Non-Hermitian systems with $\mathcal{PT}$ symmetry

Author: David Viedma Palomo
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.

Advisor: Oleg Bulashenko

Abstract: We discuss the possibility to build the formalism of quantum mechanics based on Parity-Time ($\mathcal{PT}$) symmetry, rather than on the Hermiticity of operators. We consider the simple analytically tractable case of a $\mathcal{PT}$-symmetric 2×2 Hamiltonian matrix and analyse the conditions for the eigenvalues to be real and for time evolution to be unitary. Lastly, we review and reproduce the results of an experiment involving $\mathcal{PT}$ symmetry in optics.

I. INTRODUCTION

Quantum mechanics (QM) is a profound and well established discipline, with implications in almost every field of study in physics. QM has proven itself to be a robust theory that produces precise results, and so we will not question it in this project. Nonetheless, one of its basic principles has been a matter of study and discussion in recent years, namely, the requirement of the Hermiticity of the observables. This condition has its justification in the fact that it ensures the reality of their eigenvalues, which must be real if they are to be measurable.

However, since the discovery of certain non-Hermitian Hamiltonians with $\mathcal{PT}$ symmetry having entirely real spectra [1], it has been argued that while Hermiticity is surely sufficient, it may not be a necessary condition, and thus one could build a consistent theory without relying on it. Non-Hermitian Hamiltonians have been used before in open quantum systems and in effective descriptions, and $\mathcal{PT}$-symmetric ones have more recently appeared in systems that one might not necessarily relate directly to QM, such as in optics [2]. Knowing the implications of this symmetry will enable one to understand new Hamiltonians potentially describing systems of interest, which were previously treated as unphysical.

II. ORDINARY QUANTUM MECHANICS

The mathematical foundations of QM are based on postulates, which we will introduce briefly. We assume purely discrete non-degenerate states for simplicity, but one could extend them for more general cases [3].

Postulate 1: Every physical system is described by a state vector or ket, $|\psi\rangle$, that belongs to a Hilbert space, $\mathcal{H}$, and that contains all of the information of the system. This vector is normalised, i.e.: $\langle\psi|\psi\rangle = 1$, where $\langle\psi| = |\psi\rangle^\dagger$ is the Hermitian conjugate of the ket, and is called bra. The inner product is defined by:

$$\langle\phi|\psi\rangle \equiv \int dx\, \phi^*(x)\psi(x), \tag{1}$$

where $\psi(x) = \langle x|\psi\rangle$ and $\phi(x) = \langle x|\phi\rangle$ are the wave functions associated with their respective states, and $|x\rangle$ is an eigenvector of the position operator.

Postulate 2: Every physical quantity is associated with a linear Hermitian operator, which we call an observable. An operator is Hermitian, $A = A^\dagger$, if:

$$\langle\phi|A\psi\rangle = \langle A\phi|\psi\rangle\,. \tag{2}$$

Postulate 3: The only possible result of a measurement of a physical variable is an eigenvalue, a_n, of its associated observable: $A\,|\psi_n\rangle = a_n\,|\psi_n\rangle$, where $|\psi_n\rangle$ is an eigenvector of A. These form a complete set, so we can expand any state in terms of them: $|\psi\rangle = \sum_{i=1}^{N} a_i\,|\psi_i\rangle$. A Hermitian operator has entirely real eigenvalue spectra. The probability of obtaining a result in a measurement is $P(a_n) = |\langle\psi_n|\psi\rangle|^2$. QM is, thus, probabilistic.

Two eigenstates associated with different eigenvalues are orthogonal, i.e., $\langle\psi_n|\psi_m\rangle = 0$ for $a_n \neq a_m$. If the states are properly normalised, the condition is:

$$\langle\psi_n|\psi_m\rangle = \delta_{nm}, \tag{3}$$

where δ_{nm} is the Kronecker delta. It is clear that the orthogonality of the eigenstates depends on the definition of the inner product.

Postulate 4: Immediately after a measurement with result a_n, the state vector collapses into the normalised eigenvector associated with that value, $|\psi_n\rangle$.

