Time to Switch: Different regimes and competitive management of natural resources *

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Abstract

The right management of the resources has become one of the hottest topics in the social debate. In order to give some insights on how to manage them correctly, we have extended a switching natural resources model. A Piecewise Closed-loop Nash Equilibrium differential game is studied, where agents decide the amount of the resource they extract and when to switch to a new and more efficient technology. We investigate different models introducing a renewable resource and then, they are extended under a more general utility function. Interior solutions are analysed. Our results are simulated, and a sensitivity analysis is performed to identify the effect of the renewable resource and the constant elasticity of substitution on the relevant variables of the model.

Keywords: Optimal timing of switching; Markovian Strategies; Renewable and nonrenewable resources; Differential games.

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1 Introduction

How to manage natural resources has become one of the most important problems for the civilization. If we continue consuming nonrenewable resources a fast as we have done in the past, we may be in trouble in the future. In a similar way, an incorrect management of renewable resources can affect their survival. Oil, gas, fisheries, forests, etc., are examples of both kinds of resources. Thus, a better understanding of the management of natural resources may help us for a more efficient and responsible exploitation. Moreover, in many situations in Economics, the timing of switching between alternatives and consecutive regimes is of important interest. One could think in many real-life examples, such as when it is optimal to adopt a new technology or to stick with the old one (Boucekkine et al., 2004), or when a country has to decide whether or not to join an international agreement. These models are also used in order to study the optimal management of natural resources (Boucekkine et al., 2013), or when to phase out capital controls in a given economy as studied by Makris (2001). Consequently, all these changes are of interesting utility in economics.

Will introducing a renewable resource, in contrast with the nonrenewable case, make individuals consume more quickly because they know that the resource will regenerate itself? Will they exhaust the resource sooner, and consume more in total? Or on the contrary, will they continue extracting for a longer period or shorter period of time? Will the agents change the technology later because they are under a renewable framework? In this paper, we try to provide some answers to these questions by studying a differential game problem where agents can switch between different regimes, under an infinite horizon. While there is a rich literature on exogenous changes in regimes (see e.g. (van der Ploeg, 2017) and (Zemel, 2015)), there are few papers focusing on the (endogenous) optimal timing of switching. Papers studying this last case are Tomiyama (1985), Amit (1986) and Makris (2001) where one agent decides when to switch. Furthermore, Long et al. (2017) analyze the interaction with two agents. In all these models, the switching decision involves a trade-off, since such a change entails immediate costs and potential future benefits. The decision makers now decide when to switch and how much to extract.

Previous work related to this is relatively recent. In Reinganum (1981), it is assumed that firms adopt pre-commitment (open loop) strategies on when to adopt a new technology. Then, Fudenberg and Tirole (1985) study when firms adopt preemptively a new technology to prevent or delay adoption by their opponent (for a survey of the related literature see Long (2011)). In the paper by Boucekkine et al. (2011), they analyze the trade-off between environmental quality and economic performance using a differential game with two players where they may switch to a cleaner technology that is environmentally “efficient” but economically less productive. In this previous work, they paid attention to the open-loop Nash equilibrium. As Long et al. (2017) argue, this requirement is too strong to explain when players should change to the new regime.

The only study to our knowledge that deals with the feedback effect in a player’s switching strategies operating through the state of the system is the recent paper by Long et al. (2017). A general differential game is developed with two players having two strategies. The first involves an action that affects the evolution of the state equation (how much to extract). The second concerns the timing of switching between alternatives and consecutive regimes. The novel part of the previous paper lies in the existence of some sort of feedback in player’s switching strategies operating through the state of the system (the resources left at that time). The authors develop the necessary conditions for the general framework. They define a new concept (and therefore a new methodology) in differential games, the *Piecewise Closed-loop Nash Equilibrium* (PCNE hereafter). Notice that the researchers analyze a non-renewable case. For this reason, here, we introduce a renewable resource. Moreover, we proceed with a further extension and the utility function is modified to a more general one, where the logarithmic case is a particular case. We analyze how the decision makers react to the mentioned modifications.

In the present work, we analyze a renewable resource, following the work by Long et al. (2017) where a nonrenewable resource is studied. In addition, we are also interested in how the agents react when their constant elasticity of substitution changes. For that reason, building upon the mentioned paper, we extend it for the renewable resources and with a more general utility function. Therefore, the mentioned changes in the structure of the model, may allow us to a better understanding of the management of the natural resources. These contributions are developed in Sections 3 and 4.

The organization of the paper is as follows. Section 2 introduces the paper by Makris (2001) and Long et al. (2017), which have been fundamental in order to develop the mathematical techniques that will be used in this paper. The optimal control problem (one decision maker) is developed first, and then we explain the differential game problem. Section 3 studies how to manage a nonrenewable resource where both player have a CES utility function. Moreover, a simulation is performed to the analyze how the players react under this framework. Additionally, the study of how the introduction of a renewable resource affects the behavior of the players is shown in Section 4. This section is divided for the case of the logarithmic and the CES utility function. In the latter case , we perform a simulation of the model and a an extensive study of the sensitivity analysis. Finally, Section 5 concludes and points out the directions for further research. Further mathematical developments can be found in the Appendix.

2 Previous Models

In this section, we present what have been studied in the past. Firstly, we expose the ideas by Makris (2001) for an optimal control problem, and then we present the differential game analyzed in Long et al. (2017). Hence, before we get into the differential game itself, let us study a particular case. Such a case is determined when the set of decision makers consists of a single element, one player, $N = \{1\}$, where N is the set of players. Thus, the *"Differential Game"* is called an optimal control problem in this particular case.

In order to achieve such a goal, in this section we will follow the ideas firstly introduced by Tomiyama (1985) and Amit (1986) to solve a finite horizon two-stage optimal control problem. The former is interested in an optimal investment decision of a firm whose capital goods face delivery lag. As Tomiyama (1985) has shown, this is reducible to a two-stage optimal control problem. The latter focuses on the problem of a producer who considers switching from a primary to secondary petroleum recovery process. Consequently, the problem is defined as a two-phase optimal control problem. In order to study the problems presented by Tomiyama

(1985) and Amit (1986), both authors derive the necessary conditions for the finite horizon two-stage optimal control problem.

However, as Makris (2001) justified, in many cases in economics, it seems more appropriate to consider a problem with an infinite horizon. The author studied the problem of an infinite horizon representative agent with an open economy interacting with the rest of the world through the world capital market. The problem is defined with capital restrictions at the initial time $t = 0$. The government can abolish capital controls from an instant t_1^* and on; t_1^* is such that $t_0 \leq t_1^* < \infty$. Then, in such an economy, there are two stages. The first stage is defined for $t \in [0, t_1^*)$ where capital controls are in place. The second stage $t \in (t_1^*, \infty)$ describes the case where the capital accounts are liberalized by the government. Therefore, in such a model, one is interested in the optimal instant where the government will abolish the capital controls.

The goal of this section is to introduce the concepts used in the literature. We now focus on the the two-stage optimal control problem, and we show the switching conditions, where the programming horizon is infinite, $t \in [0, \infty)$. Makris (2001) developed such conditions, in contrast to the ones presented in Tomiyama (1985) and Amit (1986) for the finite horizon.

Building upon the work by Michel (1982), Tomiyama (1985), Amit (1986) and standard optimal control theory techniques, Makris (2001) presents a complete theory of necessary conditions for an infinite-horizon discounted two-stages optimal control problem.

The problem analyzed in Makris (2001) is the following.

$$\underset{u(t)}{Max}\, J(u(\cdot)) = \int_0^{t_1} e^{-\rho t} F^1\left(x(t), u(t)\right) dt + \int_{t_1}^{\infty} e^{-\rho t} F^2\left(x(t), u(t)\right) dt - \Omega_1\left(t_1, x(t_1)\right), \tag{1}$$

where $\Omega_1\left(t_1, x(t_1)\right)$ is defined as $e^{-\rho t_1}\omega(x(t_1))$ and it can be interpreted as a cost from switching regimes. In addition, ρ is the discount rate. The corresponding differential equation describing the evolution of the state variable $x(t)$ in both regimes is

$$\dot{x} = \begin{cases} f^1(x(t), u(t)) & if \quad t \in [0, t_1) \\ f^2(x(t), u(t)) & if \quad t \in [t_1, \infty) \end{cases} \tag{2}$$

We will define s as the state of the system, in this optimal control problem, $s = 1, 2$, where $s = 1$ corresponds to $t \in [0, t_1)$ and $s = 2$ corresponds to $t \in [t_2, \infty)$. Moreover, $\omega(x(t_1))$ is a real valued once continuously differentiable function. In the present paper, we will focus on the interior solutions, meaning $0 < t_1^* < +\infty$. Therefore, the theorem in Makris (2001) is :

Theorem 1. *Consider the problem described above. Then, for $(x^*(t), u^*(t), t_1^*)$, $t \in [0, \infty)$, to be an optimal path for the problem, the necessary conditions are:*

$$\lambda^1(t_1^*) + \lambda_0^1 e^{-\rho t_1^*} \frac{d\omega(x^*(t_1^*))}{dx} = \lambda^2(t_1^*) \tag{3}$$

$$H^{1*}(t_1^*) + \lambda_0^1 \rho e^{-\rho t_1^*} \omega(x^*(t_1^*)) = H^{2*}(t_1^*) \tag{4}$$

We now expose a two-player differential game, $N = \{1, 2\}$. The problem is defined by the instantaneous payoff function and the differential equation describing the evolution of the state variable, which depends on the regime the system is in. Throughout this paper, we work with a discrete finite set of regimes $\mathcal{S}$, indexed by s. Each player will decide her strategy profile, deciding how much to extract and when to change to the second regime (incuring in a lumpy cost). We assume that each player can switch only once. The general analysis with any finite number of switching is studied in Makris (2001). It can be seen, however, that the main difference between the work mentioned above and Long et al. (2017) is that the latter studies an optimal control problem with $n > 1$, which results in a differential game. If there were only a single state $s \in \mathcal{S}$, we would be working under the optimal control theory framework developed above. The characteristic in Long et al. (2017) lies in the fact that the dynamic system can switch between different states where the state variable renews itself and the study of a more general utility function. For instance, in Long et al. (2017), a particular case is studied.

Let $t_i \in \mathbb{R}_{++}$ be the regime change action for player $i \in N$, which means that she decides when to move from her state 1 to 2. Making such a change, she incurs in a lumpy cost $\Omega_i\left(t_i, x(t_i)\right)$. In this paper, we will assume interior solutions, which means $0 < t_i < t_j < +\infty$. Assume, for instance $0 < t_1 < t_2 < +\infty$, therefore, the payoff for player 1 is given by the equation (5)

$$\begin{aligned} U_1 = \int_0^{t_1} F_1^{11}\left(u_1(t), u_2(t), x(t)\right) e^{-\rho t} dt + \int_{t_1}^{t_2} F_1^{21}\left(u_1(t), u_2(t), x(t)\right) e^{-\rho t} dt + \\ \int_{t_2}^{\infty} F_1^{22}\left(u_1(t), u_2(t), x(t)\right) e^{-\rho t} dt - \Omega_1\left(t_1, x(t_1)\right), \end{aligned} \tag{5}$$

where ρ is the discount rate. Observe how both agents have the same discount rate. If this were not the case, a problem of time inconsistency would appear as De-Paz et al. (2013) studied. The corresponding differential equation describing the evolution of the state variable $x(t) \in \mathbb{R}_+$ in any regime is

$$\frac{dx}{dt} \equiv \dot{x} = f^s\left(u_1(t), u_2(t), x(t)\right). \tag{6}$$

The "Strategy profile" is constructed by a set of two type of controls. This control set is defined as $\mathcal{C}_i = \{u_i(t), t_i\}$. A strategy consists of an *action policy* Φ_i, and a *timing strategy* θ_i, for both players.

The *action policy* Φ_i is characterized by the actions each player chooses at every possible state of the system, $(x, s) \in \mathbb{R}_{++} \times \mathcal{S}$. Therefore, the *action policy* for player i is a mapping Φ_i from the state space $\mathbb{R}_{++} \times \mathcal{S}$ to the set $\mathbb{R}^n$, that is $\Phi_i : \mathbb{R}_{++} \times \mathcal{S} \longrightarrow \mathbb{R}^n$. As far as the *timing strategy* is concerned, we will need the following pedagogical example to understand its meaning. Suppose player 1 knows that the other player (player 2) knows she is in regime 21 at a date $\tau \in [t_1, t_2)$, with a certain amount of resources left. This means that player 1 has switched before at $t_1 \leq \tau$. Therefore, as it is expected, player 2 will change at $t_2 \geq \tau$. Knowing this scenario, player 1 knows that the interval of time between when she has changed (t_1) and when her opponent will change (t_2) is a function not of the stock, but of the value of the stock at which player 1's regime change takes place ($x(t_1) = x_1$). Consequently, the timing strategy for player i, given that $s \in \mathcal{S}_i$, is a mapping θ_i from $\mathbb{R}_{++} \times \mathcal{S}$ to $\mathbb{R}_+ \cup \{\infty\}$. Observe that in our study case, interior solutions, the image of θ_i excludes $\{\infty\} \cup \{0\}$, due to if $t_i = \infty$ or $t_i = 0$ we would be studying corner solutions. Thus, the mapping is $\theta_i : \mathbb{R}_{++} \times \mathcal{S} \to \mathbb{R}_+$. Each player chooses her switching time taking the one of her opponent as given. This leads to the following definition.

Definition 1. The strategy vector and profile for both players are determined by the following definitions.

- A strategy vector for player i is a pair $\psi_i \equiv (\Phi_i, \theta_i),\ i = 1, 2$.
- A strategy profile is a pair of strategy vectors (ψ_1, ψ_2).
- A strategy profile (ψ_1^*, ψ_2^*) is called a piecewise closed-loop (Nash) equilibrium (PCNE), if given that player i adopts the strategy vector ψ_i^*, the payoof for the other player j is maximized by the strategy vector ψ_j^*, such that $i, j = 1, 2$.

