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Gravitational wave pulse and soliton wave collision

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We study the collision of a gravitational wave pulse and a soliton wave on a spatially homogeneous background. This collision is described by an exact solution of Einstein's equations in a vacuum which is generated from a nondiagonal seed by means of a soliton transformation. The effect produced by the soliton on the amplitude and polarization of the wave is considered.

I. INTRODUCTION

In this paper we study the collision of a gravitational wave pulse and a soliton wave.

Gravitational solitons are found in general relativity. They have features similar to the hydrodynamical solitons, such as a peculiar behavior under collisions. Its presence is found in some exact solutions of Einstein's equations which are generated by means of the so-called soliton transformation of Belinskii and Zakharov.[1]

Their properties were studied in Refs. 2 and 3 in a cosmological context: the solitons are localized perturbations of the gravitational field propagating on a spatially homogeneous background. They have no dispersion; a velocity of propagation can be associated to them; after collisions they keep their individuality and although their amplitudes decrease with time, this is because of the expansion of the background. The solitons show a particlelike behavior initially, but resemble gravitational waves at later stages.

Because of this peculiar behavior of the soliton waves, it is of some physical interest to study the collision of solitons with other waves, in particular with gravitational waves. For this purpose we will construct exact solutions of Einstein's equations able to describe such a situation.

First we shall describe the way in which such solutions can be obtained. The solution transformation of Belinskii and Zakharov[1] can be used to generate new exact solutions of Einstein's equations, in vacuum or stiff matter, with two commuting Killing vectors by using a known solution (seed solution) to start with. A number of solutions have been studied in recent years.[2-8] The degree of complexity of the new solutions depends on the seed solution and the number of parameters of the "pole trajectories" which characterize the transformation. The main steps in a soliton transformation are the evaluation of a generating function, which requires an integration, and the election of the pole trajectories. Usually the seed solution is taken to be a diagonal metric because the generating functions are then easily found.[9]

Nondiagonal seeds have also been considered.[10-12] In the cosmological context, however, such solutions are not truly nondiagonal from the point of view of the soliton transformation technique because the generating function can be found from a diagonal seed, Bianchi type I, followed by an Ehlers transformation.[13]

By suitably choosing the pole trajectories one may find solutions (soliton solutions) which can be interpreted as gravitational solitons propagating on the background of the seed metric.[4] Consequently, in order to find solutions describing the collision of solitons with gravitational waves, we must choose a seed solution containing gravitational waves.

Some solutions with gravitational waves are found in spatially homogeneous models: the Lukash Bianchi type VII_h (Ref. 14) or Siklos plane waves of types IV, VI_h, and VII_h (Refs. 15 and 16). The Lukash solution is rather complicated and Siklos plane waves can be integrated to find the generating functions, but they are given in terms of a rather complicated combination of hypergeometric functions.[17] Moreover, plane-wave solutions have more symmetries than required, they do not admit canonical coordinates, and the soliton transformation transforms them into new plane-wave solutions. So they are a class of their own.[18]

It is best then to directly look for inhomogeneous solutions. Wainwright and Marshman[19,20] found a family of inhomogeneous nondiagonal solutions which depend on an arbitrary function of one null coordinate. Those solutions can be interpreted as cosmological models with gravitational waves and seem appropriate to our purpose. Furthermore, Kitchingham[13] has found the generating function for such solutions. One solution of the Wainwright and Marshman family has been interpreted as a gravitational wave pulse[21] propagating on a Kasner background. In it the pulse has been constructed by appropriately restricting the arbitrary function. It turns out that this solution is the simplest solution with a gravitational wave that can be constructed with the Kasner background.

Although the gravitational wave solution which we shall use as a seed solution is relatively simple, the final solution is rather complex. Two reasons for this are that the seed being nondiagonal the generating function is not simple and that in order to get a solution with localized soliton waves we need at least four complex pole trajectories. Consequently, the final metric elements look rather complicated.

Fortunately, the fact that the soliton waves and the gravitational wave pulses are localized in finite regions in the z direction (the propagation direction) and that they can be interpreted as finite perturbations on a Kasner background, simplifies the analysis. Many properties can be found by just considering the metric components and their values on several asymptotic regions.

In particular we shall use the polarization functions defined by Adams and co-workers[22,23] to analyze the metric. These authors define, in cosmological metrics with two commuting Killing vectors, a wave amplitude and a polarization angle associated with a certain frame. These functions do not give intrinsic properties of the metric but if the frame of reference has some physical sense they have an unambiguous physical interpretation. Since our frame is associated to the spatially homogeneous Kasner background in the canonical coordinates[24] we can adopt the usual cosmological interpretation of the coordinates as the background reference frame.

