# Random-walk model for valuing path-dependent financial instruments

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**Abstract**: In this paper we shall model the evolution of a market evolving within the framework of the non-arbitrage binomial pricing asset model using a Monte Carlo-based algorithm. Our goal is to study the value of an actual path-dependent structured financial product, so we can create a commercial strategy and commercialize it. To do this we study the sensibility of the product when we vary its defining parameters, so we understand how its price depends on them and we can adjust the parameters to profit.

## I. INTRODUCTION

We all have experienced the economic crisis. We know prices are related to the law of supply and demand. We have heard about futures, options and portfolios. But few of us know what economy is about. And even fewer know that many tools and mathematical approaches we use in physics are helpful in the study and description of economic and financial systems.

Roughly speaking, economy studies how people exchange resources and the consequences of this actions. Broadly, economy has three domains: microeconomics (that deals with the behavior of individuals and firms regarding the allocation of limited resources), macroeconomics (that studies the performance, structure, behavior and decision-making of an economy as a whole) and finance, that is the field related to the study of investments: it includes the dynamics of assets and liabilities over time under conditions of different uncertainties and risks.

We can see economics' area of study is vast and has deep consequences in humans' life, so it may be surprising that most of economics is based in the following hypothesis: the decisions of any participant in any economic system are made rationally, meaning these choices are made to maximize the actor's satisfaction (utility) [1] .

Moreover, in finance another hypothesis is made: markets are efficient. This one implies all the information about a product that provides the opportunity of a risk-free profit makes the market evolve to a situation where this possibility disappears (the so-called no-arbitrage condition).

Without a doubt, the succeeds of economic models so far are unquestionable. Nevertheless, it is clear there still are problems to be solved, as evince the recent economic crisis and the incapacity of predicting the crisis itself and its impact in both local and global scale. Economists have realized the foundations where the economic theory lies may not be entirely true. In fact, to a greater or lesser extent, all these foundations have been questioned [2]. So, taking all this into account, economy might draw on a fresher point of view coming from other areas of science, as physics.

The interest of physicists in economy and, notably, in the financial markets starts in the 80's when physicists and mathematicians were hired to study the large amount of data coming from these markets. However, the connection between economy and physics is way older [3]. In fact, it was Daniel Bernoulli who introduced the concept of utility to explain people's preferences. On the other hand, in 1900 Louis Bachelier introduced a probabilistic model to describe the evolution of financial markets [4], the same mathematical model used by Einstein five years later to explain the Brownian motion, which, as we know, is a stochastic macroscopic process due to microscopic interactions.

The connection between the microscopic and macroscopic phenomena is the field of study of statistical physics. This stochastic description of nature involves random processes to emulate the unpredictable effect of the countless interactions between the particles of a system. This approach goes beyond equilibrium systems and can be used to also describe systems where phase transitions happen, which are usually not solvable analytically. The path down this road leads us to what nowadays is known as complex systems.

In complex systems, small perturbations may become huge perturbations due collective effects. Moreover, these systems frequently exhibit extreme events, which could be understood as an earthquake studying tectonics or as a global crisis studying finance [5].

## II. PRICING DERVATIVES

We shall study the behavior of a market which we consider evolves randomly. More precisely, the price of a stock that evolves within the binomial no-arbitrage pricing model by which, at each time step, this price goes up some quantity $u$ with probability $p$ or it goes down an amount $d$ with probability $q = 1 - p$.

We can view the behavior of our stock's price by imagining at each time step we toss a biased coin. Then if the outcome is head the price goes up, else the price goes down.

In this context we shall study the mean payoff of a structured financial product using a computer simulation, based on the Monte Carlo method, that allows us to perform a path-dependent study of the system's evolution.