Postulate 5: The time evolution of the state vector of a system is determined by the Schrödinger equation: $i\hbar\frac{\partial}{\partial t}\,|\psi\,(t)\rangle = H\,|\psi\,(t)\rangle$, H being the Hamiltonian. We can define an evolution operator, U, so that:

$$|\psi\,(t)\rangle = U\,(t, t_0)\,|\psi\,(t_0)\rangle\,. \tag{4}$$

It can be shown that this evolution operator must be unitary, i.e. $U^\dagger = U^{-1}$, in order for it to conserve probabilities, since unitary operators are norm preserving. This condition is always fulfilled if H is Hermitian. For a time-independent Hamiltonian:

$$U\,(t, t_0) = \exp\left(-iH(t - t_0)/\hbar\right). \tag{5}$$

The Hamiltonian determines the dynamics of a system. It is also an observable, its eigenvalues represent the energy, and so it must be Hermitian, according to Postulate 2. This imposes a rather restrictive condition on the Hamiltonians that can describe physical systems, and rules out infinitely many of them. A question then

arises: can we find an alternative for Hermiticity, but still maintain real energy spectra and unitary evolution? A possible candidate could be the requirement of $\mathcal{PT}$ symmetry. As we will see, both objectives can be met, but not without solving some problems first.

III. $\mathcal{PT}$ SYMMETRY

We define $\mathcal{P}$ and $\mathcal{T}$ by establishing how they act on the position and momentum operators [4]. $\mathcal{P}$ is the space reflection (parity) operator: $\mathcal{P}\hat{x}\mathcal{P} = -\hat{x}$, $\mathcal{P}\hat{p}\mathcal{P} = -\hat{p}$. And $\mathcal{T}$ is the time reversal operator: $\mathcal{T}\hat{x}\mathcal{T} = \hat{x}$, $\mathcal{T}\hat{p}\mathcal{T} = -\hat{p}$, $\mathcal{T}i\mathcal{T} = -i$. That last condition is for both operators to leave the fundamental commutation relation invariant: $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar\mathbb{I}$, with $\mathbb{I}$ being the identity. The square of both operators is also the identity, and they commute between them:

$$\mathcal{P}^2 = \mathcal{T}^2 = \mathbb{I}, \qquad [\mathcal{P}, \mathcal{T}] = 0. \tag{6}$$

$\mathcal{P}$ is an unitary operator, whereas $\mathcal{T}$ is antilinear, $\mathcal{T}(a_1 |\psi_1\rangle + a_2 |\psi_2\rangle) = a_1^* \mathcal{T} |\psi_1\rangle + a_2^* \mathcal{T} |\psi_1\rangle$; and antiunitary, $\langle \mathcal{T}\phi | \mathcal{T}\psi\rangle = \langle \phi | \psi\rangle^*$; and both of them are Hermitian. Having defined the relevant operators, we can start talking about $\mathcal{PT}$ symmetry. A $\mathcal{PT}$-symmetric Hamiltonian is one that commutes with the $\mathcal{PT}$ operator:

$$[H, \mathcal{PT}] = 0. \tag{7}$$

Hermiticity and $\mathcal{PT}$ symmetry are independent, i.e., a Hermitian Hamiltonian may or may not be $\mathcal{PT}$-symmetric, and vice versa; but some of them may be both [4]. Even if a Hamiltonian is $\mathcal{PT}$-symmetric, its eigenstates may not be so. Since $\mathcal{PT}$ is not a linear operator, the condition in (7) is not sufficient to guarantee it. Also, $\mathcal{PT}$ symmetry is not a sufficient condition to ensure the completeness of the eigenvectors of H, unlike in the Hermitian case of Postulate 3 [5].

We talk about unbroken $\mathcal{PT}$ symmetry if all of the eigenfunctions of H are also eigenfunctions of $\mathcal{PT}$, that is, if they are also $\mathcal{PT}$-symmetric. Otherwise we say that the symmetry is broken. This distinction is important, since we can only ensure the reality of the energy eigenvalues if the symmetry is unbroken, as we will prove now.

