 154]) asserts that

$$M : X \longrightarrow X \text{ is bounded} \quad \iff \quad \alpha_X < 1,$$

where  $M$  is the classical Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f(y)| dy,$$

(the supremum is taken over all intervals  $I$  containing  $x \in \mathbb{R}$ ) and  $\alpha_X$  is the upper Boyd index defined ([6] see also [5, p. 149]) as follows:

$$\alpha_X := \lim_{t \rightarrow \infty} \frac{\log \|D_t\|_X}{\log t},$$

with

$$\|D_t\|_X = \sup_{\|f\|_X \leq 1} \|D_t f\|_X,$$

the norm of the dilation operator  $D_t f(s) = f(s/t)$ .

Similarly, the classical Boyd theorem shows [5, p. 154] that

$$H : X \longrightarrow X \quad \text{is bounded} \quad \iff \quad \alpha_X < 1 \quad \text{and} \quad \beta_X > 0,$$

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where  $H$  is the Hilbert transform

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy,$$

whenever this limit exists almost everywhere and  $\beta_X$  is the lower Boyd index defined by

$$\beta_X := \lim_{t \rightarrow 0^+} \frac{\log \|D_t\|_X}{\log t}.$$

In [14] the Lorentz-Shimogaki and Boyd theorems were extended to the case of r.i. quasi-Banach spaces.

In a recent paper [10], the upper Boyd index for a general quasi-Banach function space  $X$ , not necessarily r.i., was defined using the so-called local maximal operator. With such definition the classical Lorentz-Shimogaki theorem was extended to this more general class of spaces.

This paper is a continuation of the work initiated in [10] for a concrete class of quasi-Banach spaces, namely, for weighted Lorentz spaces  $\Lambda_u^p(w)$  defined by (see [11], [12])

$$\Lambda_u^p(w) = \left\{ f \in \mathcal{M}(\mathbb{R}) : \|f\|_{\Lambda_u^p(w)} = \left( \int_0^\infty (f_u^*(t))^p w(t) dt \right)^{1/p} < \infty \right\}.$$

Here,  $\mathcal{M}(\mathbb{R})$  is the class of Lebesgue measurable functions on  $\mathbb{R}$  (we work in dimension one since we shall be concerned with the Hilbert transform),  $u$  is a positive and locally integrable function on  $\mathbb{R}$  (we call it weight),  $w$  will also be a weight but defined in  $(0, \infty)$ ,  $f_u^*$  is the decreasing rearrangement of  $f$  with respect to the weight  $u$  (see [5]),

$$f_u^*(t) = \inf \{ s > 0 : u(\{x \in \mathbb{R} : |f(x)| > s\}) \leq t \},$$

with  $u(E) = \int_E u(x) dx$  and  $0 < p < \infty$ . We would like to mention that these spaces include as particular cases the weighted Lebesgue spaces  $L^p(u)$  (with  $w = 1$ ), the classical Lorentz spaces  $\Lambda^p(w)$  (with  $u = 1$ ), and the Lorentz spaces  $L^{q,p}(u)$  (with  $w(t) = t^{p/q-1}$ ). We shall also need to work with the weak-type space

$$\Lambda_u^{p,\infty}(w) = \left\{ f \in \mathcal{M}(\mathbb{R}) : \|f\|_{\Lambda_u^{p,\infty}(w)} = \sup_{t>0} f_u^*(t) W^{1/p}(t) < \infty \right\},$$

where  $W(t) = \int_0^t w(s) ds$ .

As usual, we shall use the symbol  $A \lesssim B$  to indicate that there exists a universal constant  $C$ , independent of all important parameters, such that  $A \leq CB$ .  $A \approx B$  will indicate that  $A \lesssim B$  and  $B \lesssim A$ . If  $E$  is a measurable set and  $u = 1$ , we

write  $u(E) = |E|$ . We also recall that a weight  $u$  is in the Muckenhoupt class  $A_1$  if  $Mu(x) \lesssim u(x)$ , at almost every point  $x \in \mathbb{R}$ . For other definitions (like the  $A_\infty$  class) and further properties about Muckenhoupt weights we refer to the book [9].

It is known that the space  $\Lambda_u^p(w)$  is a quasi-normed space if and only if  $w \in \Delta_2$  [7]; that is,

$$W(2r) \lesssim W(r).$$

This condition will be assumed all over the paper.

Concerning the upper Boyd index for these spaces, it was proved in [10] that

$$(1.1) \quad \alpha_{\Lambda_u^p(w)} = \lim_{t \rightarrow \infty} \frac{\log \overline{W}_u^{1/p}(t)}{\log t},$$

where, for every  $t > 1$ ,

$$\overline{W}_u(t) := \sup \left\{ \frac{W \left( u \left( \bigcup_j I_j \right) \right)}{W \left( u \left( \bigcup_j S_j \right) \right)} : S_j \subseteq I_j \text{ and } |I_j| < t|S_j|, \text{ for every } j \right\},$$

with  $I_j$  disjoint intervals,  $S_j$  measurable subsets, and all unions are finite. To see (1.1), the following result was used:

**Theorem 1.1.** [8] *If  $0 < p < \infty$ ,*

$$M : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w)$$

*is bounded if and only if there exists  $q \in (0, p)$  such that, for every finite family of disjoint intervals  $(I_j)_{j=1}^J$ , and every family of measurable sets  $(S_j)_{j=1}^J$ , with  $S_j \subset I_j$ , for every  $j$ , we have that*

$$(1.2) \quad \frac{W \left( u \left( \bigcup_{j=1}^J I_j \right) \right)}{W \left( u \left( \bigcup_{j=1}^J S_j \right) \right)} \lesssim \max_{1 \leq j \leq J} \left( \frac{|I_j|}{|S_j|} \right)^q.$$

**Remark 1.2.** For later purposes, it is important to mention that, by regularity and continuity,

$$\overline{W}_u(t) := \sup \left\{ \frac{W \left( u \left( \bigcup_j I_j \right) \right)}{W \left( u \left( \bigcup_j S_j \right) \right)} : S_j \subseteq I_j \text{ and } |I_j| = t|S_j|, \text{ for every } j \right\},$$

where, for every  $j$ ,  $S_j$  is a finite union of intervals.

