# Gauge-invariant actions from constraint Hamiltonian dynamics

J. Gomis and K. Rafanelli
*Queens College of the City University of New York, Flushing, New York 11367-0904*
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Dirac's constraint Hamiltonian formalism is used to construct a gauge-invariant action for the massive spin-one and -two fields.

## I. INTRODUCTION

Recently there has been much interest in the construction of gauge-invariant actions for string fields[1,2] and ordinary fields.[2] A paper by West[3] indicates one way to proceed for the several field types. He begins with a Lorentz-invariant action whose variation gives the usual free-field equations of motion and which contains the requisite auxiliary Lagrange multiplier fields so that spin-restricting constraints appear as equations of motion. This action, which has no particular gauge symmetry, is transformed into a gauge-invariant action through the introduction of auxiliary dynamical (ghostlike) fields. The gauge invariance is with respect to the joint variations of all fields. Ramond presented an alternate gauge-invariance construction at the level of the equations of motion.[2]

In this paper we use the formalism of Dirac's constraint Hamiltonian dynamics[4] to arrive at the gauge-invariant actions for the massive boson fields of spin one and spin two. We start with the original Lagrangian density $\mathscr{L}_0$ containing only auxiliary Lagrange multiplier fields. The momenta conjugate to these fields vanish, and are primary constraints. To test the stability of these constraints we construct the Dirac Hamiltonian $H_D^0$. Stability of the primary constraints generates secondary constraints.

In order to obtain gauge invariance at this level we require the system of primary and secondary constraints to be first class.[5] We are led in this way to systematic, step-by-step modifications of $H_D^0$, and hence $\mathscr{L}_0$, such that the constraints are rendered first class. The introduction of auxiliary dynamical (ghostlike) fields is required.

Stability of the secondary constraints generates tertiary constraints. Again guided by the principle of gauge invariance, we require the system of primary, secondary, and tertiary constraints to be first class. This lead to further modifications of $H_D$ and hence $\mathscr{L}$. These modifications require other dynamical auxiliary fields and/or extra coupling terms.

Finally when the system of primary, secondary, and tertiary constraints is rendered first class, the constraint analysis ends since stability of the tertiary constraints generates no new constraints. The final action in each case is the gauge-invariant action of West.[3] The final number of phase-space degrees of freedom, i.e., the number of fields plus conjugate momenta minus twice the number of first-class constraints, is correct in each case.

## II. SPIN ONE

The four-vector field $A_\mu$ describing the massive spin-one particle obeys the equation of motion

$$(\Box+m^2)A_\mu=0 \tag{1}$$

subject to the condition

$$\partial^\mu A_\mu=0\ . \tag{2}$$

The Lorentz-invariant action whose variation yields (1) and (2) as Euler-Lagrange (EL) equations of motion is

$$S_0=\int dx\ \mathscr{L}_0\ , \tag{3}$$

where

$$\mathscr{L}_0=\tfrac{1}{2}A^\mu(\Box+m^2)A_\mu+\chi(\partial^\mu A_\mu)\ , \tag{4}$$

with $\chi$ treated as an auxiliary, Lagrange multiplier field. As discussed by West,[3] $S_0$ has no particular gauge symmetry but can be modified to include another auxiliary field $\phi$, such that under the joint gauge variations of $A_\mu$, $\chi$, and $\phi$, $S_0\rightarrow S$ which is gauge invariant.

The Hamiltonian analysis for this case proceeds as follows. From (4) we obtain the four-momentum conjugate to the boson field[5]

$$\pi^\mu=\frac{\partial\mathscr{L}_0}{\partial\dot{A}_\mu}=-\dot{A}^\mu+\chi g^{0\mu}\ , \tag{5}$$

while the momentum conjugate to the multiplier field

$$\pi_\chi=\frac{\partial\mathscr{L}_0}{\partial\dot{\chi}}=0 \tag{6}$$

is a primary constraint. To test the stability of this constraint we must construct the Dirac Hamiltonian corresponding to $\mathscr{L}_0$:[4]

$$H_D^0=\int d^3\mathbf{x}(\mathscr{H}_c^0+\lambda_\chi\pi_\chi)\ , \tag{7}$$

where $\mathscr{H}_c^0=\pi_\mu\dot{A}^\mu-\mathscr{L}_0$ is explicitly

$$\mathscr{H}_c^0=-\tfrac{1}{2}\pi^\mu\pi_\mu-\tfrac{1}{2}\chi^2+\chi[g^{0\mu}\pi_\mu-g^{i\mu}(\partial_i A_\mu)]+\tfrac{1}{2}(\partial^i A^\mu)(\partial_i A_\mu)-\frac{m^2}{2}A^\mu A_\mu\ . \tag{8}$$

