

## CONTRACTIVE INEQUALITIES FOR BERGMAN SPACES AND MULTIPLICATIVE HANKEL FORMS

FREDÉRIC BAYART, OLE FREDRIK BREVIG, ANTTI HAIMI,  
JOAQUIM ORTEGA-CERDÀ, AND KARL-MIKAEL PERFEKT

ABSTRACT. We consider sharp inequalities for Bergman spaces of the unit disc, establishing analogues of the inequality in Carleman's proof of the isoperimetric inequality and of Weissler's inequality for dilations. By contractivity and a standard tensorization procedure, the unit disc inequalities yield corresponding inequalities for the Bergman spaces of Dirichlet series. We use these results to study weighted multiplicative Hankel forms associated with the Bergman spaces of Dirichlet series, reproducing most of the known results on multiplicative Hankel forms associated with the Hardy spaces of Dirichlet series. In addition, we find a direct relationship between the two types of forms which does not exist in lower dimensions. Finally, we produce some counterexamples concerning Carleson measures on the infinite polydisc.

### 1. INTRODUCTION

Hardy spaces of the countably infinite polydisc,  $H^p(\mathbb{D}^\infty)$ , have in recent years received considerable interest and study, emerging from the foundational papers [16, 23]. Partly, the attraction is motivated by the subject's link with Dirichlet series, realized by identifying each complex variable with a prime Dirichlet monomial,  $z_j = p_j^{-s}$  (see [5]). Hardy spaces of Dirichlet series,  $\mathcal{H}^p$ , are defined by requiring this identification to induce an isometric, multiplicative isomorphism. The connection to Dirichlet series gives rise to a rich interplay between operator theory and analytic number theory — we refer the interested reader to the survey [37] or the monograph [38] as a starting point.

One aspect of the theory is the study of multiplicative Hankel forms on  $\ell^2 \times \ell^2$ . A sequence  $\varrho = (\varrho_1, \varrho_2, \dots)$  generates a multiplicative Hankel form by the formula

$$(1) \quad \varrho(a, b) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \varrho_{mn},$$

defined at least for finitely supported sequences  $a$  and  $b$ . Helson [24] observed that multiplicative Hankel forms are naturally realized as (small) Hankel operators on  $H^2(\mathbb{D}^\infty)$ , and went on to ask whether every symbol  $\varrho$  which generates a bounded multiplicative Hankel form on  $\ell^2 \times \ell^2$  also induces a bounded linear functional on the Hardy space  $H^1(\mathbb{D}^\infty)$ . In other words, he asked whether there is an analogue of Nehari's theorem [32] in this context.

---

Received by the editors January 31, 2017, and, in revised form, May 9, 2017 and May 10, 2017.  
2010 *Mathematics Subject Classification*. Primary 30H20; Secondary 47B35, 30B50.

The second-named author was supported by Grant 227768 of the Research Council of Norway. The third-named author was supported by Lise Meitner grant of Austrian Science Fund (FWF). The fourth-named author was supported by the MTM2014-51834-P grant by the Ministerio de Economía y Competitividad, and by the Generalitat de Catalunya (project 2014 SGR 289).

Helson's question inspired several papers [9, 11, 25, 26, 35, 36]. Following the program outlined in [26], it was established in [35] that there are bounded Hankel forms that do not extend to bounded functionals on  $H^1(\mathbb{D}^\infty)$ . In the positive direction, it was proved in [25] that if the Hankel form (1) instead satisfies the stronger property of being Hilbert–Schmidt, then its symbol does extend to a bounded functional on  $H^1(\mathbb{D}^\infty)$ . Briefly summarizing the most recent development, the result of [35] was generalized in [9], in [11] an analogue of the classical Hilbert matrix was introduced and studied, and in [36] the boundedness of the Hankel form (1) was characterized in terms of Carleson measures in the special case that the form is positive semi-definite.

Very recently, a study of Bergman spaces of Dirichlet series  $\mathcal{A}^p$  began in [3]. In analogy with the Hardy spaces of Dirichlet series,  $\mathcal{A}^p$  was constructed from the corresponding Bergman space,  $A^p(\mathbb{D}^\infty)$ . New difficulties appeared while trying to put this theory on equal footing with its Hardy space counterpart. One of them is the lack of contractive inequalities for Bergman spaces in the unit disc. In the Hardy space of the unit disc there is a comparative abundance of such inequalities, each immediately implying a corresponding inequality for  $\mathcal{H}^p$ . For example, the result of [25] on Hilbert–Schmidt Hankel forms relies essentially on the classical Carleman inequality,

$$\|f\|_{A^2(\mathbb{D})} \leq \|f\|_{H^1(\mathbb{D})}.$$

A second example is furnished by Weissler's inequality: defining for  $0 < r \leq 1$  the map  $P_r: H^p(\mathbb{D}) \rightarrow H^q(\mathbb{D})$ , by  $P_r f(w) = f(rw)$ , then  $P_r$  is contractive if and only if  $r \leq \sqrt{p/q} \leq 1$ . Since both of these inequalities are contractive, they carry on to the infinite polydisc by tensorization (see Section 3), thus yielding results for  $\mathcal{H}^p$ .

We derive analogues of the mentioned inequalities for Bergman spaces of the unit disc in Section 2. Our proofs involve certain variants of the Sobolev inequalities from [4] and [6]. Then, in Section 3, we follow the by now standard tensorization scheme to deduce the corresponding contractive inequalities for the Bergman spaces of Dirichlet series.

Section 4 is devoted to the weighted multiplicative Hankel forms related to the Bergman space, defined by the formula

$$(2) \quad \varrho_d(a, b) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \frac{\varrho_{mn}}{d(mn)}, \quad a, b \in \ell_d^2.$$

In (2),  $d(k)$  denotes the number of divisors of the integer  $k$ , and  $\ell_d^2$  denotes the corresponding weighted Hilbert space. Note that the divisor function  $d(k)$  counts the number of times  $\varrho_k$  appears in (2). In the same way that the forms (1) are realized as Hankel operators on the Hardy space  $H^2(\mathbb{D}^\infty)$ , the weighted forms (2) are naturally realized as (small) Hankel operators on the Bergman space of the infinite polydisc,  $A^2(\mathbb{D}^\infty)$ . Equipped with the inequalities from Sections 2 and 3 we successfully obtain the Bergman space counterparts of results from [11, 25, 26, 35].

In Section 4 we will also point out a surprising property of multiplicative Hankel forms. We first observe that  $A^2(\mathbb{D}^\infty)$  may be naturally isometrically embedded in the Hardy space  $H^2(\mathbb{D}^\infty)$ , since the same is true for  $A^2(\mathbb{D})$  with respect to  $H^2(\mathbb{D}^2)$ . Then, we notice that this embedding lifts to the level of Hankel forms, giving us a natural map taking weighted Hankel forms (2) to Hankel forms (1). The striking aspect is that this map preserves the singular numbers of the Hankel form, in particular preserving both the uniform and the Hilbert–Schmidt norm.

Finally, in Section 5 we come back to harmonic analysis on the Hardy spaces  $H^p(\mathbb{D}^\infty)$ . We produce two counterexamples for Carleson measures, again pointing out phenomena that do not exist in finite dimension.

*Notation.* We will use the notation  $f(x) \lesssim g(x)$  if there is some constant  $C > 0$  such that  $|f(x)| \leq C|g(x)|$  for all (appropriate)  $x$ . If  $f(x) \lesssim g(x)$  and  $g(x) \lesssim f(x)$ , we write  $f(x) \simeq g(x)$ . As above,  $(p_j)_{j \geq 1}$  will denote the increasing sequence of prime numbers.

## 2. INEQUALITIES OF CARLEMAN AND WEISSLER FOR BERGMAN SPACES

**2.1. Preliminaries.** Let  $\alpha > 1$  and  $0 < p < \infty$ , and define the Bergman space  $A_\alpha^p(\mathbb{D})$  as the space of analytic functions  $f$  in the unit disc

$$\mathbb{D} = \{z : |z| < 1\}$$

that are finite with respect to the norm

$$\|f\|_{A_\alpha^p(\mathbb{D})} = \left( \int_{\mathbb{D}} |f(w)|^p (\alpha - 1)(1 - |w|^2)^{\alpha-2} dm(w) \right)^{\frac{1}{p}}.$$

Here  $m$  denotes the Lebesgue area measure, normalized so that  $m(\mathbb{D}) = 1$ . It will be convenient to let  $dm_\alpha(w) = (\alpha - 1)(1 - |w|^2)^{\alpha-2} dm(w)$  for  $\alpha > 1$ , and to let  $m_1$  denote the normalized Lebesgue measure on the torus

$$\mathbb{T} = \{z : |z| = 1\}.$$

The Hardy space  $H^p(\mathbb{D})$  is defined as closure of analytic polynomials with respect to the norm

$$\|f\|_{H^p(\mathbb{D})} = \left( \int_{\mathbb{T}} |f(w)|^p dm_1(w) \right)^{\frac{1}{p}}.$$

The Hardy space  $H^p(\mathbb{D})$  is the limit of  $A_\alpha^p(\mathbb{D})$  as  $\alpha \rightarrow 1^+$ , in the sense that

$$\lim_{\alpha \rightarrow 1^+} \|f\|_{A_\alpha^p(\mathbb{D})} = \|f\|_{H^p(\mathbb{D})}$$

for every analytic polynomial  $f$ . We therefore let  $A_1^p(\mathbb{D}) = H^p(\mathbb{D})$ . Our main interest is in the distinguished case  $\alpha = 2$ , when  $m_\alpha = m$  is simply the normalized Lebesgue measure. Therefore, we also let  $A^p(\mathbb{D}) = A_2^p(\mathbb{D})$ . We will only require some basic properties of  $A_\alpha^p(\mathbb{D})$  in what follows, and refer generally to the monographs [18, 22].

Let  $c_\alpha(j)$  denote the coefficients of the binomial series

$$(3) \quad \frac{1}{(1-w)^\alpha} = \sum_{j=0}^\infty c_\alpha(j)w^j, \quad c_\alpha(j) = \binom{j+\alpha-1}{j}.$$

It is evident from (3) that

$$(4) \quad \sum_{j+k=l} c_\alpha(j)c_\beta(k) = c_{\alpha+\beta}(l).$$

If  $\alpha$  is an integer, then  $c_\alpha(j)$  denotes the number of ways to write  $j$  as a sum of  $\alpha$  non-negative integers. Furthermore, if  $f(w) = \sum_{j \geq 0} a_j w^j$ , then

$$(5) \quad \|f\|_{A_\alpha^2(\mathbb{D})} = \left( \sum_{j=0}^\infty \frac{|a_j|^2}{c_\alpha(j)} \right)^{\frac{1}{2}}.$$

Functions  $f$  in  $A_\alpha^p(\mathbb{D})$  satisfy for  $w \in \mathbb{D}$  the sharp pointwise estimate

$$(6) \quad |f(w)| \leq \frac{1}{(1 - |w|^2)^{\alpha/p}} \|f\|_{A_\alpha^p(\mathbb{D})}.$$

For the sake of completeness, we will state and prove the results in this section for as general  $\alpha > 1$  as we are able, even though we will only make use of the results for  $\alpha = 2$  in the following sections.

**2.2. Contractive inclusions of Bergman spaces.** It is well known that, if  $0 < p \leq q$  and  $\alpha, \beta \geq 1$ , then  $A_\alpha^p(\mathbb{D})$  embeds continuously into  $A_\beta^q(\mathbb{D})$  if and only if  $q/\beta \leq p/\alpha$  (see, e.g., [45, Exercise 2.27]). By tensorization, this statement extends to the Bergman spaces on the polydiscs of finite dimension. However, in order for such embeddings to exist on the infinite polydisc, it is necessary that the inclusion map in one variable is contractive.