Definition 2. The game of this paper is defined as $G \equiv (N, (\psi_i)_{i\in N}, (U_i)_{i\in N})$ which is played $\forall t$, and N is the non-empty set of players. U_i is a real function, defined on $\prod_{i\in N} \psi_i \equiv \psi \quad \forall i \in N$.

Assume that there exists a solution $(u_1^*(t), u_2^*(t), x^*(t))$ to the differential game explained before for a given set of switching time (t_1, t_2). We define the present value Hamiltonian for a player $i \in N$ as $H_i^s = F_1^{11}\left(u_i(t), \Phi_{-i}(x(t), s), x(t)\right) e^{-\rho t} + \lambda_i^s f^s\left(u_i(t), \Phi_{-i}(x(t), s), x(t)\right)$, for every state where λ_i^s is the co-state variable, evaluated at the solution denoted by H_i^{s*}. Similarly to what we have explained for the optimal control problem in the previous section, when the set of decision makers is $N = \{1, 2\}$, the optimal conditions are given by the next theorem

Theorem 2. *The necessary optimality conditions for the existence of a PCNE for the switching timing* $0 < t_1 < t_2 < \infty$ *are:*

For player 2:

$$H_2^{21*}(t_2) - \frac{\partial \Omega_2(t_2, x^*(t_2))}{\partial t_2} = H_2^{22*}(t_2) \tag{7a}$$

$$\lambda_2^{21}(t_2) + \frac{\partial \Omega_2(t_2, x^*(t_2))}{\partial x_2} = \lambda_2^{22}(t_2 \tag{7b}$$

For player 1:

$$H_1^{11*}(t_1) - \frac{\partial \Omega_1(t_1, x^*(t_1))}{\partial t_1} = H_2^{21*}(t_1) - \left[H_1^{21*}(t_2) - H_1^{22*}(t_2)\right] \tag{8a}$$

$$\lambda_1^{11}(t_1) + \frac{\partial \Omega_1(t_1, x^*(t_1))}{\partial x_1} = \lambda_1^{21}(t_1 + \theta_2'(x^*(t_1), 21)\left[H_1^{21*}(t_2) - H_1^{22*}(t_2)\right]. \tag{8b}$$

As we mentioned before, we next review the nonrenewable resource model proposed in Long et al. (2017), in order to introduce some extensions to both, the utility function and the differential equation, that defines the state of the resource. Then, we modify the utility function for the nonrenewable case. Most articles in the literature, do not take into account the possibility of switch for players. Moreover, it is normally assumed that consumption is a fixed fraction of the extraction level. The authors of the previous paper consider that players can adopt a new technology whenever they decide. Therefore, both players will decide their consumption and when to adopt the more efficient technology.

The following differential game consists of two players. Let $u_i(t)$ denote the consumption rate for both players in any regime. Furthermore, the extraction rate from the resource is defined as $e_i(t)$. The amount of resource both players have drawn is converted into consumption according to the following technology: $\gamma_i u_i(t) = e_i(t)$, where γ_i^{-1} is a positive number that defines the efficiency in transforming the extracted resource into consumption.

The efficiency of the technology is given by the parameter γ_i available to player i from $t = 0$ onwards. Due to the fact that players may have different technologies, we need to differentiate between the technologies in the different states. Player 1 starts with technology $l = 1$, and has to decide when to change to the technology $l = 2$. For player 2 instead, we use the label k. The technological state is defined therefore as $s = lk$, $\forall l, k = 1, 2$, which shows what technology each player is using. The parameters satisfy $\gamma_i^1 > \gamma_i^2$. This inequality means that the second technology is more efficient. When the ratio $\frac{\gamma_i^2}{\gamma_i^1} \in (0, 1)$ is smaller, the higher the gain.

The initial stock at $t = 0$ is given by $x(0) = x_0$. As we mentioned above, t_1 and t_2 are the switching times. Besides, we focus on the interior solutions $(0 < t_1 < t_2 < +\infty)$. The evolution of the stock is given by the following differential equation

$$\dot{x} = \begin{cases} -\gamma_1^1 u_1 - \gamma_2^1 u_2 \quad if \quad t \in [0, t_1) \\ -\gamma_1^2 u_1 - \gamma_2^1 u_2 \quad if \quad t \in [t_1, t_2) \\ -\gamma_1^2 u_1 - \gamma_2^2 u_2 \quad if \quad t \in [t_2, \infty) \end{cases} \tag{9}$$

When the players decide to change to a new regime, they incur a cost that is defined in terms of the level of the state variable at which the adoption occurs, $x(t_i) = x_i$. We define such a cost as $\omega_i(x(t_i))$ with $\omega_i'(.) \geq 0$. It takes the form $\omega_i(x_i) = \chi_i + \beta_i x_i$, where $\chi_i \geq 0$, $\beta_i \geq 0$.

χ_i is the fixed cost related to the technology investment and reflects for instance, the outlay for machinery. Moreover, β_i reflects the sensitivity of adoption cost to the level of the resource left at the instant player i decides to switch. One assumption is that such a cost is increasing in x_i. The idea behind this is that when the resource is scarce, the cost of adopting a new technology is lower. This reflects the fact that the scientific progress related to the installation of new technology to save on the extraction of resources is more intense, as the scarcity becomes more acute. The switching cost is discounted at rate ρ. That means, that if the switch occurs at t_i, the *discounted* cost is $e^{-\rho t_i}\omega_i(x_i)$, that is defined in the Theorem as $\Omega_i(t_i, x(t_i))$. The general game G is described as a sequence of three subgames, 11, 21 and 22. Therefore, the whole game G is solved backwards. Hereafter, this definition will be used in the following sections.

We restrict our attention to linear feedback strategies $u_j \equiv \Phi_j(x(t), s) = a_j^s + b_j^s x(t)$. After a tedious computation, we obtain that player's extraction strategies are the same for all regimes [1]

$$\gamma_i^l \Phi_i(x(t), s) = \gamma_j^k \Phi_j(x(t), s). \tag{10}$$

This means that both players will extract exactly the same amount of resource. Therefore, the extraction rates under regime $s = 22$ are given by

$$\gamma_i^2 \Phi_i(x(t), 22) = \gamma_j^2 \Phi_i(x(t), 22) = \rho x(t). \tag{11}$$

On the contrary, the extraction rate for the regime $s = 21$ is determined by

$$\gamma_1^2 \Phi_1^{21} = \gamma_2^1 \Phi_2^{21} = \frac{\rho^2 \beta_2 (x_2)^2}{1 - \beta_2 \rho x_2} + \rho x(t) = \Gamma(x_2) + \rho x(t), \tag{12}$$

where $\Gamma(x_2) = \frac{\rho^2 \beta_2 (x_2)^2}{1 - \beta_2 \rho x_2}$. The switching point for player 2 is given by the solution x_2 of the following equation, that has been obtained combining the equations for player 2 in the Theorem 2 and the related FOCs

$$\rho \omega_2(x_2) + \ln\left(\frac{\gamma_2^1}{\gamma_2^1}\right) = \ln(1 - \beta_2 \rho x_2). \tag{13}$$

This equation shows the optimal moment of switching x_2^*, and it is independent from the switching time of player 1. Using the consumption expression given by eq. (12), and solving the differential equation with the boundary condition $x(t_1^*) = x_1^*$ we obtain the expression for the state variable $\forall t \in [t_1, t_2)$

$$x^{21*}(t) = \left[x_1^* + \frac{\rho \beta_2 (x_2^*)^2}{1 - \beta_2 \rho x_2^*}\right] e^{-2\rho(t - t_1)} - \frac{\rho \beta_2 (x_2^*)^2}{1 - \beta_2 \rho x_2^*}\ . \tag{14}$$

[1] In the next sections, we develop the computations of the extensions proposed in Long et al. (2017).

Evaluating this equation at t_2 and solving for the timing of waiting of player 2 after player 1 has switched, $\theta_2 = t_2 - t_1$, we obtain

$$\theta_2(x_1^*, 21) = \frac{1}{2\rho} \ln \left[(1 - \rho\beta_2 x_2^*) \frac{x_1^*}{x_2^*} + \rho\beta_2 x_2^* \right] . \tag{15}$$

Remember that this represents the time-to-go, that is the waiting time between when player 1 has switched and when player 2 decides to switch. Observe, how the above equations define the behavior of player 2 as a function of what player 1 would do. The results have been derived using Theorem 2 for player 2. Now, we write the equations needed to define the behavior of player 1. Let us define

$$\xi(x_1; x_2^*) = 1 - \theta_2'[\Gamma(x_2^*) + \rho x_1] e^{-\rho\theta_2} \ln(1 - \rho\beta_2 x_2^*) - \beta_1[\Gamma(x_2^*) + \rho x_1]. \tag{16}$$

Notice in this case that $\theta_2'[\Gamma(x_2^*) + \rho x_1] = 1/2$, where $\theta_2' \equiv \frac{\partial\theta}{\partial x_1^*}$. Knowing $\gamma_1^1 u_1^1(t_1)$ at time t_1, and solving for the Markovian strategies in regime $s = 11$ we obtain

$$\gamma_1^1 \Phi_1(x, 11) = \gamma_2^1 \Phi_2(x, 11) = \frac{\Gamma(x_2^*) + \rho x_1[1 - \xi(x_1; x_2^*)]}{\xi(x_1; x_2^*)} + \rho x(t) = \Lambda(x_1; x_2^*) + \rho x(t). \tag{17}$$

The optimality condition for player 1 can be rewritten as

$$\ln \left(\frac{\gamma_2^1}{\gamma_2^1} \right) + \rho\omega_1(x_1) = e^{-\rho\theta_2} \ln(1 - \rho\beta_2 x_2^*) + \ln\left[\xi(x_1; x_2^*)\right]. \tag{18}$$

Equation (18) characterizes player 1's switching point x_1^*, which is independent of t_2. At the PCNE, it has to be evaluated for x_2. Moreover, $\theta_2(.)$ is defined in eq. (15). Thus, with (x_1^*, x_2^*) determined above, the switching time for player 1, $t_1 = \theta_1(x_0, 11)$, is given by

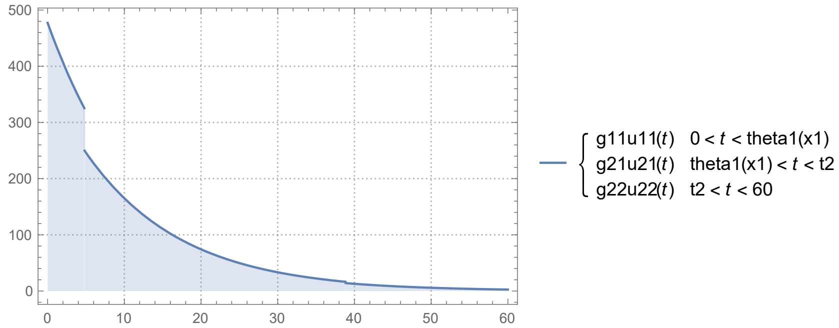
$$t_1 = \theta_1(x_0, 11) = \frac{1}{2\rho} \ln \left(\frac{x_0 + \frac{\Lambda(x_1; x_2^*)}{\rho}}{x_1^* + \frac{\Lambda(x_1; x_2^*)}{\rho}} \right) . \tag{19}$$

In this section, we have seen how Long et al. (2017) obtained analytically the expressions for the relevant variables of the game. In order to make it easier for the reader, to understand the true meaning of them, we will perform a simulation for each model we develop.

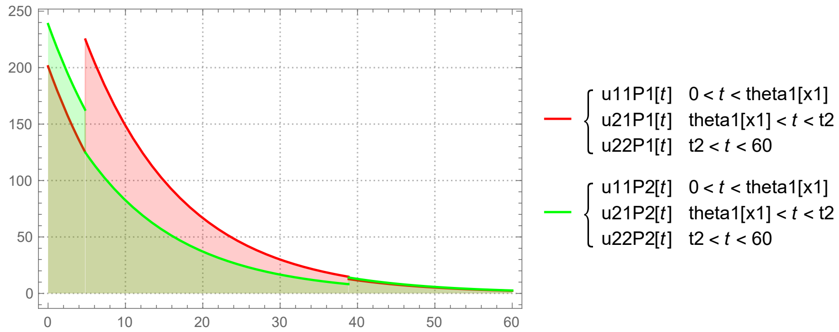
Simulations

Parameters	β_1	β_2	γ_1^1	γ_1^2	γ_2^1	γ_2^2	ρ	χ_1	χ_1	x_0
Values	0.001	0.01	2	1.11	2	1	0.04	1	10	10000

Table 1: Values for the simulations.



(a) $e_1 = e_2$ when $t_1 < t_2$



(b) u_1 and u_2 when $t_1 < t_2$ when $t_1 < t_2$

Figure 1: Evolution of the extraction (above) and consumption rate (below) at the PCNE.

Using the values given in Table 1 for the simulations [2], we can plot the behavior for the extraction rate and the consumption in each state as a function of time. Derived from the equations above and observing the graphics, one can notice that the extraction rates are the same for both players in each state, but, unsurprisingly, different consumption rates, due to the players having different efficient technologies. The optimal response of the agents is determined in the Table 2:

	x_1	x_2	$\theta_2(x_1^*, 21)$	t_1	t_2
Values	6,193.14	352.686	34.04	4.8	38.8

Table 2: Optimal response of the agents.

Intuitively, from Fig. 1b one can see how the consumption rate for player 1 (who has changed first) is higher obviously, due to her changing to a more efficient technology first. Player 2 is consuming less with a lower γ compared to player 1. Player 1 is consuming more extracting

[2] The parameters are given in Long et al. (2017), but we modify $\gamma_1^2 = 1.715$ to the new $\gamma_1^2 = 1.11$ in this paper. This change is motivated to achieve solutions in the model with renewable function. Observe how with a smaller γ_1^2, player 1 becomes more efficient than with the parameters in Long et al. (2017).

less, and player 2 is consuming less due to she now is extracting less (this is an obvious result because player 2 keeps the same technology.). For that reason, the extraction rate decreases, where the effect of the diminished γ_i prevails over the increase extraction rate effect. Observe the small downwards jump in the extraction rate in Fig. 1a at t_2. The intuition behind this behavior, is explained as follows: With the new technology, the player needs less resources to produce a given amount of the consumption. The other face of the same coin is shown in Fig. 1b, where player 2 jumps onward. This piecewise change is explained by the fact that now she can extract more (otherwise, she would not make the switch).