Then with the equal-time Poisson brackets (PB) for scalar fields

$$(\chi(\mathbf{x},t),\pi(\mathbf{y},t))=\delta(\mathbf{x}-\mathbf{y}) \tag{9}$$

 

we obtain

$$\dot{\pi}_X=(\pi_X,H_D^0)=X-[\pi^0-(\partial_i A^i)] \ . \quad (10)$$

As this is not identically zero it is a secondary constraint.

Now $(\pi_X,\dot{\pi}_X)=-\delta(\mathbf{x}-\mathbf{y})$ and therefore the two constraints $\pi_X\approx 0$ and $\dot{\pi}_X\approx 0$ as they stand are second class, and the stability of $\dot{\pi}_X$ with respect to $H_D^0$ will remove the arbitrariness of $\lambda$ and destroy gauge invariance.[6] The only way to avoid this is to modify $H_D^0$ so that $\dot{\pi}_X$ does not contain $X$. This is done via

$$\mathscr{H}_c^0\to\mathscr{H}_c^1=\mathscr{H}_c^0+\tfrac{1}{2}X^2 \ , \quad (11)$$

which implies

$$\mathscr{L}_0\to\mathscr{L}_1=\mathscr{L}_0-\tfrac{1}{2}X^2 \ . \quad (12)$$

Now with respect to $H_D^1$, the secondary constraint,

$$\dot{\pi}_X=(\pi_X,H_D^1)=-\pi^0+\partial_i A^i\approx 0 \ , \quad (13)$$

and the primary constraint are first class.

Next we must test the stability of the secondary constraint (13) with respect to $H_D^1$. Using the equal-time PB for vector fields

$$(A^\mu(\mathbf{x},t),\pi^\nu(\mathbf{y},t))=-g^{\mu\nu}\delta(\mathbf{x}-\mathbf{y}) \quad (14)$$

we obtain

$$\ddot{\pi}_X=(\dot{\pi}_X,H_D^1)=+\partial^i\partial_i A^0+m^2A^0+\partial_i\pi^i \quad (15)$$

which is a tertiary constraint. We then have $(\pi_X,\ddot{\pi}_X)=0$ but $(\dot{\pi}_X,\ddot{\pi}_X)=-m^2\delta(\mathbf{x}-\mathbf{y})$ so that to reinstate gauge invariance at this level, we must modify $\dot{\pi}_X$ to read

$$\dot{\pi}_X=-\pi^0+\partial_i A^i+m\phi \quad (16)$$

where $\phi$ is an auxiliary field such that

$$\dot{\phi}=\pi_\phi \ , \quad (17)$$

where $\pi_\phi$ is the momentum conjugate to $\phi$. Then the three constraints $\pi_X,\dot{\pi}_X,\ddot{\pi}_X$ will be first class.

To obtain (16) let

$$\mathscr{H}_c^1\to\mathscr{H}_c^2=\mathscr{H}_c^1-Xm\phi \ , \quad (18)$$

and, therefore,

$$\mathscr{L}_1\to\mathscr{L}_2=\mathscr{L}_1+Xm\phi \ ; \quad (19)$$

to obtain (17) we need only add the term $\frac{1}{2}\pi_\phi{}^2$ to $\mathscr{H}_c^2$ but this would destroy the Lorentz invariance of the action. Therefore let

$$\mathscr{H}_c^2\to\mathscr{H}_c^3=\mathscr{H}_c^2+\tfrac{1}{2}\pi_\phi{}^2-\tfrac{1}{2}(\partial^i\phi)(\partial_i\phi) \ ; \quad (20)$$

thus

$$\mathscr{L}_2\to\mathscr{L}_3=\mathscr{L}_2+\tfrac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) \ . \quad (21)$$

Now test the stability of $\ddot{\pi}_X$ with respect to $H_D^3$. We obtain

$$\dddot{\pi}_X=(\ddot{\pi}_X,H_D^3)=-(\partial^i\partial_i+m^2)\pi_X+m^3\phi \ . \quad (22)$$