The first result of the type we are looking for was given by Carleman [13]. For  $f \in H^1(\mathbb{D})$  it holds that

$$(7) \quad \|f\|_{A^2(\mathbb{D})} = \|f\|_{A_2^2(\mathbb{D})} \leq \|f\|_{A_1^1(\mathbb{D})} = \|f\|_{H^1(\mathbb{D})}.$$

A modern and natural way to prove (7) can be found in [43]. First, it is easy to verify that

$$\|gh\|_{A^2(\mathbb{D})} \leq \|g\|_{H^2(\mathbb{D})} \|h\|_{H^2(\mathbb{D})},$$

for example by computing by coefficients. If  $f$  is a non-vanishing function of  $H^1(\mathbb{D})$ , writing  $f = gh$  with  $g = h = f^{1/2}$  now leads to (7). For a general function  $f \in H^1(\mathbb{D})$ , we first factor out the zeros through a Blaschke product. This is possible by what seems to be a coincidence: multiplication by a Blaschke product decreases the norm on the left hand side of (7) but preserves the norm on the right hand side.

The ability to factor out zeros and take roots implies that Carleman’s inequality (7) holds for arbitrary  $0 < p < \infty$ ,

$$\|f\|_{A^{2p}(\mathbb{D})} \leq \|f\|_{H^p(\mathbb{D})}.$$

In [12], Burbea generalized Carleman’s inequality, showing that for every  $0 < p < \infty$  and every non-negative integer  $n$ , it holds that

$$(8) \quad \|f\|_{A_{1+n}^{p(1+n)}(\mathbb{D})} \leq \|f\|_{H^p(\mathbb{D})}.$$

Let

$$\alpha_0 = \frac{1 + \sqrt{17}}{4} = 1.280776\dots$$

We offer the following extension of Carleman’s inequality.

**Theorem 1.** *Let  $\alpha \geq \alpha_0$  and  $0 < p < \infty$ . For every  $f \in A_\alpha^p(\mathbb{D})$ ,*

$$\|f\|_{A_{\alpha+1}^{p(\alpha+1)/\alpha}(\mathbb{D})} \leq \|f\|_{A_\alpha^p(\mathbb{D})}.$$

*Moreover, if  $\alpha > \alpha_0$ , we have equality if and only if there exists constants  $C \in \mathbb{C}$  and  $\xi \in \mathbb{D}$  such that*

$$f(w) = \frac{C}{(1 - \bar{\xi}w)^{2\alpha/p}}.$$

Let us give two corollaries. The first is mainly decorative, but it illustrates that (8) gets weaker as  $n$  increases.

**Corollary 2.** *Let  $f \in H^1(\mathbb{D}) = A_1^1(\mathbb{D})$ . Then*

$$\|f\|_{A_1^1(\mathbb{D})} \geq \|f\|_{A_2^2(\mathbb{D})} \geq \|f\|_{A_3^3(\mathbb{D})} \geq \|f\|_{A_4^4(\mathbb{D})} \geq \dots$$

We also have the following corollary, which will be important in the next section.

**Corollary 3.** *Let  $p = 2/(1 + n/2)$  for a non-negative integer  $n$  and suppose that  $f(w) = \sum_{j \geq 0} a_j w^j$  is in  $A^p(\mathbb{D})$ . Then*

$$\|f\|_{A_{n+2}^2(\mathbb{D})} = \left( \sum_{j=0}^{\infty} \frac{|a_j|^2}{c_{n+2}(j)} \right)^{\frac{1}{2}} \leq \|f\|_{A^p(\mathbb{D})}.$$

*Proof.* This follows from  $n$  successive applications of Theorem 1, starting from  $p = 2/(1 + n/2)$  and  $\alpha = 2$ . □

We now begin the proof of Theorem 1. A version of it was announced in [4],<sup>1</sup> following a scheme designed in [7]. Observe also that an analogous result in the Fock space was proved by Carlen [14] using a logarithmic Sobolev inequality. We follow the general strategy of [4, 7], replacing [4, Sec. 5] with a result from [31]. We include many additional details in an attempt to make the scheme used in [4, 7, 14] available to a wider audience.

We shall use two structures on the disc, the Euclidean and the hyperbolic. The usual gradient and Laplacian of  $u$  will be denoted by  $\nabla u$  and  $\Delta u$ , while the hyperbolic gradient and the hyperbolic Laplacian are denoted by  $\nabla_{\mathbb{H}} u$  and  $\Delta_{\mathbb{H}} u$ . They are connected by the following formulas:

$$\nabla_{\mathbb{H}} u(w) = \left( \frac{1 - |w|^2}{2} \right) \nabla u(w) \quad \text{and} \quad \Delta_{\mathbb{H}} u(w) = \left( \frac{1 - |w|^2}{2} \right)^2 \Delta u(w).$$

We shall also use the Möbius invariant measure

$$d\mu(w) = \frac{dm(w)}{(1 - |w|^2)^2}.$$

We begin with an integral identity (essentially [4, Thm. 3.1]). An analogous result was proven for the Fock space in [14], and a similar result also appears in [7].

**Lemma 4.** *Let  $p > 0$  and  $\beta > 1/2$ . For an analytic function  $f$  in  $\overline{\mathbb{D}}$ , set  $u(w) = |f(w)|^p (1 - |w|^2)^\beta$ . Then*

$$\int_{\mathbb{D}} |\nabla_{\mathbb{H}} u(w)|^2 d\mu(w) = \frac{\beta}{2} \int_{\mathbb{D}} |u(w)|^2 d\mu(w).$$

*Proof.* Integrating by parts gives

$$(9) \quad \int_{\mathbb{D}} |\nabla_{\mathbb{H}} u|^2 d\mu = \frac{1}{4} \int_{\mathbb{D}} |\nabla u|^2 dm = -\frac{1}{4} \int_{\mathbb{D}} u \Delta u dm.$$

It follows from the assumption  $\beta > 1/2$  that boundary terms do not appear here. We compute the Laplacian now. At any point where  $f$  does not vanish, we can write

$$\frac{\partial u}{\partial w} = \frac{p}{2} |f|^{p-2} f \overline{f'} (1 - |w|^2)^\beta - \beta w |f|^p (1 - |w|^2)^{\beta-1},$$

---

<sup>1</sup>Theorem 3.2 in [4] is stated for  $kq > 2$ , but there seems to be a mistake in the proof of uniqueness on p. 1083. The argument in its entirety seems to apply only when  $kq > 3$ .

so that

$$\begin{aligned} \frac{\partial^2 u}{\partial w \bar{\partial} w} &= \frac{p^2}{4} |f'|^2 |f|^{p-2} (1 - |w|^2)^\beta - \beta \frac{p}{2} |f|^{p-2} f \bar{f}' \bar{w} (1 - |w|^2)^{\beta-1} \\ &\quad - \beta |f|^p (1 - |w|^2)^{\beta-1} - \frac{\beta p}{2} |f|^{p-2} f' \bar{f} (1 - |w|^2)^{\beta-1} \\ &\quad + \beta(\beta - 1) |w|^2 |f|^p (1 - |w|^2)^{\beta-2}. \end{aligned}$$

We see that

$$\begin{aligned} -u\Delta u &= -p^2 |f'|^2 |f|^{2p-2} (1 - |w|^2)^{2\beta} + 2\beta p |f|^{2p-2} f \bar{f}' \bar{w} (1 - |w|^2)^{2\beta-1} \\ &\quad + 4\beta |f|^{2p} (1 - |w|^2)^{2\beta-2} + 2\beta p |f|^{2p-2} f' \bar{f} w (1 - |w|^2)^{2\beta-1} \\ &\quad - 4\beta^2 |w|^2 |f|^{2p} (1 - |w|^2)^{2\beta-2}. \end{aligned}$$

Coming back to the expression of  $\partial u / \bar{\partial} w$ , we find that

$$-\frac{1}{4} u \Delta u = \beta \frac{u^2}{(1 - |w|^2)^2} - \left| \frac{\partial u}{\bar{\partial} w} \right|^2 = \beta \frac{u^2}{(1 - |w|^2)^2} - \frac{|\nabla_{\mathbb{H}} u|^2}{(1 - |w|^2)^2}.$$

Integrating with respect to  $d\mu$  and using (9) gives the result. □

*Proof of Theorem 1.* We set  $q = p(\alpha + 1)/\alpha$ ,  $A = (\alpha - 2)/(\alpha - 1)$  and  $B = 1/(\alpha - 1)$ , so that  $A + B = 1$ . We want to find the infimum of

$$(\alpha - 1) \int_{\mathbb{D}} |f(w)|^p (1 - |w|^2)^\alpha d\mu(w)$$

under the constraint

$$\alpha \int_{\mathbb{D}} |f(w)|^q (1 - |w|^2)^{\alpha+1} d\mu(w) = 1.$$

Equivalently, using Lemma 4 with

$$(10) \quad u(w) = |f(w)|^{p/2} (1 - |w|^2)^{\alpha/2},$$

we want to find the infimum of

$$(11) \quad A \int_{\mathbb{D}} |u(w)|^2 d\mu(w) + \frac{4B}{\alpha} \int_{\mathbb{D}} |\nabla_{\mathbb{H}} u(w)|^2 d\mu(w)$$

under the constraint

$$(12) \quad \alpha \int_{\mathbb{D}} |u(w)|^{2q/p} d\mu(w) = 1.$$

We now solve the latter minimization problem for real-valued  $u$  belonging to the Sobolev space  $W^{1,2}(\mathbb{D})$ , i.e., functions  $u$  such that

$$\int_{\mathbb{D}} |\nabla_{\mathbb{H}} u(w)|^2 d\mu(w) < \infty.$$

By the well-known inequality for the bottom of the spectrum of the Laplace–Beltrami operator (see, e.g., [31]) we know that for any  $u \in W^{1,2}(\mathbb{D})$ ,

$$\int_{\mathbb{D}} |u(w)|^2 d\mu(w) \leq 4 \int_{\mathbb{D}} |\nabla_{\mathbb{H}} u(w)|^2 d\mu(w).$$

Hence

$$N(u) = \left( A \int_{\mathbb{D}} |u(w)|^2 d\mu(w) + \frac{4B}{\alpha} \int_{\mathbb{D}} |\nabla_{\mathbb{H}} u(w)|^2 d\mu(w) \right)^{1/2}$$

is a norm on  $W^{1,2}(\mathbb{D})$  equivalent to the usual norm, since  $A > -B/\alpha$ . By the Rellich–Kondrakov theorem [30, Ch. 11], which asserts that the inclusion map from  $W^{1,2}(\mathbb{D})$  into  $L^s(\mathbb{D}, d\mu)$  is compact for any finite  $s$ , the problem of finding the infimum of (11) for  $u \in W^{1,2}(\mathbb{D})$  satisfying (12) is well-posed. Moreover, this also ensures that minimizers do exist. Indeed, let us take any sequence  $(u_n)$  realizing the infimum. This sequence is bounded in the reflexive space  $W^{1,2}(\mathbb{D})$ , so we may assume that it converges weakly to some  $u \in W^{1,2}(\mathbb{D})$ . Then  $(u_n)$  converges to  $u$  in  $L^{2q/p}(\mathbb{D}, d\mu)$  so that  $\|u\|_{L^{2q/p}}^{2q/p} = 1/\alpha$  whereas  $N(u) \leq \liminf_n N(u_n)$ .