Suppose that we have unbroken symmetry, and that $|\psi\rangle$ is an eigenvalue of both H and $\mathcal{PT}$: $\mathcal{PT} |\psi\rangle = \lambda |\psi\rangle$, $H |\psi\rangle = E |\psi\rangle$. Considering that $\mathcal{T}$ is antilinear:

$$\mathcal{PT}H |\psi\rangle = \mathcal{PT}E |\psi\rangle = E^* \mathcal{PT} |\psi\rangle = E^* \lambda |\psi\rangle , \tag{8}$$

$$H\mathcal{PT} |\psi\rangle = H\lambda |\psi\rangle = \lambda H |\psi\rangle = E\lambda |\psi\rangle . \tag{9}$$

Knowing (7), both are equal, so we get $E^*\lambda = E\lambda$. To see that λ is non-zero, we do the following:

$$|\psi\rangle = (\mathcal{PT})^2 |\psi\rangle = \mathcal{PT}\lambda |\psi\rangle = \lambda^* \lambda |\psi\rangle , \tag{10}$$

since $(\mathcal{PT})^2 = \mathbb{I}$. Hence, $|\lambda|^2 = 1$, and λ is a pure phase. Coming back to (8) and (9), we get $E = E^*$ for any $|\psi\rangle$, and thus the energy eigenvalues are all real. It can be similarly proven that, in the broken phase, complex eigenvalues come as conjugate pairs.

Despite the importance of this result, we also encounter here the first of the problems that we talked about before. In general, it is not trivial to show whether a Hamiltonian has unbroken symmetry, as one needs to find a complete set of $\mathcal{PT}$-invariant eigenvectors of H.

In ordinary QM, the inner product is associated with Hermiticity, since the Hamiltonians are always Hermitian. In terms of matrix elements, it just involves a transpose and complex conjugation. Since we consider $\mathcal{PT}$-symmetric Hamiltonians, we define a possible inner product associated with that symmetry [4]:

$$(\phi, \psi) \equiv \int dx\, [\mathcal{PT}\phi(x)]\, \psi(x) = \int dx\, \phi^*(-x)\psi(x). \tag{11}$$

This choice of inner product is equivalent to changing the bras in the following way: $\langle\psi| \to \langle\psi| \mathcal{P}$, since $\mathcal{T}$ is just complex conjugation in this case. Hence [6]:

$$(\phi, \psi) = \langle \phi | \mathcal{P} | \psi \rangle . \tag{12}$$

It should be noted that properties such as the orthogonality relation in (3) are now altered due to the change in the inner product. Although this choice of inner product seems sensible, we will shortly show that it brings us problems even in the simplest of examples. In spite of that, we can easily see that its value is conserved under time evolution. The time evolution of a state was determined in (4), and the wave function evolves in the same way. If we assume a time-independent Hamiltonian, equation (5) still holds, and then:

$$(\phi(t), \psi(t)) = \int dx\, \left[\mathcal{PT}e^{-iHt}\phi(x)\right] e^{-iHt}\psi(x),$$

where we have taken $t_0 = 0$ and $\hbar = 1$ for ease of notation. Since $\mathcal{PT}$ and H commute, we can write: $\left[\mathcal{PT}e^{-iHt}\phi(x)\right] = [\mathcal{PT}\phi(x)]\, e^{iHt}$, which in turn implies:

$$(\phi(t), \psi(t)) = (\phi(0), \psi(0)) . \tag{13}$$

This result tells us that even though we have detoured from Hermitian Hamiltonians, if we use this non-standard inner product, norms are still conserved under time evolution, and that evolution is unitary. With this, we have met both of the conditions we set at the end of sec. II. Nonetheless, we are oblivious to an important flaw in the formalism for the time being.

A. Revealing the flaw via an example: A simple 2×2 matrix

Consider the following $\mathcal{PT}$-symmetric non-Hermitian 2×2 matrix:

$$H = \begin{pmatrix} i\gamma & \kappa \\ \kappa & -i\gamma \end{pmatrix}. \tag{14}$$

In this case, we define the parity operator as $\mathcal{P} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $\mathcal{T}$ is just complex conjugation. The eigenvalues and eigenvectors of (14) are:

$$E_\pm = \pm\sqrt{\kappa^2 - \gamma^2}, \tag{15}$$

$$|\psi^\pm\rangle = \begin{pmatrix} i\gamma/\kappa \pm \sqrt{1-\gamma^2/\kappa^2} \\ 1 \end{pmatrix}. \tag{16}$$