Let $S_n$ be the price of our stock at the time step $n$. Consequently, its initial price is $S_0$. As already said, this price evolves at every time step. Consider now the price if the outcome of the coin toss is head, $S_1(H)$, and the price if it is tail, $S_1(H)$. Then we can define the up and down factors as

$$u = \frac{S_1(H)}{S_0}, \qquad d = \frac{S_1(T)}{S_0}.$$

Moreover, we introduce the interest rate $r$, which tells us how the value of the money change from one time-step to the next one. For instance, one euro invested in the money market at time $n$ would yield $1 + r$ euros at time $n + 1$.

A feature of a market is that if a trading strategy can generate some profit, then it must also contemplate the risk of loss; otherwise there would be an arbitrage. More specifically, an arbitrage is a trading strategy that has zero

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probability of losing money and a positive probability of making it.

In the binomial model, to rule out possible arbitrages we must assume

$$0 < d < 1 + r < u. \tag{1}$$

The inequality $d > 0$ follows from the stock prices' positivity. The other two inequalities in (1) follow from the absence of arbitrage, as we shall explain now. Imagine $d \geq 1 + r$ and we begin with zero money, then at time zero we borrow money from the market to buy stock. Even in the worst case, i.e. the stock's price going down, its price at time one will be enough to pay off out money market debt; besides, there is a positive probability the stock is worth more since $u > d > 1 + r$. This provides an arbitrage. On the other hand, if $u \leq 1 + r$, we could sell the stock short and invest in the money market. Even in the case where the stock is worth the most, the cost of replacing it at time one will be less than or equal to the value of the money market investment, and since $d < u \leq 1 + r$ there is a finite probability that the cost of replacing the stock will be strictly less than the value of the money market investment. This again provides an arbitrage.

In addition to the no-arbitrage conditions, we have assumed that:

(i) shares of stock can be subdivided for sale or purchase,
(ii) the interest rate for investing is the same as the interest rate for borrowing,
(iii) the purchase price of stock in the same as the selling price,
(iv) at any time, the stock can take only two possible values in the next period.

So far, we have introduced the ratios in which our system will evolve, i.e. the amount the price will change at every time step, and we have put some constraints at them. Nonetheless, to fully describe the binomial model we still must find the probabilities of the unfair coin. These are derived using financial arguments.

On one hand, assume we have an initial wealth $X_0$ and we buy $\Delta_0$ shares worth $S_0$ each. This leaves us with a cash position $X_0 - \Delta_0 S_0$. Then, at time one, our cash position will be $(1+r)(X_0 - \Delta_0 S_0)$. Moreover, also at time one, we will have a stock worth $\Delta_0 S_1$. In particular, if the stock's price goes up, the value of our portfolio and our money market account at time one will be

$$X_1(H) = \Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0). \tag{2}$$

Otherwise, if the stock's price goes down, our cash position at time one will be

$$X_1(T) = \Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0). \tag{3}$$

On the other hand, we define a derivative security to be a security that pays some amount $V_1(H)$ at time one if the coin toss results in head and pays a possibly different amount $V_1(T)$ at time one if the coin toss results in tail. For a given $X_0$ and $\Delta_0$, we want that $X_1 = V_1$ regardless the outcome of the coin toss. Thus, from equations (2) and (3)

$$\begin{aligned} X_0 + \Delta_0\left(\frac{1}{1+r}S_1(H) - S_0\right) &= \frac{1}{1+r}V_1(H), \\ X_0 + \Delta_0\left(\frac{1}{1+r}S_1(T) - S_0\right) &= \frac{1}{1+r}V_1(T). \end{aligned} \tag{4}$$

Solving the system of equations (4) for $X_0$ and $\Delta_0$ we find

$$X_0 = \frac{1}{1+r}[pV_1(H) + qV_1(T)] = V_0, \tag{5}$$

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}. \tag{6}$$

Let's focus on equation (5), where it appears two quantities $p$ and $q$ we have found solving the system. These are given by the expressions

$$p = \frac{1+r-d}{u-d}, \qquad q = \frac{u-1-r}{u-d}. \tag{7}$$

Due the no arbitrage condition (1), both $p$ and $q$ are positive-defined. Besides, they sum up to one, so we can consider them a probability measure. In fact, they probabilities of head and tail for the biased coin, respectively. They are not the actual probabilities, which we denote by $\tilde{p}$ and $\tilde{q}$, but the so-called risk-neutral probabilities.