Next we compute the Euler–Lagrange equation corresponding to the constrained variational problem given by (11) and (12). By standard arguments, we find that any local minimum of the problem is a weak solution of

$$(13) \quad Au - \frac{4B}{\alpha} \Delta_{\mathbb{H}} u = \lambda u^{\frac{2q}{p}-1}$$

for some  $\lambda \in \mathbb{R}$ . By Lemma 5 below, there are minimizers that are actually  $C^2(\mathbb{D})$ . Multiplying by  $u$  and integrating with respect to  $\mu$ , we find from (9) that  $\lambda > 0$ . We now rescale (13) by setting  $u = \kappa v$  with

$$\kappa^{2q/p-2} = \frac{4B}{\alpha\lambda}.$$

Then  $v \in W^{1,2}(\mathbb{D}) \cap C^2(\mathbb{D})$  satisfies

$$(14) \quad \Delta_{\mathbb{H}} v - \frac{(\alpha - 2)\alpha}{4} v + v^{\frac{2q}{p}-1} = 0.$$

We now investigate (13) for our candidate solution  $u_0(w) = (1 - |w|^2)^{\alpha/2}$ . Since

$$\Delta_{\mathbb{H}} u_0(w) = -\frac{\alpha}{2}(1 - |w|^2)^{\alpha/2} \left(1 - \frac{\alpha}{2}|w|^2\right)$$

we have that

$$Au_0 - \frac{4B}{\alpha} \Delta_{\mathbb{H}} u_0 = \frac{\alpha}{\alpha - 1} (1 - |w|^2)^{\frac{\alpha}{2}+1} = \lambda_0 u_0^{\frac{2q}{p}-1},$$

where  $\lambda_0 = \alpha/(\alpha - 1)$ . Hence, if we let  $u_0 = \kappa_0 v_0$  with

$$\kappa_0^{2q/p-2} = \frac{4B}{\alpha\lambda_0},$$

then  $v_0 \in W^{1,2}(\mathbb{D})$  is a solution of (14). However, by [31, Thm. 1.3] we know that the solution of (14) is unique up to a Möbius transformation, as long as

$$\frac{\alpha(2 - \alpha)}{4} < \frac{4q}{p \left(\frac{2q}{p} + 2\right)^2}.$$

Replacing  $q/p$  by its value, we find that this inequality is satisfied if and only if  $\alpha > \alpha_0$ . Both the Euler–Lagrange equation and our constraint problem are invariant under Möbius transformations, so we have found all minimizers. Coming back to analytic functions via (10), we have shown that we have equality if and only if there exists  $\xi \in \mathbb{D}$  and  $\tilde{C} \in \mathbb{R}$  such that

$$|f(w)|^{p/2} = \tilde{C} \left| 1 - \left| \frac{\xi - w}{1 - \bar{\xi}w} \right|^2 \right|^{\alpha/2} |1 - |w|^2|^{-\alpha/2} = \tilde{C} \frac{(1 - |\xi|^2)^{\alpha/2}}{|1 - \bar{\xi}w|^\alpha}.$$

This shows that  $f$  has to be a multiple of  $(1 - \bar{\xi}w)^{-2\alpha/p}$  for some  $\xi \in \mathbb{D}$ . Finally, the assertion of the theorem for  $\alpha = \alpha_0$  is obtained by taking the limit as  $\alpha \rightarrow \alpha_0^+$ .  $\square$

The following is the regularity result that was used in the proof of the previous theorem.

**Lemma 5.** *There are minimizers of the variational constrained variational problem given by (11) and (12) that are  $C^2$  smooth in  $\mathbb{D}$ .*

*Proof.* Let  $u$  be a minimizer. Then it is a weak solution of the Euler-Lagrange equation (13). We also know that  $u \in L^{2q/p}(\mathbb{D}, d\mu)$ . Since the radial rearrangement decreases the Dirichlet norm (by the Polya–Szegő inequality [30, Thm. 16.17]) there is a minimizer  $u$  that is positive, radially symmetric, and decreasing. Therefore,  $F(u)$  is bounded in the unit disc, where

$$F(u) := \frac{\alpha}{4B} (Au - \lambda u^{\frac{2q}{p}-1}).$$

Consider any solution  $v$  to the Poisson equation:

$$\Delta v(z) = \frac{F(u(z))}{(1 - |z|^2)^2};$$

then  $u - v$  satisfies  $\Delta(u - v) = 0$  weakly. Therefore,  $u = v + h$ , where  $h$  is a harmonic function. One explicit solution to the Poisson equation is given by

$$v(z) = \int_{\mathbb{D}} K(z, w) \frac{F(u(w))}{(1 - |w|^2)^2} dm(w),$$

where

$$K(z, w) = \frac{1}{2\pi} \left\{ \log \left| \frac{w - z}{1 - \bar{w}z} \right|^2 + \frac{(1 - |w|^2)(1 - |z|^2)}{|1 - \bar{w}z|^2} + |z|^2 \left( \frac{1 - |w|^2}{|1 - \bar{w}z|} \right)^2 \right\}.$$

It was shown in [1] that  $K(z, w)$  satisfies the estimate

$$|K(z, w)| \lesssim \frac{(1 - |w|^2)^2}{|1 - \bar{w}z|^2} \left( 1 + \log \left| \frac{1 - \bar{w}z}{w - z} \right| \right), \quad z, w \in \mathbb{D}.$$

The difference between  $u$  and  $v$  is harmonic, thus the regularity of  $u$  follows from the regularity of  $v$ . □

*Remark.* The constants  $A$  and  $B$ , with  $A + B = 1$ , were chosen in the proof so that  $u(w) = (1 - |w|^2)^{\alpha/2}$  would be a solution of the Euler–Lagrange equation for some  $\lambda \in \mathbb{R}$ . This is only possible if  $\beta = \alpha + 1$ , and thus explains why this relationship is imposed in the statement of Theorem 1. The condition  $\alpha \geq \alpha_0$  comes from [31, Thm. 1.3], but we do not know if it is necessary for the uniqueness of (14).

**Question.** For any  $0 < p \leq q$  and  $\alpha, \beta \geq 1$  such that  $q/\beta \leq p/\alpha$ , does the contractive inequality

$$\|f\|_{A_\beta^q(\mathbb{D})} \leq \|f\|_{A_\alpha^p(\mathbb{D})}$$

hold? By Carleman’s inequality and Theorem 1, this is true when  $\beta = \alpha + n$  for some integer  $n$ , and either  $\alpha = 1$  or  $\alpha \geq \alpha_0$ . We remark that it is easy to show, for example by computing with coefficients, that

$$\|f\|_{A_{2\alpha}^4(\mathbb{D})} \leq \|f\|_{A_\alpha^2(\mathbb{D})}$$

holds for every  $\alpha \geq 1$ .



**2.3. Hypercontractivity of the Poisson kernel.** For  $r \in [0, 1]$ , let  $P_r$  denote the operator defined on analytic functions in  $\mathbb{D}$  by  $P_r f(w) = f(rw)$ . Clearly, if  $r < 1$  it follows from (6) that  $P_r$  maps any  $A_\alpha^p(\mathbb{D})$  into every  $A_\beta^q(\mathbb{D})$ . We are interested in knowing when this map is contractive.

**Theorem 6.** *Let  $0 < p \leq q < \infty$  and let  $\alpha = (n + 1)/2$  for some  $n \in \mathbb{N}$ . Then  $P_r$  is a contraction from  $A_\alpha^p(\mathbb{D})$  to  $A_\alpha^q(\mathbb{D})$  if and only if  $r \leq \sqrt{p/q}$ .*

Weissler [44] proved Theorem 6 when  $\alpha = 1$ . The case  $\alpha = 3/2$  is also known, see [21, Remark 5.14] or [28], but it appears that these are the only two previously demonstrated cases. To prove Theorem 6 we will use a classical argument of complex analysis to transfer results from Hardy spaces to Bergman spaces in smaller dimensions. This will be accomplished through the following lemma.

**Lemma 7** ([40], Sec. 1.4.4). *Let  $\mathbb{S}^n$  denote the real unit sphere of dimension  $n \geq 1$ , and let  $\sigma_n$  denote its normalized surface measure. Extend the function  $h: \mathbb{D} \rightarrow \mathbb{C}$  to  $\mathbb{S}^n$  by  $\tilde{h}(x) = h(x_1 + ix_2)$  for  $x = (x_1, x_2, \dots, x_{n+1}) \in \mathbb{S}^n$ . Then*

$$\int_{\mathbb{S}^n} \tilde{h}(x) d\sigma_n(x) = \int_{\mathbb{D}} h(w) dm_{(n+1)/2}(w).$$

We can now demonstrate how Theorem 6 follows from a result of Beckner [6] concerning the unit sphere.

*Proof of Theorem 6.* Let  $\mathcal{P}_r$  denote the Poisson kernel on  $\mathbb{S}^n$ , defined by

$$\mathcal{P}_r(\xi, \eta) = \frac{1 - r^2}{|r\xi - \eta|^{n+1}}, \quad \xi, \eta \in \mathbb{S}^n.$$

For a function  $g$  on  $\mathbb{S}^n$ , let

$$(\mathcal{P}_r g)(\xi) = \int_{\mathbb{S}^n} \mathcal{P}_r(\xi, \eta) g(\eta) d\sigma_n(\eta).$$

It is proved in [6] that  $\mathcal{P}_r$  defines a contraction from  $L^s(\mathbb{S}^n)$  to  $L^t(\mathbb{S}^n)$ ,  $1 \leq s \leq t < \infty$ , if and only if  $r \leq \sqrt{(s - 1)/(t - 1)}$ .

Let us now start with  $0 < p \leq q < \infty$  and  $r < \sqrt{p/q}$ . Let  $m$  be a large number such that  $mp > 1$  and such that

$$r \leq \sqrt{\frac{mp - 1}{mq - 1}}.$$

Given an analytic polynomial  $f$ , we define  $g$  on  $\mathbb{S}^n$  by

$$g(x_1, x_2, \dots, x_{n+1}) = |f(x_1 + ix_2)|^{1/m}.$$

Since  $f$  is analytic, it follows that  $g$  is subharmonic and hence for any  $(x_1, \dots, x_{n+1}) \in \mathbb{S}^n$  we get that

$$g(rx_1, \dots, rx_{n+1}) \leq \mathcal{P}_r g(x_1, \dots, x_{n+1}).$$

Using Beckner’s result with  $s = mp$  and  $t = mq$  we get that

$$\left( \int_{\mathbb{S}^n} g(rx_1, \dots, rx_{n+1})^{mq} d\sigma_n(x) \right)^{1/q} \leq \left( \int_{\mathbb{S}^n} g(x_1, \dots, x_{n+1})^{mp} d\sigma_n(x) \right)^{1/p}.$$

By Lemma 7, this is the same as

$$\left( \int_{\mathbb{D}} |f(rw)|^q dm_{(n+1)/2}(w) \right)^{\frac{1}{q}} \leq \left( \int_{\mathbb{D}} |f(w)|^p dm_{(n+1)/2}(w) \right)^{\frac{1}{p}}.$$

It follows that the condition  $r \leq \sqrt{p/q}$  is sufficient (by a limiting argument in the endpoint case  $r = \sqrt{p/q}$ ). Conversely, for fixed  $r > 0$  and small  $\varepsilon > 0$  we have that

$$\left( \int_{\mathbb{D}} |1 + \varepsilon rw|^q dA_{\alpha}(w) \right)^{\frac{1}{q}} = 1 + \frac{qr^2}{4\alpha} \varepsilon^2 + O(\varepsilon^4).$$

Letting  $\varepsilon \rightarrow 0$  shows that  $qr^2 \leq p$  is also necessary for any value of  $\alpha \geq 1$ . □

*Remark.* As in the previous subsection, we conjecture that Theorem 6 is true for all values of  $\alpha \geq 1$ . Several other positive results can be deduced from Theorem 1. For instance, if  $\alpha \geq \alpha_0$ , then

$$\|P_r f\|_{A_{\alpha}^2(\mathbb{D})} \leq \|f\|_{A_{\alpha}^{2\alpha/(\alpha+1)}(\mathbb{D})}$$

for every analytic polynomial  $f$  if and only if  $r^2 \leq (\alpha + 1)/\alpha$ . In fact, it follows from Theorem 1 that

$$\|f\|_{A_{\alpha+1}^2(\mathbb{D})} \leq \|f\|_{A_{\alpha}^{2\alpha/(\alpha+1)}(\mathbb{D})}.$$