We should emphasize that the solutions considered here are vacuum Solutions. Therefore we do not have a fluid flow to prove the spacetime. Solutions with stiff fluid are easily found[25,26] and, in fact, Wainwright[21] gives the pulse-wave solution for stiff fluid. We have chosen, however, to generate vacuum solutions because they contain only pure gravitational effects of the spacetime without matter influence.

In Sec. II we consider the gravitational pulse-wave solution and give the particular function we have used in the seed metric. In Sec. III the polarization functions are reviewed, in Sec. II the final solution representing the collision of a pulse wave with a soliton wave is given and their parameters are classified and interpreted. Finally, in Sec. V the solution is analyzed: we examine the asymptotic regions analytically, in particular the behavior along different light cones reveals properties of the pulse-wave–soliton-wave interaction. We see that as a result of the interaction both waves get strongly polarized. The wave amplitudes and polarization angles are shown to change under the collision in a peculiar way. The pulse wave and the soliton waves keep their individuality after colliding in a way similar to the pure soliton collision.

II. GRAVITATIONAL WAVE PULSE SOLUTION

We start with the gravitational wave pulse solution given by Wainwright and Marshman. The metric can be written as[19,21]

$$ds^2=t^{-3/8}e^n(dz^2-dt^2)+t^{1/2}[dx^2+(t+w^2)dy^2+2w\,dx\,dy]\ , \qquad (1)$$

where $w(t+z)$ is an arbitrary function and $n'=(w')^2$. The coordinate range is $0\le t\le\infty,\ -\infty\le x,y,z\le\infty$.

This is the simplest solution with a gravitational wave that can be constructed with the Kasner background because it is the simplest solution which depends on an arbitrary function of one null coordinate (implying speed-of-light propagation). In fact, if we try to modify the family of Kasner solutions with a function in the diagonal coefficients one finds, from Einstein's equations, that such a function must verify a linear wave equation of the cylindrical type which does not admit as solution a function of a unique null coordinate. If we then try to keep the Kasner solution in the diagonal part and consider a nondiagonal coefficient, Einstein's equations give nonlinear equations for the nondiagonal coefficient and it is found that they admit, as a solution, an arbitrary function of one null coordinate, $t+z$ or $t-z$, provided we restrict the Kasner family to just one of its members. Stachel has shown this in the cylindrical wave context[27] which is obtained from the cosmological one by considering t as the cylindrical radial coordinate and z as the time coordinate. In such context the Kasner solution is interpreted as the static gravitational background produced by an infinite line of matter. The abovementioned solution is, perhaps, of no physical interest because it corresponds to a line with negative energy density. This problem does not arise in the cosmological context.

When $w=0$ (or constant) the metric (1) reduces to a member of the Kasner family. To define a pulse wave we choose w localized in a small region of the spacetime in the following way:

$$w(u)=\begin{cases} H, & u\le u_F\ ,\\ H-A\left[1-\cos\left[2\pi\dfrac{u-u_F}{u_B-u_F}\right]\right], & u_F\le u\le u_B\ ,\\ H, & u_B\le u\ ,\end{cases} \qquad (2)$$

where $u\equiv t+z$ and H, A, u_F, and u_B are arbitrary constants: A may be interpreted as the amplitude of the pulse wave and u_B-u_F as its width. In the spacetime regions $u\le u_B$ the metric is the spatially homogeneous Kasner, and Wainwright[21] has shown that in the region $u_F\le u\le u_B$ we have a gravitational wave. Thus the interpretation of this solution as a gravitational-wave pulse propagating at the speed of light on a homogeneous Kasner background is clear. As an example, in Fig. 1 the metric coefficient g_{12} has been represented for the pulse wave (2) with $H=0$.

In what follows we shall take $H=0$ so that the coordinates are adapted to the Kasner solution, if H is nonnull we can recover the usual coordinates with the change $dy'=dy$, $dx'=dx+H\,dy$.

III. WAVE AMPLITUDE AND POLARIZATION

The seed metric and the new metric we generate can be seen as a generalization of a Bianchi type-I metric in which we break the homogeneity in the z direction. Following Adams *et al.*[22] they can be written generically as

$$ds^2=f(dz^2-dt^2)+e^{2b}\left[\left[\cosh\phi+\frac{\psi}{\phi}\sinh 2\phi\right]dx^2+\left[\cosh 2\phi-\frac{\psi}{\phi}\sinh 2\phi\right]dy^2+\frac{\gamma}{\phi}\sinh 2\phi\,dx\,dy\right]\ , \qquad (3)$$