Why player 1 has switched dramatically onward at t_1? As player 1 knows that she will be worse off after player 2 changes to the new technology at t_2 (due to the downward jump we have already explained), she will decide to compensate that cost before, at t_1. This may induce her to compensate this future anticipated costly change by increasing the extraction rate at t_1 . As depicted in Fig.1, even with a decrease of extraction, player 1 can consume much more at t_1 due to to her being more efficient now, and with less extraction, she can consume much more. This will be compared in Section 4.2.

As it has been mentioned above, the switching strategies are defined by eq. (13) and (18). Furthermore, eq. (15) defines the time-to-go before x_1 has been defined. For that reason, $\theta_2(x_1^*, 21)$ is a function of x_1, that is the player 1's switching point. Consequently, player 1 can affect player 2's switching strategies, and this influence will be taken into account by the former. Note also that the waiting time for player 2 (after player 1 has switched), $\theta_2(x_1^*, 21)$, is increasing in x_1^*. This last result means that the larger the resource stock at t_1, the later the adoption of player 2. Intuitively, this means that when player 1 decides to switch promptly, player 2 tends to delay the adoption of the new technology.

It is also worth seeing the evolution of the stock. This is shown in Fig. 2. Where the areas below the function, is the amount of resources that have been consumed up to a period of time, such that $t \in \mathbb{R}_{++}$. As far as sensitivity analysis is concerned, we leave it for the last section.

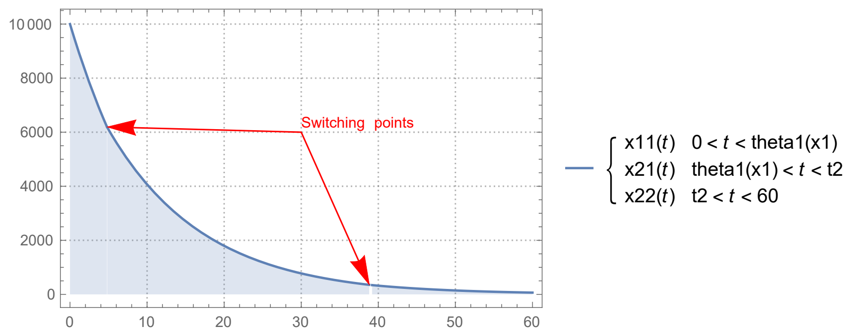


Figure 2: Evolution of the resource.

In the next Sections 3 and 4, we extend the model analyzed by Long et al. (2017) to the case first of a CES utility function, and latter with the introduction of a renewable resource. The renewable extensions is analyzed first with a logarithmic and then with a CES utility function.

3 A Differential Game with CES Utility under a Nonrenewable Resource

This section extends the model presented previously towards a more general utility function. The function is defined as a Constant Elasticity of Substitution (CES) Utility Function. The elasticity of substitution between two different periods in time is constant. Therefore, the utility function for this section is

$$U_i(t) = \frac{u_i^s(t)^\sigma}{\sigma},\ \forall i \in N, s \in \mathcal{S}. \tag{20}$$

Which means that it is used for both players at each state of the game. The present value Hamiltonian for player $i \in N$, in every state of the game is given by

$H_i^s = \frac{u_i^s(t)^\sigma}{\sigma} e^{-\rho t} - \lambda_i^s \left(\gamma_i^l u_i^s(t) + \gamma_j^k \Phi_j(x(t), s)\right)$ and guessing a linear feedback strategy, the Hamiltonian becomes:

$$H_i^s = \frac{u_i^s(t)^\sigma}{\sigma} e^{-\rho t} - \lambda_i^s \left(\gamma_i^l u_i^s(t) + \gamma_j^k (a_j^s + b_j^s x(t))\right). \tag{21}$$

The FOCs are given by the maximum principle

$$\frac{\partial H_i^s}{\partial u_i^s} = 0 \Longrightarrow \lambda_i^s = \frac{1}{\gamma_i^l}(u_i^s)^{\sigma-1} e^{-\rho t} \tag{22a}$$

$$\frac{d\lambda_i^s}{dt} \equiv \dot{\lambda}_i^s = \gamma_j^k b_j^s \lambda_i^s \tag{22b}$$

$$\dot{x} = -\gamma_i^l u_i^s(t) - \gamma_j^k (a_j^s + b_j^s x(t)) \tag{22c}$$

As we proceeded previously, we solve the differential game backwards, i.e., we first study the state $s = 22$, equivalent to the period of time $t \in [t_2, \infty)$. For this purpose, we derive the results from the HJB equation for both players. The FOCs have to be combined with the suitable transversality conditions in each state. Remember that we could differentiate two type of states, terminal or not. Solving (22), we conclude that player's extraction strategies are the same for every state

$$\gamma_i^l \Phi_i(x(t), s) = \gamma_j^k \Phi_j(x(t), s). \tag{23}$$

See Appendix (A.1) for the development of the result. Observe, that if $\sigma \to 0$, the result for the logarithmic non-renewable part appears. As a result of (A-3), for the terminal state $s = 22$ we obtain

$$\gamma_i^2\Phi_i(x(t),22) = \gamma_j^2\Phi_i(x(t),22) = -\frac{\rho}{2\sigma-1}x(t). \tag{24}$$

which means that both players consume the resource accordingly to this equation. With the result (24) and the FOC (22), the Hamiltonian reduces to

$$H_i^s = u_i^s(t)^\sigma e^{-\rho t}\left[\frac{1}{\sigma} + \frac{2\rho}{\gamma_i^s u_i^s(t)(2\sigma-1)}x(t)\right].$$

The evolution of the stock $x^{22}(t)$ at state $s = 22$, that is $\forall t \in [t_2, \infty)$, is given by

$$x^{22*}(t) = x_2^* e^{(\frac{2\rho}{2\sigma-1})(t-t_2)}. \tag{25}$$

3.1 Player 2's switching problem

The goal of this subsection is to study the behavior of player 2. To achieve such a purpose, we proceed as in the previous sections. As player 1 has switched at $t_1 \in (0,\infty)$, there is an amount of resources left x_1. Using (22), (23) and knowing that s is a terminal regime, $s = 22$ and by (7a) and (7b) we obtain

$$[u_2^{21}(t_2)]^\sigma\left[\frac{1}{\sigma} + \frac{2\rho x_2}{\gamma_2^1 u_2^{21}(t_2)(2\sigma-1)}\right] + \rho\omega_2(x_2) = \frac{1-2\sigma}{\sigma}\left(-\frac{\rho}{\gamma_2^1(2\sigma-1)}\right)x_2^\sigma \tag{26a}$$

$$u_2^{21}(t_2) = \left(\frac{\gamma_2^1}{\gamma_2^2}\left(\frac{-\rho x_2}{\gamma_2^2(2\sigma-1)}\right)^{\sigma-1} - \beta_2\gamma_2^1\right)^{\frac{1}{\sigma-1}}, \tag{26b}$$

Knowing eq. (26b) and the value of $\gamma_2^1\Phi_2$ at $t = t_2$ we obtain the consumption strategy in regime $s = 21$

$$\gamma_1^2\Phi_1^{21} = \gamma_2^1\Phi_2^{21} = \gamma_2^1\left(\frac{\gamma_2^1}{\gamma_2^2}\left(\frac{-\rho x_2}{\gamma_2^2(2\sigma-1)}\right)^{\sigma-1} - \beta_2\gamma_2^1\right)^{\frac{1}{\sigma-1}} + \frac{\rho x_2}{2\sigma-1} - \frac{\rho}{2\sigma-1}x(t)$$
$$= \Gamma(x_2) - \frac{\rho}{2\sigma-1}x(t), \tag{27}$$

where in this section,

$$\Gamma(x_2) = \gamma_2^1\left(\frac{\gamma_2^1}{\gamma_2^2}\left(\frac{-\rho x_2}{\gamma_2^2(2\sigma-1)}\right)^{\sigma-1} - \beta_2\gamma_2^1\right)^{\frac{1}{\sigma-1}} + \frac{\rho x_2}{2\sigma-1}.$$

Therefore, here one can see how much both players are extracting in regime 21. From eq. (22) - (26b) and (27), eq. (26a) can be rewritten as

$$\rho\omega_2(x_2) + \left[\left(\frac{1}{\gamma_2^1}\left(\frac{-\rho x_2}{\gamma_2^2(2\sigma-1)}\right)^{\sigma-1} - \beta_2\gamma_2^1\right)^{\sigma-1}\right]^{\sigma}\left[\frac{1}{\sigma} + \frac{2\rho x_2\left(\frac{1}{\gamma_2^1}\left(\frac{-\rho x_2}{\gamma_2^2(2\sigma-1)}\right)^{\sigma-1} - \beta_2\gamma_2^1\right)^{\sigma-1}}{\gamma_2^1(2\sigma-1)}\right] = \frac{(x_2)^{\sigma}\rho}{\sigma\gamma_2^2}. \tag{28}$$

This equation shows the optimal moment of switching x_2^*, and it is independent of the switching time of player 1.

Replacing the consumption given by the expression (27) in the state equation, and solving the resulting differential equation, with the condition x_1 at time $t = t_1$, one obtains the state variable for any $t \in [t_1, t_2)$

$$x^{21*}(t) = \left[x_1^* - \frac{\Gamma(x_2^*)}{\frac{2\rho}{2\sigma-1}}\right] e^{\frac{2\rho}{2\sigma-1}(t-t_1)} - \frac{\Gamma(x_2^*)}{\frac{2\rho}{2\sigma-1}}. \tag{29}$$

One can check how when $\sigma \to 0$, the same result is obtained when the logarithmic utility is considered. Evaluating this equation at t_2 and solving the timing of waiting of player 2 after player 1 has switched $\theta_2 = t_2 - t_1$, we obtain

$$\theta_2(x_1^*, 21) = -\frac{1}{\frac{2\rho}{2\sigma-1}} \ln\left[\frac{x_1^* - \frac{\Gamma(x_2^*)}{\frac{\rho}{2\sigma-1}}}{x_2^* - \frac{\Gamma(x_2^*)}{\frac{\rho}{2\sigma-1}}}\right], \tag{30}$$

which gives the time-to-go (before switching) strategy of player 2, as a function of the equilibrium switching point of player 1, x_1^*. This is the same interpretation of $\theta_2(x_1^*, 21)$ for all sections, but obviously, $\theta_2(.)$ will be different in each section.

3.2 Player 1's switching problem

Now we take a step back to solve the problem of player 1. Player 2's regime switching takes place at some $t_2 \in (0, \infty)$, with the amount of resource at period t_2 defined as $x(t_2^*) = x_2^*$. Making use of the Theorem (8a), (8b), and eq. (22), (24), (27) and (30) we obtain

$$[u_1^{11}(t_1)]^{\sigma}\left[\frac{1}{\sigma}+\frac{2\rho x_1}{\gamma_1^1 u_1^{11}(t_1)(2\sigma-1)}\right]+\rho\omega_1(x_1)=\sum(x_1,x_2^*)+e^{-\rho\theta_2}\Psi(x_2^*) \tag{31}$$

$$\gamma_1^1 u_1^{11}(t_1)=\gamma_1^1\xi(x_1;x_2^*)^{\frac{1}{\sigma-1}}, \tag{32}$$

where $\sum(x_1,x_2^*)$, $\Psi(x_2^*)$ in eq. (31) and $\xi(x_1;x_2^*)$ in eq. (32) are defined in the Appendix (A.3). Knowing $\gamma_1^1 u_1^1(t_1)$ by (32) at the time t_1, and solving for the Markovian strategies in regime $s=11$ we can obtain

$$\gamma_1^1\Phi_1(x,11)=\left(\gamma_1^1\right)^{\frac{\sigma}{\sigma-1}}\xi(x_1;x_2^*)^{\frac{1}{\sigma-1}}+\frac{\rho x_1}{2\sigma-1}-\frac{\rho}{2\sigma-1}x(t)=\Lambda(x_1;x_2^*)-\frac{\rho}{2\sigma-1}x(t). \tag{33}$$

This is now the consumption rate for both players when they are in regime 11. Making use of $u_1^{11}(t_1)$ in (32), and (28), the optimality condition (31) can be rewritten as

$$\left(\frac{\Lambda(x_1,x_2^*)}{\gamma_1^1}-\frac{\rho x_1}{\gamma_1^1(2\sigma-1)}\right)\left[\frac{1}{\sigma}+\frac{2\rho x_1}{(2\sigma-1)\left[\Lambda(x_1,x_2^*)-\frac{\rho}{2\sigma-1}x_1\right]}\right]+\rho\omega_1(x_1)=\sum(x_1,x_2^*)+e^{-\rho\theta_2}\Psi(x_2^*). \tag{34}$$

Equation (34) characterizes player 1's switching point x_1^* which is independent of t_2. At the PCNE, it has to be evaluated for x_2 and $\theta_2(.)$, defined in (28) and (30), respectively.