Computing the norms as in (5), we have that

$$\|P_r f\|_{A_{\alpha}^2(\mathbb{D})} \leq \|f\|_{A_{\alpha+1}^2(\mathbb{D})}$$

if and only if, for any  $k \geq 1$ ,

$$r^{2k} \leq \frac{c_{\alpha+1}(k)}{c_{\alpha}(k)} = \frac{\alpha + k}{\alpha}.$$

### 3. INEQUALITIES ON THE POLYDISC AND IN THE HALF-PLANE

For  $\alpha > 1$ , consider the following product measure on  $\mathbb{D}^{\infty}$ :

$$\mathbf{m}_{\alpha}(z) = m_{\alpha}(z_1) \times m_{\alpha}(z_2) \times m_{\alpha}(z_3) \times \cdots,$$

and for  $0 < p < \infty$  the corresponding Lebesgue space  $L^p_{\alpha}(\mathbb{D}^{\infty})$ . We define the Bergman spaces of the infinite polydisc, denoted  $A^p_{\alpha}(\mathbb{D}^{\infty})$ , as the closure in  $L^p_{\alpha}(\mathbb{D}^{\infty})$  of the space of analytic polynomials in an arbitrary number of variables. The Hardy spaces  $H^p(\mathbb{D}^{\infty})$  are defined as the closure of analytic polynomials with respect to the norm given by the product  $m_1 \times m_1 \times \cdots$  on  $\mathbb{T}^{\infty}$ , so that

$$\|f\|_{H^p(\mathbb{D}^{\infty})}^p = \int_{\mathbb{T}^{\infty}} |f(z)|^p d\mathbf{m}_1(z).$$

As before,  $H^p(\mathbb{D}^{\infty})$  is the limit as  $\alpha \rightarrow 1^+$  of  $A^p_{\alpha}(\mathbb{D}^{\infty})$ , in the sense that

$$\lim_{\alpha \rightarrow 1^+} \|f\|_{A^p_{\alpha}(\mathbb{D}^{\infty})} = \|f\|_{H^p(\mathbb{D}^{\infty})}$$

for every analytic polynomial  $f$ . We distinguish the case  $\alpha = 2$  by writing  $A^p(\mathbb{D}^{\infty}) = A^p_2(\mathbb{D}^{\infty})$ . Applying the point estimate (6) repeatedly we find that if  $f$  is a polynomial in  $A^p_{\alpha}(\mathbb{D}^{\infty})$ , then

$$(15) \quad |f(z)| \leq \left( \prod_{j=1}^{\infty} \frac{1}{1 - |z_j|^2} \right)^{\alpha/p} \|f\|_{A^p_{\alpha}(\mathbb{D}^{\infty})},$$

which implies that elements of  $A^p_\alpha(\mathbb{D}^\infty)$  are analytic functions on  $\mathbb{D}^\infty \cap \ell^2$ . Every  $f$  in  $A^p_\alpha(\mathbb{D}^\infty)$  has a power series expansion convergent in  $\mathbb{D}^\infty \cap \ell^2$ ,

$$(16) \quad f(z) = \sum_{\kappa \in \mathbb{N}_0^\infty} a_\kappa z^\kappa,$$

where  $\mathbb{N}_0^\infty$  denotes the set of all finite non-negative multi-indices.

Finally, when  $p = 2$  we can compute the norm explicitly. Suppose that  $f$  is of the form (16). Then

$$(17) \quad \|f\|_{A^2_\alpha(\mathbb{D}^\infty)} = \left( \sum_{\kappa \in \mathbb{N}_0^\infty} \frac{|a_\kappa|^2}{c_\alpha(\kappa)} \right)^{\frac{1}{2}}, \quad \text{where} \quad c_\alpha(\kappa) = \prod_{j=1}^\infty c_\alpha(\kappa_j).$$

Note that the final product contains only a finite number of factors not equal to 1, since  $\kappa$  is a finite multi-index.

The contractive inequalities of Section 2 can now be extended to  $\mathbb{D}^\infty$  using Minkowski’s inequality in the following formulation: if  $X$  and  $Y$  are measure spaces,  $g$  a measurable function on  $X \times Y$ , and  $p \geq 1$ , then

$$\left( \int_X \left( \int_Y |g(x, y)| dy \right)^p dx \right)^{\frac{1}{p}} \leq \int_Y \left( \int_X |g(x, y)|^p dx \right)^{\frac{1}{p}} dy.$$

It is sufficient to prove the contractive results on the finite polydiscs  $\mathbb{D}^d$ ,  $d < \infty$ , as this allows us to conclude by the density of analytic polynomials. This is done by iteratively applying the one dimensional result to each of the variables, and applying Minkowski’s inequality in each step. This procedure has been repeated many times (for instance in [5, 8, 25] or in [38, Sec. 6.5.3]) and we do not include the details here.

In particular, Corollary 3 for  $n = 2$  yields the next result on the polydisc. Helson [25] proved the corresponding result for the Hardy spaces  $H^p(\mathbb{D}^\infty)$ , which he used to study Hilbert–Schmidt multiplicative Hankel forms. We shall carry out the analogous study for weighted multiplicative Hankel forms associated with the Bergman space in the next section.

**Lemma 8.**  $\|f\|_{A^2_4(\mathbb{D}^\infty)} \leq \|f\|_{A^1(\mathbb{D}^\infty)}$ .

Let  $\mathbf{r} = (r_1, r_2, \dots)$  with  $r_j \in [0, 1]$  and define  $P_{\mathbf{r}}f(z) = f(r_1z_1, r_2z_2, \dots)$ . Following [5] and using Theorem 6 (with  $\alpha = 2$ ), we get the next result.

**Lemma 9.** *Let  $0 < p \leq q < \infty$ . The map  $P_{\mathbf{r}}$  is a contraction from  $A^p(\mathbb{D}^\infty)$  to  $A^q(\mathbb{D}^\infty)$  if and only if  $r_j \leq \sqrt{p/q}$ . Moreover,  $P_{\mathbf{r}}$  is bounded from  $A^p(\mathbb{D}^\infty)$  to  $A^q(\mathbb{D}^\infty)$  as soon as  $r_j \leq \sqrt{p/q}$  for all but a finite set of  $j$ s.*

When working with multiplicative Hankel forms and Dirichlet series, it is often convenient to recast the expansion (16) in multiplicative notation. Each integer  $n \geq 1$  can be written in a unique way as a product of prime numbers,

$$n = \prod_{j=1}^\infty p_j^{\kappa_j}.$$

This factorization associates  $n$  uniquely to the finite non-negative multi-index  $\kappa(n)$ . Setting  $a_n = a_{\kappa(n)}$ , we rewrite (16) as

$$(18) \quad f(z) = \sum_{n=1}^{\infty} a_n z^{\kappa(n)}.$$

For  $\alpha \geq 1$  we define the general divisor function  $d_\alpha(n)$  as the coefficients of the Dirichlet series given by  $\zeta^\alpha$ , where  $\zeta(s) = \sum_{n \geq 1} n^{-s}$  is the Riemann zeta function. Using the Euler product of the Riemann zeta function, say for  $\text{Re}(s) > 1$ , we find that

$$(19) \quad \zeta(s)^\alpha = \left( \prod_{j=1}^{\infty} \frac{1}{1 - p_j^{-s}} \right)^\alpha = \prod_{j=1}^{\infty} \left( \sum_{k=0}^{\infty} c_\alpha(k) p_j^{-ks} \right) = \sum_{n=1}^{\infty} d_\alpha(n) n^{-s}.$$

It follows that  $c_\alpha(\kappa(n)) = d_\alpha(n)$ . In multiplicative notation, we restate (17) as

$$\left\| \sum_{n=1}^{\infty} a_n z^{\kappa(n)} \right\|_{A_\alpha^2(\mathbb{D}^\infty)} = \left( \sum_{n=1}^{\infty} \frac{|a_n|^2}{d_\alpha(n)} \right)^{\frac{1}{2}}.$$

When  $\alpha \geq 1$  is an integer, it is clear that  $d_\alpha(n)$  denotes the number of ways to write  $n$  as a product of  $\alpha$  non-negative integers. In particular,  $d_2$  is the usual divisor function  $d$ . It also follows from (19) that

$$(20) \quad \sum_{mn=l} d_\alpha(m) d_\beta(n) = d_{\alpha\beta}(l),$$

in analogy with (4).

The Bohr lift of a Dirichlet series  $f(s) = \sum_{n \geq 1} a_n n^{-s}$  is the power series defined by

$$\mathcal{B}f(z) = \sum_{n=1}^{\infty} a_n z^{\kappa(n)},$$

realizing the identification  $z_j = p_j^{-s}$ . The Bergman space of Dirichlet series  $\mathcal{A}^p$  is defined as the completion of Dirichlet polynomials in the norm

$$\|f\|_{\mathcal{A}^p} = \|\mathcal{B}f\|_{A^p(\mathbb{D}^\infty)}.$$

Inequality (15) implies that  $\mathcal{A}^p$  is a space of analytic functions in the half-plane  $\mathbb{C}_{1/2}$ , and that  $f$  in  $\mathcal{A}^p$  enjoys the sharp pointwise estimate

$$(21) \quad |f(s)| \leq \zeta(2 \text{Re } s)^{2/p} \|f\|_{\mathcal{A}^p}.$$

Let  $\mathcal{T}$  denote the conformal map of  $\mathbb{D}$  to  $\mathbb{C}_{1/2}$  given by

$$\mathcal{T}(z) = \frac{1}{2} + \frac{1-z}{1+z}.$$

The conformally invariant Bergman space of the half-plane  $\mathbb{C}_{1/2}$ , denoted  $A_{\alpha,i}^p(\mathbb{C}_{1/2})$ , is the space of analytic functions  $f$  in  $\mathbb{C}_{1/2}$  with the property that  $f \circ \mathcal{T} \in A_\alpha^p(\mathbb{D})$ . A computation shows that

$$\|f\|_{A_{\alpha,i}^p(\mathbb{C}_{1/2})}^p = \int_{\mathbb{C}_{1/2}} |f(s)|^p (\alpha - 1) \left( \text{Re}(s) - \frac{1}{2} \right)^{\alpha-2} \frac{4^{\alpha-1}}{|s + 1/2|^{2\alpha}} dm(s).$$

By Lemma 8 we have the following version of Carleman’s inequality for Dirichlet series in the half-plane.

**Theorem 10.** *Suppose that  $f(s) = \sum_{n \geq 1} a_n n^{-s}$  is in  $\mathcal{A}^1$ . Then*

$$(22) \quad \left( \sum_{n=1}^{\infty} \frac{|a_n|^2}{d_4(n)} \right)^{\frac{1}{2}} \leq \|f\|_{\mathcal{A}^1}.$$

Moreover, there is a constant  $C \geq 1$  such that  $\|f\|_{A_{4,i}^2(\mathbb{C}_{1/2})} \leq C \|f\|_{\mathcal{A}^1}$ .

*Proof.* The inequality (22) is Lemma 8 in multiplicative notation. The second statement follows from the first and Example 2 in [33]. □

For  $\varepsilon > 0$ , define the translation operator  $T_\varepsilon$  by  $T_\varepsilon f(s) = f(s + \varepsilon)$ . Here is a sharp and general version of [3, Prop. 9], which we interpret as Weisler’s inequality for Dirichlet series in the half-plane. The corresponding result for  $\mathcal{H}^p$  can be found in [5].