**Remark 1.3.**

(i) We observe that (1.2) is equivalent to saying that there exists  $q \in (0, p)$  such that, for every  $t > 1$ ,

$$\overline{W}_u(t) \lesssim t^q.$$

(ii) It was also proved in [8] that, if  $0 < p < \infty$  and

$$(1.3) \quad M : \Lambda_u^p(w) \longrightarrow \Lambda_u^{p,\infty}(w)$$

is bounded, then  $\overline{W}_u(t) \lesssim t^p$ . Moreover, if  $0 < p < 1$ , this condition is sufficient for (1.3), although this is not the case for other values of  $p$ . In this paper we shall also give a characterization, in the case  $p > 1$ , of the weights  $u$  and  $w$  for which

$$M : \Lambda_u^p(w) \longrightarrow \Lambda_u^{p,\infty}(w)$$

is bounded solving an open problem left in [8], (see Theorem 3.2.11).

We now describe the main goals of this work:

(i) We give a new proof of the Lorentz-Shimogaki theorem for weighted Lorentz spaces, without using the local maximal operator (we shall define the upper Boyd index by (1.1)).

(ii) We study whether the corresponding generalization of the classical Boyd theorem for the Hilbert transform:

$$H : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w) \text{ is bounded} \quad \iff \quad \beta_{\Lambda_u^p(w)} > 0 \text{ and } \alpha_{\Lambda_u^p(w)} < 1$$

holds true, where the generalized lower Boyd index  $\beta_{\Lambda_u^p(w)}$  will be defined later on.

Concerning (ii), we shall prove that this is the case if  $p > 1$  and, as a consequence, we shall give the complete characterization of the boundedness

$$H : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w)$$

in the case  $p > 1$  in terms of geometric conditions on the weights  $u$  and  $w$ .

Finally, we shall show that, for every  $p > 0$ ,

$$\beta_{\Lambda_u^p(w)} > 0 \text{ and } \alpha_{\Lambda_u^p(w)} < 1 \implies H : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w) \implies \beta_{\Lambda_u^p(w)} > 0.$$

2. THE CASE  $u = 1$ :  $\Lambda^p(w)$ 

Let us start by analyzing the case  $u = 1$  since it was the starting point for our results. Note that the space  $\Lambda^p(w)$  is, in fact, a rearrangement invariant function space. In particular, a simple computation of  $\|D_t\|_{\Lambda^p(w)}$  gives us the following result.

**Proposition 2.1.** [6, 14] *For every  $0 < p < \infty$ ,*

$$\alpha_{\Lambda^p(w)} := \lim_{t \rightarrow \infty} \frac{\log \overline{W}^{1/p}(t)}{\log t},$$

and

$$\beta_{\Lambda^p(w)} := \lim_{t \rightarrow 0^+} \frac{\log \overline{W}^{1/p}(t)}{\log t},$$

where

$$\overline{W}(t) := \sup_{s \in (0, +\infty)} \frac{W(st)}{W(s)}.$$

Then, the Lorentz-Shimogaki theorem [14] applied to  $\Lambda^p(w)$  says that

$$M : \Lambda^p(w) \longrightarrow \Lambda^p(w) \quad \text{is bounded} \quad \iff \quad \lim_{t \rightarrow \infty} \frac{\log \overline{W}^{1/p}(t)}{\log t} < 1.$$

On the other hand, we recall the following result of Ariño and Muckenhoupt [3]:

**Theorem 2.2.** *For every  $0 < p < \infty$ ,*

$$M : \Lambda^p(w) \longrightarrow \Lambda^p(w) \quad \text{is bounded} \quad \iff \quad w \in B_p;$$

that is, for every  $r > 0$ ,

$$r^p \int_r^\infty \frac{w(t)}{t^p} dt \lesssim \int_0^r w(s) ds.$$

Consequently, we have the following corollary:

**Corollary 2.3.** *For every  $0 < p < \infty$ ,*

$$\lim_{t \rightarrow \infty} \frac{\log \overline{W}^{1/p}(t)}{\log t} < 1 \quad \text{if and only if} \quad w \in B_p.$$

We shall give a direct proof of this result using the following lemma about submultiplicative functions. Observe that  $\overline{W}$  is submultiplicative; that is, for every  $t, s > 0$ ,

$$\overline{W}(ts) \leq \overline{W}(t)\overline{W}(s).$$

**Lemma 2.4.** *For every submultiplicative increasing function  $\varphi$  defined in  $[1, \infty)$ ,*

$$\lim_{t \rightarrow \infty} \frac{\log \varphi(t)}{\log t} < 1,$$

*if and only if there exists  $\gamma < 1$  such that  $\varphi(x) \lesssim x^\gamma$ , for every  $x > 1$ .*

*Proof.* By hypothesis, there exists  $t_0 > 1$  such that  $\varphi(t_0) < t_0$ . Now, given  $x > 1$ , there exists  $k \in \mathbb{N}$  such that  $x \in (t_0^k, t_0^{k+1})$  and hence, since  $\varphi$  is increasing and submultiplicative,

$$\varphi(x) \leq \varphi(t_0^{k+1}) \leq \varphi(t_0)^{k+1} \leq t_0 \left( \frac{\varphi(t_0)}{t_0} \right)^{k+1} x.$$

Using that  $c = \frac{\varphi(t_0)}{t_0} < 1$ , we have that  $c^{k+1} \leq c^{\frac{\log x}{\log t_0}} = x^{\frac{\log c}{\log t_0}}$  with  $\log c < 0$ . Hence,

$$\varphi(x) \leq t_0 x^{1 + \frac{\log c}{\log t_0}} \approx x^\gamma,$$

with  $\gamma < 1$ . Conversely, if  $\varphi(t) \leq Ct^\gamma$ , for every  $t > 1$ ,

$$\lim_{t \rightarrow \infty} \frac{\log \varphi(t)}{\log t} \leq \lim_{t \rightarrow \infty} \frac{\log(Ct^\gamma)}{\log t} = \gamma < 1.$$

□

*Proof of Corollary 2.3.* It is enough to apply Lemma 2.4 to the function  $\overline{W}^{1/p}$  and recall that [3]:

$$w \in B_p \quad \Longleftrightarrow \quad \overline{W}(t) \lesssim t^q, \text{ for some } q < p \text{ and every } t > 1.$$