Under the actual probabilities, the average growth of the stock's price is typically strictly greater than the rate of growth of the money market; otherwise no one would want to take the risk associated to investing in the stock. Hence, $\tilde{p}$ and $\tilde{q}$ should satisfy

$$S_0 < \frac{1}{1+r}[\tilde{p}S_1(H) + \tilde{q}S_1(T)],$$

whereas $p$ and $q$ satisfy

$$S_0 = \frac{1}{1+r}[pS_1(H) + qS_1(T)].$$

If the average rate of growth for the stock were equal to the rate of growth of the money market investment, then investors would take no risk meaning that they do not require any compensation for assuming it, nor were they willing to pay extra for it. This is not simply the case when one invests, so $p$ and $q$ cannot be the actual probabilities.

Since the study we are about to do involves more than one time-step, now we need to extend these ideas to multiple periods. We can do so defining the value of our portfolio in a recursive way, beginning with $X_0$, via the wealth equation

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n),$$

which is a non-anticipating magnitude, involving quantities at different time steps.

Then, using the expressions for $p$ and $q$ given by (7), defining recursively backward in time the random variable

$$V_n = \frac{1}{1+r}[pV_{n+1}(H) + qV_{n+1}(T)], \tag{9}$$

and defining

$$\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)},$$

one can prove by induction that we will have $X_N = V_N$ for all possible outcomes of the biased-coin toss. Moreover, the random variable (9) is defined to be the price of the derivative security for all $n = 1,2 \dots, N$ [5].

Last, but not least, from (9) we can find a formula to price the options in the present time in terms of the price at a time step $n$

$$X_0 = \frac{1}{(1+r)^n}\mathbb{E}[X_n]. \tag{10}$$

## III. THE VANILLA CASE

Now that we have introduced our model, the simplest option to analyze is the so-called Vanilla option, which is a financial instrument that gives the holder he right, but not the obligation, to buy or sell an underlying asset, security or currency at a predetermined price within a given timeframe. A concrete case is the European call option, that gives the owner the right to buy stock for a stake price $K$. Obviously, the holder will exercise the option if he obtains any profit, so the payoff of this option is $(S_N - K)^+ = \max(S_N - K, 0)$.

Since the payoff of this kind of option is not path-dependent, the expected value of the payoff can be computed analytically within the binomial pricing model using the formulas of the binomial distribution. Of course, it can also be computed with a method based in random numbers, such as the Monte Carlo method we shall introduce later, as a verification for the non-analytical one.

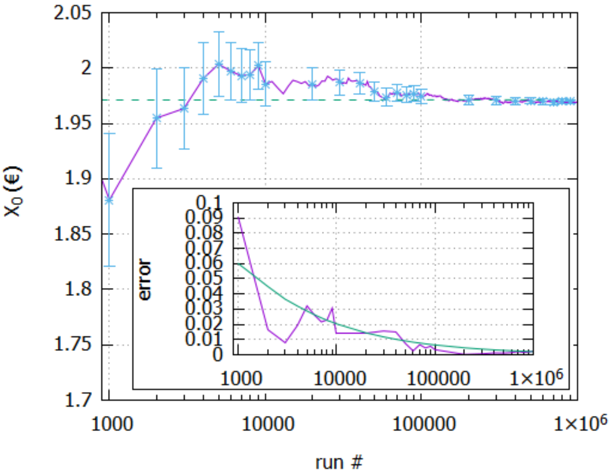


**FIG. 1**: Figure comparing the payoff's mean value of a European call option, using both analytical binomial formulas and the random number generation method, in terms of the number of times we run the Monte Carlo algorithm. The dashed line represents the analytical result and the one with the error bars goes for the Monte Carlo. We have used $S_0 = K = 10$ €, $n = 250$, $\sigma = 0.014$, $r = 1{,}54 \cdot 10^{-5}$.