Looking at Eq. (15), one can distinguish two different regimes. For $\kappa > \gamma$ we have real eigenvalues, and we are in the unbroken regime. On the other hand, for $\kappa < \gamma$, the energies become complex conjugate pairs and we are in the broken regime. Precisely at $\kappa = \gamma$, we only have one eigenvalue, which is degenerate. These points in the (κ, γ) plane are called exceptional points [5]. We can check the $\mathcal{PT}$ symmetry of the states in both regimes. For the unbroken (ub) regime: $\mathcal{PT}\,|\psi^\pm\rangle_{(ub)} = \left(\begin{smallmatrix} 1 \\ -i\gamma/\kappa \pm \sqrt{1-\gamma^2/\kappa^2} \end{smallmatrix}\right)$. By direct analysis, we can check that:

$$\mathcal{PT}\,|\psi^\pm\rangle_{(ub)} = \left(-i\gamma/\kappa \pm \sqrt{1-\gamma^2/\kappa^2}\right)|\psi^\pm\rangle_{(ub)}\,. \tag{17}$$

By taking $\gamma/\kappa = \sin(\theta)$ we can immediately see that the eigenvalue is just a pure phase, as we proved in (10). As expected, in the unbroken regime the eigenvectors of H are also eigenvectors of $\mathcal{PT}$. On the contrary, in the broken (b) regime, one can check that $\mathcal{PT}\,|\psi^\pm\rangle_{(b)} \neq \lambda\,|\psi^\pm\rangle_{(b)}$. Everything seems to be working as expected until we calculate the $\mathcal{PT}$-norms of these eigenstates. For the unbroken regime:

$${}_{(ub)}\langle\psi^\pm|\mathcal{P}|\psi^\pm\rangle_{(ub)} = \pm 2\sqrt{1-\gamma^2/\kappa^2}. \tag{18}$$

This tells us two things. First, we were lacking a normalisation constant in (16), but that is a trivial thing to fix. The major problem here is that we are getting a state with a negative norm, which completely breaks the probabilistic interpretation of quantum mechanics! Norms in QM represent probabilities, and these must always be positive. We have to change something in order for the theory to have any physical meaning at all. Also, we can similarly check that the norms of the eigenstates vanish in the broken regime.

B. Fixing the mess: The C operator

In order to solve the negative-norm problem, we need to modify our definition of inner product. One of the possible choices is to introduce a $\mathcal{C}$ operator in the unbroken regime, resembling the charge conjugation operator [4]. $\mathcal{C}$ is a linear operator that commutes with both H and $\mathcal{PT}$, $[H, \mathcal{C}] = [\mathcal{PT}, \mathcal{C}] = 0$; and whose square is the identity, $\mathcal{C}^2 = \mathbb{I}$. We don't define it in the broken regime, since $\mathcal{C}$ and $\mathcal{PT}$ don't commute there [7]. The idea behind this operator is that its eigenvalues must precisely be the $\mathcal{PT}$-norms of its normalised eigenvectors, calculated using (11). Thus, the inner product defined by

$$\langle\phi, \psi\rangle_{CPT} \equiv \int dx\; [\mathcal{CPT}\phi(x)]\,\psi(x) = \langle\phi\,|\mathcal{PC}|\,\psi\rangle \tag{19}$$

should always be positive. Furthermore, since the eigenvalues of $\mathcal{C}$ will be ± 1, and $\mathcal{C}$ commutes with H, this inner product will also be norm preserving, in the same sense as in (13). We also see that if we make $\mathcal{C} = \mathcal{P}$, we recover the product in (1). The $\mathcal{C}$ operator will be defined in position space, in general, by:

$$\mathcal{C}(x, y) = \sum_n \phi_n(x)\phi_n(y), \tag{20}$$

where ϕ_n are the eigenfunctions of H. Since $\mathcal{C}$ depends on H, the inner product does as well. We say that the inner product is dynamically determined [4], since it depends on the system it is used on. One must obtain the eigenfunctions of H before knowing the inner product. We should mention that this is not the only possible choice of inner product nor the most general one, as seen in [7].