Replace the extraction u_i for both players into the expression $\Lambda(x_1;x_2^*)+\rho x(t)$, and then plugging it into the state equation and solve the resulting differential equation with the condition $x(t_0)=x_0$. Remember by eq. (23) that both players behave symmetrically in each state. Particularly, at $s=11$, and using eq. (33), we obtain the following differential equation

$$\dot{x}=-\Lambda(x_1;x_2^*)+\frac{\rho}{2\sigma-1}x(t)-\Lambda(x_1;x_2^*)+\frac{\rho}{2\sigma-1}x(t).$$

We then solve $x^{11}(t)=e^{-\int(-\frac{2\rho}{2\sigma-1})dt}\left[C_0+\int e^{\int(-\frac{2\rho}{2\sigma-1})dt}\left[-2\Lambda(x_1;x_2^*)\right]dt\right]$ combined with the given value at t_0 and we obtain

$$x^{11}(t)=e^{\frac{2\rho}{2\sigma-1}t}\left[x_0-\frac{\Lambda(x_1;x_2^*)}{\frac{\rho}{2\sigma-1}}\right]+\frac{\Lambda(x_1;x_2^*)}{\frac{\rho}{2\sigma-1}}. \tag{35}$$

This equation reflects the evolution of the resource for the state $s=11$. Evaluating eq.(35) at the end of its state, which means at $t=t_1$, the timing of switching for player 1 is defined as

$$t_1 = \theta_1(x_0, 11) = \frac{1}{\frac{\rho}{2\sigma - 1}} \ln \left(\frac{x_1 - \frac{\Lambda(x_1; x_2^*)}{\frac{\rho}{2\sigma-1}}}{x_0^* - \frac{\Lambda(x_1; x_2^*)}{\frac{\rho}{2\sigma-1}}} \right). \tag{36}$$

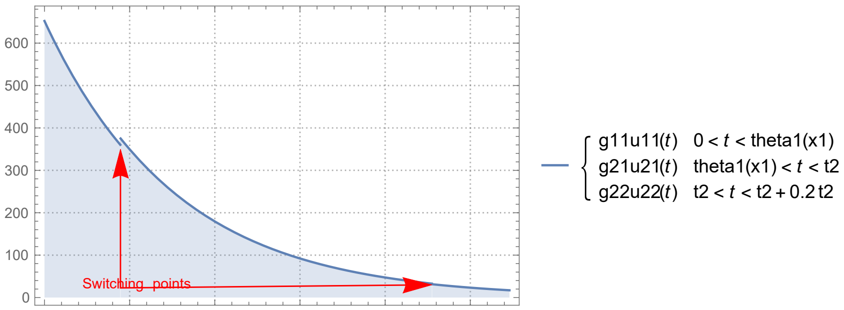
where it depends of the discount rate, elasticity of substitution, the resources available at the beginning and when player 2 switches.

3.3 Simulations

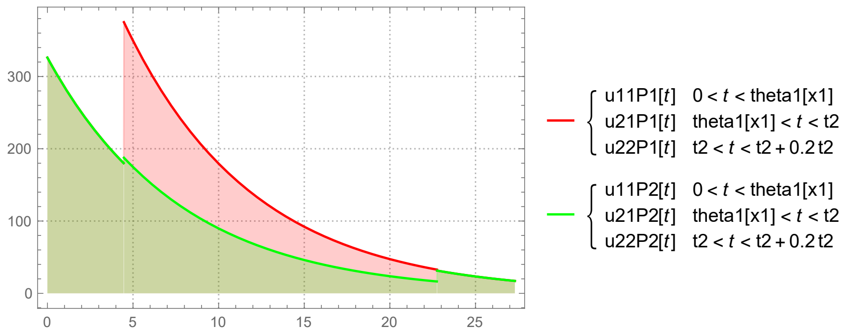
Simulations for the non-renewable CES case is a particular case of the renewable CES model in the next section. Working with the same parameters established in the previous sections, (Table 1) we get the following results for the relevant variables presented in Table 3. The evolution of the extraction and the consumption is shown in Fig. 3.

	x_1	x_2	$\theta_2(x_1^*, 21)$	t_1	t_2
Values	5,613.55	471.864	18.2878	4.46117	22.749

Table 3: Optimal response of the agents under a CES Non-Renewable differential game.



(a) Extraction: $e_1 = e_2$ when $t_1 < t_2$



(b) Consumption: u_1 and u_2 when $t_1 < t_2$ when $t_1 < t_2$

Figure 3: Evolution of the extraction (above) and consumption rate (below) at the PCNE.

In order to get the results from the simulations, we set $\sigma = 0.2$. This will be an increasing function at a decreasing rate, similar to the logarithmic one. Under this scenario, we observe how player 1 has changed slightly a little earlier ($t_1 = 4.46 < 4.8 = t_1^{log}$), where the variables with *log* reflect the logarithmic case. The same analysis is performed as in previous sections. Player 1, who has switched first, extracts more resource because she is more efficient ($\gamma_1^2 = 1.11 < \gamma_1^1 = 2$). At the timing of switching, t_1, now player 2 consumes just a little bit more. She is less efficient in comparison with player 2, but she tries to reduce the gap between her opponent. For that reason, the gap is similar (a little bigger), to the logarithmic case. [3] As in the previous sections, during the state $s = 21$, where player 1 is more efficient than player 2, the former consumes more compared to the latter. It is also worth mentioning that the amount of resources left at t_2 is higher ($x_2 = 471.864 > 352.686 = x_2^{log}$) in the CES non-renewable section. But in this case, the reserves at t_1 are lower, which means that there are less resources when player 1 has switched ($x_1^* = 5613.55 < 6193.14 = x_1^{log*}$). The following fact holds as before, where $\theta_2(.)$ is an increasing function of x_1, which means that when player 1 switches and leaves more resources available, player 2 will wait longer to change to the new regime. Under this framework, player 2's time-to-go reduces to $\theta_2(.) = 18.28 < 34.04 = \theta_2^{log}(.)$. See Appendix (A.2) for the evolution of the resource.

The following section analyzes how the introduction of a renewable resource affect the behavioral of the players. As before, this scenario is studied under two utility functions.

4 Renewable Cases

Will individuals consume more quickly in this section due to the renewable natural resource? Will they consume more? Can we expect the resource to last longer? In this section we try to give some insights to the previous questions.

4.1 A Differential Game with Logarithmic Utility under a Renewable Resource

We now are interested on how a renewable resource affect the behavior of the players for the logarithmic case. To achieve that, we introduce a renewable function to the differential equation of the form

$$\dot{x} = rx(t) - \gamma_i^l u_i^s(t) - \gamma_j^k u_j^s(t). \tag{37}$$

As we proceeded in the previous sections, we study the linear feedback strategies. The Hamiltonian for both player at every state is

$$H_i^s = \ln(u_i^s(t))e^{-\rho t} + \lambda_i^s \left(rx(t) - \gamma_i^l u_i^s(t) - \gamma_j^k (a_j^s + b_j^s x(t)) \right). \tag{38}$$

[3]If player 2 would have jumped downward, she would increase even more the gap.

The FOCs are given by the maximum principle [4]

$$\frac{\partial H_i^s}{\partial u_i^s} = 0 \Longrightarrow \frac{1}{u_i^s} e^{-\rho t} = \gamma_i^l \lambda_i^s \tag{39a}$$

$$\dot{\lambda_i^s} = -\left[\lambda_i^s (r - \gamma_j^k b_j^s)\right] \tag{39b}$$

$$\dot{x} = rx(t) - \gamma_i^l u_i^s(t) - \gamma_j^k u_j^s(t) \tag{39c}$$

Using the same guessing for the HJB equation than in the section without the removable function, we obtain the same extraction strategies as in (10) (see Appendix (A.4)).

$$\gamma_i^l \Phi_i(x(t), s) = \gamma_j^k \Phi_j(x(t), s). \tag{40}$$

The extraction rate, for state $s = 22$ is given by

$$\gamma_i^2 \Phi_i(x(t), 22) = \gamma_j^2 \Phi_i(x(t), 2) = \rho x(t). \tag{41}$$

which is the same extraction rate than under the nonrenewable case for the last regime. With the result (41), and the FOCs (39), the Hamiltonian reduces to

$$H_i^s = e^{-\rho t}\left[\ln(u_i^s(t)) - 2 + \frac{rx(t)}{u_i^s \gamma_i^l}\right]. \tag{42}$$

Thus, using the extraction rate at the terminal stat, the evolution of the stock $x^{22}(t)$ is given by

$$x^{22*}(t) = x_2^* e^{(r-2\rho)(t-t_2)}. \tag{43}$$

4.1.1 Player 2's switching problem

In this subsection, we use the Theorem 2, to determine the behavior of player 2. Making use of the theorem mentioned above for player 1, that is eq. (7a), we obtain

$$\ln\left(u_2^{21}(t_2)\right) + \frac{rx_2}{\gamma_2^1 u_2^{21}(t_2)} + \rho\omega(x_2) = \ln\left(\frac{\rho x_2}{\gamma_2^2}\right) + \frac{r}{\rho}\ . \tag{44}$$

By the same Theorem 2, but now with eq. (7b) , one obtains the same condition as with non-removable differential equation

[4]For a rigorous study of the derivations of the FOC see Sydsæter et al. (2008) and Wlde (2008).

$$u_2^{21}(t_2) = \frac{\rho x_2}{\gamma_2^1(1-\beta_2\rho x_2)}. \tag{45}$$

Knowing $\gamma_2^1\Phi_2^{21}$ at the value t_2, and u_2^{21} from (45), one obtains the consumption strategies in regime 21

$$\gamma_1^2\Phi_1^{21} = \gamma_2^1\Phi_2^{21} = \frac{\rho^2\beta_2(x_2)^2}{1-\beta_2\rho x_2} + \rho x(t) = \Gamma(x_2) + \rho x(t). \tag{46}$$

From (39), (45), and (46), we can rewrite (44) as

$$\rho\omega_2(x_2) + \ln\left(\frac{\gamma_2^2}{\gamma_2^1}\right) - \frac{r}{\rho}(\rho x_2\beta_2) = \ln(1-\beta_2\rho x_2). \tag{47}$$

This equation defines the optimal level of switching, x_2^*, which is independent of the switching time of player 1. For a relationship between r and x_2, see Appendix (A.5).

Replacing consumption with the eq. (46) in the differential equation, and solve it for the boundary condition $x(t_1^*) = x_1^*$, the evolution of the state variable for any $t \in [t_1, t_2)$ is determined by

$$x^{21*}(t) = \left[x_1^* - \frac{2\left(\frac{\rho^2\beta_2(x_2^*)^2}{1-\beta_2\rho x_2^*}\right)}{r-2\rho}\right] e^{(r-2\rho)[t-t_1]} + \frac{2\rho^2\beta_2(x_2^*)^2}{(r-2\rho)(1-\beta_2\rho x_2^*)}\ . \tag{48}$$

Evaluating this equation at t_2 and solving for the timing of waiting for player 2 after player 1 has switched, $\theta_2 = t_2 - t_1$, we obtain

$$\theta_2(x_1^*, 21) = \frac{1}{2\rho - r}\ln\left[\frac{(r-2\rho)(1-\rho\beta_2 x_2^*)x_1^* - 2\rho^2\beta_2(x_2^*)^2}{(r-2\rho)(1-\rho\beta_2 x_2^*)x_2^* - 2\rho^2\beta_2(x_2^*)^2}\right]. \tag{49}$$

4.1.2 Player 1's switching problem

Following the same philosophy as before, player 2's regime switching takes place at some $t_2 \in \mathbb{R}_{++}$, with the amount of resource at period t_2 defined as $x(t_2^*) = x_2^*$. Making use of (8a) (8b) from the Theorem 2, and eq. (39), (41), (46) and (49) one can obtain

$$\ln\left(u_1^{11}(t_1)\right) + \frac{rx_1}{\gamma_1^1 u_1^{11}} + \rho\omega_1(x_1) = \ln\left(\frac{\Gamma(x_2^*)+\rho x_1}{\gamma_1^2}\right) + \frac{rx_1}{\Gamma(x_2^*)+\rho x_1} + e^{-\rho\theta_2}\left[\ln\left(\frac{\rho x_2}{\Gamma(x_2^*)+\rho x_2^*}\right) + r\left[\frac{\Gamma(x_2^*)}{\rho\left(\Gamma(x_2^*)+\rho x_2^*\right)}\right]\right] \tag{50}$$

$$\gamma_1^1 u_1^{11}(t_1) = \frac{\Gamma(x_2^*)+\rho x_1}{\xi(x_1;x_2^*)}, \tag{51}$$

where

$$\xi(x_1;x_2^*) = 1 - \theta_2'[\Gamma(x_2^*)+\rho x_1]e^{-\rho\theta_2}\left[\ln(1-\rho\beta_2 x_2^*) - \frac{rx_2}{\Gamma(x_2^*)+\rho x_2^*} + \frac{r}{\rho}\right] - \beta_1[\Gamma(x_2^*)+\rho x_1] \tag{52}$$

Observe that $\Gamma(.)$ is defined in this model by eq. (46), and $\theta_2'[\Gamma(x_2^*)+\rho x_1] \equiv \partial\theta_2/\partial x_1^*[\Gamma(x_2^*)+\rho x_1] = 1/2$ as we obtained in the non-renewable section. Solving for the Markovian consumption strategies in regime $s = 11$, one finds

$$\gamma_1^1\Phi_1(x,11) = \gamma_2^1\Phi_2(x,11) = \frac{\Gamma(x_2^*)+\rho x_1[1-\xi(x_1;x_2^*)]}{\xi(x_1;x_2^*)} + \rho x(t) = \Lambda(x_1;x_2^*) + \rho x(t). \tag{53}$$

which determines the consumption rate for both players at regime 11. An interesting detail to keep in mind is that we have made the necessary manipulations to make it look like the non-renewable model, although the expressions themselves are completely different. Notice that $\Lambda(x_1;x_2^*)$ is different due to $\xi(x_1;x_2^*)$ is different. Substituting $u_1^{11}(t_1)$ with the expression in (51), using (47), and $\Gamma(x_2^*) = \gamma_1^2 u_1^{21} - \rho x_1$, the optimality condition (50) can be rewritten as

$$\left(\frac{\gamma_2^1}{\gamma_1^1}\right) + \rho\omega_1(x_1) = e^{-\rho\theta_2}\left[\ln\left(\frac{\rho x_2^*}{\frac{\rho^2\beta_2(x_2^*)^2}{1-\beta_2\rho x_2^*} + \rho x_2^*}\right) + r\left[\frac{\frac{\rho^2\beta_2(x_2^*)^2}{1-\beta_2\rho x_2^*}}{\rho\left[\frac{\rho^2\beta_2(x_2^*)^2}{1-\beta_2\rho x_2^*} + \rho x_2\right] - \rho^2 x_1 + \rho^2 x_2^*}\right]\right] + \\ + \ln\left[\xi(x_1;x_2^*)\right] - \frac{rx_1}{\frac{\rho^2\beta_2(x_2^*)^2}{1-\beta_2\rho x_2^*} + \rho x_1}\left[\xi(x_1;x_2^*) - 1\right]. \tag{54}$$

Eq. (54) characterizes player 1's switching point x_1^* which is independent of t_2. Replacing the extraction u_i, for both players with the expression $\Lambda(x_1;x_2^*) + \rho x(t)$, and then plugging it into the state equation and solve the resulting differential equation with the condition $x(t_0) = x_0$. Remember that both players play symmetrically in each state. Particularly, at $s = 11$ we use eq. (53).

$$\dot{x} = rx(t) - \Lambda(x_1; x_2^*) - \rho x(t) - \Lambda(x_1; x_2^*) - \rho x(t)$$

We then solve $x^{11}(t) = e^{-\int(-(r-2\rho))dt}\left[C_0 + \int e^{\int(-(r-2\rho))dt}\left[-2\Lambda(x_1; x_2^*)\right] dt\right]$ combined with the given value x_0 at t_0 obtaining

$$x^{11}(t) = e^{(r-2\rho)t}\left[x_0 - \frac{2\Lambda(x_1; x_2^*)}{r-2\rho}\right] + \frac{2\Lambda(x_1; x_2^*)}{r-2\rho}. \tag{55}$$

Evaluations eq.(55) (which shows the evolution of the resource at the first regime) at the end of the state $s = 11$, which means at $t = t_1$, we obtain the switching time for player 1

$$t_1 = \theta_1(x_0, 11) = \frac{1}{(r-2\rho)} \ln\left(\frac{x_1^* - \frac{2\Lambda(x_1; x_2^*)}{r-2\rho}}{x_0 - \frac{2\Lambda(x_1; x_2^*)}{r-2\rho}}\right). \tag{56}$$

Observe, how in this section the recovery factor appears in the equation explicitly and in the expression $\Lambda(x_1; x_2^*)$.