Thus the constraint analysis would end if the $m^3\phi$ term were removed from $\dddot{\pi}_X$. For this to occur, let

$$\mathscr{H}_c^3\to\mathscr{H}_c^4=\mathscr{H}_c^3+\tfrac{1}{2}m^2\phi^2 \quad (23)$$

and

$$\mathscr{H}_3\to\mathscr{L}_4=\mathscr{L}_3-\tfrac{1}{2}m^2\phi^2 \ . \quad (24)$$

Thus the constraint analysis ends and $S_4=\int dx\,\mathscr{L}_4$ is the gauge-invariant action of West,[3] derived via Dirac's constraint Hamiltonian formalism with one auxiliary scalar multiplier field and one dynamical scalar field. There are three first-class constraints. Therefore the final number of phase-space degrees of freedom is $12-6=6$, the appropriate number.

The explicit variations under which $S_4$ is invariant are

$$\delta A^\mu=\partial^\mu\Lambda \ ,$$

$$\delta X=(\Box+m^2)\Lambda \ ,$$

$$\delta\phi=m\Lambda \ .$$

Further the equations of motion can be used to eliminate the field $X$ which reduces $\mathscr{L}_4$ to the Stueckelberg form.[7]

## III. SPIN TWO

The symmetric tensor field $\phi_{\mu\nu}$, describing the massive spin-two particle, obeys the equation of motion

$$(\Box+m^2)\phi_{\mu\nu}=0 \quad (25)$$

subject to the conditions

$$\partial^\nu\phi_{\mu\nu}=0 \quad (26)$$

and

$$\phi_\mu^\mu=0 \ . \quad (27)$$

The Lorentz-invariant action whose variation yields (25), (26), and (27) as EL equations of motion is (3) with

$$\mathscr{L}_0=\tfrac{1}{2}\phi^{\mu\nu}(\Box+m^2)\phi_{\mu\nu}+B^\mu(\partial^\nu\phi_{\mu\nu})+X(\phi_\mu^\mu) \quad (28)$$

with $B^\mu$ and $X$ treated as auxiliary, Lagrange multiplier fields.

The Hamiltonian constraint analysis which results in a gauge-invariant action for this case is as follows. From (28) we obtain the symmetric tensor momentum conjugate to the boson field

$$\pi^{\mu\nu}=\frac{\partial\mathscr{L}_0}{\partial\dot{\phi}_{\mu\nu}}=-\phi^{\mu\nu}+\tfrac{1}{2}(B^\mu g^{0\nu}+B^\nu g^{0\mu}) \ , \quad (29)$$

while the momenta conjugate to the multiplier fields

$$\pi_B^\mu=\frac{\partial\mathscr{L}_0}{\partial\dot{B}_\mu}=0 \quad (30)$$

and

$$\pi_X=\frac{\partial\mathscr{L}_0}{\partial\dot{X}}=0 \quad (31)$$

are primary constraints. To test the stability of these constraints we must construct

$$H_D^0=\int d\mathbf{x}(\mathscr{H}_c^0+\lambda_\mu\pi_B^\mu+\lambda\pi_X) \ , \quad (32)$$

where $\mathscr{H}_c^0=\pi_{\mu\nu}\dot{\phi}^{\mu\nu}-\mathscr{L}_0$ is, explicitly,

$$\mathscr{H}_c^0=-\tfrac{1}{2}\pi^{\mu\nu}\pi_{\mu\nu}+\tfrac{1}{2}(B^\mu g^{0\nu}+B^\nu g^{0\mu})\pi_{\mu\nu}-\frac{m^2}{2}\phi^{\mu\nu}\phi_{\mu\nu}-\tfrac{1}{2}(B^\mu g^{i\nu}+B^\nu g^{i\mu})(\partial_i\phi_{\mu\nu})-\chi\phi_\mu^\mu+\tfrac{1}{2}(\partial^i\phi^{\mu\nu})(\partial_i\phi_{\mu\nu})-\tfrac{1}{4}(B^\mu B_\mu+B_0{}^2)\ . \tag{33}$$