Moreover, in the inset we can see the convergence of the real error between the two payoffs and the statistical one (monotone line), in terms of the number of times we run the algorithm.

Then, we can use the Monte Carlo method for other options or compound products which cannot be studied analytically or whose payoff has path dependent conditions.

## IV. THE PATH-DEPENDENT CASE

As already said, we use a Monte Carlo-based algorithm to simulate the evolution of our market. Basically, the stock's price behaves as a random walker, moving forward or backward a fixed quantity. To emulate the random-walk dynamics we generate random numbers uniformly distributed between zero and one, $x \in [0,1]$. Then, if $x \in [0, p]$, we take it as the unfair coin tossing head and the stock's price goes up. Otherwise, we interpret it as a tail and the price goes down.

The value $p$ is the risk-free probability, defined in (7). To compute it we need to fix the values for $u$ and $r$ which are realistic, meaning they must be like the ones we find in real markets.

### A. Setting the parameters

Let's begin with the interest rate $r$. This is a tricky parameter to set, since it changes from epoch to epoch. In the present scenario of low interest rates, we shall choose the annual interest as $r_a = 0.385\%$. We can connect it with the interest rate for a time horizon $T$ using the relation $r_T = r_a/\mathrm{T}$ Hence, considering we check how the market evolve every trading day, $T = 250$ and

$$r_T \equiv r = 0.0000154. \tag{11}$$

We carry on with the up factor. Typical values for the annual volatility for a market's share prices are around $\sigma_a \approx 20\,\%$ and for our purposes we shall use $\sigma_a \approx 23\,\%$. For a random-walk alike market the volatility for a time horizon $T$ is given by $\sigma_T = \sigma_a/\sqrt{T}$. Taking the same time horizon as before we get

$$\sigma_T \equiv \sigma = 0.0145464. \tag{12}$$

Moreover, we know that

$$\sigma^2 = \mathbb{E}\left[\left(\frac{S_{n+1}}{S_n}\right)^2\right] - \left(\mathbb{E}\left[\frac{S_{n+1}}{S_n}\right]\right)^2,$$

where

$$\mathbb{E}\left[\frac{S_{n+1}}{S_n}\right] = p\frac{S_{n+1}(H)}{S_n} + (1-p)\frac{S_{n+1}(T)}{S_n} = 1 + r,$$

$$\mathbb{E}\left[\left(\frac{S_{n+1}}{S_n}\right)^2\right] = (1+r)\left(u + \frac{1}{u}\right) - 1.$$

Consequently,

$$\sigma^2 = (1+r)\left[\left(u + \frac{1}{u}\right) - 1 - (1+r)\right], \tag{13}$$

which is an equation for u.

Since $\sigma^2 \ll 1$, $u$ will be slightly higher than 1 and therefore we can expand $u = 1 + \epsilon + \epsilon^2$. Thus, from (11) we find an equation for $\epsilon$ that reads

$$\sigma^2 = (1+r)(2+\epsilon^2) - 1 - (1+r)^2.$$

Its solution is

$$\epsilon^2 = \frac{r^2 + \sigma^2}{1+r}. \tag{14}$$

Since, as we have said, typical values of this parameters satisfy $1 \gg \sigma^2 \gg r$ we can take $\epsilon \approx \sigma$, meaning in practice we shall consider $r \approx 0$.

Concerning the down factor, it is common to have $d = 1/u$, and this will be our case, so the evolution of our system evolves as a one-dimensional random walk.