C. Back to the 2×2 case

After having introduced $\mathcal{C}$, let us check if it fixes the problems in our simple example. We can construct it using: $\mathcal{C} = |\psi^+\rangle\langle\psi^+| + |\psi^-\rangle\langle\psi^-|$, where the states have been normalised according to $|\psi\rangle \to N^{-1}|\psi\rangle$, with $N = \sqrt{2}(1-\gamma^2/\kappa^2)^{1/4}$, so that we get ± 1 in (18). We should also stress that the bras have been altered as before with the parity operator. Doing this we obtain:

$$\mathcal{C} = \frac{1}{\sqrt{1-\gamma^2/\kappa^2}} \begin{pmatrix} \frac{i\gamma}{\kappa} & 1 \\ 1 & \frac{-i\gamma}{\kappa} \end{pmatrix}. \tag{21}$$

Using this operator we can check the eigenvalues of $|\psi^\pm\rangle$ and calculate their $\mathcal{CPT}$-norms:

$$\mathcal{C}\,|\psi^\pm\rangle = \pm\,|\psi^\pm\rangle\;,\; \langle\psi^\pm, \psi^\pm\rangle_{CPT} = 1. \tag{22}$$

We see that $\mathcal{C}$ indeed returns the $\mathcal{PT}$-norm of a state, and fixing the inner product according to (19) corrects the negative-norm problem while keeping the eigenvectors orthonormal, $\langle\psi^\pm, \psi^\mp\rangle_{CPT} = 0$. We will now comment on a couple more elements of this formalism before introducing pseudo-Hermiticity.

D. Treatment of operators

Observables are Hermitian (2) in regular QM, and that ensures that their expectation values are real. We could argue that the corresponding condition for the $\mathcal{CPT}$ inner product would be $A = \mathcal{CPT}\,A\,\mathcal{CPT}$. However, since in the Heisenberg picture $A(t) = e^{iHt}Ae^{-iHt}$, the condition would not be maintained in time. If $H = H^T$, where ^{T} is

the transpose, a suitable condition is $A^T = \mathcal{CPT}\,A\,\mathcal{CPT}$ [4], but that imposes a restriction on the Hamiltonian. For a general one, the condition is to be Hermitian with respect to the new inner product, that is [8]:

$$\langle \phi, A\psi\rangle_{CPT} = \langle A\phi, \psi\rangle_{CPT}\,. \tag{23}$$

This implies that, in general, $\hat{x}$ and $\hat{p}$ are not observables now. After trying to avoid Hermiticity, we end up finding it again, but with respect to the new inner product.

E. The continuity equation

The continuity equation relates the probability density, $P(x,t) = |\psi(x,t)|^2$, and the probability current, $J(x,t) = \frac{\hbar}{2mi}\left[\psi^*(x,t)\frac{\partial\psi(x,t)}{\partial x} - \psi(x,t)\frac{\partial\psi^*(x,t)}{\partial x}\right]$, in the following way [9]: $\frac{\partial P(x,t)}{\partial t} + \frac{\partial J(x,t)}{\partial x} = 0$.

This equation is obtained from Schrödinger's equation and its Hermitian conjugate. It can be shown that if instead one takes its $\mathcal{PT}$-conjugate, for two arbitrary wave functions ψ_1 and ψ_2, the resulting equation is [9]:

$$\frac{\partial\psi_2^*\psi_1}{\partial t} + \frac{\hbar}{2mi}\frac{\partial}{\partial x}\left[\psi_2^*\frac{\partial\psi_1}{\partial x} - \psi_1\frac{\partial\psi_2^*}{\partial x}\right] = 0, \tag{24}$$

where every ψ_1, ψ_2^* is a function of (x,t) and $(-x,t)$, respectively. This equation leads to the conservation of the quantity: $Q = \int dx\,\psi_2^*(-x,t)\psi_1(x,t)$. For $\psi_1 = \psi_2$, this implies the conservation of the $\mathcal{PT}$ norm, as in (13).

IV. PSEUDO-HERMITICITY

$\mathcal{PT}$ symmetry can be placed in a more general frame known as pseudo-Hermiticity. A Hamiltonian is said to be η-pseudo-Hermitian [5] if: $H^\dagger = \eta H \eta^{-1}$, where η is Hermitian. It is clear that if $\eta = \mathbb{I}$, we recover regular Hermiticity. Also, the inner product defined by $\langle\phi|\eta|\psi\rangle$ is time independent if H is pseudo-Hermitian. A pseudo-Hermitian operator either has real eigenvalues, or they come in complex conjugate pairs. In the special case where $\eta = OO^\dagger$, O being a linear operator, their entire spectrum is real, just like those with unbroken $\mathcal{PT}$ symmetry. We see that both formalisms share several properties. The reason behind it is that as long as it has a complete orthonormal basis, every $\mathcal{PT}$-symmetric Hamiltonian is pseudo-Hermitian [10].