4.1.3 Simulations

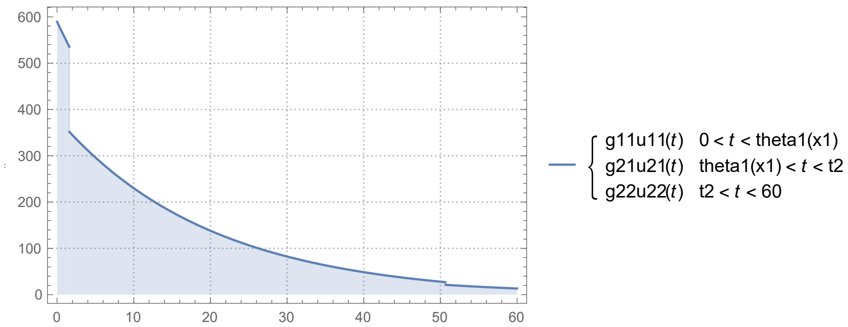
This section analyses the effect of the renewable function on the values of the important variables of the model. We base this analysis in the frameworks presented before (with the same values as in Table 1) but now observe that $r > 0$. In this case we set $r = 0.03$, which means that the resource renews itself by three percent each period. Under this framework, the behavior for the extraction is given in the Table 4. [5]

	x_1	x_2	$\theta_2(x_1^*, 21)$	t_1	t_2
Values	8,651.09	533.108	49.057	1.597	50.65

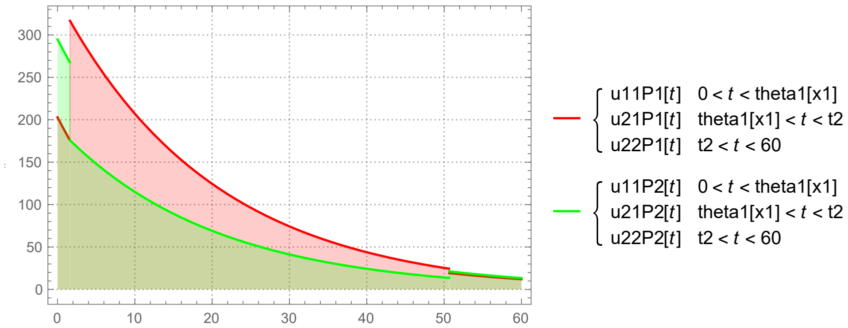
Table 4: Optimal response of the agents.

Under the renewable framework, one can easily see how player 1 switches earlier ($t_1 = 1.597$) compared to the model without renewable function ($t_1^{nr} = 4.8$). Both players continue extracting the same at each state, but as in the section with a nonrenewable recourse with a logarithmic utility, they have different consumption rates, due to both players have different technologies at each state. Player 1, who has switched first, extracts more resource because she is more efficient ($\gamma_1^2 = 1.11 < \gamma_1^1 = 2$). At the timing of switching, t_1, player 2 consumes less, due to her moving downwards (she is less efficient compared to player 1). During the state $s = 21$, where player 1 is more efficient than player 2, the former consumes more compared to the latter. It is also worth

[5] We have written the code of the model in *Mathematica* in order to see how it reacts to a change in a parameter.



(a) Extraction: $e_1 = e_2$ when $t_1 < t_2$



(b) Consumption: u_1 and u_2 when $t_1 < t_2$ when $t_1 < t_2$

Figure 4: Evolution of the extraction (above) and consumption rate (below) at the PCNE.

mentioning that the amount of resources left at t_2 is higher ($x_2^* = 533.108 > 352.686 = x_2^{*nr}$) in the renewable section as well as the reserves at t_1 ($x_1^* = 8,651.09 > 6,193.14 x_1^{*nr}$). This is determined by the introduction of a new resource, a self-generated one. One may be wondering why this resource is exhausted when $t \to \infty$. To study that, one should ensure that the equation of the resources at $s = 22$ converge to some $\alpha \in \mathbb{R}_{++}$, and therefore, it would be stable. The condition for this to happen is determined by the exponential part $(r - 2\rho)$.

- If $(r > 2\rho)$, therefore $x^{22*}(t)$ explodes, and the resource would be infinite.
- If $(r < 2\rho)$, $x^{22*}(t)$ will be exhausted when times tends to infinite, but latter than in the case with $r = 0$
- If $(r = 2\rho)$, one may think that the resource will stay at x_2^* forever. Under this framework, there is a big problem, the model cannot be solved. The first stone on the road is determined by the time-to-go for player 2. When $(r = 2\rho)$, there is not a direct value, and therefore one should apply l'Hopital rule. But the big problem arises, when we try to find x_1^*. Under this scenario, there is no solution. As a consequence of this big problem, we get that there is no evolution of the resource in the state $s = 11$, nor the time to change for player 1.

With all the above cases in mind, the most realistic scenario will be when $(r < 2\rho)$, but very slightly below. We will pay attention to this case in a few lines below, as it will present certain peculiarities.

But first, study the case analyzed before, where $(r < 2\rho)$ but not very slightly below, where $r = 0.3$. The same philosophy is used as in the section that shows the work by Long et al. (2017). Player 1 is consuming more extracting less, and player 2 is consuming less because she is extracting lees (and furthermore she is using the inefficient technology at $s = 21$). The same behavior at t_2 is observed for player 2 moving downwards. [6] The waiting time between the two changes is longer in this model ($\theta_2 = 49.057 > 38.839 = \theta_2^{nr}$), which affects player 2 to change later ($t_2 = 50.65 > 38.83 = t_2$), even if the player 1 has changed her technology quickly. The evolution of the stock under the $(r < 2\rho)$ case is shown in Fig. 5

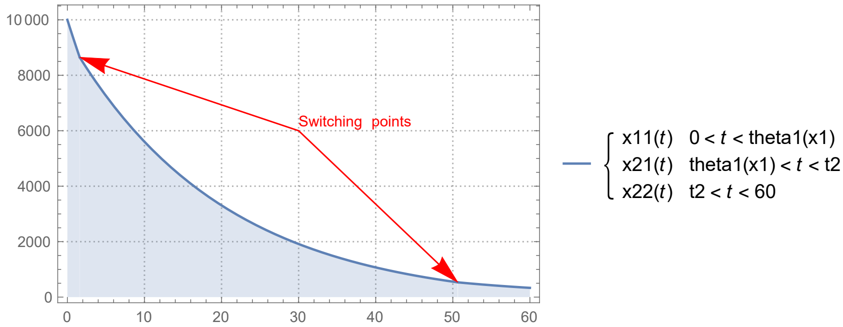


Figure 5: Evolution of the resource under a renewavle framework with r=0.3.

Observe how now, not surprisingly, both players could extract more resource due to the fact that the resource is self-generated.

As mentioned earlier, it does not seem reasonable for players to exhaust the challenge when time tends to infinite. For that reason, if we simulate the model, when $r = 0.0799999999999999$, which is very close to $r = 0.08$, we obtain the results in Table 5

	x_1	x_2	$\theta_2(x_1^*, 21)$	t_1	t_2
Values	10,000	1460.14	51.88	0	51.88

Table 5: Optimal response of the agents for a r close to 0.08.

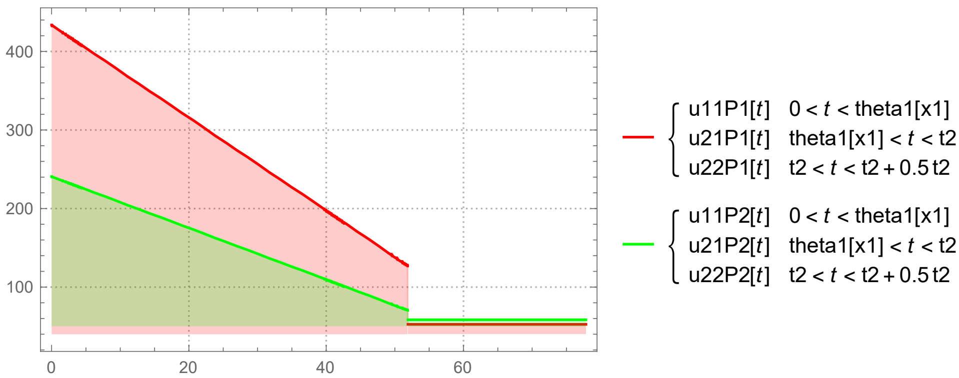


Figure 6: Evolution of the resource under a renewavle framework with $r \to 0.08^-$, lower bound .

[6]See Section ?? for the intuition.

Observe that, trying to make the resource last as long as possible, we found a corner solution, which is beyond the scope of the paper, because we are studying interior solutions. As one can see, from t_2 onwards, the function is almost parallel to the x-axis. The resource will disappear at the time 4×10^{15}. See Appendix (A.6) for the evolution of the resource.

4.2 A Differential Game with CES Utility under a Renewable Resource

In this section we study how a more general utility function affects the behavior of both players. Following the same philosophy that has been used throughout the paper, the present value Hamiltonian for player $i \in N$ and for every state is given by

$H_i^s = \frac{u_i^s(t)^\sigma}{\sigma} e^{-\rho t} + \lambda_i^s \left(rx(t) - \gamma_i^l u_i^s(t) - \gamma_j^k \Phi_j(x(t), s) \right)$. and using our linear feedback strategy, the Hamiltonian becomes

$$H_i^s = \frac{u_i^s(t)^\sigma}{\sigma} e^{-\rho t} + \lambda_i^s \left(rx(t) - \gamma_i^l u_i^s(t) - \gamma_j^k (a_j^s + b_j^s x(t)) \right) \tag{57}$$

The FOCs are given by the maximum principle

$$\frac{\partial H_i^s}{\partial u_i^s} = 0 \Longrightarrow \lambda_i^s = \frac{1}{\gamma_i^l} (u_i^s)^{\sigma - 1} e^{-\rho t} \tag{58a}$$

$$\frac{d\lambda_i^s}{dt} \equiv \dot{\lambda}_i^s = -\left(\lambda_i^s \left(r - \gamma_j^k b_j^s \right) \right) \tag{58b}$$

$$\dot{x} = rx(t) - \gamma_i^l u_i^s(t) - \gamma_j^k (a_j^s + b_j^s x(t)), \tag{58c}$$

As we proceeded previously, we solve the differential game backwards. Thus, we pay attention to the state $s = 22$, equivalent to the period of time $t \in [t_2, \infty)$. For this purpose, we derive the results from the HJB equation for both players. The FOCs have to be combined with the suitable transversality conditions in each state. Remember that we could differentiate two type of states, terminal or not. Solving (58), we conclude that player's extraction strategies are the same for every state

$$\gamma_i^l \Phi_i(x(t), s) = \gamma_j^k \Phi_j(x(t), s). \tag{59}$$

See Appendix (A.7) to the development of the HJB equation. Therefore, for the terminal state $s = 22$, we obtain

$$\gamma_i^2 \Phi_i(x(t), s) = \gamma_j^k \Phi_i(x(t), s) = \frac{r\sigma - \rho}{2\sigma - 1} x(t). \tag{60}$$

This is now the consumption rate for both players when they are at regime 22. Notice, that if we set $r = 0$, we obtain the same result as in the non-renewable framework. With this result, (60) and the FOCs (58), the Hamiltonian reduces to

$$H_i^s = u_i^s(t)^\sigma e^{-\rho t}\left[\frac{1}{\sigma} + \frac{r - 2\left(\frac{r\sigma-\rho}{2\sigma-1}\right)}{\gamma_i^s u_i^s(t)} x(t)\right].$$

Accordingly, using eq. (60) and the boundary condition $x(t_2) = x_2$, the evolution of the stock $x^{22}(t)$ at state $s = 22$, that is $\forall t \in [t_2, \infty)$, is given by

$$x^{22*}(t) = x_2^* e^{\left(r - \frac{2(r\sigma-\rho)}{2\sigma-1}\right)(t-t_2)} \tag{61}$$