**Theorem 11.** *Let  $0 < p \leq q < \infty$ . The operator  $T_\varepsilon: \mathcal{A}^p \rightarrow \mathcal{A}^q$  is bounded for every  $\varepsilon > 0$ , and contractive if and only if  $2^{-\varepsilon} \leq \sqrt{p/q}$ .*

*Proof.* This follows from Lemma 9, using the fact that  $T_\varepsilon$  corresponds to  $P_{\mathbf{r}}$  with  $r_j = p_j^{-\varepsilon}$ . □

We end this section by demonstrating that Lemma 9 also implies a weak generalization of Theorem 10 to more general exponents. In the Hardy space context, it was proven in [8] that if  $f(s) = \sum_{n \geq 1} a_n n^{-s}$  and  $0 < p \leq 2$ , then

$$\left( \sum_{n=1}^{\infty} |a_n|^2 \frac{|\mu(n)|}{d_{2/p}(n)} \right)^{\frac{1}{2}} \leq \|f\|_{\mathcal{H}^p}.$$

The Möbius factor  $|\mu(n)|$  is 1 if  $n$  is square-free and 0 if not. From (8), it follows that this factor may actually be replaced by 1 if  $p = 2/(1+n)$  for some non-negative integer  $n$ . We have the following extension to Bergman spaces in mind.

**Theorem 12.** *Let  $0 < p \leq 2$  and suppose that  $f(s) = \sum_{n \geq 1} a_n n^{-s}$  is in  $\mathcal{A}^p$ . Then*

$$\left( \sum_{n=1}^{\infty} |a_n|^2 \frac{|\mu(n)|}{d_{4/p}(n)} \right)^{\frac{1}{2}} \leq \|f\|_{\mathcal{A}^p}.$$

If  $p = 2/(1+n/2)$  for some non-negative integer  $n$ , then

$$\left( \sum_{n=1}^{\infty} |a_n|^2 \frac{1}{d_{4/p}(n)} \right)^{\frac{1}{2}} \leq \|f\|_{\mathcal{A}^p}.$$

*Proof.* Let  $\Omega(n)$  denote the number of prime factors of  $n$  (counting multiplicity). Using Lemma 9 with  $r_j = \sqrt{p/2}$ , we have that

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} a_n n^{-s} \right\|_{\mathcal{A}^p} &\geq \left\| \sum_{n=1}^{\infty} a_n \left(\frac{p}{2}\right)^{\Omega(n)/2} n^{-s} \right\|_{\mathcal{A}^2} = \left( \sum_{n=1}^{\infty} |a_n|^2 \frac{1}{(2/p)^{\Omega(n)} d(n)} \right)^{\frac{1}{2}} \\ &\geq \left( \sum_{n=1}^{\infty} |a_n|^2 \frac{|\mu(n)|}{(2/p)^{\Omega(n)} d(n)} \right)^{\frac{1}{2}} = \left( \sum_{n=1}^{\infty} |a_n|^2 \frac{|\mu(n)|}{d_{4/p}(n)} \right)^{\frac{1}{2}}. \end{aligned}$$

In the final equality we used that  $d_\alpha(n) = \alpha^{\Omega(n)}$  when  $n$  is square-free. When  $p = 2/(1+n/2)$  for a non-negative integer  $n$ , tensorizing Corollary 3 (by appealing

to Minkowski’s inequality) yields that the Möbius factor is actually unnecessary; see Lemma 8 and Theorem 10.  $\square$

*Remark.* Considering the square-free terms only of a Dirichlet series is in many cases sufficient to obtain sharp results; see for example [8]. Often, the reason for this is related to the fact that the square-free zeta function has the same behaviour as the zeta function  $\zeta(s)$  near  $s = 1$ , since

$$\sum_{n=1}^{\infty} |\mu(n)|n^{-s} = \prod_{j=1}^{\infty} (1 + p_j^{-s}) = \prod_{j=1}^{\infty} \frac{1 - p_j^{-2s}}{1 - p_j^{-s}} = \frac{\zeta(s)}{\zeta(2s)}.$$

#### 4. MULTIPLICATIVE HANKEL FORMS

The multiplicative Hankel form (2) is said to be bounded if there is a constant  $C < \infty$  such that

$$(23) \quad |\varrho(a, b)| = \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \frac{\varrho_{mn}}{d(mn)} \right| \leq C \left( \sum_{m=1}^{\infty} \frac{|a_m|^2}{d(m)} \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} \frac{|b_n|^2}{d(n)} \right)^{\frac{1}{2}}.$$

The smallest such constant is the norm of  $\varrho$ . The symbol of the form  $\varrho$  is the Dirichlet series  $\varphi(s) = \sum_{n \geq 1} \overline{\varrho_n} n^{-s}$ . If  $f$  and  $g$  are Dirichlet series with coefficient sequences  $a$  and  $b$ , respectively, then (23) can be rewritten as  $|H_\varphi(fg)| \leq C \|f\|_{\mathcal{A}^2} \|g\|_{\mathcal{A}^2}$ , where we define

$$H_\varphi(fg) = \langle fg, \varphi \rangle_{\mathcal{A}^2} = \sum_{l=1}^{\infty} \left( \sum_{mn=l} a_m b_n \right) \frac{\varrho_l}{d(l)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \frac{\varrho_{mn}}{d(mn)}.$$

Hence, the multiplicative Hankel form is bounded if and only if  $H_\varphi$  is a bounded form on  $\mathcal{A}^2 \times \mathcal{A}^2$ .

We begin with the following example, giving the Bergman space analogue of the multiplicative Hilbert matrix studied in [11]. Let  $\mathcal{A}_0^2$  denote the subspace of  $\mathcal{A}^2$  consisting of Dirichlet series  $f(s) = \sum_{n \geq 1} a_n n^{-s}$  such that  $a_1 = f(+\infty) = 0$ . As in [11], it is natural to work with Dirichlet series without constant term for convergence reasons. We consider the form

$$(24) \quad H(fg) = \int_{1/2}^{\infty} f(\sigma)g(\sigma) \left( \sigma - \frac{1}{2} \right) d\sigma, \quad f, g \in \mathcal{A}_0^2.$$

**Theorem 13.** *The bilinear form (24) is a multiplicative Hankel form with symbol*

$$\varphi(s) = \int_{1/2}^{\infty} (\zeta(s + \sigma)^2 - 1) \left( \sigma - \frac{1}{2} \right) d\sigma = \sum_{n=2}^{\infty} \frac{d(n)}{\sqrt{n}(\log n)^2} n^{-s}.$$

*The form  $H_\varphi$  is bounded, but not compact, on  $\mathcal{A}_0^2 \times \mathcal{A}_0^2$ .*

*Proof.* To see that  $\varphi$  is the symbol, one can either compute  $H(fg)$  at the level of coefficients or use that  $\zeta(s + \overline{w})^2 - 1$  is the reproducing kernel of  $\mathcal{A}_0^2$ . To see that  $H$  is bounded, we first use the Cauchy–Schwarz inequality,

$$|H(fg)| \leq \left( \int_{1/2}^{\infty} |f(\sigma)|^2 \left( \sigma - \frac{1}{2} \right) d\sigma \right)^{\frac{1}{2}} \left( \int_{1/2}^{\infty} |g(\sigma)|^2 \left( \sigma - \frac{1}{2} \right) d\sigma \right)^{\frac{1}{2}}.$$

By symmetry, we only need to consider one of the factors. We split the integral at  $\sigma = 1$ .

$$\int_{1/2}^{\infty} |f(\sigma)|^2 \left(\sigma - \frac{1}{2}\right) d\sigma = \left(\int_{1/2}^1 + \int_1^{\infty}\right) |f(\sigma)|^2 \left(\sigma - \frac{1}{2}\right) d\sigma.$$

The first integral is bounded by a constant multiple of  $\|f\|_{\mathcal{A}^2}^2$ , as follows from [33, Thm. 3 and Example 4]. For the second integral, we have by the pointwise estimate (21) that

$$|f(\sigma)|^2 \leq \|f\|_{\mathcal{A}^2}^2 \left(\sum_{n=2}^{\infty} d(n)n^{-2\sigma}\right) \leq (2 + o(1))4^{-\sigma}\|f\|_{\mathcal{A}^2}^2,$$

where we in the final inequality used that  $\sigma \geq 1$ . To show that  $H_\varphi$  is not compact, let  $k_\varepsilon(s)$  denote the normalized reproducing kernel of  $\mathcal{A}_0^2$  at the point  $1/2 + \varepsilon/2$ ,

$$k_\varepsilon(s) = \frac{\zeta^2(s + 1/2 + \varepsilon/2) - 1}{\sqrt{\zeta^2(1 + \varepsilon) - 1}}.$$

The functions  $k_\varepsilon$  converge weakly to 0 as  $\varepsilon \rightarrow 0$ , since they converge to 0 on every compact subset of  $\mathbb{C}_{1/2}$ . By the fact that

$$\zeta(s) = \frac{1}{s - 1} + O(1)$$

for  $\text{Re}(s) > 1$  close to 1, we get for, say  $1/2 < \sigma < 1$ , that

$$k_\varepsilon(\sigma) = \frac{(\sigma + 1/2 + \varepsilon/2 - 1)^{-2} + O(1)}{(1 + \varepsilon - 1)^{-1} + O(1)} = \varepsilon \left(\frac{1}{(\sigma - 1/2 + \varepsilon/2)^2} + O(1)\right).$$

Setting  $f = g = k_\varepsilon$ , we find that

$$H(fg) = \varepsilon^2 \left(\int_{1/2}^1 \left(\frac{1}{(\sigma - 1/2 + \varepsilon/2)^4} + O(1)\right) \left(\sigma - \frac{1}{2}\right) d\sigma + O(1)\right) \gtrsim 1,$$

showing that  $H$  is not compact. □

Since the Bohr lift is multiplicative, it holds that

$$\langle fg, \varphi \rangle_{\mathcal{A}^2} = \langle \mathcal{B}f\mathcal{B}g, \mathcal{B}\varphi \rangle_{A^2(\mathbb{D}^\infty)}.$$

For the remainder of this section we will work in the polydisc, and we therefore tacitly identify the Dirichlet series  $f$  with its Bohr lift  $\mathcal{B}f$ . Hence, we consider symbols of the form

$$\varphi(z) = \sum_{n=1}^{\infty} \overline{q}_n z^{\kappa(n)},$$

and define  $H_\varphi(fg) = \langle fg, \varphi \rangle_{A^2(\mathbb{D}^\infty)}$  for  $f, g \in A^2(\mathbb{D}^\infty)$ .

If  $\varphi$  defines a bounded functional on  $A^1(\mathbb{D}^\infty)$ , then it follows from the Cauchy–Schwarz inequality that

$$|H_\varphi(fg)| = |\langle fg, \varphi \rangle_{A^2}| \leq \|\varphi\|_{(A^1)^*} \|fg\|_{A^1} \leq \|\varphi\|_{(A^1)^*} \|f\|_{A^2} \|g\|_{A^2},$$

i.e., the Hankel form  $H_\varphi$  is bounded on  $A^2(\mathbb{D}^\infty) \times A^2(\mathbb{D}^\infty)$  in this case. Our first goal is to show that the converse does not hold. We define the weak product

$A^2(\mathbb{D}^\infty) \odot A^2(\mathbb{D}^\infty)$  as the closure of all finite sums  $f = \sum_k g_k h_k$ ,  $g_k, h_k \in A^2(\mathbb{D}^\infty)$ , under the norm

$$\|f\|_{A^2(\mathbb{D}^\infty) \odot A^2(\mathbb{D}^\infty)} = \inf \sum_k \|g_k\|_{A^2(\mathbb{D}^\infty)} \|h_k\|_{A^2(\mathbb{D}^\infty)}.$$

Here the infimum is taken over all finite representations  $f = \sum_k g_k h_k$ . Note that  $\|f\|_{A^1(\mathbb{D}^\infty)} \leq \|f\|_{A^2(\mathbb{D}^\infty) \odot A^2(\mathbb{D}^\infty)}$ .