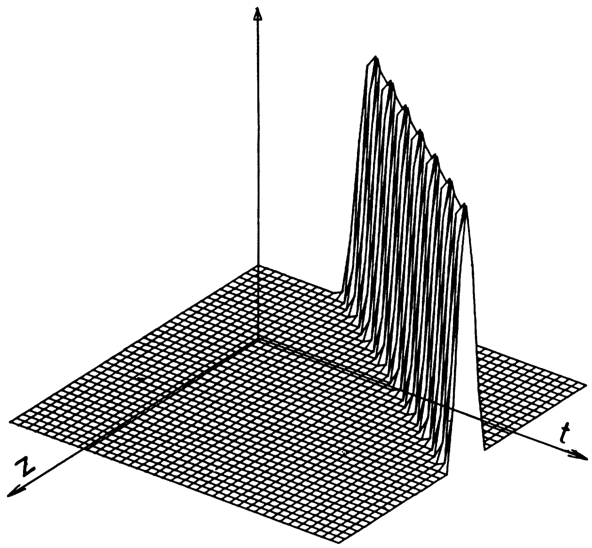

FIG. 1. Nondiagonal metric coefficient for the seed metric: $(g_0)_{12}/t^{1/2}$ in the t-z plane. This shows essentially the pulse wave (2) with the parameters $A=8\times10^{-5}$, $u_B=2.4$, $u_F=1.8$.

where

$$\phi\equiv(\psi^2+\gamma^2)^{1/2} \tag{4}$$

and all functions f, b, ψ, and γ depend on t and z. The two-dimensional metric with dx and dy (∂_x and ∂_y are the two Killing vectors) has only two independent components ψ and γ. These will be identified as the two independent polarizations of the gravitational waves: ψ corresponding to the $+$ mode and γ corresponding to the $\times$ mode, relative to the invariant basis ∂/∂_x and ∂/∂_y.

We can now define a phase angle θ

$$\tan2\theta=\gamma/\psi \tag{5}$$

and then we may use the functions ϕ,θ instead of ψ,γ since $\gamma=\phi\sin2\theta$ and $\psi=\phi\cos2\theta$. It is possible to give a physical meaning to the ϕ and θ functions defined in (4) and (5). In fact, performing a rotation of the invariant basis at any spacetime point with angle θ:

$$dx'=\cos\theta\,dx+\sin\theta\,dy\ ,$$

$$dy'=-\sin\theta\,dx+\cos\theta\,dy\ ,$$

the two-dimensional metric becomes

$$e^{2\phi}(dx')^2+e^{-2\phi}(dy')^2\ ,$$

which has the form of a pure $+$ wave of amplitude ϕ. It is therefore clear that ϕ in (4) represents the total amplitude of the gravitational wave while θ in (5) is the physical angle between dx and the direction of polarization of the wave.

For the pulse-wave solution (1) the polarization angle θ and the wave amplitude are given by

$$\tan2\theta=\frac{2w}{1-w^2-t},\quad \phi=\operatorname{arccosh}\left[\frac{1+w^2+t}{2t^{1/2}}\right]. \tag{6}$$

For the value of w taken in (2), where we make $H=0$, the polarization angle is null except along the null rays $u\in(u_B,u_F)$. It is interesting to see how θ changes along the null ray $u=0$, say, from $t=0$, where it takes a finite value, to $t\to\infty$, where it goes like

$$\tan2\theta\xrightarrow[u=0,t\to\infty]{}-\frac{2w}{t}\ . \tag{7}$$

This indicates that the metric approaches the Kasner background when $t\to\infty$; i.e., it becomes diagonal.

The wave amplitude ϕ decreases like $t^{-1/2}$ as it is typical of gravitational waves in homogeneous backgrounds.[4]

IV. A SOLUTION WITH A GRAVITATIONAL WAVE PULSE AND SOLITON WAVES

This solution is generated from the seed solution (1) by a soliton transformation[1,4] with four complex pole trajectories. As remarked in the Introduction the essential step in performing the soliton transformation is to find the generating function by means of an integration. For the metric (1) this has been found by Kitchingham[13] and, after correcting some misprints in the published version, it reads

$$\psi(\lambda)=(\lambda/k)^{1/4}\begin{bmatrix}\cos(Y) & k^{-1}\sin(Y)\\ w\cos(Y)-\lambda^{1/2}\sin(Y) & (\lambda/k)^{1/2}[\cos(Y)+w\lambda^{-1/2}\sin(Y)]\end{bmatrix}, \tag{8a}$$

where λ is an arbitrary complex parameter,

$$Y(\lambda,t,z)\equiv k^{1/2}\int w'(1-2uk)^{-1/2}du\ , \tag{8b}$$

and

$$k(\lambda,t,z)\equiv\lambda(\lambda^2+2z\lambda+t^2)^{-1}\ . \tag{8c}$$

Now the new solutions (soliton solutions) can be generated by means of purely algebraic operations.[1,4] We define four pole trajectories