4.2.1 Player 2's switching problem

Now, we analyze the behavior of player 2, and her decision variables. As player 1 has switched at $t_1 \in (0, \infty)$ there is an amount of resources left x_1. Using the FOC for this section, (60) and knowing that s is a terminal regime, $s = 22$ and by the Theorem 2, that is, equations (7a) and (7b) we obtain

$$[u_2^{21}(t_2)]^\sigma \left[\frac{1}{\sigma} + \frac{x_2\left(r - \frac{2(r\sigma-\rho)}{2\sigma-1}\right)}{\gamma_2^1 u_2^{21}(t_2)}\right] + \rho\omega_2(x_2) = (x_2)^\sigma \left(\frac{r\sigma-\rho}{\gamma_2^2(2\sigma-1)}\right)^\sigma \left(\frac{1}{\sigma} + \frac{r - \frac{2(r\sigma-\rho)}{2\sigma-1}}{\frac{r\sigma-\rho}{2\sigma-1}}\right) \tag{62a}$$

$$u_2^{21}(t_2) = \left\{\left[\frac{1}{\gamma_2^2}\left[\left(\frac{r\sigma-\rho}{2\sigma-1}\right)\frac{x_2}{\gamma_2^2}\right]^{\sigma-1} - \beta_2\right]\gamma_2^1\right\}^{\frac{1}{\sigma-1}}, \tag{62b}$$

Knowing eq. (62b) and the value of $\gamma_2^1\Phi_2$ at $t = t_2$ we obtain the consumption strategy in regime $s = 21$

$$\gamma_1^2\Phi_1^{21} = \gamma_2^1\left(\gamma_2^1\left[\frac{\left(\frac{x_2(r\sigma-\rho)}{\gamma_2^2(2\sigma-1)}\right)^{\sigma-1}}{\gamma_2^2} - \beta_2\right]\right)^{\frac{1}{\sigma-1}} - \frac{(r\sigma-\rho)x_2}{2\sigma-1} + \frac{r\sigma-\rho}{2\sigma-1}x(t)$$

$$= \Gamma(x_2) + \frac{r\sigma-\rho}{2\sigma-1}x(t), \tag{63}$$

where in this section,

$$\Gamma(x_2) = \gamma_2^1 \left(\gamma_2^1 \left[\frac{\left(\frac{x_2(r\sigma-\rho)}{\gamma_2^2(2\sigma-1)}\right)^{\sigma-1}}{\gamma_2^2} - \beta_2 \right] \right)^{\frac{1}{\sigma-1}} - \frac{(r\sigma-\rho)x_2}{2\sigma-1}.$$

From the FOCs of this section, (62b) and (63), eq. (62a) can be rewritten as

$$\left(\frac{x_2\left(r - \frac{2(r\sigma-\rho)}{2\sigma-1}\right)}{\gamma_2^1 \left(\gamma_2^1 \left(\frac{\left(\frac{x_2(r\sigma-\rho)}{\gamma_2^2(2\sigma-1)}\right)^{\sigma-1}}{\gamma_2^2} - \beta_2 \right) \right)^{\frac{1}{\sigma-1}}} + \frac{1}{\sigma} \right) \left(\left(\gamma_2^1 \left(\frac{\left(\frac{x_2(r\sigma-\rho)}{\gamma_2^2(2\sigma-1)}\right)^{\sigma-1}}{\gamma_2^2} - \beta_2 \right) \right)^{\frac{1}{\sigma-1}} \right)^{\sigma} + \rho\omega_2(x_2)$$

$$= (x_2)^{\sigma} \left(\frac{1}{\sigma} + \frac{r - \frac{2(r\sigma-\rho)}{2\sigma-1}}{\frac{r\sigma-\rho}{2\sigma-1}} \right) \left(\frac{r\sigma-\rho}{\gamma_2^2(2\sigma-1)} \right)^{\sigma}. \quad (64)$$

This equation shows the optimal moment of switching x_2^*, and it is independent of the switching time of player 1.

Replacing consumption given by the expression (63) in the state equation, and solving the resulting differential equation, with the condition x_1 at time $t = t_1$, one obtains the state variable for any $t \in [t_1, t_2)$

$$x^{21*}(t) = \left[x_1^* - \frac{2\Gamma(x_2^*)}{\left(r - 2\frac{r\sigma-\rho}{2\sigma-1}\right)} \right] e^{\left(r-2\frac{r\sigma-\rho}{2\sigma-1}\right)(t-t_1)} + \frac{2\Gamma(x_2^*)}{\left(r - 2\frac{r\sigma-\rho}{2\sigma-1}\right)}. \quad (65)$$

One can check how when $\sigma \to 0$, and $r = 0$, the same result is obtained when the logarithmic utility with the non-renewable resource is considered. Evaluating this equation at t_2 and solving the timing of waiting of player 2 after player 1 has switched $\theta_2 = t_2 - t_1$, we obtain

$$\theta_2(x_1^*, 21) = \frac{1}{\left(r - 2\frac{r\sigma-\rho}{2\sigma-1}\right)} \ln \left[\frac{x_2^* - \frac{2\Gamma(x_2^*)}{\left(r-2\frac{r\sigma-\rho}{2\sigma-1}\right)}}{x_1^* - \frac{2\Gamma(x_2^*)}{\left(r-2\frac{r\sigma-\rho}{2\sigma-1}\right)}} \right], \quad (66)$$

which gives the time-to-go strategy of player 2 as a function of the equilibrium switching point of player 1, x_1^*.

4.2.2 Player 1's switching problem

Player 2's regime switching takes place at some $t_2 \in (0, \infty)$, with the amount of resource at period t_2 defined as $x(t_2^*) = x_2^*$. Making use of the Theorem (8a), (8b), and eq. (58), (60), (63) and (66) we obtain

$$[u_1^{11}(t_1)]^\sigma \left[\frac{1}{\sigma} + \frac{r - 2\left(\frac{r\sigma-\rho}{2\sigma-1}\right)x_1}{\gamma_1^1 u_1^{11}(t_1)} \right] + \rho\omega_1(x_1) = \sum(x_1, x_2^*) + e^{-\rho\theta_2}\Psi(x_2^*) \tag{67}$$

$$\gamma_1^1 u_1^{11}(t_1) = \left(\gamma_1^1\right)^{\frac{\sigma}{\sigma-1}} \xi(x_1; x_2^*)^{\frac{1}{\sigma-1}}, \tag{68}$$

where $\sum(x_1, x_2^*)$, $\Psi(x_2^*)$ in eq. (67) and $\xi(x_1; x_2^*)$ in eq. (68) are defined in the Appendix (A.8). Knowing $\gamma_1^1 u_1^1(t_1)$ by (68) at time t_1, and solving for the Markovian strategies in regime $s = 11$ we obtain

$$\gamma_1^1 \Phi_1(x, 11) = \left(\gamma_1^1\right)^{\frac{\sigma}{\sigma-1}} \xi(x_1; x_2^*)^{\frac{1}{\sigma-1}} - \left(\frac{r\sigma-\rho}{2\sigma-1}\right)x_1 + \frac{r\sigma-\rho}{2\sigma-1}x(t) = \Lambda(x_1; x_2^*) + \frac{r\sigma-\rho}{2\sigma-1}x(t). \tag{69}$$

Making use of $u_1^{11}(t_1)$ in (68), and (64), the optimality condition (67) can be rewritten as

$$\left[\Lambda(x_1, x_2^*) + \left(\frac{r\sigma-\rho}{2\sigma-1}\right)x_2\right]^\sigma \left[\frac{1}{\sigma} + \frac{\left[r - 2\left(\frac{r\sigma-\rho}{2\sigma-1}\right)\right]x_1}{\Lambda(x_1, x_2^*) + \left(\frac{r\sigma-\rho}{2\sigma-1}\right)x_1} \right] + \rho\omega_1(x_1) = \sum(x_1, x_2^*) + e^{-\rho\theta_2}\Psi(x_2^*). \tag{70}$$

Equation (70) characterizes player 1's switching point x_1^* which is independent of t_2. Replace the extraction u_i for both players into the expression $\Lambda(x_1; x_2^*) + \rho x(t)$, and then plug it into the state equation with the condition $x(t_0) = x_0$.

Remember by eq. (59) that both players behave symmetrically in each state. Particularly, at $s = 11$, and using eq. (69), we obtain the following differential equation

$$\dot{x} = rx(t) - \Lambda(x_1; x_2^*) - \left(\frac{r\sigma-\rho}{2\sigma-1}\right)x(t) - \Lambda(x_1; x_2^*) - \left(\frac{r\sigma-\rho}{2\sigma-1}\right)x(t).$$

We then solve $x^{11}(t) = e^{-\int -(r-2\frac{r\sigma-\rho}{2\sigma-1})dt}\left[C_0 + \int e^{\int -(r-2\frac{r\sigma-\rho}{2\sigma-1})dt}\left[-2\Lambda(x_1; x_2^*)\right]dt\right]$ combined with the given value at t_0

$$x^{11}(t) = e^{\left[r-2\frac{r\sigma-\rho}{2\sigma-1}\right]t}\left[x_0 - \frac{2\Lambda(x_1;x_2^*)}{\left[r-2\frac{r\sigma-\rho}{2\sigma-1}\right]}\right] + \frac{2\Lambda(x_1;x_2^*)}{\left[r-2\frac{r\sigma-\rho}{2\sigma-1}\right]}. \tag{71}$$

Evaluations eq.(71) at the end of the state $s = 11$, what means, $t = t_1$, we obtain the switching time for player 1

$$t_1 = \theta_1(x_0, 11) = \frac{1}{\left(r-2\frac{r\sigma-\rho}{2\sigma-1}\right)} \ln\left(\frac{x_1 - \frac{2\Lambda(x_1;x_2^*)}{\left(r-2\frac{r\sigma-\rho}{2\sigma-1}\right)}}{x_0^* - \frac{2\Lambda(x_1;x_2^*)}{\left(r-2\frac{r\sigma-\rho}{2\sigma-1}\right)}}\right). \tag{72}$$

Observe how now the recovery factor appears in the equation explicitly and in the expression $\Lambda(x_1;x_2^*)$. Eq. (72) defines when player 1 switches to her second regime, and therefore, to the regime $s = 21$ in the game.

4.2.3 Simulation and Sensitivity Analysis

Finally, we proceed to perform simulations and a sensitivity analysis under the umbrella of the present model. Performing the simulations, the results of the relevant variables of the model are shown in Table 7.

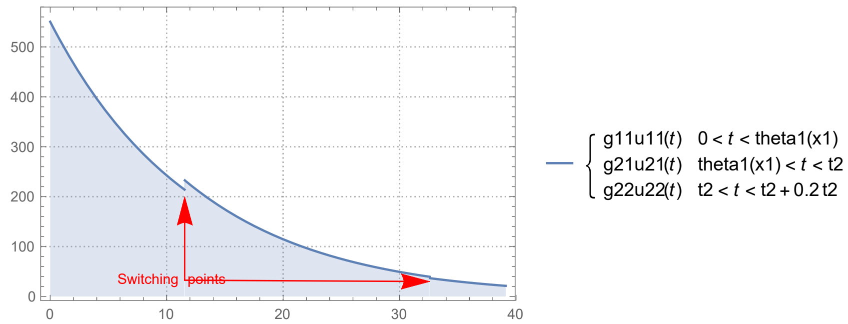
Parameters	β_1	β_2	γ_1^1	γ_1^2	γ_2^1	γ_2^2	ρ	χ_1	χ_1	x_0	r	σ
Values	0.001	0.01	2	1.11	2	1	0.04	1	10	10000	0.03	0.2

Table 6: Values for the simulations under the CES-Renewable model.

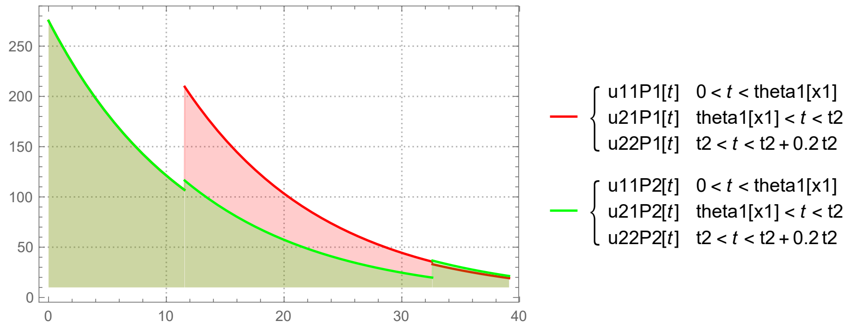
	x_1	x_2	$\theta_2(x_1^*, 21)$	t_1	t_2
Values	4,061.13	648.275	21.0413	11.5476	32.5889

Table 7: Optimal response of the agents under a CES-Renewable differential game.

The same behavior described at Table 7 is shown in Fig. 7. It is worth showing the evolution of the resource over the time horizon. This information is displayed in Fig. 8. As it was proved by the equations, both players have the same extraction rate in each regime. This is easily seen in Fig. 7a. However, they differ in their consumption shown in Fig. 7b as they have different γ (technologies). In this scenario, both players decide to extract more, which means that player 2 consumes a little more, while player 1 consumes much more, due to she is now more efficient. Slightly increasing the extraction rate for player 1, she benefits from a higher consumption. The impact for player 2 after she has adopted the new technology at time t_2 is positive (she jumps upwards), or otherwise, she would never switch. As it was considered in Long et al. (2017), the movement of the extraction rate at t_1 depends on whether $\xi(x_1^*;x_2^*) \lesseqgtr 1$, which is unclear in general. This expression presents information on the magnitude and direction of the adjustment of extraction (and the consumption for player 2, since she is still using the same



(a) Extraction: $e_1 = e_2$ when $t_1 < t_2$



(b) Consumption: u_1 and u_2 when $t_1 < t_2$ when $t_1 < t_2$

Figure 7: Evolution of the extraction (above) and consumption rate (below) at the PCNE for the CES-Renewable differential game.

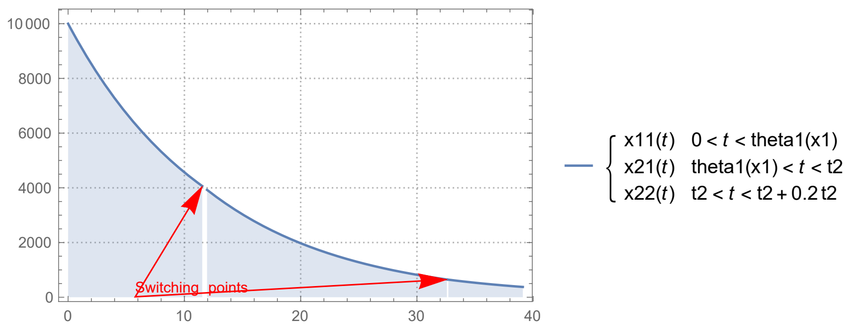


Figure 8: Evolution of the resource under the CES-Renewable differential game.

technology). This adjustment occurs when players move from regime 11 to regime 21. For this reason ($\xi(x_1^*; x_2^*)$), both players jumped downwards in the previous section. Player 1 knows that she will be worse off when player 2 decides to change, because she will endure the decrease in extraction rates while she will not be able to compensate by changing his own technology again. This explains why player 1 rewards herself by increasing the extraction rate at t_1.