Then

$$\dot{\pi}_B^\mu=-\pi^{\mu 0}+\partial_i\phi^{\alpha i}+\tfrac{1}{2}B^\alpha+g^{0\alpha}B^0 \tag{34}$$

and

$$\dot{\pi}_\chi=\phi_\mu^\mu \tag{35}$$

are secondary constraints which can only be made first class with respect to the $\pi_B^\mu$ by removal of the $B^\mu B_\mu$ and $B_0{}^2$ terms from $\mathscr{H}_c^0$. Therefore let

$$H_c^0\to H_c^1=H_c^0+\tfrac{1}{2}B^\mu B_\mu\ ; \tag{36}$$

hence

$$\mathscr{L}_0\to\mathscr{L}_1=\mathscr{L}_0-\tfrac{1}{4}B^\mu B_\mu\ . \tag{37}$$

We cannot simply add $\frac{1}{4}B_0{}^2$ to the $\mathscr{H}_c^1$ as that would destroy the Lorentz invariance of the action. Therefore we must introduce a new auxiliary field $\Sigma$, which must couple to $B^\mu$ derivatively to cancel the $B_0{}^2$ term in $\mathscr{H}_c^1$. Further, so as not to introduce new constraints we must include kinetic energy terms for $\Sigma$. Therefore let

$$\mathscr{H}_c^1\to\mathscr{H}_c^2=\mathscr{H}_c^1+B^i(\partial_i\Sigma)-(\partial_i\Sigma)(\partial^i\Sigma)+\frac{B_0\pi_\Sigma}{2}-\pi_\Sigma\pi_\Sigma+\tfrac{1}{4}B_0{}^2\ , \tag{38}$$

which obtains from

$$\mathscr{L}_1\to\mathscr{L}_2=\mathscr{L}_1-B^\mu(\partial_\mu\Sigma)+(\partial^\mu\Sigma)(\partial_\mu\Sigma)\ . \tag{39}$$

Thus, with respect to $H_D^2$, we have

$$\dot{\pi}_B^\mu=-\pi^{\mu 0}+\partial_i\phi^{\mu i}-\tfrac{1}{2}g^{\mu 0}\pi_\Sigma-g^{\mu i}(\partial_i\Sigma) \tag{40}$$

and

$$\dot{\pi}_\chi=\phi_\mu^\mu\ . \tag{41}$$

Among the primary and secondary constraints, only

$$(\dot{\pi}_B^\mu,\dot{\pi}_\chi)=g^{\mu 0}\delta(\mathbf{x}-\mathbf{y}) \tag{42}$$

is nonzero. Thus if $\dot{\pi}_\chi$ becomes

$$\dot{\pi}_\chi=\phi_\mu^\mu-2\Sigma\ , \tag{43}$$

the constraint system becomes first class to the secondary constraint level. This is accomplished by

$$\mathscr{H}_c^2\to\mathscr{H}_c^3=\mathscr{H}_v^2+\chi(2\Sigma) \tag{44}$$

with

$$\mathscr{L}_2\to\mathscr{L}_3=\mathscr{L}_2-\chi(2\Sigma)\ . \tag{45}$$

Now we must consider the stability of the secondary constraints with respect to $H_D^3$. For (43) and (40) we obtain, explicitly,

$$\ddot{\pi}_\chi=(\pi_\chi,H_D^3)=-\pi_{B\mu}^\mu-\pi_\Sigma \tag{46}$$

and

$$\ddot{\pi}_B^\mu=-(\partial^i\partial_i+m^2)\phi^{\mu 0}-\partial_i\pi^{\mu i}+\tfrac{1}{2}g^{\mu 0}\partial^i\partial_i\Sigma-\tfrac{1}{2}g^{\mu i}\partial_i\pi_\Sigma\ , \tag{47}$$

where we have used the equal-time PB

$$(\phi^{\mu\nu}(\mathbf{x},t)\pi^{\alpha\beta}(\mathbf{y},t))=\tfrac{1}{2}(g^{\mu\alpha}g^{\nu\beta}+g^{\nu\alpha}g^{\mu\beta})\delta(\mathbf{x}-\mathbf{y}) \tag{48}$$

for symmetric tensor fields. Equations (46) and (47) are tertiary constraints

$$(\ddot{\pi}_\chi,\dot{\pi}_\chi)=(d-2)\delta(\mathbf{x}-\mathbf{y})\ , \tag{49}$$

$$(\ddot{\pi}_B^\mu,\ddot{\pi}_\chi)=g^{\mu 0}(-\tfrac{3}{2}\partial^i\partial_i+m^2)\delta(\mathbf{x}-\mathbf{y})\ , \tag{50}$$

$$(\ddot{\pi}_B^\mu\ddot{\pi}_B^\nu)=\tfrac{1}{2}(g^{\mu\nu}+g^{\mu 0}g^{\nu 0})(\partial^i\partial_i+m^2)\delta(\mathbf{x}-\mathbf{y})-\tfrac{1}{2}(g^{\mu\nu}+\tfrac{1}{2}g^{\mu 0}g^{\nu 0})\partial^i\partial_i\delta(\mathbf{x}-\mathbf{y})\ , \tag{51}$$

where $d=g_\mu^\mu$ is the space-time dimension.