□

Similarly, the Boyd theorem applied to  $\Lambda^p(w)$  says that

$$H : \Lambda^p(w) \longrightarrow \Lambda^p(w) \text{ is bounded} \quad \Longleftrightarrow \quad \lim_{t \rightarrow \infty} \frac{\log \overline{W}^{1/p}(t)}{\log t} < 1 \quad \text{and} \quad \lim_{t \rightarrow 0^+} \frac{\log \overline{W}^{1/p}(t)}{\log t} > 0.$$

On the other hand, we now have the following result [17, 16]:

**Theorem 2.5.** *For every  $0 < p < \infty$ ,*

$$H : \Lambda^p(w) \longrightarrow \Lambda^p(w) \text{ is bounded} \quad \Longleftrightarrow \quad w \in B_p \cap B_\infty^*,$$

where the  $B_\infty^*$  class is defined by the following condition: for every  $r > 0$ ,

$$\int_0^r \frac{1}{t} \int_0^t w(s) ds dt \lesssim \int_0^r w(s) ds.$$

In order to describe the conditions of Theorem 2.5 in terms of  $\overline{W}$ , and in view of Corollary 2.3, it suffices to prove the following result.

**Proposition 2.6.**  *$w \in B_\infty^*$  if and only if*

$$(2.1) \quad \lim_{t \rightarrow 0^+} \frac{\log \overline{W}^{1/p}(t)}{\log t} > 0.$$

The proof of this result is based on the following lemma:

**Lemma 2.7.** *If  $\varphi : (0, 1] \rightarrow [0, 1]$  is an increasing submultiplicative function, then the following statements are equivalent:*

- (i)  $\varphi(\lambda) < 1$  for some  $\lambda \in (0, 1)$ ,
- (ii)  $\varphi(x) \lesssim \frac{1}{1 + \log(1/x)}$ ,
- (iii)  $\lim_{t \rightarrow 0^+} \frac{\log \varphi(t)}{\log t} > 0$ .

*Proof.* Clearly (ii) implies (i) and (iii) implies (i) as well.

(i)  $\Rightarrow$  (ii) Since  $0 < \lambda < 1$ , given  $x \in (0, 1)$ , there exists  $k \in \mathbb{N}$  such that  $x \in (\lambda^{k+1}, \lambda^k)$  and hence, since  $\varphi(\lambda) < 1$ , we have that

$$A = \sup_{k \in \mathbb{N}} \varphi(\lambda)^k (1 + (k+1) \log(1/\lambda)) < \infty.$$

Therefore,

$$\varphi(x) \leq \varphi(\lambda^k) \leq \varphi(\lambda)^k \leq \frac{A}{1 + (k+1) \log(1/\lambda)} \lesssim \frac{1}{1 + \log(1/x)},$$

as we wanted to see.

(i)  $\Rightarrow$  (iii) If  $x \in (\lambda^{k+1}, \lambda^k)$ , we have that  $\log \varphi(x) \leq k \log \varphi(\lambda)$ , and since  $(k+1) \log \lambda \leq \log x$ , we get that

$$\frac{\log \varphi(x)}{\log x} \geq \frac{k}{k+1} \frac{\log \varphi(\lambda)}{\log \lambda} \geq \frac{\log \varphi(\lambda)}{2 \log \lambda},$$

from which the result follows. □

*Proof of Proposition 2.6.* If  $w \in B_\infty^*$ , for every  $s \leq r$ ,

$$W(s) \log \frac{r}{s} \leq \int_s^r \frac{W(t)}{t} dt \lesssim W(r),$$

and since  $W$  is increasing, we deduce that  $W(s)(1 + \log \frac{r}{s}) \lesssim W(r)$ , which implies that

$$\overline{W}(y) \lesssim \frac{1}{1 + \log \frac{1}{y}},$$

for every  $0 < y \leq 1$ . Thus,  $\overline{W}^{1/p}$  satisfies the hypothesis of Lemma 2.7 and (2.1) follows.

Conversely, if (2.1) holds, and we write  $c = \lim_{t \rightarrow 0} \frac{\log \overline{W}^{1/p}(t)}{\log t}$ , it is easy to see that, for  $t$  small enough,  $\overline{W}^{1/p}(t) \leq t^{c/2}$ , and thus there exists  $\lambda < 1$  satisfying that  $\overline{W}^{1/2}(\lambda) < 1$ . Hence, by Lemma 2.7,

$$\int_0^r \frac{W(t)}{t} dt \lesssim W(r) \int_0^r \left(1 + \log(r/t)\right)^{-2} \frac{dt}{t} \lesssim W(r),$$

and therefore,  $w \in B_\infty^*$ . □

**Remark 2.8.** Concerning the function  $\overline{W}_u$ , we observe that, if  $u = 1$  then, for every  $t > 1$ ,

$$\overline{W}(t) = \overline{W}_u(t).$$

Indeed, it is enough to note that, given any finite family of disjoint intervals  $(I_j)_{j=1}^J$  and measurable sets  $(S_j)_{j=1}^J$ , such that  $S_j \subset I_j$  and  $|I_j| = t|S_j|$ , for every  $j$ , it holds that

$$W\left(\left|\bigcup_j S_j\right|\right) = W\left(t\left|\bigcup_j I_j\right|\right).$$

Since  $|\bigcup_j I_j|$  can be any positive real number, by Remark 1.2 and the definition of  $\overline{W}(t)$ , it follows that  $\overline{W}_u(t)$  and  $\overline{W}(t)$  have to coincide.

### 3. THE LORENTZ-SHIMOGAKI THEOREM FOR $\Lambda_u^p(w)$

As mentioned in the introduction, it was proved in [10] that

$$(3.1) \quad \alpha_{\Lambda_u^p(w)} = \lim_{t \rightarrow \infty} \frac{\log \overline{W}_u^{1/p}(t)}{\log t}.$$

To justify the existence of the limit, the authors show that  $\overline{W}_u$  is pointwise equivalent to a submultiplicative function involving the local maximal function. In the following proposition, we will prove that the function  $\overline{W}_u$  is in fact submultiplicative, which gives a direct proof of this result. With this aim, we need the following technical lemma.