### B. The product

We have already introduced our model, chosen a working line using Monte Carlo-based simulations and set realistic parameters for our algorithm, thereupon we shall study the payoff of an option which is path-dependent, as we describe here below. It is inspired by an actual structured financial product recently commercialized by MAPFRE VIDA S.A. [7], where we find to types of conditions:

1. The final condition: regardless the stock price's evolution path, if the final price is above a certain upper-limit percentage, $L_u$, with respect the initial price, the payoff is the initial inversion plus the initial capital times the upper limit percentage. In other words,
$$S_N > (1 + L_u)S_0 \Longrightarrow X_N = (1 + L_u)S_0$$
2. The path-dependent condition: if the final prince is below this fix percentage, i.e. if
$$S_N < (1 + L_u)S_0,$$
then

(i) if at any time the price of the stock has been below a certain value, $1-L_d$, then the payoff is the final price of the stock, that is to say, $X_N = S_N$.

(ii) if the stock's price never reaches the lower limit $L_d$ then its value is the maximum between the final price and the initial one plus a certain percentage $L_i$, i.e
$$X_N = \max(S_N, (1+L_i)S_0).$$

Since the payoff is path-dependent, the Monte Carlo method has here an extra advantage because the method itself works generating random paths. Hereunder we present different simulated paths with different payoff

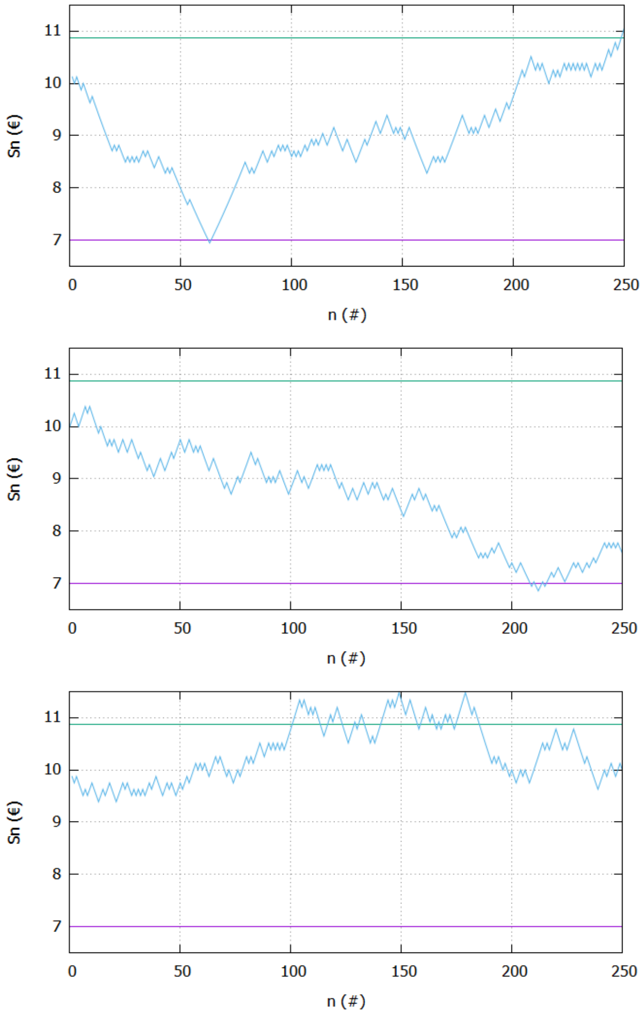


**FIG. 2**: Different random walks for the stock's price. We have chosen $u$ and $r$ as discussed before and $S_0 = 10$ €, $L_d = 30\%$, $L_u = 8{,}75\%$ and $L_i = 2\%$. Consequently

a) The price reaches both the lower and the upper limit. Since condition 1. is fulfilled the payoff in this case is $X_N = 10{,}875$ €.

b) In this case the price also reaches the lower limit, hence condition 2. $ii$ is satisfied and the payoff is $X_N = S_N = 7.585$ €.

c) The last plot shows how the stock's price reaches repeatedly the upper limit, but in the end it's below this value. Hence the payoff is $X_N = \max(S_N, 1{,}02\ S_0) = 10{,}2$ €.