To restore gauge invariance at this level is a bit more complex than for the spin-one case. A new auxiliary field $\psi$ is required which should separately eliminate the $d$ and the 2 from (44). From (43) and (46) we see that this can be done by adding the term $-d\psi$ to $\dot{\pi}_\chi$ and requiring that $(2\dot{\Sigma}+d\dot{\psi})=\pi_\psi$. Therefore first add $\chi(d\psi)$ to $\mathscr{H}_c^3$ which means subtract $\chi(d\psi)$ from $\mathscr{L}_3$, then further modify $\mathscr{L}_3$ by Lorentz-invariant kinetic terms so that in total we have

$$\mathscr{L}_3\to\mathscr{L}_4=\mathscr{L}_3-\chi(d\psi)+2(\partial^\mu\Sigma)(\partial_\mu\psi)+\frac{d}{2}(\partial^\mu\psi)(\partial_\mu\psi)\ . \tag{52}$$

Because of the $\partial^\mu\Sigma$ term in $\mathscr{H}_c^3$ we cannot simply additively amend $\mathscr{H}_c^2$ as was done earlier. Rather, we must return to the $\mathscr{H}_c^1$ stage to obtain, from $\mathscr{L}_4$,

$$\begin{aligned}\mathscr{H}_c^4=\mathscr{H}_c^1&+\left[\frac{\pi_\Sigma}{2}+\frac{B^0}{2}-\left[\frac{\pi_\psi-\pi_\Sigma}{d-2}\right]-\frac{B^0}{d-2}\right]\pi_\Sigma+\left[\left[\frac{\pi_\psi-\pi_\Sigma}{d-2}\right]-\frac{B^0}{d-2}\right]\pi_\psi+B^0\left[\frac{\pi_\Sigma}{2}+\frac{B^0}{2}-\left[\frac{\pi_\psi-\pi_\Sigma}{d-2}\right]-\frac{B^0}{d-2}\right]\\&+B^i(\partial_i\Sigma)-\left[\left[\frac{\pi_\Sigma+B^0}{2}\right]-\left[\frac{\pi_\psi-\pi_\Sigma}{d-2}\right]-\frac{B^0}{d-2}\right]^2-(\partial^i\Sigma)(\partial_i\Sigma)\\&+\chi(2\Sigma+d\psi)-\frac{2}{d-2}\left[\left[\frac{\pi_\Sigma+B^0}{2}\right]-\left[\frac{\pi_\psi-\pi_\Sigma}{d-2}\right]-\frac{B^0}{d-2}\right](\pi_\pi-\pi_\Sigma-B^0)\\&-2(\partial^i\Sigma)(\partial_i\Sigma)-\frac{d}{2}(\partial^i\psi)(\partial_i\psi)-\frac{d}{2}\left[\left[\frac{\pi_\psi-\pi_\Sigma}{d-2}\right]-\frac{B^0}{d-2}\right]^2\ .\end{aligned} \tag{53}$$

To retain first-class status with respect to the primary $\pi_B^\mu$ constraint, the uncanceled $\pi_\Sigma {B_0}^2$ term in $\mathscr{H}_c^4$ must be eliminated; hence,

$$\pi_\psi = \frac{\partial \mathscr{L}_4}{\partial \dot\psi} = 2\dot\sigma + d\dot\psi \to 2\dot\sigma + d\dot\psi - B^0 \;. \quad (54)$$