**Lemma 3.1.** *Let  $I$  be an interval and let  $S = \cup_{k=1}^N (a_k, b_k)$  be the union of disjoint intervals such that  $S \subset I$ . Then, for every  $t \in [1, |I|/|S|]$  there exists a collection of disjoint subintervals  $\{I_n\}_{n=1}^M$  satisfying that  $S \subset \cup_n I_n$  and such that, for every  $n$ ,*

$$(3.2) \quad t|S \cap I_n| = |I_n|.$$

*Proof.* Without loss of generality we can assume that  $I = (0, |I|)$  and also that  $a_1 < a_2 < \dots < a_N$ . First observe that if  $J = \cup_n I_n$  we should in particular obtain  $t|S| = |J|$  applying (3.2). We use induction in  $N$ . Clearly it is true for  $N = 1$ . Indeed, it suffices to consider  $0 \leq c \leq a_1 < b_1 \leq d \leq |I|$  such that  $t(b_1 - a_1) = d - c$ . Suppose that the results holds for all  $k < N$ . We will prove that it also holds for  $k = N$ .

Case I: If  $|I| - t|S| \leq a_1$ , then it suffices to consider  $I_1 = (|I| - t|S|, |I|) = J$ .  
Case II: If  $a_1 < |I| - t|S|$ , set  $\bar{I} = (a_1, |I|)$ . Observe that  $t|S| < |\bar{I}|$  and  $S \subset \bar{I}$ . Hence in this case we could assume, without loss of generality, that  $a_1 = 0$ . Let now  $I_1 = (0, c)$  such that  $b_1 \leq c \leq |I|$  and  $t|S \cap I_1| = c = |I_1|$ . Note that  $c \notin S$ . In fact, suppose that there exists  $S_m = (a_m, b_m)$  such that  $c \in S_m$ . Then,  $t|S \cap [0, a_m]| > |[0, a_m]|$  which implies that  $t|S \cap [0, c]| > |[0, c]| = |I_1|$ , which is a contradiction. Therefore, we obtain that

$$t|S \cap I_1| + t|S \cap [c, |I|)| = t|S| < |I| = |I_1| + |[c, |I|)| = t|S \cap I_1| + |[c, |I|)|,$$

and consequently  $t|S \cap [c, |I|)| < |[c, |I|)|$ . Then, since  $[c, |I|)$  is the union of at most  $N - 1$  intervals  $\{(a_k, b_k)\}_k$ , we apply the inductive hypothesis to the intervals  $[c, |I|)$  and the set  $S \cap [c, |I|)$  and we obtain the intervals  $I_2, \dots, I_M$  such that (3.2) is satisfied. □

**Lemma 3.2.** *The function  $\overline{W}_u$  is submultiplicative on  $[1, \infty)$ .*

*Proof.* Consider a finite family of intervals  $I_j$ , and measurable sets  $S_j \subseteq I_j$  which are finite union of intervals such that  $|I_j| = \lambda\mu|S_j|$ . Then, we can apply Lemma 3.1 and for each  $j$  obtain a set  $J_j$  such that it is a union of a finite number of pairwise disjoint intervals, that we call  $J_{ji}$ :

$$S_j \subseteq J_j, \quad \lambda|S_j \cap J_{ji}| = |J_{ji}|, \quad J_j \subseteq I_j, \quad \text{and} \quad \mu|J_j| = |I_j|.$$

So, we have that

$$\frac{W\left(u\left(\bigcup_j I_j\right)\right)}{W\left(u\left(\bigcup_j S_j\right)\right)} = \frac{W\left(u\left(\bigcup_j I_j\right)\right)}{W\left(u\left(\bigcup_j S_j\right)\right)} \frac{W\left(u\left(\bigcup_j J_j\right)\right)}{W\left(u\left(\bigcup_j J_j\right)\right)} \leq \overline{W}_u(\lambda)\overline{W}_u(\mu).$$

Therefore, taking supremum over all possible choices of intervals  $I_j$  and measurable subsets  $S_j$  such that  $S_j \subseteq I_j$  and  $|I_j| = \lambda\mu|S_j|$ , we get that  $\overline{W}_u(\lambda\mu) \leq \overline{W}_u(\lambda)\overline{W}_u(\mu)$ .  $\square$

**Definition 3.3.** Let us define the upper Boyd index for the space  $\Lambda_u^p(w)$  as

$$\alpha_{\Lambda_u^p(w)} = \lim_{t \rightarrow \infty} \frac{\log \overline{W}_u^{1/p}(t)}{\log t}.$$

**Remark 3.4.** In [10] the generalized upper Boyd index was introduced in terms of the local maximal operator. Recall that, the local maximal operator  $m_\lambda$  of a measurable function  $f$  is defined by

$$m_\lambda f(x) = \sup_{x \in I} (f\chi_I)^*(\lambda|I|),$$

where  $\lambda \in (0, 1)$ . In terms of this operator, the upper Boyd index was defined by:

$$\lim_{\lambda \rightarrow 0^+} \frac{\log \|m_\lambda\|_{\Lambda_u^p(w)}}{\log 1/\lambda}.$$

In the original definition for r.i. spaces, the function  $t \mapsto \|D_t\|$  is submultiplicative, which justifies the existence of the limit. In the case of the local maximal operator, it is not known whether or not the function  $\lambda \mapsto \|m_\lambda\|$  is submultiplicative. In the case of weighted Lorentz spaces, it can be proved that (see [10, Lemma 5.1] and the proof of Theorem 3.2.4 in [8])

$$\overline{W}_u(1/\lambda) \leq \|m_\lambda\|_{\Lambda_u^p(w)}^p \leq \overline{W}_u(2/\lambda).$$

We can now prove the following extension of the Lorentz-Shimogaki theorem:

**Theorem 3.5.** [10] *If  $0 < p < \infty$ , then*

$$M : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w)$$

*is bounded if and only if  $\alpha_{\Lambda_u^p(w)} < 1$ .*

*Proof.* By Theorem 1.1 and Remark 1.3, the boundedness of  $M$  is equivalent to  $\overline{W}_u(t) \lesssim t^q$  for some  $q < p$ , and since  $\overline{W}_u$  is increasing and submultiplicative, the result follows from Lemma 2.4.  $\square$

4. THE BOYD THEOREM FOR  $\Lambda_u^p(w)$ 

In what follows

$$H : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w)$$

will indicate that, for every  $f \in \Lambda_u^p(w)$ ,  $Hf(x)$  is well defined at almost every point  $x \in \mathbb{R}$ , and

$$\|Hf\|_{\Lambda_u^p(w)} \lesssim \|f\|_{\Lambda_u^p(w)},$$

and similarly for  $H : \Lambda_u^p(w) \longrightarrow \Lambda_u^{p,\infty}(w)$ .