We shall now use the algorithm to simulate the behavior of a certain market and study the average value of the option's payoff for different values of the model's parameters and the ones from the option. Unless it is the varying parameter, the values which we shall use are the ones in (11) and (12) and those used while plotting FIG 2.

As it has been said before, in practice we consider $r \approx 0$, but let's see what happens if we don't:

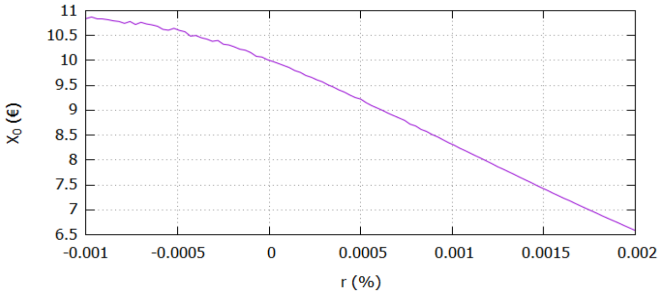


**FIG. 3**: Value of the product's initial payoff, $X_0$, in terms of the interest rate, $r$. To find $X_0$ we use (10)

We see that if the interest rate is negative, i.e. money is worth less at the end than it was at the beginning, then the value of our stock has decreased. Therefore, we must increase the initial price to compensate this devaluation.

We go on studying the variation of the payoff as we vary the market's volatility

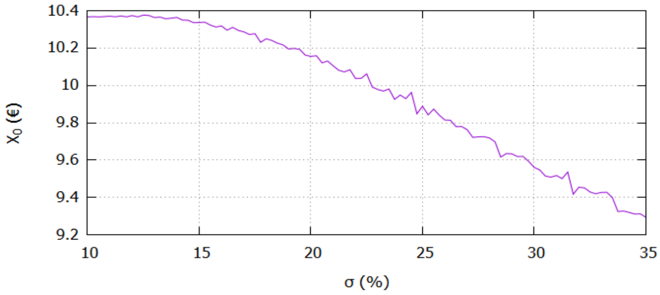


**FIG. 4**: Value of the product's initial payoff, $X_0$, in terms of the volatility, $\sigma$.

This time we observe the average value of the payoff goes down as volatility increases. One can show that, for a fixed interest rate $r$, taking $u = 1+\sigma$ and $d = 1/u$, the risk-neutral probability is given by
$$p \approx \frac{1}{2}\left(\frac{r}{\sigma}+1\right),$$
meaning the probability of the stock's price going up rises as $\sigma$ drops, and vice versa.

Moreover, from a qualitative point of view, reducing the volatility means investing in a less risky market hence, even the amount we would win may be smaller, so is the probability of losing our investment.

The following situation we consider is the one where we change the value of the upper-limit.

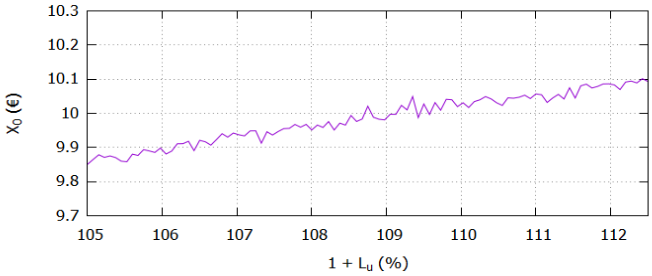


**FIG. 5**: Value of the product's initial payoff, $X_0$, in terms of the upper limit, $L_u$.

We observe the expected value of the payoff increases as $L_u$ rises, since the product is worth more every time the stock's price overcomes the upper limit