**Lemma 14.** *Suppose that  $\varphi$  generates a Hankel form on  $A^2(\mathbb{D}^\infty) \times A^2(\mathbb{D}^\infty)$ . Then*

$$\|H_\varphi\| = \|\varphi\|_{(A^2(\mathbb{D}^\infty) \odot A^2(\mathbb{D}^\infty))^*}.$$

*Every bounded Hankel form  $H_\varphi$  extends to a bounded functional on  $A^1(\mathbb{D}^\infty)$  if and only if there is a constant  $C_\infty < \infty$  such that for any  $f \in A^1(\mathbb{D}^\infty)$ ,*

$$\|f\|_{A^2(\mathbb{D}^\infty) \odot A^2(\mathbb{D}^\infty)} \leq C_\infty \|f\|_{A^1(\mathbb{D}^\infty)}.$$

*Proof.* The first statement is a tautology. The weak product space  $A^2(\mathbb{D}^\infty) \odot A^2(\mathbb{D}^\infty)$  is a Banach space, and therefore the second statement follows from the closed graph theorem and duality (see [9, 25]). □

Factorization and weak factorization of Hardy and Bergman spaces have a long history. Strong factorization for  $H^1(\mathbb{D})$  was treated by Nehari [32], and the analogous factorization for  $A^1(\mathbb{D})$  was given by Horowitz [27]. Every  $f$  in  $H^1(\mathbb{D})$  or  $A^1(\mathbb{D})$  can be written as a single product  $f = gh$ , for  $g, h$  in  $H^2(\mathbb{D})$  or  $A^2(\mathbb{D})$ , respectively. In Nehari’s theorem it is even possible to choose  $g$  and  $h$  such that  $\|f\|_{H^1(\mathbb{D})} = \|g\|_{H^2(\mathbb{D})} \|h\|_{H^2(\mathbb{D})}$ . The same is not possible in the factorization of  $A^1(\mathbb{D})$ , a simple observation we do not find recorded in the literature.

Factorization on the polydisc  $\mathbb{D}^d$  is a much subtler matter, even when  $1 < d < \infty$ . Strong factorization is certainly not possible, but in [20, 29] it was shown that the corresponding weak factorization holds,

$$H^1(\mathbb{D}^d) = H^2(\mathbb{D}^d) \odot H^2(\mathbb{D}^d), \quad d < \infty.$$

The Bergman space analogue was established in [17],

$$A^1(\mathbb{D}^d) = A^2(\mathbb{D}^d) \odot A^2(\mathbb{D}^d), \quad d < \infty.$$

In [35] it was shown that the best constant  $C_d$  in the factorization,

$$\|f\|_{H^2(\mathbb{D}^d) \odot H^2(\mathbb{D}^d)} \leq C_d \|f\|_{H^1(\mathbb{D}^d)},$$

satisfies growth estimate  $C_d \geq (\pi^2/8)^{d/4}$  when  $d$  is an even integer. This immediately implies that the weak factorization  $H^1(\mathbb{D}^\infty) = H^2(\mathbb{D}^\infty) \odot H^2(\mathbb{D}^\infty)$  is impossible. By tensorization, it is explained in [9, Sec. 3] that  $C_{kd} \geq C_d^k$  for every positive integer  $k$ , a result which effortlessly carries over to the context of Bergman spaces. Hence we have the following.

**Theorem 15.** *Let  $C_d$  denote the best constant in the inequality*

$$\|f\|_{A^2(\mathbb{D}^d) \odot A^2(\mathbb{D}^d)} \leq C_d \|f\|_{A^1(\mathbb{D}^d)}$$

*for  $d = 1, 2, \dots$ . Then*

$$C_d \geq \left(\frac{9}{8}\right)^{d/2}.$$



In particular, the factorization in the unit disc is not norm-preserving, and therefore the weak factorization

$$A^1(\mathbb{D}^\infty) = A^2(\mathbb{D}^\infty) \odot A^2(\mathbb{D}^\infty)$$

does not hold.

*Proof.* In view of the discussion preceding the theorem, it is sufficient to prove that  $C_1 \geq 3/(2\sqrt{2})$ . For every polynomial  $\varphi$ , we get from duality that

$$C_1 \geq \frac{\|\varphi\|_{(A^1(\mathbb{D}))^*}}{\|\varphi\|_{(A^2(\mathbb{D}) \odot A^2(\mathbb{D}))^*}} \geq \frac{\|\varphi\|_{A^2(\mathbb{D})}^2}{\|\varphi\|_{A^1(\mathbb{D})} \|\varphi\|_{(A^2(\mathbb{D}) \odot A^2(\mathbb{D}))^*}},$$

where we have estimated the  $(A^1(\mathbb{D}))^*$ -norm by testing  $\varphi$  against itself. As in Lemma 14, we have that

$$\|\varphi\|_{(A^2(\mathbb{D}) \odot A^2(\mathbb{D}))^*} = \|H_\varphi\|_{A^2(\mathbb{D}) \times A^2(\mathbb{D})}.$$

We choose  $\varphi(w) = \sqrt{2}w$ . Clearly  $\|\varphi\|_{A^2(\mathbb{D})} = 1$ . The matrix of  $H_\varphi$  with respect to the standard basis of  $A^2(\mathbb{D})$  is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so we find that  $\|H_\varphi\|_{A^2(\mathbb{D}) \times A^2(\mathbb{D})} = 1$ . We are done, since

$$\|\varphi\|_{A^1(\mathbb{D})} = 2\sqrt{2} \int_0^1 r^2 dr = \frac{2\sqrt{2}}{3}. \quad \square$$

It would be interesting to decide if the symbol of the Hilbert-type form considered in Theorem 13, which lifts to

$$(25) \quad \varphi(z) = \sum_{n=2}^\infty \frac{d(n)}{\sqrt{n}(\log n)^2} z^{\kappa(n)},$$

defines a bounded linear functional on  $H^1(\mathbb{D}^\infty)$ . We are unable to settle this problem, but offer the following two observations. First, if  $f$  is an analytic polynomial on  $\mathbb{D}^\infty$  such that  $f(0) = 0$ , we may write

$$\langle f, \varphi \rangle_{A^2(\mathbb{D}^\infty)} = \int_{1/2}^\infty (\mathcal{B}^{-1}f)(\sigma + it) \left( \sigma - \frac{1}{2} \right) d\sigma.$$

If we could prove the embedding  $\|f\|_{A^1_1(\mathbb{C}_{1/2})} \leq \tilde{C}\|f\|_{\mathcal{A}^1}$ , which is a stronger version of the second statement in Theorem 10, then it would follow by simple Carleson measure argument that (25) defines a bounded linear functional on  $H^1(\mathbb{D}^\infty)$ , through the (inverse) Bohr lift.

Our second observation is contained in the following result.

**Theorem 16.** *Let  $\varphi$  be as in (25). Then  $\varphi$  defines a bounded functional on  $A^p(\mathbb{D}^\infty)$  for every  $1 < p < \infty$ .*

*Proof.* This is trivial when  $p \geq 2$ , since  $\varphi \in H^2(\mathbb{D}^\infty)$ . Let us therefore fix  $1 < p < 2$ , and suppose that  $f(z) = \sum_{n \geq 1} a_n z^{\kappa(n)}$  is in  $A^p(\mathbb{D}^\infty)$ . Then it follows from the

Cauchy–Schwarz inequality and Lemma 9 with  $r_j = \sqrt{p/2}$  that

$$\begin{aligned} |\langle f, \varphi \rangle_{A^2(\mathbb{D}^\infty)}| &= \left| \sum_{n=2}^\infty a_n \frac{1}{\sqrt{n}(\log n)^2} \right| \\ &\leq \left( \sum_{n=2}^\infty \frac{|a_n|^2}{d(n)} \left(\frac{p}{2}\right)^{\Omega(n)} \right)^{\frac{1}{2}} \left( \sum_{n=2}^\infty \left(\frac{2}{p}\right)^{\Omega(n)} \frac{d(n)}{n(\log n)^4} \right)^{\frac{1}{2}} \\ &\leq \|f\|_{A^p(\mathbb{D}^\infty)} \left( \sum_{n=2}^\infty \left(\frac{2}{p}\right)^{\Omega(n)} \frac{d(n)}{n(\log n)^4} \right)^{\frac{1}{2}}, \end{aligned}$$

where again  $\Omega(n)$  denotes the number of prime factors of  $n$ . We may conclude if we can show that

$$\sum_{n=2}^\infty \frac{d(n)\alpha^{\Omega(n)}}{n(\log n)^4} < \infty$$

if  $1 < \alpha < 2$ . This follows at once from Abel summation and the estimate

$$(26) \quad \frac{1}{x} \sum_{n \leq x} d(n)\alpha^{\Omega(n)} = C_\alpha(\log x)^{2\alpha-1} + O((\log x^{2\alpha-2})).$$

To demonstrate (26), we consider the associated Dirichlet series, for say  $\operatorname{Re}(s) > 1$ , and factor out an appropriate power of the zeta function

$$\begin{aligned} f_\alpha(s) &= \sum_{n=1}^\infty d(n)\alpha^{\Omega(n)}n^{-s} = \prod_{j=1}^\infty \left( \frac{1}{1 - \alpha p_j^{-s}} \right)^2 \\ &= \zeta^{2\alpha}(s) \prod_{j=1}^\infty \left( \frac{(1 - p_j^{-s})^\alpha}{1 - \alpha p_j^{-s}} \right)^2 =: \zeta^{2\alpha}(s)g_\alpha(s). \end{aligned}$$

Note that since

$$\left( \frac{(1 - p_j^{-s})^\alpha}{1 - \alpha p_j^{-s}} \right)^2 = 1 + (\alpha - 1)\alpha p_j^{-2s} + O(p_j^{-3s}),$$

the Dirichlet series  $g_\alpha$  is absolutely convergent for

$$\operatorname{Re}(s) > \max(1/2, \log_2 \alpha).$$

A standard residue integration argument (see, e.g., [42, Ch. II.5]) now gives (26) with  $C_\alpha = g_\alpha(1)/\Gamma(2\alpha)$ . □

Next, we investigate Hilbert–Schmidt Hankel forms (2), following [25]. Recall that on the finite polydisc  $\mathbb{D}^d$ ,  $d < \infty$ , a symbol  $\varphi$  generates a Hilbert–Schmidt Hankel form on  $H^2(\mathbb{D}^d) \times H^2(\mathbb{D}^d)$  if and only if it generates a Hilbert–Schmidt Hankel form on  $A^2(\mathbb{D}^d) \times A^2(\mathbb{D}^d)$ . On the infinite polydisc we have the following result. Theorem 10 is its essential ingredient.