Our next goal is to introduce the definition of the lower Boyd index for  $\Lambda_u^p(w)$  and prove the corresponding Boyd theorem.

**Definition 4.1.** If  $t \in (0, 1]$ , we define

$$\underline{W}_u(t) := \sup \left\{ \frac{W\left(u\left(\bigcup_j S_j\right)\right)}{W\left(u\left(\bigcup_j I_j\right)\right)} : S_j \subseteq I_j \text{ and } |S_j| < t|I_j|, \text{ for every } j \right\},$$

where  $I_j$  are disjoint intervals and all unions are finite.

As in Remark 1.2 we can substitute  $|S_j| < t|I_j|$  by an equality  $|S_j| = t|I_j|$  and assume that  $S_j$  is the finite union of intervals.

With this definition, we can prove the following result (similar to Lemma 3.2).

**Proposition 4.2.** *The function  $\underline{W}_u$  is submultiplicative in  $[0, 1]$ .*

**Lemma 4.3.** *If  $u \in A_\infty$ , there exist  $C_u > 0$  and  $\alpha > 0$  such that, for every  $0 < t < 1$ ,*

$$\overline{W}(t) \leq \underline{W}_u(C_u t^\alpha).$$

*Proof.* It is known that if  $u \in A_\infty$ , there exist  $C_u > 0$  and  $\alpha > 0$  such that, for every interval  $I$  and every measurable set  $E \subset I$ ,

$$\frac{|E|}{|I|} \leq C_u \left( \frac{u(E)}{u(I)} \right)^\alpha.$$

Now, let  $0 < t < 1$  and let  $s > 0$ . Let  $I$  be such that  $u(I) = s$  and set  $E \subset I$  such that  $u(E) = ts$ . Then,

$$\frac{W(ts)}{W(s)} = \frac{W(u(E))}{W(u(I))} \leq \underline{W}_u\left(\frac{|E|}{|I|}\right) \leq \underline{W}_u(C_u t^\alpha),$$

and the result follows taking the supremum in  $s > 0$ .  $\square$

By analogy with the case of the upper index, we give the following definition (which agrees in the case  $u = 1$  with the classical one).

**Definition 4.4.** We define the generalized lower Boyd index associated to  $\Lambda_u^p(w)$  as

$$\beta_{\Lambda_u^p(w)} := \lim_{t \rightarrow 0^+} \frac{\log W_u^{1/p}(t)}{\log t}.$$

**Theorem 4.5.** *Let  $0 < p < \infty$ . If*

$$H : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w)$$

*is bounded, then  $\beta_{\Lambda_u^p(w)} > 0$ .*

*Proof.* It was proved in [2] that the boundedness of  $H$  on  $\Lambda_u^p(w)$  implies that  $u \in A_\infty$  and  $w \in B_\infty^*$ , and hence the result now follows from Proposition 2.6 and Lemma 4.3.  $\square$

**Theorem 4.6.** *Let  $0 < p < \infty$ . If*

$$\alpha_{\Lambda_u^p(w)} < 1 \quad \text{and} \quad \beta_{\Lambda_u^p(w)} > 0,$$

*then*

$$H : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w)$$

*is bounded.*

*Proof.* If  $\beta_{\Lambda_u^p(w)} > 0$ , we have that necessarily  $\underline{W}_u(0+) = 0$  and hence, for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $\underline{W}_u(t) < \varepsilon$ , for every  $t < \delta$ . Consequently, for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $W(u(S)) \leq \varepsilon W(u(I))$ , provided  $S \subseteq I$  and  $|S| \leq \delta|I|$ . But this condition was proved in [2] to be equivalent to  $u \in A_\infty$  and  $w \in B_\infty^*$ . Now, if  $u \in A_\infty$  [4],

$$(H^*f)_u^*(t) \lesssim (Q(Mf)_u^*)(t/4),$$

whenever the right hand side is finite,  $H^*$  is the Hilbert maximal operator

$$H^*f(x) = \sup_{\varepsilon > 0} \left| \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy \right|,$$

and

$$Qf(t) = \int_t^\infty f(s) \frac{ds}{s}$$

is the conjugate Hardy operator.

Then (see [2] for the details), using the facts that, under the condition  $w \in B_\infty^*$  we have that  $Q$  is bounded on the cone of decreasing functions on  $L^p(w)$ , and  $M$  is bounded on  $\Lambda_u^p(w)$  since  $\alpha_{\Lambda_u^p(w)} < 1$ , we obtain that  $H^*$  is bounded on  $\Lambda_u^p(w)$ . Hence, standard techniques show that, for every  $f \in \Lambda_u^p(w)$ , there exists  $Hf(x)$  at almost every  $x \in \mathbb{R}$  and, by Fatou's lemma, we obtain the result.  $\square$

Let us now see that, if  $p > 1$ , then we have the converse of the previous result, and so, the Boyd theorem in the context of weighted Lorentz spaces.

**Theorem 4.7.** *If  $p > 1$ , then*

$$H : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w)$$

*is bounded if and only if*

$$\alpha_{\Lambda_u^p(w)} < 1 \quad \text{and} \quad \beta_{\Lambda_u^p(w)} > 0.$$

Note that, it only remains to prove that  $\alpha_{\Lambda_u^p(w)} < 1$ . By Theorem 3.5, it is equivalent to prove

$$H : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w) \implies M : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w).$$

In [2], it was proved that

$$H : \Lambda_u^p(w) \longrightarrow \Lambda_u^{p,\infty}(w) \implies M : \Lambda_u^p(w) \longrightarrow \Lambda_u^{p,\infty}(w).$$

Since the strong boundedness implies the weak boundedness, Theorem 4.7 will be proved if, for  $p > 1$ ,

$$M : \Lambda_u^p(w) \longrightarrow \Lambda_u^{p,\infty}(w) \implies M : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w).$$

This is a problem of independent interest and it was left open in [8]. We dedicate the next section to the proof of this result, which will conclude the proof of Theorem 4.7.