For every Hamiltonian with unbroken symmetry there exists a similarity transformation that maps it to a Hermitian Hamiltonian [4], $h = e^{-Q/2}He^{Q/2}$, where $e^Q = \mathcal{CP}$. This new Hamiltonian has the same eigenvalues as the original, so one could ask the reason for building the new formalism when one can recover Hermiticity using this transformation. First of all, computing h can be very complicated in general, since it may involve non-local terms; and secondly, a similarity transformation does not keep the eigenvectors invariant, so the relations between them may be altered [4].

V. A PHYSICAL SYSTEM DESCRIBED BY A $\mathcal{PT}$-SYMMETRIC HAMILTONIAN

In coupled optical systems, one can achieve gain and loss, i.e., increasing or decreasing amplitudes, with complex potentials. These can be implemented by complex refractive indexes: $n = n_R + in_I$, and the system can be described using a $\mathcal{PT}$-symmetric Hamiltonian. [2] If we consider paraxial conditions, and only one relevant transverse direction, they are governed by the equation:

$$i\frac{\partial\epsilon}{\partial z} + \frac{1}{2k}\frac{\partial^2\epsilon}{\partial x^2} + k_0\left[n_R(x) + in_I(x)\right]\epsilon = 0, \tag{25}$$

where ϵ is the electric field envelope, $k_0 = 2\pi/\lambda$ and $k = k_0 n_0$, with λ being the wavelength and n_0 the substrate index. It is straightforward to see that this equation is equivalent to Schrödinger's equation, $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi$, if we identify $\epsilon \leftrightarrow \psi$ and $z \leftrightarrow t$. A necessary condition for the equation to be $\mathcal{PT}$-symmetric is that the potential obeys $V(x) = V^*(-x)$. Hence, $n_R(x) = n_R(-x)$ and $n_I(x) = -n_I(-x)$. A system that fulfills this condition is one with two coupled parallel channels, one of them pumped with light to obtain gain, and with the other experiencing loss. The coupled equations that the field amplitudes obey in each channel are [2], [11]:

$$i\frac{\mathrm{d}\epsilon_1}{\mathrm{d}z} - i\frac{\gamma_g'}{2}\epsilon_1 + \kappa\epsilon_2 = 0 \quad i\frac{\mathrm{d}\epsilon_2}{\mathrm{d}z} + i\frac{\gamma_l}{2}\epsilon_2 + \kappa\epsilon_1 = 0, \tag{26}$$

where κ is the coupling constant, $\gamma_{g,l}$ are the gain and loss coefficients and $\gamma_g' = \gamma_g - \gamma_l > 0$. The condition for $\mathcal{PT}$ symmetry is then $\gamma_g' = \gamma_l = \gamma$. Considering this, we can put (26) in matrix form to get:

$$i\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix}\epsilon_1\\ \epsilon_2\end{pmatrix} = \begin{pmatrix}i\gamma/2 & -\kappa\\ -\kappa & -i\gamma/2\end{pmatrix}\begin{pmatrix}\epsilon_1\\ \epsilon_2\end{pmatrix}. \tag{27}$$

We see that the matrix in (27) is very similar to (14), so this system should have the same features: a broken and an unbroken phase, and an exceptional point between them, at $\gamma = 2\kappa$. One can solve (27) by diagonalising, obtaining for the eigenvectors:

$$\left|\psi^\pm(z)\right\rangle = e^{-iE_\pm z}\left|\psi^\pm(0)\right\rangle = e^{\mp iE_+ z}\left|\psi^\pm(0)\right\rangle, \tag{28}$$

since $E_+ = -E_-$. Using this, one can obtain the propagation along z of a general initial state. In particular, of an excitation in each of the channels, $|1\rangle$ and $|2\rangle$, as well as their associated intensities ($I_i \sim |\epsilon_i|^2$).

We depict the behaviour of such a system in Fig. 1. For $\gamma = 0$, we have a regular Hermitian coupled system with neither gain nor loss, for which we have reciprocal light propagation. This means that we get a mirrored intensity output if we change the excitation between channels. For $\gamma < 2\kappa$, in the unbroken phase, the propagation ceases to be reciprocal. And in the broken phase, for $\gamma > 2\kappa$, energies become complex and only one of the states in (28) effectively survives, favoring the channel with gain and resulting in channel 1 outputting more light.