$$\mu_i(t,z)\equiv u_i-z\pm[(u_i-z)^2-t^2]^{1/2},\quad i=1,\ldots,4 \tag{9a}$$

which must verify that

$$\mu_3=\bar{\mu}_1, \quad \mu_4=\bar{\mu}_2 , \tag{9b}$$

and take μ_1 and μ_2 with opposite signs in the square root of (9a) to avoid spacelike singularities.[4] Then we define two vectors,

$$m_a^{(i)}=(m_0)_b^{(i)}[\psi_0^{-1}(\mu_i,t,z)]_{ba}, \quad a,b=1,2 , \tag{10a}$$

with $(m_0)_e^{(i)}$ arbitrary complex vectors and construct the 4×4 complex matrix,

$$\Gamma_{ij}=[m_c^{(i)}(g_0)_{cb}m_b^{(j)}](\mu_i\mu_j-t^2)^{-1} , \tag{10b}$$

where $(g_0)_{cb}$ are the four coefficients of the seed metric (1) written in the form of the next equation (11).

Finally the new solution can be written as

$$ds^2=f(t,z)(dz^2-dt^2)+g_{ab}(t,z)dx^a dx^b , \tag{11}$$

with

$$f=f_0\left[\frac{\mu_1\mu_2}{u_1u_2}\right]^{3/2}\frac{(\mu_2-\mu_1)^{-2}}{(\mu^2-t^2)(\mu^2-t^2)}\left[\frac{t(\mu_1-\mu_2)^2}{(\mu_1 m_2-t^2)^2}[tc_1c_2+(\mu_1\mu_2)^{1/2}s_1s_2]^2+[(\mu_2)^{1/2}c_1s_2-(\mu_1)^{1/2}s_1c_2]^2\right] , \tag{12a}$$

where f_0 is the corresponding coefficient of the seed metric,

$$s_i(t,z)\equiv\sin[Y(\mu_i,t,z)+\phi_i], \quad c_i(t,z)\equiv\cos[Y(\mu_i,t,z)+\phi_i] , \tag{12b}$$

and the complex parameters ϕ_i are introduced instead of $(m_0)_c^{(i)}$:

$$(2u_i)^{1/2}(m_0)_1^{(i)}\equiv\epsilon_i\sin\phi_i, \quad (m_0)_2^{(i)}\equiv\epsilon_i\cos\phi_i . \tag{12c}$$

Now, Einstein's equations determine f up to multiplication with an arbitrary constant. The parameters ϵ_i can be absorbed in such a constant; consequently they do not play an essential role and may be ignored. The remaining coefficients are

$$g_{ab}=(|\mu_1||\mu_2|)^2t^{-4}\left[(g_0)_{ab}-\sum_{i,j=1}^{4}(\Gamma^{-1})_{ij}\phi_a^{(i)}\phi_b^{(j)}(\mu_i\mu_j)^{-1}\right] , \tag{13a}$$

where

$$\begin{aligned}\phi_1^{(i)}(t,z)&\equiv t^{1/2}(2u_i\mu_i)^{-3/4}m_i^{1/2}s_i ,\\ \phi_2^{(i)}(t,z)&\equiv t^{1/2}(2u_i\mu_i)^{-3/4}(w\mu_i^{1/2}s_i+tc_i) .\end{aligned} \tag{13b}$$

This final solution has a rather complex structure, especially since one has to invert the 4×4 matrix (10b) to obtain (13a). For this reason, to get explicit solutions with more than four poles is impractical. Fortunately, the main features of more general solutions can be seen from those with fewer parameters.

As we have remarked in the Introduction, in order to get soliton waves we require at least four complex pole trajectories. They have to be complex to avoid light-cone singularities;[1] note that if we take real poles (9a) with u_i real, then $\mu_i(t,z)$ are not defined on the whole range of t,z coordinates. We need two independent poles to get localized solutions, as we shall see shortly, and four complex poles in all because in order to get a real metric solution[1] for each complex pole its complex conjugate must also be a pole.

In spite of the complexity of the solution (11) we may classify and give meaning to its parameters in the following way.