## 5. WEAK AND STRONG BOUNDEDNESS OF HARDY-LITTLEWOOD MAXIMAL OPERATOR IN WEIGHTED LORENTZ SPACES

The main result of this section is the following theorem:

**Theorem 5.1.** *If  $p > 1$ , then*

$$M : \Lambda_u^p(w) \longrightarrow \Lambda_u^{p,\infty}(w)$$

*is bounded if and only if, for some  $q < p$ ,  $\overline{W}_u(t) \lesssim t^q$  for every  $t > 1$ . Consequently,*

$$M : \Lambda_u^p(w) \longrightarrow \Lambda_u^{p,\infty}(w) \text{ is bounded} \iff M : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w) \text{ is bounded.}$$

The idea of the proof is inspired by the article of Neugebauer [15]. Firstly, we need the following lemma:

**Lemma 5.2.** *Given an interval  $I$  and a set  $S = \cup_{j=1}^m S_j$ , with  $S_j$  pairwise disjoint intervals, there exists a positive function  $f_{S,I}$  supported in  $I$  satisfying the following conditions:*

- (i)  $f_{S,I}(x) = 1$ , for every  $x \in S$ .
- (ii)  $f_{S,I}(x) \geq \frac{|S|}{|I|}$ , for every  $x \in I$ .
- (iii) For every  $\frac{|S|}{|I|} < \lambda \leq 1$ , the level set

$$\{x : f_{S,I}(x) \geq \lambda\} = \cup_k J_{k,\lambda},$$

where  $\{J_{k,\lambda}\}_k$  are pairwise disjoint intervals satisfying

$$|S \cap J_{k,\lambda}| = \lambda |J_{k,\lambda}|,$$

and there exists  $L_{k,\lambda} \subset \{1, \dots, m\}$  such that

$$J_{k,\lambda} \cap S = \cup_{l \in L_{k,\lambda}} S_{j_l}.$$

*Proof.* For simplicity we shall use the following notation: if we have a collection of sets  $\{F_j\}_{j=1}^N$ , we write  $\cup^* F_j$  to indicate the union of a subcollection, whenever it is not important which subcollection is. Similarly, we write  $\sum^* |F_j|$  to indicate that we are summing the measures of the sets of a certain subcollection. We emphasize that the symbols  $\cup^*$  or  $\sum^*$  in two different places may refer to two different subcollections.

The proof is done by induction in  $m$ . The case  $m = 1$  is easy since, in this case, if  $I = (a, d)$  and  $S = (b, c)$  with  $a < b < c < d$ , we take, for every  $\frac{|S|}{|I|} < \lambda < 1$ ,  $x_\lambda \in (a, b)$  and  $y_\lambda \in (c, d)$  such that

$$\frac{b - x_\lambda}{b - a} = \frac{y_\lambda - c}{d - c} \quad \text{and} \quad y_\lambda - x_\lambda = \frac{1}{\lambda}(c - b).$$

Then, if we define

$$J_{1,\lambda} = [x_\lambda, y_\lambda], \quad \text{if } \frac{|S|}{|I|} < \lambda \leq 1,$$

$$J_{1,\lambda} = \bar{I}, \quad \text{if } \lambda \leq \frac{|S|}{|I|},$$

and  $J_{1,\lambda} = \emptyset$ , if  $\lambda > 1$ , one can immediately see that if  $\lambda_1 \leq \lambda_2$ ,  $J_{1,\lambda_2} \subset J_{1,\lambda_1}$ , and

$$J_{1,\lambda} = \bigcap_{\mu < \lambda} J_{1,\mu}.$$

Hence, if we define

$$f_{S,I}(x) = \sup\{\lambda > 0 : x \in J_{1,\lambda}\},$$

we obtain that  $\{x : f_{S,I}(x) \geq \lambda\} = J_{1,\lambda}$  and the rest of the properties are easy to verify. The cases where  $a = b$  or  $c = d$  are done similarly (see Figure 1).

Let us now assume that the result is true for  $m = n$  and let us prove it for  $m = n + 1$ . Let then  $S = \cup_{j=1}^{n+1} S_j$ , with  $S_j$  pairwise disjoint intervals. Let us define  $f_k = f_{S_k,I}$  and  $f_0(x) = \max\left(\sup_k f_k(x), \frac{|S|}{|I|}\right)$ , and let

$$\lambda_0 = \inf \left\{ \frac{|S|}{|I|} \leq r < 1 : \{f_j(x) \geq r\} \cap \{f_k(x) \geq r\} = \emptyset, \forall j \neq k \right\}.$$

Let  $E_{\lambda_0} = \{x \in I : f_0(x) \geq \lambda_0\}$  and observe that  $E_{\lambda_0} = \cup_{j \in J} E_{\lambda_0,j}$ , where  $\text{card } J < n + 1$ ,  $E_{\lambda_0,j}$  are pairwise disjoint intervals such that

$$(5.1) \quad E_{\lambda_0,j} \cap S = \cup^* S_j,$$

$\lambda_0 |E_{\lambda_0}| = |S|$  and, in fact, for every  $j$ ,

$$(5.2) \quad \lambda_0 |E_{\lambda_0,j}| = |E_{\lambda_0,j} \cap S|.$$

Now, by induction hypothesis, there exists a positive function  $g$  supported in  $I$  such that:

(i')  $g(x) = 1$ , for every  $x \in E_{\lambda_0}$ .

(ii')  $g(x) \geq \frac{|E_{\lambda_0}|}{|I|}$ , for every  $x \in I$ .