Therefore,

$$\mathscr{L}_4 \to \mathscr{L}_5 = \mathscr{L}_4 - B^\mu(\partial_\mu \psi) \;, \quad (55)$$

and it is now easiest to write

$$\mathscr{H}_c^5 = \mathscr{H}_c^0 + B^i\partial_i(\Sigma+\psi) + \chi(2\Sigma + d\psi) - (\partial^i\Sigma)(\partial_i\Sigma) - 2(\partial^i\Sigma)(\partial_i\psi) - \frac{d}{2}(\partial^i\psi)(\partial_i\psi) + \tfrac{1}{2}B^0\pi_\Sigma + \tfrac{1}{4}\pi_\Sigma\pi_\Sigma + \frac{1}{2}\frac{(\pi_\psi - \pi_\Sigma)^2}{d-2} \;. \quad (56)$$

Now with respect to $H_D^5$ we have

$$\dot\pi_B^\mu = -\pi^{\mu 0} + \partial_i\partial^{\mu i} - g^{\mu i}\partial_i(\Sigma+\psi) - \tfrac{1}{2}g^{\mu 0}\pi_\Sigma \quad (57)$$

and

$$\dot\pi_\chi = \phi_\mu^\mu - (2\Sigma + d\psi) \quad (58)$$

for the secondary constraints. Together with the primary constraints, these form a first-class system.

We must now test the stability of the secondary constraints with respect to $H_D^5$. We have

$$\ddot\pi_B^\mu = -(\partial^i\partial_i + m^2)\phi^{\mu 0} + g^{\mu 0}\partial^i\partial_i(\Sigma+\psi) - \partial_i(\pi^{\mu i} + \tfrac{1}{2}g^{\mu i}\pi_\Sigma) \quad (59)$$

and

$$\ddot\pi_\chi = -\pi_\mu^\mu - \pi_\psi \quad (60)$$

and

$$(\ddot\pi_B^\mu, \ddot\pi_\chi) = m^2 g^{\mu 0}\delta(\mathbf{x}-\mathbf{y}) \;. \quad (61)$$

Therefore we need a mass term in $\ddot\pi_B^\mu$ such as $g^{\mu 0}m^2\psi$. Therefore if

$$\mathscr{H}_c^5 \to \mathscr{H}_c^6 = \mathscr{H}_c^5 + 2m^2\Sigma\psi \quad (62)$$

and

$$\mathscr{L}_5 \to \mathscr{L}_6 = \mathscr{L}_5 - 2m^2\Sigma\psi \;, \quad (63)$$

then $(\ddot\pi_B^\mu, \ddot\pi_\chi) = 0$. Further, since

$$(\ddot\pi_B^\mu, \dot\pi_B^\nu) = \tfrac{1}{2}m^2(g^{\mu\nu} + g^{\mu 0}g^{\nu 0})\delta(\mathbf{x}-\mathbf{y}) \;, \quad (64)$$

we need a mass term in $\ddot\pi_B^\mu$ such as $g^{\mu 0}m^2\Sigma$. Therefore if

$$\mathscr{H}_c^6 \to \mathscr{H}_c^7 = \mathscr{H}_c^6 + m^2\Sigma^2 \quad (65)$$

and

$$\mathscr{L}_6 \to \mathscr{L}_7 = \mathscr{L}_6 - m^2\Sigma^2 \quad (66)$$

then

$$(\ddot\pi_B^\mu, \dot\pi_B^\nu) = \tfrac{1}{2}m^2 g^{\mu\nu}\delta(\mathbf{x}-\mathbf{y}) \quad (67)$$

alone survives as the nonzero PB.

To render the set of primary, secondary, and tertiary constraints first class we must introduce another auxiliary field, a four-vector field $A_\mu$, such that

$$\dot\pi_B^\mu \to \dot\pi_B^\mu + mA^\mu \quad (68)$$

and

$$\ddot\pi_B^\mu \to \ddot\pi_B^\mu - \tfrac{1}{2}m\pi_A^\mu \;. \quad (69)$$

Therefore

$$\mathscr{H}_c^7 \to \mathscr{H}_c^8 = \mathscr{H}_c^7 - m\rlap{\phantom{x}}_B^\mu A_\mu - \tfrac{1}{4}\pi_A^\mu \pi_{A_\mu} \quad (70)$$

or

$$\mathscr{L}_7 \to \mathscr{L}_8 = \mathscr{L}_7 + mB^\mu A_\mu + (\partial^\mu A^\nu)(\partial_\mu A_\nu) \;. \quad (71)$$

Thus the system of constraints is first class to this tertiary level.

Now test the stability of the tertiary constraints with respect to $H_D^8$.