The parameters A, u_F, u_B are the pulse-wave parameters. They characterize the seed solution and the pulse wave in the new solution. A is the *amplitude of the pulse wave* and $|u_B-u_F|$ the *width of the pulse wave.*

The parameters u_i are the soliton parameters. They appear in the pole trajectories (9) and characterize the spatial origin, width, and amplitude of the soliton waves. Thus $\mathrm{Re}(u_i)$ gives the *soliton origins* on the z axis, $\mathrm{Im}(u_1-u_2)$ gives the *soliton width,* and $\mathrm{Im}(u_1-u_2)[|\mathrm{Im}(u_1)|+|\mathrm{Im}(u_2)|]^{-1}$ the *soliton amplitude.*

The pole trajectories appear explicitly in (9a), we note that g_{ab} is given by the seed metric $(g_0)_{ab}$ minus a "perturbation" and all modulated by the pole trajectories in the combination $|\mu_1||\mu_2|t^{-2}$ which contain the main soliton contribution. In Fig. 2 this combination of pole trajectories has been represented in the t-z plane, it gives essentially a localized function along the light cones $|z-\mathrm{Re}(u_i)|^2-t^2$. This localization is achieved because we take μ_1 and μ_2 with opposite signs in (9a); it cannot be achieved with one independent pole only [since it leads to metric singularities at $|z|\to\infty$ (Ref. 4)]. Thus g_{ab} can be seen roughly (i.e., ignoring the perturbation which may play an important role in some coefficients) as a superposition of the seed solution, whose $(g_0)_{12}$ component has been represented in Fig. 1, and the soliton waves shown in Fig. 2. It is important to note that the two soliton waves take the same values for $z\to\pm\infty$.

The remaining parameters ϕ_i may be called *polarization parameters,*[4] because they are related to the nondiagonality of the matrix g_{ab} and therefore, the polarization of the associated waves. In fact when the seed is diagonal one may choose ϕ_i such that the vectors $m_a^{(i)}$ in (10a) have one null component, then the new solution is

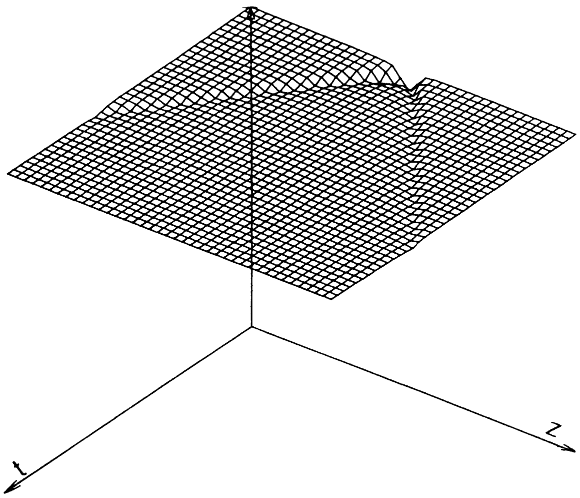

FIG. 2. Soliton waves: $(|\mu_1||\mu_2|)^2 t^{-4}$ in the t-z plane. The parameters in (9) are $\mathrm{Re}(u_1)=R(u_2)=-0.8$, $\mathrm{Im}(u_1)=0.043$, $\mathrm{Im}(u_2)=0.045$. There is reflection symmetry in the z axis at $z\to\pm\infty$.

also diagonal. In the present case the seed solution is already nondiagonal and the soliton solution is nondiagonal anyway.

V. SOLUTION ANALYSIS

A detailed analysis of the soliton solution (11) may be performed analytically by studying the asymptotic behavior of g_{ab} and the polarization functions introduced in Sec. III.

The coefficient f is, according to Einstein's equations,[1] entirely determined by g_{ab}. We shall therefore ignore it and concentrate on the metric coefficients g_{ab}.

We shall consider three asymptotic regions: timelike infinity ($|z| \ll t\to\infty$), spacelike infinity ($t\ll|z|\to\infty$), and null infinity ($|z|-t\to\infty$).

At timelike infinity the solution approaches the seed solution $(g_0)_{ab}$; this is typical of all soliton solutions independently of the number of pole trajectories[4] the reason being that $|\mu_i|t^{-1}$ approaches unity in this region. We have

$$\begin{aligned} g_{11} &\to (g_0)_{11}[1+O(t^{-1})]\ , \\ g_{12} &\to (g_0)_{12}[1+O(t^{-1/2})]\ , \\ g_{22} &\to (g_0)_{22}[1+O(t^{-1})]\ . \end{aligned} \tag{14}$$

The polarization angle and wave amplitude (4) and (6) also approach the seed values (6):

$$\begin{aligned} \tan 2\theta &\to -2wt^{-1}[1+O(t^{-1/2})]\ , \\ \phi &\to \mathrm{arccosh}\{t^{1/2}[1+O(t^{-1/2})]\}\ , \end{aligned} \tag{15}$$

where we recall that $w=H(=0)$ in this region [see (2)].