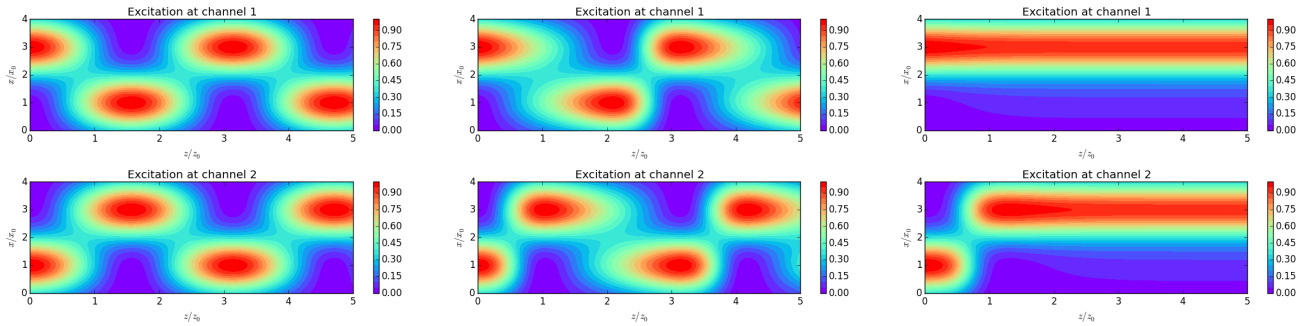

Figure 1: Normalised intensity in each channel after individual excitations for $\gamma = 0$ (L), $\gamma = \kappa$ (M) and $\gamma = 4\kappa$ (R).

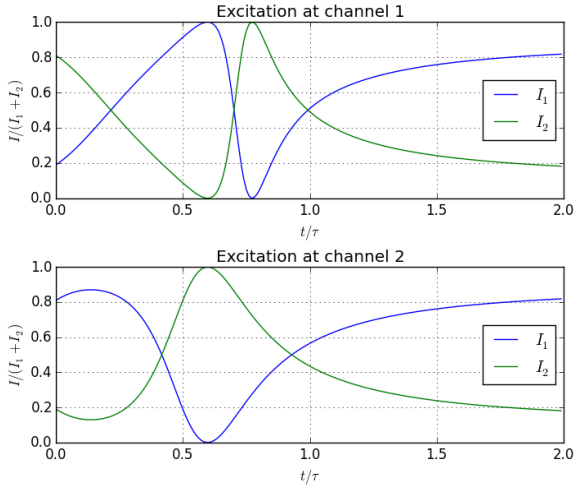

Figure 2: Normalised intensity output in each channel after individual excitations for $\gamma = 2\kappa$, $\gamma_m = 3\gamma$.

This broken-unbroken transition can be experimentally achieved, as was shown in [2]. The authors achieved gain that had a temporal build-up, and was assumed to be exponential, $\gamma_g = \gamma_m\,(1 - \exp(-t/\tau))$. This means that the system should cross the exceptional point as time progresses, and its effects should be observable. We solve Eqs. (26) numerically, and represent the output intensity with respect to time in Fig. 2. We see that for $t/\tau = 0$ we get a reciprocal response, since we only have loss for both channels. As time goes on, reciprocity is lost, and for $t/\tau \approx 1.0986$ we get the phase transition, after which the output intensity starts to favor channel 1. Experimental results matched this behaviour [2].

VI. CONCLUSIONS

We have seen that one can build a consistent formalism of quantum mechanics based on $\mathcal{PT}$ symmetry. By changing the inner product, one can solve the negative-norm problem whilst maintaining real eigenvalues and unitary evolution. However, one cannot get rid of Hermiticity, since observables need to be Hermitian with respect to the new inner product.

This new formalism allows one to study new Hamiltonians involving complex valued potentials, but it is not a generalisation of regular quantum mechanics, since not all Hermitian operators are $\mathcal{PT}$-symmetric. It is also limited, since it only provides the desired results in the unbroken regime. Both formalisms can however be placed in the more general frame of pseudo-Hermiticity.

Lastly, we have seen that $\mathcal{PT}$-symmetric Hamiltonians can be achieved experimentally. In systems with balanced gain and loss, stationary solutions can be found as long as the symmetry is unbroken. $\mathcal{PT}$ symmetry can be found in several fields of study, not only in optics, but also in cosmology, Bose-Einstein condensates, superconductivity, and many others. [5], [12]

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