(iii') For every  $\frac{|E_{\lambda_0}|}{|I|} < \lambda \leq 1$ , the level set

$$\{x : g(x) \geq \lambda\} = \cup_k J'_{k,\lambda},$$

satisfying that  $\{J'_{k,\lambda}\}_k$  are pairwise disjoint intervals,

$$(5.3) \quad |E_{\lambda_0} \cap J'_{k,\lambda}| = \lambda |J'_{k,\lambda}|, \quad E_{\lambda_0} \cap J'_{k,\lambda} = \cup^* E_{\lambda_0,j},$$

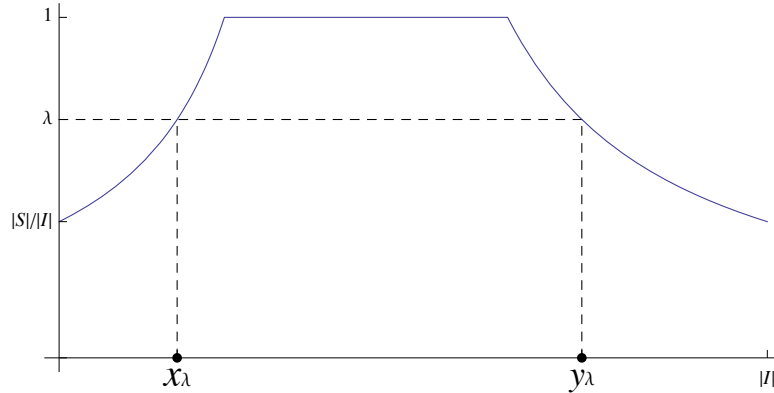


FIGURE 1.  $f_{S,I}$  when  $m = 1$ .

and

$$(5.4) \quad \{x : g(x) \geq \lambda\} \cap E_{\lambda_0} = \cup^* E_{\lambda_0, j}.$$

Then, we claim that the function  $f_{S,I}$  defined by

$$f_{S,I}(x) = f_0(x), \quad \text{if } x \in E_{\lambda_0}, \quad f_{S,I}(x) = \lambda_0 g(x), \quad \text{if } x \in I \setminus E_{\lambda_0},$$

satisfies all the required conditions (see Figure 2). Clearly (i) and (ii) hold true.

To see (iii) we divide it in two cases:

Case 1.- If  $\lambda_0 \leq \lambda \leq 1$ ,

$$\{x : f_{S,I}(x) \geq \lambda\} = \{x \in E_{\lambda_0} : f_0(x) \geq \lambda\} = \bigcup_{k=1}^{n+1} \{x \in I : f_k(x) \geq \lambda\} = \bigcup_k J_{k,\lambda},$$

and the result follows easily.

Case 2.-  $\frac{|S|}{|I|} < \lambda < \lambda_0$ . In this case

$$\{x : f_{S,I}(x) \geq \lambda\} = \{x : g(x) \geq \lambda/\lambda_0\},$$

and since  $\frac{|E_{\lambda_0}|}{|I|} = \frac{|S|}{\lambda_0|I|} < \frac{\lambda}{\lambda_0} < 1$ , we can apply (iii') and the properties of  $E_{\lambda_0}$  to conclude that

$$\{x : f_{S,I}(x) \geq \lambda\} = \bigcup_k J'_{k,\lambda/\lambda_0},$$

with  $\{J'_{k,\lambda/\lambda_0}\}_k$  pairwise disjoint intervals satisfying

$$|E_{\lambda_0} \cap J'_{k,\lambda/\lambda_0}| = \frac{\lambda}{\lambda_0} |J'_{k,\lambda/\lambda_0}|.$$

So we have to prove that

$$|S \cap J'_{k,\lambda/\lambda_0}| = \lambda_0 |E_{\lambda_0} \cap J'_{k,\lambda/\lambda_0}|.$$

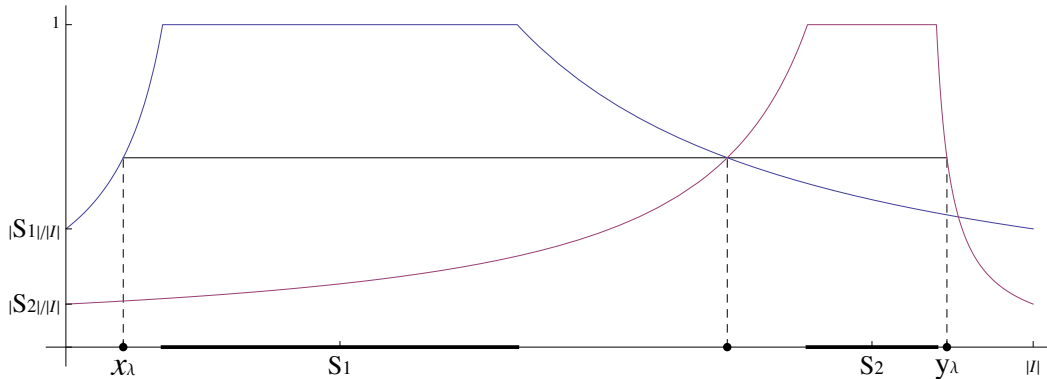


FIGURE 2.  $f_{S,I}$  when  $m = 2$ .



Now, from (5.2) and (5.3), we obtain that

$$\lambda_0 |E_{\lambda_0} \cap J'_{k,\lambda/\lambda_0}| = \lambda_0 \sum^* |E_{\lambda_0,j}| = |\cup^* E_{\lambda_0,j} \cap S| = |S \cap J'_{k,\lambda/\lambda_0}|.$$

Finally, using (5.1) and (5.4), we obtain that

$$\{x : g(x) > \lambda\} \cap S = \{x : g(x) > \lambda\} \cap E_{\lambda_0} \cap S = \cup^* E_{\lambda_0,j} \cap S = \cup^* S_i,$$

and the result follows.  $\square$

**Lemma 5.3.** *Let  $S$  be a subset of the interval  $I$  such that it is a union of pairwise disjoint intervals:  $S = \cup_{k=1}^N S_k$ . If  $s = \frac{|I|}{|S|}$ , then*

$$\frac{1}{|I|} \int_I f_{S,I}(x) dx = \frac{1 + \log s}{s}.$$

*Proof.* We observe that by construction of the function  $f_{S,I}$  we have that

$$|\{x : f_{S,I}(x) \geq \lambda\}| = \begin{cases} |I|, & \text{if } \lambda \in (0, 1/s) \\ |S|/\lambda, & \text{if } \lambda \in [1/s, 1] \\ 0, & \text{if } \lambda > 1. \end{cases}$$

Then

$$\frac{1}{|I|} \int_I f_{S,I}(x) dx = \frac{1}{|I|} \int_0^\infty |\{x : f_{S,I}(x) \geq \lambda\}| d\lambda = \frac{1 + \log s}{s}.$$