However, a question that has been hovering in this paper and, we now develop in this more general model, is how player 2 chooses the switching point x_2^*. According to eq. (64), she equalizes the net marginal gain of adoption to the direct marginal switching cost (this is just $\rho\omega_2(x_2)$). Furthermore, eq. (66) verifies that the waiting time for player 2 before player 1 has switched, is defined in terms of the switching point of player 1, x_1^*, and other parameters that define regime 21. As we mentioned in the section with the nonrenewable resource with the logarithmic utility function, player 1 affects the moment of change of player 2, and the former will take this advantage into account in the first regime problem.

We now focus on a comparative analysis. Using this model as the most general one, it is compared to the non-renewable model. For that purpose, Tables 3 and 7 should be compared. Firstly, player 1 switches later compared to the CES model with the non-renewable resource ($t_1 = 11.55 > 4.46 = t_1^{nr}$), where the nonrenewable variables are denoted with nr. Then, player 2 decides to switch later under the renewable umbrella ($t_2 = 32.59 > 22.749 = t_2^{nr}$). This phenomenon is determined by the fact that player 2 waits longer to change, once player 1 has already changed to regime 21 ($\theta_2(.) = 32.59 > 22.749 = \theta_2(.)^{nr}$). Comparing the switching point for player 1, she decides to switch leaving less resources available ($x_1^* = 4,061.13 < 5,613.55 = x_1^{*nr}$). Finally, player 2 decides to switch to regime 22, by making available an amount of resources equal to ($x_2^* = 648.275 > 471.86 = x_2^{*nr}$), which is more than with the model presented in the previous section.

After performing this analysis, we proceed to study how sensible the model is. For this end, we modify all the parameters, increasing and decreasing their values in the next table, showing therefore, how the relevant variables are affected. These simulations should be compared with the CES-renewable model presented in this section.

One can observe how sensible the model is when the renewable parameter is modified. When this approaches to 0.08 from the left, player 1 and player 2 decide to switch really late. For instance, player 1 switches 27 times later than under the model presented in the first row. [7] This answers one question in the introduction. Under this renewable resource, players decide to extract more, due to the fact that the resource is auto generated, and they will enjoy more from it than under a non-renewable framework. When player 1 is more sensible for the adoption cost at the instant of switching (β_1), she will decide to switch later leaving less resources available for the rest of the game (which means that they have been consuming more at the first stage). The same behavior is applied for player 2, but taking into account her variables. As far as the analysis of the efficiency of the players' technology is concerned, when player 1 is less efficient at stage 11 ($\gamma_1^1 = 2.3$) she will decide to switch earlier and enjoy the new technology. The same philosophy is observed for player 2. On the contrary, when they become more efficient before they change to the new technology (lower γ_1), they will delay the moment of change, due to the fact that the technology gap between the two regimes is lower now. Regarding the adoption of their new technology, when they become more efficient in the future, obviously they want to adopt it earlier.

Additionally, when the discount rate increases, that is, they do not consider the future so much anymore, and when both players switch, they leave less resources than under the baseline model, what means that they have consumed more. A remarkable fact is that player 1 does not change her switching time much, while player 2 reduces it considerably (due to when player 2 switches, it is considered very distant in time from the perspective of $t = 0$). Moreover,

[7] We show the regular model presented in Table 7 in the simulation Table 8 to make the comparison easier for the reader.

	$x2$	$x1$	$\theta_2(.)$	t_1	t_2
Regular	648.275	4.061,13	21.041	11.547	32.5889
r =0,02	577.042	3,971.520	18.679	9.731	28.410
r=0,078	1.375,950	3.590,990	49.434	300.709	350.143
β_1 = 0.003	648.275	1,213.060	6.987	26.448	33.436
β_1 =0.0008	648.275	5,233.530	24.041	8.350	32.392
β_2 =0.03	141.845	3,477.580	37,724	13,380	51.104
β_2 =0.007	1,048.430	4,384.700	16,197	10,621	26.818
γ_1^1 =2.3	648.275	5,343.390	24.288	8.141	32.428
γ_1^1 =1.7	648.275	270.720	16.243	16.535	32.770
γ_1^2 =1.3	648.275	1,379.000	8.451	25.342	33.782
γ_1^2 =1	648.275	6,382.740	26.396	5.689	32.086
γ_2^1 =2.3	863.882	4,584.830	18.618	10.103	28.721
γ_2^1 =1.7	398.066	3,530.590	25.782	13.197	38.979
γ_2^2=1.1	337.842	3,429.540	27.548	13.537	41.084
γ_2^2=0.8	1,064.120	5,018.060	16.921	9.015	25.936
ρ =0.06	298.347	2,257.430	13.396	10.299	23.695
ρ =0.02	2,394.540	9,744.730	64.940	2.043	66.983
χ_1 =2	648.275	3,846.700	20.401	12.281	32.682
χ_1 =0.5	648.275	4,167.900	21.348	11.198	32.545
χ_2 =15	505.983	3,534.410	22.981	13.182	36.163
χ_2 =5	779.487	4,581.160	17.724	10.118	29.842
x_0 =20000	648.275	4,061.130	21.041	25.051	46.093
x_0 =8000	648.275	4,061.130	21.041	8.470	29.789
σ =0.25	838.409	7,531.830	21.186	3.073	24.259
σ =0.15	498.628	2,073.400	18.823	23.213	42.042

Table 8: Values of the relevant variables for different set of parameters.

analyzing the fixed costs related to the technology investment (χ_i, $\forall i \in N$), when it is bigger for one player, such a player delays her moment of change. When the cost is lower, they decide to change a bit earlier. Notice now how when the game starts with less amount of available resource, both players decide to switch earlier, to be able to enjoy the few resources that remain, extracting it more easily earlier. When both agents decrease their sigma, that means that they prefer to smooth their consumption, as it is explained in Barro and Sala-i Martin (2004). The greater the σ, the lower the desire to smooth consumption over time. For that reason, with a lower σ, players 1 and 2 will change later, and therefore, they have consumed more in the previous regimes.

5 Conclusion

This paper has extended the basic model studied by Long et al. (2017) in several directions. The role of the renewable resource has pointed out how the agents react to its introduction. Both players have two types of strategies at their fingertips, which affect the strategies of the other player. At each point in the game, players decide their actions that will influence the evolution of the state equation. Furthermore, they decide on the timing of switching, which takes the game to a new stage. Our models have been developed under the piecewise close-loop Nash equilibrium (PCNE) technique, developed in the paper mentioned above, where the switching strategy is a function of the state of the system, which means that both players decide when to change taking into account the resources available.

The interaction that emerges from the game shows how one player's decision to switch affects the other player's strategy and, in addition, the second player's switching strategy will affect the welfare of the first player. We have tried to give some insights to the questions formulated in the introduction. For this purpose, we have computed numerically how the introduction of a renewable resource affects the relevant variables of the model such as the timing of switching (t_i), the available resource at those moments (x_i^*), and how long the second player waits until she decides to move to the last stage of the game ($\theta_2(.)$). The simulations of the proposed models have shown that players enjoy higher consumption rates and stay in the regimes longer. This is determined by the fact that the resource is self-generated and thus, it gives them the option to extract more. A complete sensitivity analysis has been performed for each parameter of the model. Therefore, both players enjoy a higher extraction rate and hence, higher consumption, which is translated into a higher welfare under this kind of resource. This paper has been built upon the work by Long et al. (2017), where the authors develop a new methodology called PCNE for differential games with regime strategies. However, just interior solutions have been analyzed. Wherefore, corner solutions will be considered in the future.

This paper is only the first step towards a more elaborate framework, where a Dynamic Stochastic General Equilibrium model is introduced, with a generalization of the set of decision makers to $N = \{1, 2, ..., n\}$. Future work will be based on the introduction of the climate change problem, introducing emissions tax and transfer policies as in Hillebrand and Hillebrand (2018). Moreover, in future research, a more complicate and general natural resources, with renewable and nonrenewable resources model will be analyzed, to study how it affects the behavior of the player. These players could be countries competing for a resource and how much to pollute.

A APPENDICES

A.1

The result is driven from the guessing of the HJB equation. The guessing for the value function $\forall i \in N$ is: $V_i(x) = \frac{A_i x(t)^\sigma}{\sigma}$. The derivative to respect x of our guessing is $V_i'(x) = A_i x(t)^{\sigma-1}$. Therefore, the HJB becomes:

$$\rho V(x(t)) = \underset{u_i^{22}(t)}{Max} \left\{ \frac{u_i(t)^\sigma}{\sigma} + V_i'(x) \left[-\gamma_i^2 u_i^{22}(t) - \gamma_j^2 u_j^{22}(t) \right] \right\} \tag{A-1}$$

Maximizing the RHS of eq. (A-1) to respect $u_i^{22}(t)$ we obtain $u_i^{22}(t) = \left(\gamma_i^2 A_i\right)^{\frac{1}{\sigma-1}} x(t)$, $\forall i \in N$. Plugging $u_i^{22}(t)$ into the HJB equation with the guessing, and we get:

$$\rho \frac{A_i x(t)^\sigma}{\sigma} = \underset{u_i^{22}(t)}{Max} \left\{ \frac{\left(\left(\gamma_i^2 A_i\right)^{\frac{1}{\sigma-1}} x(t) \right)^\sigma}{\sigma} + A_i x(t)^{\sigma-1} \left[-\gamma_i^2 \left(\left(\gamma_i^2 A_i\right)^{\frac{1}{\sigma-1}} x(t) \right) - \gamma_j^2 \left(\left(\gamma_j^2 A_j\right)^{\frac{1}{\sigma-1}} x(t) \right) \right] \right\} \tag{A-2}$$

Regrouping the terms with x, we get a system of non-linear equations that, through various manipulations and variable changes, has as a solution:

$$A_i = \gamma_i^{-\sigma} \left(-\frac{\rho}{2\sigma - 1} \right)^{\sigma-1}, \forall i \in N \tag{A-3}$$

Observe, that if $\sigma \to 0$, the result for the logarithmic non-renewable part appears. As a result of (A-3), for the terminal state $s = 22$, we obtain:

$$\gamma_i^2 \Phi_i(x(t), 22) = \gamma_j^2 \Phi_i(x(t), 22) = -\frac{\rho}{2\sigma - 1} x(t) \tag{A-4}$$

A.2

The evolution of the resources along the CES non-renewable differential game is shown in the Fig. (9)

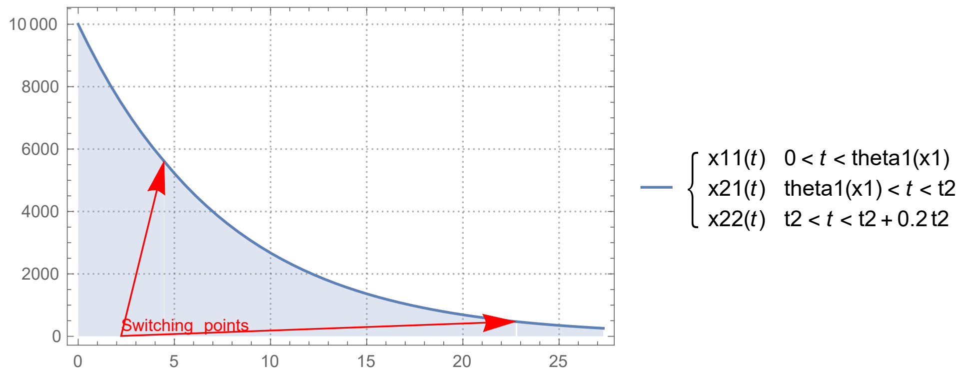


Figure 9: Evolution of the resource under non-renewable CES model.