The last parameter left to study is the lower limit

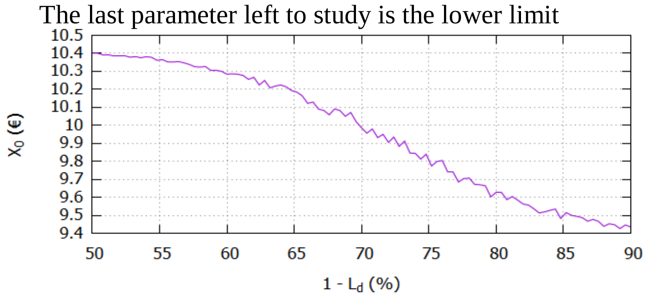


**FIG. 6**: Value of the product's initial payoff, $X_0$, in terms of the lower limit, $L_d$.

In this case the product's price drops as the lower limit goes down, since our hedging position becomes smaller, meaning it's more likely we lose money.

Finally, let's study how $X_0$ depends on the most relevant parameters, i.e. $\sigma$ and $L_d$.

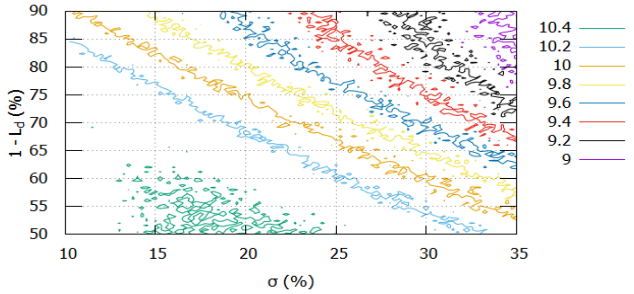


**FIG. 7**: Contour curves of the value of the product's initial payoff, $X_0$, in terms of the lower limit, $L_d$, and the volatility, $\sigma$.

One can see from this contour plot that, for the parameters considered, the mean value of the initial payoff verifies

$$9.8\ € < X_0 < 10\ €. \qquad (15)$$

### C. The commercial strategy

For this product to be attractive, we have to offer a relatively high $L_u$, so the buyer feels he can have an acceptable profit considering the risk taken. However, to compensate the rise of the product's price we must introduce some factors adding risk to the investment. These can be both external, as the volatility of the market where we invest, or internal, such as the probability of having an unhedged position if the stock's price reaches a certain lower limit, $L_d$.

Following this logic, if the product offers a limited profit it's price decreases, while offering covering for the possible loses makes the price rise. In this way, the product may make us feel like we are truly investing, since we have a positive probability of a situation where we have no hedging.

If we were about to commercialize this product, our profit would come from the initial investment of the costumer, meaning we would buy the derivative instead of the stock and our benefit would be the difference.

Therefore, we must adjust the parameters, so the estimated price of the product is slightly lower than the initial price of the stock we are buying, just as we see that happens in FIG 7.

## CONCLUSIONS

Summarizing, we have studied the expected value of a structured financial product's payoff. To do so, we have used the no-arbitrage binomial model, which states the stock's price can only go up or down some fixed quantities $u$ and $d = 1/u$ with probabilities $p$ and $q = 1 - p$, respectively. These probabilities are derived using financial arguments for the one period model, and then generalized to the multiperiod model by induction.

We have set the parameters like the typical values we find in real markets nowadays, $\sigma = 1{,}45\ \%$ and $r = 0{,}00154\ \%$, obtaining $p = 0.497$ and $q = 0.503$.

Since we analyse a structured product whose payoff is path-dependent, we study the stock's price time evolution as it was a random walker moving with probabilities $p$ and $q$. To do so, we have used a Monte Carlo-based algorithm.

To have a better understanding of the product, we have studied its sensibility, i.e. how the option's price varies when we change respect the parameters defining it.

Then we have defined a commercial strategy, we have set the product's parameters, so the initial price of the product is less than the initial stock's price and we make profit, as we can see in (15).


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