$\square$

*Proof of Theorem 5.1.* Let  $(I_j)_{j=1}^J$  be a finite family of pairwise disjoint intervals and let  $(S_j)_{j=1}^J$  be such that  $S_j \subseteq I_j$ ,  $S_j$  a finite union of pairwise disjoint intervals with  $|I_j|/|S_j| = s$ , for every  $j$ . Let

$$(5.5) \quad f(x) = \sum_{j=1}^J f_{S_j,I_j}(x).$$

By the weak-type boundedness of  $M$  we get, for every  $t > 0$ ,

$$(5.6) \quad W(u(\{x \in \mathbb{R} : Mf(x) > t\})) \lesssim \frac{1}{t^p} \|f\|_{\Lambda_u^p(w)}^p.$$

Now,

$$\begin{aligned} \|f\|_{\Lambda_u^p(w)}^p &= \int_0^\infty p\lambda^{p-1} W(u(\{x : f(x) > \lambda\})) d\lambda \\ &\leq \int_0^{1/s} p\lambda^{p-1} W(u(\{x : f(x) > \lambda\})) d\lambda \\ &\quad + \int_{1/s}^1 p\lambda^{p-1} W(u(\{x : f(x) > \lambda\})) d\lambda = I + II. \end{aligned}$$

By Remark 1.3 (ii) we have that

$$\frac{W\left(u\left(\bigcup_{j=1}^J I_j\right)\right)}{W\left(u\left(\bigcup_{j=1}^J S_j\right)\right)} \lesssim \max_{1 \leq j \leq J} \left(\frac{|I_j|}{|S_j|}\right)^p \approx s^p,$$

and so

$$I \lesssim \int_0^{1/s} \lambda^{p-1} s^p W(u(\bigcup_{j=1}^J S_j)) d\lambda \approx W(u(\bigcup_{j=1}^J S_j)).$$

On the other hand, by Lemma 5.2, if  $\lambda \in (1/s, 1)$ , the set  $J_\lambda = \{x : f(x) > \lambda\}$  is the union of disjoint intervals  $J_{\lambda,k}$  such that, for every  $k$ ,

$$\frac{|J_{\lambda,k}|}{|S \cap J_{\lambda,k}|} = \frac{1}{\lambda},$$

and  $S = \bigcup_{j=1}^J S_j \subseteq J_\lambda$ . Therefore,

$$\frac{W(u(\bigcup_k J_{\lambda,k}))}{W(u(S))} = \frac{W(u(\bigcup_k J_{\lambda,k}))}{W(u(\bigcup_k S \cap J_{\lambda,k}))} \lesssim \max_k \left(\frac{|J_{\lambda,k}|}{|S \cap J_{\lambda,k}|}\right) \approx \lambda^{-p}.$$

Hence

$$\begin{aligned} II &\lesssim \int_{1/s}^1 \lambda^{p-1} \lambda^{-p} W(u(S)) d\lambda \approx (1 + \log s) W(u(S)) \\ &\approx (1 + \log s) W(u(\bigcup_{j=1}^J S_j)). \end{aligned}$$

So, we have that

$$(5.7) \quad \|f\|_{\Lambda_u^p(w)}^p \lesssim (1 + \log s) W(u(\bigcup_{j=1}^J S_j)).$$

On the other hand, for every  $j$ ,

$$I_j \subseteq \left\{ x \in \mathbb{R} : Mf(x) > \frac{1}{2|I_j|} \int_{I_j} f(x) dx \right\},$$

and, by Lemma 5.3, for every  $j$ ,

$$\frac{1}{|I_j|} \int_{I_j} f(x) dx = \frac{1 + \log s}{s}.$$

Hence,

$$(5.8) \quad W(u(\bigcup_j I_j)) \leq W(u(\{x \in \mathbb{R} : Mf(x) > (1 + \log s)/2s\})).$$

Finally, if we fix  $t = (1 + \log s)/2s$  in (5.6), and combine (5.7) and (5.8) we obtain

$$\frac{W(u(\bigcup_j I_j))}{W(u(\bigcup_j S_j))} \lesssim (1 + \log s)^{1-p} s^p.$$

Then, taking supremum, we obtain that

$$\overline{W}_u(s) \lesssim (1 + \log s)^{1-p} s^p$$

and by Lemma 2.4, it follows that  $\overline{W}_u(t) \lesssim t^q$ , for some  $q < p$ , as we wanted to see.  $\square$

Theorem 5.1 not only concludes the proof of the Boyd Theorem in weighted Lorentz spaces. As we mentioned at the end of the Section 4, it also implies the following result:

**Theorem 5.4.** *Let  $p > 1$ . If*

$$H : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w)$$

*is bounded, then*

$$M : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w)$$

*is also bounded.*

Finally, as in the proof of the characterization of the weak-type boundedness given in [2], and using Theorem 5.1, we can also characterize the boundedness of  $H$  on  $\Lambda_u^p(w)$ , for  $p > 1$ , in terms of geometric conditions on the weights  $u$  and  $w$  as follows:

**Theorem 5.5.** *If  $p > 1$ , then*

$$H : \Lambda_u^p(w) \longrightarrow \Lambda_u^p(w)$$

*is bounded if and only if the following three conditions hold:*

- (i)  $u \in A_\infty$ .
- (ii)  $w \in B_\infty^*$ .
- (iii) Condition (1.2) holds.

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E. AGORA, DEPARTMENT OF APPLIED MATHEMATICS AND ANALYSIS, UNIVERSITY OF BARCELONA, 08007 BARCELONA, SPAIN.

*E-mail address:* `elona.adora@ub.edu`

J. ANTEZANA, DEPARTMENT OF MATHEMATICS, FACULTY OF EXACT SCIENCES, NATIONAL UNIVERSITY OF LA PLATA, 1900 LA PLATA, ARGENTINA.

*E-mail address:* `jaantezana@yahoo.com.ar`

M. J. CARRO, DEPARTMENT OF APPLIED MATHEMATICS AND ANALYSIS, UNIVERSITY OF BARCELONA, 08007 BARCELONA, SPAIN.

*E-mail address:* `carro@ub.edu`

J. SORIA, DEPARTMENT OF APPLIED MATHEMATICS AND ANALYSIS, UNIVERSITY OF BARCELONA, 08007 BARCELONA, SPAIN.

*E-mail address:* `soria@ub.edu`