A.3

In this Appendix, we develop $\sum(x_1, x_2^*)$, $\Psi(x_2^*)$ in eq. (31) and $\xi(x_1; x_2^*)$ in eq. (32) in Section 3

$$\sum(x_1, x_2^*) \equiv \left[\frac{(2\sigma-1)\Gamma(x_2^*)-\rho x_1}{\gamma_1^2(2\sigma-1)}\right]^{\sigma}\left[\frac{1}{\sigma}+\frac{2\rho x_1}{(2\sigma-1)\Gamma(x_2^*)-\rho x_1}\right] \tag{A-5}$$

and

$$\Psi(x_2^*) \equiv \left\{\left[\frac{(2\sigma-1)\Gamma(x_2^*)-\rho x_2^*}{\gamma_1^2(2\sigma-1)}\right]^{\sigma}\left[\frac{1}{\sigma}+\frac{2\rho x_2^*}{(2\sigma-1)\Gamma(x_2^*)-\rho x_2^*}\right]+\left(\frac{-\rho x_2^*}{\gamma_1^1(2\sigma-1)}\right)\left(\frac{1-2\sigma}{\sigma}\right)\right\} \tag{A-6}$$

and from eq. (32) we have renamed:

$$\xi(x_1; x_2^*) \equiv \frac{1}{\gamma_1^2}\left[\frac{(2\sigma-1)\Gamma(x_2^*)-\rho x_1}{\gamma_1^2(2\sigma-1)}\right]^{\sigma-1}+$$
$$\theta_2' e^{-\rho\theta_2}\left\{\left[\frac{(2\sigma-1)\Gamma(x_2^*)-\rho x_1}{\gamma_1^2(2\sigma-1)}\right]^{\sigma}\left[\frac{1}{\sigma}+\frac{2\rho x_2^*}{(2\sigma-1)\Gamma(x_2^*)-\rho x_2^*}\right]-\left(\frac{-\rho x_2^*}{\gamma_1^2(2\sigma-1)}\right)\left(\frac{1-2\sigma}{\sigma}\right)\right\}-\beta_1 \tag{A-7}$$

and in eq. (A-7), θ_2' is given by

$$-\frac{-1+2\sigma}{2\left(\rho x_1-\rho x_2-\gamma 1\,(-1+2\sigma)\left(-\frac{\gamma 1\left((-1+2\sigma)\left(\frac{\rho x_2}{\gamma 2-2\gamma 2\sigma}\right)^{\sigma}+\rho x_2\beta_2\right)}{\rho x_2}\right)^{\frac{1}{-1+\sigma}}\right)}$$

A.4

The FOCs in Section 4.1 have to be combined with the suitable Transversality Conditions in each state. Remember that we could differentiate two type of states, terminal or not. Solving (39), we conclude that player's extraction strategies are the same $\forall s$. After a tedious computation, we obtain that player's extraction strategies are the same $\forall s \in \mathcal{S}$:

$$\gamma_i^l \Phi_i(x(t), s) = \gamma_j^k \Phi_j(x(t), s) \tag{A-8}$$

This result is driven from the guessing of the HamiltonJacobiBellman (HJB) equation $\forall i \in N$:

$$\rho V_i(x(t)) = \underset{u_i^{22}(t) \in \mathbb{R}+}{Max} \left\{ F(x(t), u_i^{22}(t)) + V_i'(x(t)) \cdot g(x(t), u_i^{22}(t)) \right\} \tag{A-9}$$

that is: $V_i(x) = A_i \cdot \ln(x(t)) + B_i$. The derivative to respect x of our guessing is $V_i'(x) = \frac{A_i}{x(t)}$. Therefore, the HJB becomes:

$$\rho V_i(x(t)) = \underset{u_i^{22}(t) \in \mathbb{R}+}{Max} \left\{ \ln(u_i^{22}(t)) + \frac{A_i}{x(t)} \left[rx(t) - \gamma_i^2 u_i^{22}(t) - \gamma_j^2 u_j^{22}(t) \right] \right\} \tag{A-10}$$

Maximizing the RHS of eq. (A-10) to respect $u_i^{22}(t)$ we obtain $u_i^{22}(t) = \frac{x(t)}{A_i \gamma_i^2}$, $\forall i \in \{1, 2\}$.

Plug $u_i^{22}(t)$ into the HJB equation with the guessing, and we get:

$$\rho \left[A_i \cdot \ln(x(t)) + B_i \right] = \ln \left(\frac{x(t)}{A_i \gamma_i^2} \right) + \frac{A_i}{x(t)} \left[rx(t) - \gamma_i^2 \frac{x(t)}{A_i \gamma_i^2} - \gamma_j^2 \frac{x(t)}{A_j \gamma_j^2} \right] \tag{A-11}$$

Regrouping the terms with x, we get $\Phi_i(x(t), s) \equiv u_i^{22}(t) = \frac{\rho x(t)}{\gamma_i^2}$, $\forall i \in \{1, 2\}$. Therefore, for the terminal state $s = 22$, we obtain :

$$\gamma_i^l \Phi_i(x(t), s) = \gamma_j^k \Phi_i(x(t), s) = \rho x(t) \tag{A-12}$$

A.5

The relationship between r and x_2^* is given by eq. (47). From this equation, we obtain the amount of resource x_2 at switching time t_2. Solving the equation, the solution to x_2 is given by:

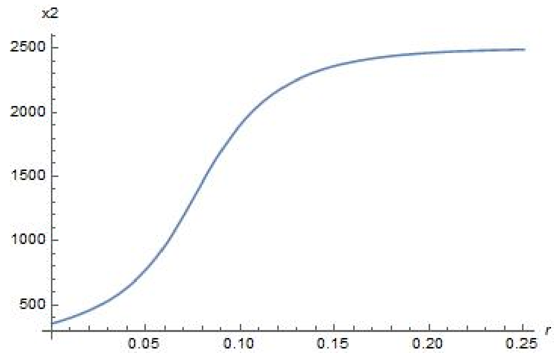


Figure 10: Relationship between x_2^* and r.

$$\text{x2} \to \frac{100W\left(-50.69e^{-25.r}(1.r-0.04)\right)+2500r-100}{r-0.04} \tag{A-13}$$

Where W is the *Lambert W function*, also called the Omega Function or Product Logarithm [8].

The graphic representation of eq. (A-13) is shown in the Figure (10). As it can bee easily seen, in order to get a coherent solution of x2, we set $r = 0.03$.

A.6

The evolution of the resources when $r \to 0.08^-$ is shown in Fig. (11).

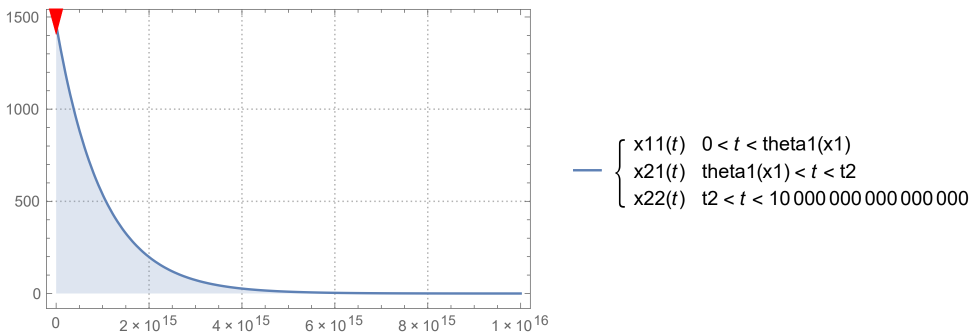


Figure 11: Evoluion of the resource after Player 2 has switched.

A.7

This result is driven from the guessing of the HJB equation. The guessing for the value function $\forall i \in N$ is: $V_i(x) = \frac{A_i x(t)^\sigma}{\sigma}$. The derivative to respect x of our guessing is $V_i'(x) =$

[8]For a further analysis of the Lambert function see Corless et al. (1996)

$A_i x(t)^{\sigma-1}$. Therefore, the HJB becomes (observe how the renewable part is introduced):

$$\rho V(x(t)) = \underset{u_i^{22}(t)}{Max} \left\{ \frac{u_i(t)^\sigma}{\sigma} + V_i'(x) \left[rx(t) - \gamma_i^2 u_i^{22}(t) - \gamma_j^2 u_j^{22}(t) \right] \right\} \tag{A-14}$$

Maximizing the RHS of eq. (A-14) to respect $u_i^{22}(t)$ we obtain $u_i^{22}(t) = \left(\gamma_i^2 A_i\right)^{\frac{1}{\sigma-1}} x(t)$, $\forall i \in N$. Plug $u_i^{22}(t)$ into the HJB equation with the guessing, and we get:

$$\rho \frac{A_i x(t)^\sigma}{\sigma} = \underset{u_i^{22}(t)}{Max} \left\{ \frac{\left(\left(\gamma_i^2 A_i\right)^{\frac{1}{\sigma-1}} x(t) \right)^\sigma}{\sigma} + A_i x(t)^{\sigma-1} \left[rx(t) - \gamma_i^2 \left(\left(\gamma_i^2 A_i\right)^{\frac{1}{\sigma-1}} x(t) \right) - \gamma_j^2 \left(\left(\gamma_j^2 A_j\right)^{\frac{1}{\sigma-1}} x(t) \right) \right] \right\} \tag{A-15}$$

Regrouping the terms with x, we get a system of non-linear equations that, through various manipulations and variable changes, has as a solution:

$$A_i = \gamma_i^{-\sigma} \left(\frac{r\sigma - \rho}{2\sigma - 1} \right)^{\sigma-1}, \forall i \in N \tag{A-16}$$

A.8

We now develop $\sum(x_1, x_2^*)$, $\Psi(x_2^*)$ in eq. (67) and $\xi(x_1; x_2^*)$ in eq. (68) in Section 4.2

$$\sum(x_1, x_2^*) \equiv \left[\frac{\Gamma(x_2) + \left(\frac{r\sigma-\rho}{2\sigma-1}\right) x_1}{\gamma_1^2} \right]^\sigma \left[\frac{1}{\sigma} + \frac{\left[r - 2\left(\frac{r\sigma-\rho}{2\sigma-1}\right) \right] x_1}{\Gamma(x_2) + \left(\frac{r\sigma-\rho}{2\sigma-1}\right) x_1} \right] \tag{A-17}$$

and

$$\Psi(x_2^*) \equiv \left\{ \left[\frac{\left(\frac{r\sigma-\rho}{2\sigma-1}\right) x_2}{\gamma_2^2} \right]^\sigma \left[\frac{1}{\sigma} + \frac{\left[r - 2\left(\frac{r\sigma-\rho}{2\sigma-1}\right) \right]}{\left(\frac{r\sigma-\rho}{2\sigma-1}\right)} \right] - \left[\frac{\Gamma(x_2) + \left(\frac{r\sigma-\rho}{2\sigma-1}\right) x_2}{\gamma_1^2} \right]^\sigma \left[\frac{1}{\sigma} + \frac{\left[r - 2\left(\frac{r\sigma-\rho}{2\sigma-1}\right) \right] x_2}{\Gamma(x_2) + \left(\frac{r\sigma-\rho}{2\sigma-1}\right) x_2} \right] \right\} \tag{A-18}$$

Notice that from eq. (68) we have renamed:

$$\xi(x_1;x_2^*) \equiv \frac{1}{\gamma_1^2}\left[\frac{\Gamma(x_2)+\left(\frac{r\sigma-\rho}{2\sigma-1}\right)x_1}{\gamma_1^2}\right]^{\sigma-1} +$$

$$[\theta_2' e^{-\rho\theta_2}\left\{\left[\frac{\Gamma(x_2)+\left(\frac{r\sigma-\rho}{2\sigma-1}\right)x_2}{\gamma_1^2}\right]^{\sigma}\left[\frac{1}{\sigma}+\frac{\left[r-2\left(\frac{r\sigma-\rho}{2\sigma-1}\right)\right]x_2}{\Gamma(x_2)+\left(\frac{r\sigma-\rho}{2\sigma-1}\right)x_2}\right]-\left[\frac{\left(\frac{r\sigma-\rho}{2\sigma-1}\right)x_2}{\gamma_1^2}\right]^{\sigma}\left[\frac{1}{\sigma}+\frac{\left[r-2\left(\frac{r\sigma-\rho}{2\sigma-1}\right)\right]}{\left(\frac{r\sigma-\rho}{2\sigma-1}\right)}\right]\right\}^{-\beta_1} \tag{A-19}$$

and θ_2' in eq. (A-22) is given by:

$$-\frac{1}{\left(r-\frac{2(-\rho+r\sigma)}{-1+2\sigma}\right)\left(x_1^*+\frac{2\Gamma x_2^*}{r-\frac{2(-\rho+r\sigma)}{-1+2\sigma}}\right)}$$

We now develop $\sum(x_1,x_2^*)$, $\Psi(x_2^*)$ in eq. (67) and $\xi(x_1;x_2^*)$ in eq. (68) in Section (4.2)

$$\sum(x_1,x_2^*) \equiv \left[\frac{\Gamma(x_2)+\left(\frac{r\sigma-\rho}{2\sigma-1}\right)x_1}{\gamma_1^2}\right]^{\sigma}\left[\frac{1}{\sigma}+\frac{\left[r-2\left(\frac{r\sigma-\rho}{2\sigma-1}\right)\right]x_1}{\Gamma(x_2)+\left(\frac{r\sigma-\rho}{2\sigma-1}\right)x_1}\right] \tag{A-20}$$

and

$$\Psi(x_2^*) \equiv \left\{\left[\frac{\left(\frac{r\sigma-\rho}{2\sigma-1}\right)x_2}{\gamma_2^2}\right]^{\sigma}\left[\frac{1}{\sigma}+\frac{\left[r-2\left(\frac{r\sigma-\rho}{2\sigma-1}\right)\right]}{\left(\frac{r\sigma-\rho}{2\sigma-1}\right)}\right]-\left[\frac{\Gamma(x_2)+\left(\frac{r\sigma-\rho}{2\sigma-1}\right)x_2}{\gamma_1^2}\right]^{\sigma}\left[\frac{1}{\sigma}+\frac{\left[r-2\left(\frac{r\sigma-\rho}{2\sigma-1}\right)\right]x_2}{\Gamma(x_2)+\left(\frac{r\sigma-\rho}{2\sigma-1}\right)x_2}\right]\right\} \tag{A-21}$$

Notice that from eq. (68) we have renamed:

$$\xi(x_1;x_2^*) \equiv \frac{1}{\gamma_1^2}\left[\frac{\Gamma(x_2)+\left(\frac{r\sigma-\rho}{2\sigma-1}\right)x_1}{\gamma_1^2}\right]^{\sigma-1} +$$

$$[\theta_2' e^{-\rho\theta_2}\left\{\left[\frac{\Gamma(x_2)+\left(\frac{r\sigma-\rho}{2\sigma-1}\right)x_2}{\gamma_1^2}\right]^{\sigma}\left[\frac{1}{\sigma}+\frac{\left[r-2\left(\frac{r\sigma-\rho}{2\sigma-1}\right)\right]x_2}{\Gamma(x_2)+\left(\frac{r\sigma-\rho}{2\sigma-1}\right)x_2}\right]-\left[\frac{\left(\frac{r\sigma-\rho}{2\sigma-1}\right)x_2}{\gamma_1^2}\right]^{\sigma}\left[\frac{1}{\sigma}+\frac{\left[r-2\left(\frac{r\sigma-\rho}{2\sigma-1}\right)\right]}{\left(\frac{r\sigma-\rho}{2\sigma-1}\right)}\right]\right\}^{-\beta_1} \tag{A-22}$$

and θ_2' in eq. (A-22) is given by:

$$-\frac{1}{\left(r-\frac{2(-\rho+r\sigma)}{-1+2\sigma}\right)\left(x_1^*+\frac{2\Gamma x_2^*}{r-\frac{2(-\rho+r\sigma)}{-1+2\sigma}}\right)}$$

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