At spacelike infinity we have a similar behavior because, as remarked in the last section, we have chosen the two independent pole trajectories with opposite signs. Therefore, we have

$$\begin{aligned} g_{11} &\to (g_0)_{11}[1+O(z^{-1})]\ , \\ g_{12} &\to (g_0)_{12}[1+O(z^{-1/2})]\ , \\ g_{22} &\to (g_0)_{22}[1+O(z^{-1})]\ , \end{aligned} \tag{16}$$

and the polarization angle and wave amplitude as expected,

$$\begin{aligned} \tan 2\theta &\to \frac{2w}{1-w^2-t}[1+O(z^{-1/2})]\ , \\ \phi &\to \mathrm{arccosh}\left[\frac{1+w^2+t}{2t^{1/2}}\right][1+O(z^{-1/2})]\ , \end{aligned} \tag{17}$$

where again $w=H(=0)$ in this region.

Of course the interesting asymptotic features are found at null infinity where the pulse wave and the soliton waves are localized. Recalling that the pulse wave travels to the right (towards positive z, see Fig. 1) and that the soliton waves travel in opposite directions (see Fig. 2) one of the soliton waves will collide with the pulse wave. The effect of the collision may be seen by analyzing the two soliton waves at null infinity; if no pulse wave were present they would be completely symmetric.

The asymptotic behavior at null infinity along different light cones may be summarized as

$$\begin{aligned} g_{11} &\to (g_0)_{11}[1-2\sqrt{2}C(\chi_{11})][1+O(t^{-1/2})]\ , \\ g_{12} &\to -2\sqrt{2}tC(\chi_{12})[1+O(t^{-1/2})]\ , \\ g_{22} &\to (g_0)_{22}[1-2\sqrt{2}C(\chi_{12})][1+O(t^{-1/2})]\ , \end{aligned} \tag{18}$$

with

$$C(\chi_{ij}) \equiv \begin{cases} \mathrm{Re}(\chi_{ij}) & \text{if } z=-t+b\ , \\ \mathrm{Im}(\chi_{ij}) & \text{if } z=t+a\ , \end{cases} \tag{19a}$$

where a,b are finite real parameters:

$$\begin{aligned} \chi_{11} \equiv{}& d_1(2u_1)^{-3/2}s_1{}^2+2d_2(4u_1u_2)^{-3/4}s_1s_2+d_5(2u_2)^{-3/2}s_2{}^2+d_3|2u_1|^{-3/2}|s_1|^2 \\ &+2d_4(4u_1\bar{u}_2)^{-3/4}s_1\bar{s}_2+d_6|2u_2|^{-3/2}|s_2|^2\ , \\ \chi_{12} \equiv{}& d_1(2u_1)^{-3/2}s_1c_1+d_2(4u_1u_2)^{-3/4}(s_1c_2+s_2c_1)+d_5(2u_2)^{-3/2}s_2c_2+\tfrac{1}{2}d_3|2u_1|^{-3/2}(s_1\bar{c}_1+\bar{s}_1c_1) \\ &+d_4(4u_1\bar{u}_2)^{-3/4}(s_1\bar{c}_2+c_1\bar{s}_2)+d_6|2u_2|^{-3/2}(s_2\bar{c}_2+\bar{s}_2c_2)\ , \\ \chi_{22} \equiv{}& d_1(2u_1)^{-3/2}c_1{}^2+2d_2(4u_1u_2)^{-3/4}c_1c_2+d_5(2u_2)^{-3/2}c_2{}^2+d_3|2u_1|^{-3/2}|c_1|^2 \\ &+2d_4(4u_1\bar{u}_2)^{-3/4}c_1\bar{c}_2+d_6|2u_2|^{-3/2}|c_2|^2\ , \end{aligned} \tag{19b}$$

where an overbar denotes complex conjugation,

$$\begin{aligned}
d_1 &\equiv -2e_1(e_1+\bar{e}_1)^2(e_1-e_2)^2(e_1-\bar{e}_2)^2(e_1-\bar{e}_1)^{-2}(e_1+\bar{e}_2)^{-2}(e_1+e_2)^{-2} ,\\
d_2 &\equiv 4e_1e_2(e_1+\bar{e}_1)(e_2+\bar{e}_2)(e_1-e_2)\,|e_1-\bar{e}_2|^2(e_1+e_2)^{-2}\,|e_1+\bar{e}_2|^{-2}\,|e_1-\bar{e}_1|^{-1}\,|e_2-\bar{e}_2|^{-1} ,\\
d_3 &\equiv 4\,|e_1|^2(e_1+\bar{e}_1)\,|e_1-e_2|^2\,|e_1-\bar{e}_2|^2\,|e_1+e_2|^{-2}\,|e_1+\bar{e}_2|^{-2}\,|e_1-\bar{e}_1|^{-2} ,\\
d_4 &\equiv -4e_1\bar{e}_2(e_1+\bar{e}_1)(e_2+\bar{e}_2)\,|e_1-e_2|^2(e_1-\bar{e}_2)\,|e_1+e_2|^{-2}(e_1+\bar{e}_2)^{-1}(e_1-\bar{e}_1)^{-1}(e_2-\bar{e}_2)^{-1} ,\\
d_5 &\equiv 2e_2(e_2+\bar{e}_2)^2(e_1-e_2)^2(\bar{e}_1-e_2)^2(e_1+e_2)^{-2}(\bar{e}_1+e_2)^{-2}(e_2+\bar{e}_2)^{-2} ,\\
d_6 &\equiv -4\,|e_2|^2(e_2+\bar{e}_2)\,|e_1-e_2|^2\,|e_1-\bar{e}_2|^2\,|e_1+e_2|^{-2}\,|e_1+\bar{e}_2|^{-2}(e_2-\bar{e}_2)^{-2} ,
\end{aligned} \tag{19c}$$