First

$$\dddot\pi_\chi = -(\partial_i\partial^i + m^2)\dot\pi_\chi - m^2 d\psi \;. \quad (72)$$

Therefore, if

$$\mathscr{H}_c^8 \to \mathscr{H}_c^9 = \mathscr{H}_c^8 + \frac{m^2 d}{2}\psi^2 \quad (73)$$

or

$$\mathscr{L}_8 \to \mathscr{L}_9 = \mathscr{L}_8 - \frac{m^2 d}{2}\psi^2 \;, \quad (74)$$

then $\dddot\pi_\chi$ is not a new constraint, and there is no change in the prior first-class status. Second, since

$$(\pi_B^\mu, H_D^9) = -(\partial^i\partial_i + m^2)\pi_B^\mu + m^3 A^\mu \quad (75)$$

if

$$\mathscr{H}_c^9 \to \mathscr{H}_c^{10} = \mathscr{H}_c^9 + \frac{m^2}{2}A^\mu A_\mu \quad (76)$$

or

$$\mathscr{L}_9 \to \mathscr{L}_{10} - \frac{m^2}{2}A_\mu A^\mu \;, \quad (77)$$

then $\dddot\pi_B^\mu$ is not a new constraint.

The constraint analysis ends and $S_{10} = \int dx\, \mathscr{L}_{10}$ is the gauge-invariant action of West,[3] derived via Dirac's constraint Hamiltonian formalism, with one auxiliary scalar plus one auxiliary four-vector multiplier fields and two auxiliary scalar plus one auxiliary four-vector dynamical fields, there are 15 first-class constraints. Therefore the number of phase-space degrees of freedom is $42-30=12$, the appropriate number.

Explicitly the final Lagrangian density $\mathscr{L}_{10}$ is

$$\mathscr{L}_{10} = \tfrac{1}{2}\phi^{\mu\nu}(\Box + m^2)\phi_{\mu\nu} + B^\mu(\partial^\nu\phi_{\mu\nu}) + \chi\phi_\mu^\mu - \tfrac{1}{4}B^\mu B_\mu + mB^\mu A_\mu - B^\mu\partial_\mu(\Sigma+\psi) - \chi(2\Sigma + d\psi) - A^\mu(\Box + m^2)A_\mu - \Sigma(\Box + m^2)\Sigma + 2(\partial^\mu\Sigma)(\partial_\mu\psi) - 2m^2\Sigma\psi - \frac{d}{2}\psi(\Box + m^2)\psi \quad (78)$$

and the joint variations with respect to which the action $S_{10}$ is invariant are

$$\delta\phi_{\mu\nu}=\partial_\mu\lambda_\nu+\partial_\nu\lambda_\mu+g_{\mu\nu}\eta \ , \tag{79a}$$

$$\delta B_\mu=2(\Box+m^2)\lambda_\mu \ , \tag{79b}$$

$$\delta\chi=-(\Box+m^2)_\eta \ , \tag{79c}$$

$$\delta A_\mu=m\lambda_\mu \ , \tag{79d}$$

$$\delta\psi=\eta \ , \tag{79e}$$

$$\delta\Sigma=\partial^\mu\lambda_\mu \ . \tag{79f}$$

Again it is possible to eliminate, via the equations of motion, the auxiliary multiplier fields $B^\mu$ and $\chi$, yielding a gauge-invariant action involving only dynamical fields which is the spin-two analog of the Stueckelberg action.

## IV. CONCLUSION

We have constructed a gauge-invariant action for the massive boson fields of spin 1 and spin 2, using Dirac's constraint Hamiltonian formalism. We begin with the non-gauge-invariant action which gives the free, mass-shell, field equations plus supplementary conditions. The momenta conjugate to the auxiliary fields vanish and are primary constraints. Primary-constraint stability, with respect to the beginning Dirac Hamiltonian, generates secondary constraints.

The guiding principle here is that all the constraints must be first class if the theory is to exhibit gauge invariance. Step-by-step modification of the Dirac Hamiltonian, and consequently the Lagrangian density, renders the primary- and second-constraint system first class. Stability of the secondary constraints generates tertiary constraints. Further step-by-step modifications of the Hamiltonian yields a first-class constraint system. There are no further constraints and we thus arrive at the gauge-invariant actions of West.

That the constraint analysis should end at the tertiary is also dictated by a counting of the number of degrees of freedom. In a future paper we will extend this analysis to arbitrary spin.

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