**Theorem 17.** *If the Hankel form generated by  $\varphi$  is Hilbert–Schmidt on  $A^2(\mathbb{D}^\infty) \times A^2(\mathbb{D}^\infty)$ , then  $\varphi$  also generates a bounded functional on  $A^1(\mathbb{D}^\infty)$ . If  $\varphi$  generates a Hilbert–Schmidt form on  $H^2(\mathbb{D}^\infty) \times H^2(\mathbb{D}^\infty)$ , then it generates a Hilbert–Schmidt form on  $A^2(\mathbb{D}^\infty) \times A^2(\mathbb{D}^\infty)$ , but the converse does not hold.*

*Proof.* First, we compute the Hilbert–Schmidt norm on  $A^2(\mathbb{D}^\infty) \times A^2(\mathbb{D}^\infty)$  of the form  $H_\varphi$  generated by the symbol  $\varphi(s) = \sum_{n \geq 1} \overline{\varrho_n} z^{\kappa(n)}$ . An orthonormal basis for  $A^2(\mathbb{D}^\infty)$  is given by

$$e_n(z) = z^{\kappa(n)} \sqrt{d(n)}.$$

Hence,

$$\begin{aligned} \|H_\varphi\|_{S_2(A^2(\mathbb{D}^\infty) \times A^2(\mathbb{D}^\infty))}^2 &= \sum_{m=1}^\infty \sum_{n=1}^\infty |H_\varphi(e_m e_n)|^2 = \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{|\varrho_{mn}|^2 d(m)d(n)}{[d(mn)]^2} \\ &= \sum_{l=1}^\infty \frac{|\varrho_l|^2}{[d(l)]^2} \sum_{mn=l} d(m)d(n) = \sum_{l=1}^\infty |\varrho_l|^2 \frac{d_4(l)}{[d(l)]^2}, \end{aligned}$$

where we have made use of (20) after recalling the convention that  $d_2 = d$ . The first statement now follows from Theorem 10, since the Cauchy–Schwarz inequality implies that

$$|\langle f, \varphi \rangle_{A^2(\mathbb{D}^\infty)}| = \left| \sum_{n=1}^\infty \frac{a_n \varrho_n}{d(n)} \right| \leq \left( \sum_{n=1}^\infty \frac{|a_n|^2}{d_4(n)} \right)^{\frac{1}{2}} \left( \sum_{n=1}^\infty |\varrho_n|^2 \frac{d_4(n)}{[d(n)]^2} \right)^{\frac{1}{2}}.$$

Similarly we have that

$$\|H_\varphi\|_{S_2(H^2(\mathbb{D}^\infty) \times H^2(\mathbb{D}^\infty))}^2 = \sum_{n=1}^\infty |\varrho_n|^2 d(n).$$

Note that when  $n$  is a prime power  $n = p_j^k$  we have that

$$d_4(n) = \frac{(k+1)(k+2)(k+3)}{6} \leq (k+1)^3 = [d(n)]^3.$$

Since both  $d_4(n)$  and  $d(n)$  are multiplicative functions, it follows that  $d_4(n) \leq [d(n)]^3$  for every  $n$ . Hence the second statement is proved.

To see that the converse of the second statement does not hold, consider the set  $\mathcal{N} = \{n_1 = 2, n_2 = 3 \cdot 5, n_3 = 7 \cdot 11 \cdot 13, \dots\}$  and define  $\varphi(s) = \sum_{n \in \mathcal{N}} \overline{\varrho_n} z^{\kappa(n)}$ . Then we have that

$$\begin{aligned} \|H_\varphi\|_{S_2(A^2(\mathbb{D}^\infty) \times A^2(\mathbb{D}^\infty))}^2 &= \sum_{j=1}^\infty |\varrho_{n_j}|^2, \\ \|H_\varphi\|_{S_2(H^2(\mathbb{D}^\infty) \times H^2(\mathbb{D}^\infty))}^2 &= \sum_{j=1}^\infty |\varrho_{n_j}|^2 2^{n_j}. \quad \square \end{aligned}$$

The final part of this section is devoted to showing that every Hankel form of the type (2) naturally corresponds to a Hankel form of the type (1) with the same singular numbers. Let  $D$  denote the diagonal operator in two variables,  $Df(w) = f(w, w)$ , for which we have the following observation.

**Lemma 18.** *The operator  $D$  is a contraction from  $H^2(\mathbb{D}^2)$  to  $A^2(\mathbb{D})$ .*

*Proof.* This is proven in [39], but in an abstract formulation it dates back at least to Aronszajn [2]. The proof of our particular case is very easy and we include it here. Consider

$$f(z_1, z_2) = \sum_{j=0}^\infty \sum_{k=0}^\infty a_{j,k} z_1^j z_2^k$$

and use the Cauchy–Schwarz inequality to conclude that

$$\|Df\|_{A^2(\mathbb{D})}^2 = \sum_{l=0}^{\infty} \frac{1}{l+1} \left| \sum_{j+k=l} a_{j,k} \right|^2 \leq \sum_{l=0}^{\infty} \sum_{j+k=l} |a_{j,k}|^2 = \|f\|_{H^2(\mathbb{D}^2)}^2. \quad \square$$

The diagonal operator  $D$  may be written as an integral operator using the reproducing kernel of  $H^2(\mathbb{D}^2)$ ,

$$Df(w) = \int_{\mathbb{T}^2} f(z_1, z_2) \frac{1}{1-w\bar{z}_1} \frac{1}{1-w\bar{z}_2} dm_1(z_1) dm_1(z_2).$$

Hence its adjoint operator  $E: A^2(\mathbb{D}) \rightarrow H^2(\mathbb{D}^2)$  is given by

$$Eg(z_1, z_2) = \int_{\mathbb{D}} g(w) \frac{1}{1-z_1\bar{w}} \frac{1}{1-z_2\bar{w}} dA(w).$$

If  $f$  and  $g$  are in  $A^2(\mathbb{D})$ , then

$$\langle Ef, Eg \rangle_{H^2(\mathbb{D}^2)} = \langle f, g \rangle_{A^2(\mathbb{D})},$$

that is,  $E$  is an isometry. Clearly, the composition  $DE$  is the identity operator on  $A^2(\mathbb{D})$ . Hence we have identified  $A^2(\mathbb{D})$  with the subspace  $X = EA^2(\mathbb{D})$  of  $H^2(\mathbb{D}^2)$  (although perhaps it would be more appropriate to think of it as the factor space induced by the map  $D$ ). The projection  $P: H^2(\mathbb{D}^2) \rightarrow X$  is given by  $P = ED$ . Note that  $P$  averages the coefficients of monomials of same degree. Precisely, if  $f(z) = \sum_{j,k \geq 0} a_{j,k} z_1^j z_2^k$ , then

$$Pf(z_1, z_2) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} A_{j+k} z_1^j z_2^k, \quad \text{where} \quad A_l = \frac{1}{l+1} \sum_{j+k=l} a_{j,k}.$$

Clearly,  $D(fg) = D(f)D(g)$ , but  $E$  does not have this property. For example, if  $g(w) = w$ , then

$$Eg(z_1, z_2) = \frac{z_1 + z_2}{2} \quad \text{and} \quad E(g^2)(z_1, z_2) = \frac{z_1^2 + z_1 z_2 + z_2^2}{3},$$

so that  $E(g)E(g) \neq E(g^2)$ .

Let us now turn to the relationship between the operator  $E$  and Hankel forms. To fix the notation, let  $Y$  be a Hilbert space with an orthonormal basis  $\{e_j\}_{j \geq 1}$ . For a bilinear form  $H: Y \times Y \rightarrow \mathbb{C}$ , let  $s_n(H)$  denote its  $n$ th singular value, i.e.,

$$s_n(H) = \inf\{\|H - K\|_{Y \times Y} : \text{rank } K \leq n\},$$

where the rank of a bilinear form  $K: Y \times Y \rightarrow \mathbb{C}$  is given by

$$\text{rank } K = \text{codim } \ker K = \text{codim}\{f \in Y : K(f, g) = 0 \text{ for all } g \in Y\}.$$

Of course,  $s_n(H)$  is the same as the  $n$ th singular value of the operator

$$\{H(e_j, e_k)\}_{j,k \geq 1} : \ell^2 \rightarrow \ell^2.$$

The  $p$ -Schatten norm of  $H$ ,  $0 < p < \infty$ , is given by

$$\|H\|_{S_p(Y \times Y)}^p = \sum_{n=0}^{\infty} |s_n(H)|^p.$$

When  $p = 2$  we obtain the Hilbert-Schmidt norm, which can also be computed as the square sum of the coefficients,

$$\|H\|_{S_2(Y \times Y)}^2 = \sum_{n=0}^{\infty} |s_n(H)|^2 = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |H(e_j, e_k)|^2.$$

We have the following result.

**Lemma 19.** *Suppose that  $\varphi \in A^2(\mathbb{D})$ . Then*

$$s_n(H_\varphi) = s_n(H_{E\varphi}), \quad n \geq 0.$$

*In particular, for  $0 < p < \infty$  we have*

$$\begin{aligned} \|H_\varphi\|_{A^2(\mathbb{D}) \times A^2(\mathbb{D})} &= \|H_{E\varphi}\|_{H^2(\mathbb{D}^2) \times H^2(\mathbb{D}^2)}, \\ \|H_\varphi\|_{S_p(A^2(\mathbb{D}) \times A^2(\mathbb{D}))} &= \|H_{E\varphi}\|_{S_p(H^2(\mathbb{D}^2) \times H^2(\mathbb{D}^2))}. \end{aligned}$$

*Proof.* Let  $J : X \times X \rightarrow \mathbb{C}$  be the restriction of  $H_{E\varphi}$  to  $X = EA^2(\mathbb{D})$ ,

$$J(f, g) = \langle fg, E\varphi \rangle_{H^2(\mathbb{D}^2)}, \quad f, g \in X.$$

For  $f, g \in H^2(\mathbb{D}^2)$  we have the identity

$$(27) \quad \langle fg, E\varphi \rangle_{H^2(\mathbb{D}^2)} = \langle D(fg), \varphi \rangle_{A^2(\mathbb{D})} = \langle DfDg, \varphi \rangle_{A^2(\mathbb{D})}.$$

Since  $D : X \rightarrow A^2(\mathbb{D})$  is unitary, this implies that  $J$  is unitarily equivalent to  $H_\varphi : A^2(\mathbb{D}) \times A^2(\mathbb{D}) \rightarrow \mathbb{C}$ . If  $K : H^2(\mathbb{D}^2) \times H^2(\mathbb{D}^2) \rightarrow \mathbb{C}$  is a rank- $n$  form, then its restriction to  $X$ ,  $K' : X \times X \rightarrow \mathbb{C}$ , has smaller rank,  $\text{rank } K' \leq n$ . Since

$$\|H_{E\varphi} - K\|_{H^2(\mathbb{D}^2) \times H^2(\mathbb{D}^2)} \geq \|J - K'\|_{X \times X}$$

it follows that

$$s_n(H_{E\varphi}) \geq s_n(J) = s_n(H_\varphi), \quad n \geq 0.$$

Conversely, if the form  $K : A^2(\mathbb{D}) \times A^2(\mathbb{D}) \rightarrow \mathbb{C}$  has rank  $n$ , then clearly  $K' : H^2(\mathbb{D}^2) \times H^2(\mathbb{D}^2) \rightarrow \mathbb{C}$  has smaller rank, where  $K'(f, g) = K(Df, Dg)$ , for  $f, g \in H^2(\mathbb{D}^2)$ . However, it follows from (27) and Lemma 18 that

$$\|H_\varphi - K\| = \|H_{E\varphi} - K'\|,$$

proving that also  $s_n(H_\varphi) \geq s_n(H_{E\varphi})$ . □

Consider  $A^2(\mathbb{D}^\infty)$  as a function space over the variables  $z = (z_1, z_2, \dots)$  and  $H^2(\mathbb{D}^\infty)$  as a function space over  $\xi = (\xi_1, \xi_2, \dots)$ . Define the extension map  $\mathcal{E}$  from  $A^2(\mathbb{D}^\infty)$  to  $H^2(\mathbb{D}^\infty)$  by its integral kernel,

$$K_\xi(z) = \prod_{j=1}^{\infty} \frac{1}{1 - \xi_{2j-1} \bar{z}_j} \frac{1}{1 - \xi_{2j} z_j}, \quad z, \xi \in \mathbb{D}^\infty \cap \ell^2,$$

so that

$$\mathcal{E}f(\xi) = \int_{\mathbb{D}^\infty} f(z) K_\xi(z) \, d\mathbf{m}(z).$$

By tensorization of Lemma 19 (the required technical details may be found in [9, Lem. 2]), we obtain the following.

**Theorem 20.** *The map  $\mathcal{E}$  has the following properties:*

- (a)  $\mathcal{E}$  defines an isometric isomorphism from the Bergman space  $A^2(\mathbb{D}^\infty)$  to a subspace of the Hardy space  $H^2(\mathbb{D}^\infty)$ .