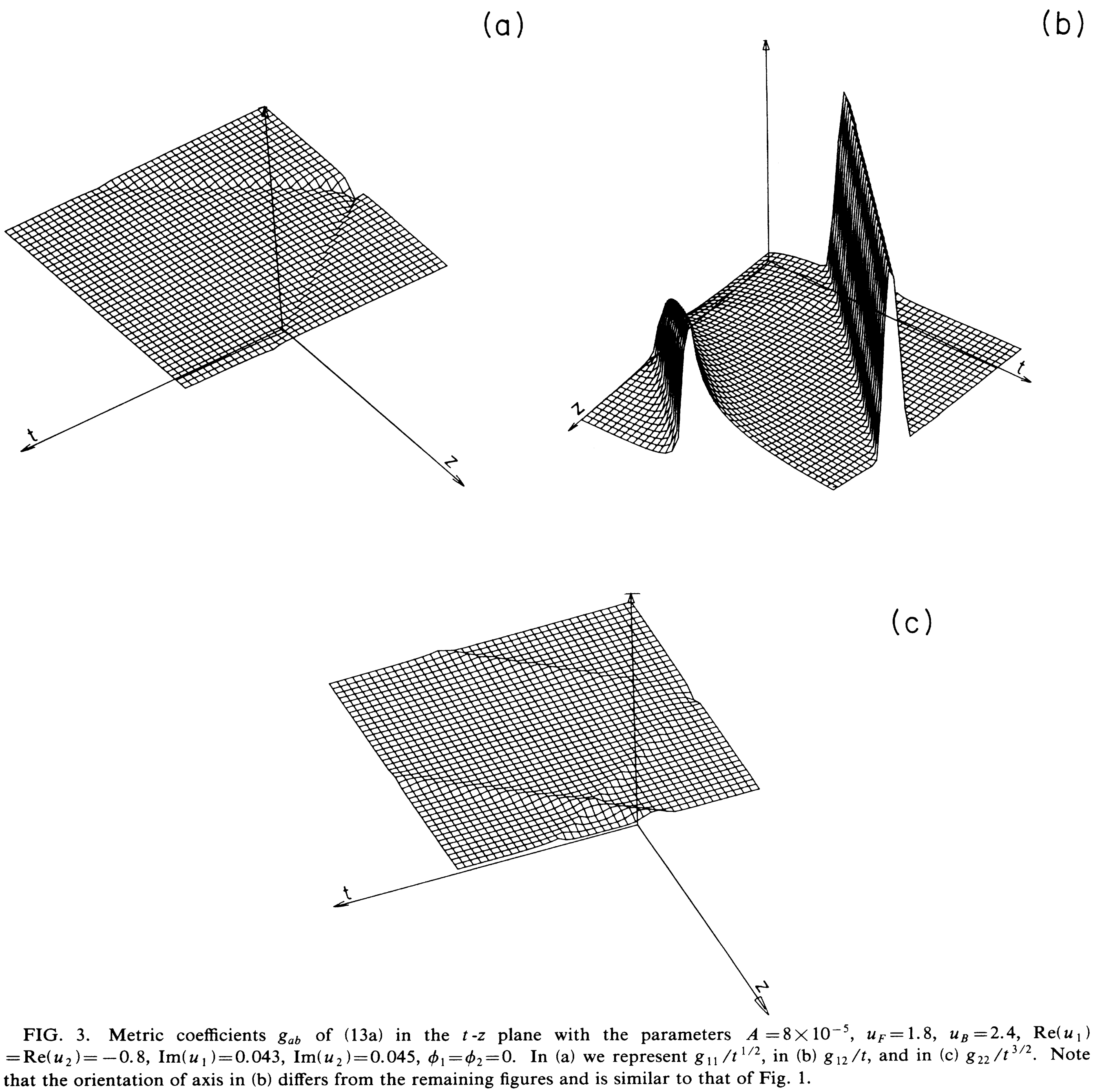

FIG. 3. Metric coefficients g_{ab} of (13a) in the t-z plane with the parameters $A=8\times10^{-5}$, $u_F=1.8$, $u_B=2.4$, $\mathrm{Re}(u_1)=\mathrm{Re}(u_2)=-0.8$, $\mathrm{Im}(u_1)=0.043$, $\mathrm{Im}(u_2)=0.045$, $\phi_1=\phi_2=0$. In (a) we represent $g_{11}/t^{1/2}$, in (b) g_{12}/t, and in (c) $g_{22}/t^{3/2}$. Note that the orientation of axis in (b) differs from the remaining figures and is similar to that of Fig. 1.

and

$$e_i = \begin{cases} -\sqrt{u_i - b} & \text{if } z = -t + b \ , \\ \sqrt{a - u_i} & \text{if } z = t + a \ . \end{cases} \tag{19d}$$

The asymptotic behavior of the pulse wave is obtained by simply taking one value for b in (19d) such that $b \in (u_F, u_B)$. The asymptotic behavior of the soliton which collides with the pulse wave is obtained by taking a of order $\mathrm{Re}(u_i)$ in (19d) and for the soliton which does not collide by taking $b \sim \mathrm{Re}(u_i)$. It is clear from (19d) and (19a) that the asymptotic values for the soliton will differ, in opposition to the soliton waves of Fig. 2, and this can be interpreted because of the interaction with the pulse wave.

An interesting feature of (18) is that the diagonal coefficients of the metric g_{11} and g_{22} grow in time, like the seed solution, although they do not reach exactly the same values due to the parameters χ_{11} and χ_{12} which differ for each light cone. Furthermore, the nondiagonal coefficient g_{12} does not grow like the seed coefficient $(g_0)_{12}$ which grows like $t^{1/2}$; this is because the perturbation term in (13a) dominates over the seed for this coefficient. Since the nondiagonal term is essential for the pulse wave we can say that, because of the soliton interaction, the pulse wave gets stronger and more polarized.

This last aspect, in fact, is seen more clearly from the polarization angle for the metric (11) which becomes, at null infinity,

$$\tan(2\theta) \to \frac{4\sqrt{2}C(\chi_{12})}{1-2\sqrt{2}C(\chi_{12})} t^{-1/2}[1+O(t^{-1/2})] \ , \tag{20}$$

which when compared with the seed value indicates that along the pulse wave the polarization angle is greater in the solution (11). Calculating along the soliton waves indicates that they give an angle of polarization comparable to that of the pulse wave. Recall that in the seed solution this angle is zero along the light cones $|z|^2 \sim t^2$.

The wave amplitude of the metric (11) goes like

$$\phi \to \operatorname{arccosh} \frac{1-2\sqrt{2}C(\chi_{12})}{2} t^{1/2}[1+O(t^{-1/2})] \ , \tag{21}$$

which grows like the seed metric but with different parameters along each null line, similar to the behavior of the diagonal coefficients.

Some of these features are illustrated in Fig. 3, where the exact metric coefficients are represented in the t-z plane.

In Fig. 3(a) the coefficient g_{11} is shown; in this coefficient the soliton waves dominate; it may be compared to the soliton waves of Fig. 2.

In Fig. 3(b) the coefficient g_{12} is represented; this coefficient shows clearly both the pulse and soliton waves. The pulse wave may be compared to the metric coefficient of the seed metric $(g_0)_{12}$ in Fig. 1, but note that there is a global $t^{1/2}$ factor between them because of the interaction with the solitons. It is also clear that the two soliton waves have different "amplitudes" reflecting the fact that one of them interacts strongly with the pulse wave.

In Fig. 3(c) the g_{22} coefficient is represented; this coefficient shows the pulse and soliton waves with comparable amplitudes and the trajectories on the t-z plane are clear.

Note that the solitons do not travel at the speed of light; it is known[2,3] that they start at zero speed like quasiparticles and only reach the speed of light at infinity. The pulse wave, however, propagates at the speed of light. Because of the singularity at $t=0$ the figures do not show very small values of t.

We can now say something about the global properties of the solution considered. The seed solution can be interpreted as a gravitational pulse propagating on a Kasner background;[21] because of the localized character of the wave the metric is globally of Petrov type I for the background. Similarly, the soliton solution (11) can be interpreted as giving the propagation and interaction of a gravitational wave pulse and two soliton waves on the same Kasner background; again the metric is globally of Petrov type I. This metric has the cosmological singularity only ($t=0$) like the corresponding Kasner metric. Thus it may be used as a cosmological model which starts highly inhomogeneous and evolves to a Kasner background with small localized waves of decreasing amplitudes propagating on it.

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