- (b) For  $\varphi \in A^2(\mathbb{D}^\infty)$ , let  $H_\varphi : A^2(\mathbb{D}^\infty) \times A^2(\mathbb{D}^\infty) \rightarrow \mathbb{C}$  be the Hankel form generated by  $\varphi$ , and let  $H_{\mathcal{E}\varphi} : H^2(\mathbb{D}^\infty) \times H^2(\mathbb{D}^\infty) \rightarrow \mathbb{C}$  be the Hankel form generated by  $\mathcal{E}\varphi$ . Then, for every  $n \geq 0$ , we have that

$$s_n(H_\varphi) = s_n(H_{\mathcal{E}\varphi}).$$

In particular,  $H_\varphi$  is bounded ( $p$ -Schatten,  $0 < p < \infty$ ) if and only if  $H_{\mathcal{E}\varphi}$  is bounded ( $p$ -Schatten), with equality of the norms.

*Remark.* In [35], the symbol  $\psi(z) = (z_1 + z_2)/2$  is used to show that the weak factorization  $H^1(\mathbb{D}^\infty) = H^2(\mathbb{D}^\infty) \odot H^2(\mathbb{D}^\infty)$  cannot hold. In Theorem 15 the symbol  $\varphi(w) = w$  is used to demonstrate the corresponding fact for the Bergman spaces. In fact the two examples considered are the same, because  $E\varphi = \psi$ .

### 5. CARLESON MEASURES ON THE INFINITE POLYDISC

We end this paper by producing two infinite dimensional counterexamples to well-known finite dimensional results for Carleson measures for the Hardy spaces  $H^p(\mathbb{D}^d)$ . Let  $\mu$  be a finite positive measure on  $\mathbb{D}^d$  (where possibly  $d = \infty$ ), i.e., a finite positive Borel measure on  $\overline{\mathbb{D}^d}$  such that  $\mu(\overline{\mathbb{D}^d} \setminus \mathbb{D}^d) = 0$ . As usual, measures on the compact space  $\overline{\mathbb{D}^d}$  correspond to linear functionals on the space of continuous functions  $C(\overline{\mathbb{D}^d})$ . We say that  $\mu$  is an  $H^p$ -Carleson measure if there exists a constant  $C = C(\mu_d, p) < \infty$  such that

$$\int_{\mathbb{D}^d} |f(z)|^p d\mu_d(z) \leq C \|f\|_{H^p(\mathbb{D}^d)}^p$$

for every analytic polynomial  $f$ . We say that  $\mu$  is an  $L^p$ -Carleson measure if there exists a constant  $C = C(\mu_d, p) < \infty$  such that

$$\int_{\mathbb{D}^d} |\mathcal{P}f(z)|^p d\mu_d(z) \leq C \|f\|_{L^p(\mathbb{T}^d)}^p$$

for every trigonometric polynomial  $f$ . Here  $\mathcal{P}f$  is the Poisson extension of  $f$ , defined for  $f \in L^p(\mathbb{T}^d)$  by

$$\mathcal{P}f(w) = \int_{\mathbb{T}^d} f(z) \mathcal{P}_w(z) d\mathbf{m}_1(z), \quad \mathcal{P}_w(z) = \prod_{j=1}^d \frac{1 - |w_j|^2}{|1 - \overline{z_j} w_j|^2}.$$

This is always well defined as long as we restrict ourselves to  $L^2(\mathbb{T}^d)$ -functions  $f$  only dependent on a finite number of variables, since we may then suppose that  $w$  is finitely supported.

The study of Carleson measures on the infinite polydisc is an important part of the theory of  $H^p$  spaces. For instance, the local embedding problem discussed in [41, Sec. 3] can be formulated in terms of Carleson measures. Let  $\mathcal{B}^{-1}$  denote the inverse Bohr lift, so that

$$(\mathcal{B}^{-1}f)(s) = f(2^{-s}, 3^{-s}, 5^{-s}, \dots, p_j^{-s}, \dots).$$

For  $0 < p < \infty$ , is it true that the measure  $\mu_\infty$  defined on  $\mathbb{D}^\infty$  by

$$\int f(z) d\mu_\infty(z) = \int_0^1 (\mathcal{B}^{-1}f)(1/2 + it) dt, \quad f \in C(\overline{\mathbb{D}^d}),$$

is an  $H^p$ -Carleson measure? A positive answer is only known for even integers. Additionally, the boundedness of positive definite Hankel forms (1) can be formulated

in terms of Carleson measures on  $\mathbb{D}^\infty$  [36], and the same is true for the Volterra operators studied in [10].

From [15], it is known that a measure  $\mu$  on  $\mathbb{D}^d$ , for  $d < \infty$ , is an  $H^p$ -Carleson measure for one  $0 < p < \infty$  if and only if it is a Carleson measure for every  $0 < p < \infty$ . We will now construct a counterexample to this statement when  $d = \infty$ . We recall that the diagonal restriction operator  $Df(w) = f(w, w)$  induces a bounded map from  $H^p(\mathbb{D}^2)$  to  $A^p(\mathbb{D})$  for every  $0 < p < \infty$  (see [19]), and offer the following clarification in the case  $0 < p < 2$ .

**Lemma 21.** *The diagonal operator  $D$  is not contractive from  $H^p(\mathbb{D}^2)$  to  $A^p(\mathbb{D})$  when  $0 < p < 2$ .*

*Proof.* Let  $0 < p < 2$  and consider  $f(z_1, z_2) = (z_1 + z_2)/2$ . Clearly

$$\|Df\|_{A^p(\mathbb{D})}^p = \int_{\mathbb{D}} |f(w, w)|^p dm(w) = \frac{2}{2+p},$$

so it is enough to verify that  $\|f\|_{H^p(\mathbb{D}^2)}^p < 2/(2+p)$ . We factor out  $z_2$  and compute using various identities for the Beta and Gamma functions, obtaining that

$$\begin{aligned} \|f\|_{H^p(\mathbb{D}^2)}^p &= \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1+e^{i\theta}}{2} \right|^p d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left| \cos \frac{\theta}{2} \right|^p d\theta \\ &= \frac{1}{\pi} \int_0^\pi \left( \cos \frac{\theta}{2} \right)^p d\theta = \frac{2}{\pi} \int_0^1 \frac{t^p}{\sqrt{1-t^2}} dt \\ &= \frac{1}{\pi} \int_0^1 t^{(p-1)/2} (1-t)^{-1/2} dt = \frac{B((p+1)/2, 1/2)}{\pi} \\ &= \frac{\Gamma(p/2 + 1/2)\Gamma(1/2)}{\pi\Gamma(p/2 + 1)} = \frac{\Gamma(p/2 + 1/2)}{\Gamma(1/2)(p/2)\Gamma(p/2)} = \frac{2}{pB(p/2, 1/2)}. \end{aligned}$$

To conclude we make use of the identity

$$B(x, y) = \sum_{n=0}^\infty \binom{n-y}{n} \frac{1}{x+n}, \quad x, y > 0.$$

The binomial coefficient is positive for every  $n$  when  $y = 1/2$ , so if  $0 < p < 2$  we have that

$$B(p/2, 1/2) > \frac{1}{p/2} + B(1, 1/2) - \frac{1}{1} = \frac{2}{p} + 1. \quad \square$$

*Remark.* Lemma 18 implies that  $D$  is a contraction from  $H^p(\mathbb{D}^2)$  to  $A^p(\mathbb{D})$  if  $p$  is an even integer. It would be interesting to know if  $D$  is a contraction for every  $p \geq 2$ .

Tensorization of Lemma 18 and Lemma 21 yields the following result.

**Theorem 22.** *Let  $\mu_\infty$  be the measure defined for  $f$  in  $C(\overline{\mathbb{D}^\infty})$  by*

$$(28) \quad \int_{\mathbb{D}^\infty} f(z_1, z_2, z_3, z_4, \dots) d\mu_\infty(z) = \int_{\mathbb{D}^\infty} f(z_1, z_1, z_3, z_3, \dots) d\mathbf{m}(z),$$

where  $\mathbf{m}$  denotes the infinite product of the unweighted normalized Lebesgue measure on  $\mathbb{D}$ . The measure  $\mu_\infty$  is an  $H^p$ -Carleson measure on  $\mathbb{D}^\infty$  if  $p$  is an even integer, but not when  $0 < p < 2$ .

Theorem 22 invites the following question.

**Question.** If  $\mu$  defines an  $H^p$ -Carleson measure on  $\mathbb{D}^\infty$  for some  $0 < p < \infty$ , does it also define an  $H^q$ -Carleson measure for every  $p < q < \infty$ ?

In [15], it is also proven that  $L^p$ -Carleson and  $H^p$ -Carleson measures coincide on  $\mathbb{D}^d$ , when  $d < \infty$ . Again, this is no longer true on  $\mathbb{D}^\infty$ , as our next two examples will demonstrate.

To obtain the first counterexample, we verify that the measure (28) of Theorem 22 does not define an  $L^2$ -Carleson measure on  $\mathbb{D}^\infty$  by replacing Lemma 21 with the following result.

**Lemma 23.** *The operator  $D \circ \mathcal{P}$  is not a contraction from  $L^2(\mathbb{T}^2)$  to  $L^2(\mathbb{D}, m)$ .*

*Proof.* Consider

$$f(e^{i\theta_1}, e^{i\theta_2}) = \frac{1}{\sqrt{3}} (e^{i\theta_1} + e^{i\theta_2} + e^{i2\theta_1}e^{-i\theta_2}),$$

for which clearly  $\|f\|_{L^2(\mathbb{T}^2)} = 1$ . Furthermore, we find that

$$\mathcal{P}f(re^{i\theta}, re^{i\theta}) = \frac{e^{i\theta}}{\sqrt{3}}(2r + r^3),$$

so it follows that

$$\int_{\mathbb{D}} |\mathcal{P}f(z, z)|^2 dm(z) = \frac{2}{3} \int_0^1 (2r + r^3)^2 r dr = \frac{43}{36} > 1. \quad \square$$

Our second counterexample is obtained through the connection with Dirichlet series. In preparation, let us recall a few properties of  $L^2(\mathbb{T}^\infty)$ . Let  $\mathbb{Q}_+$  denote the set of positive rational numbers. Each  $q \in \mathbb{Q}_+$  has a finite expansion of the form

$$q = \prod_{j=1}^{\infty} p_j^{\kappa_j},$$

where  $\kappa_j \in \mathbb{Z}$ . Hence  $\mathbb{Q}_+$  can be identified with the set of all finite multi-indices. As in (18), every function  $f \in L^2(\mathbb{T}^\infty)$  has an expansion

$$f(z) = \sum_{q \in \mathbb{Q}_+} a_q z^{\kappa(q)}, \quad \|f\|_{L^2(\mathbb{T}^\infty)}^2 = \sum_{q \in \mathbb{Q}_+} |a_q|^2.$$

Note that if  $f \in L^2(\mathbb{T}^{d'})$  for some  $d' < \infty$  and  $s = \sigma + it$ , then

$$(\mathcal{B}^{-1} \mathcal{P}f)(s) = \sum_{q \in \mathbb{Q}_+} a_q (q_+)^{-\sigma} q^{-it}, \quad \text{where} \quad q_+ = \prod_{j=1}^{\infty} p_j^{|\kappa_j|}.$$

As our final preliminary, let  $\omega(n)$  denote the number of distinct prime factors of  $n$ . It is well known that if  $\text{Re}(s) > 1$ , then

$$\frac{[\zeta(s)]^2}{\zeta(2s)} = \prod_{j=1}^{\infty} \frac{1 + p_j^{-s}}{1 - p_j^{-s}} = \sum_{n=1}^{\infty} 2^{\omega(n)} n^{-s}.$$

**Theorem 24.** *Let  $\mu_\infty$  be the measure defined for  $f$  in  $C(\overline{\mathbb{D}^\infty})$  by*

$$\int_{\mathbb{D}^\infty} f(z) d\mu_\infty(z) = \int_0^1 (\mathcal{B}^{-1}f)(1/2 + \sigma) d\sigma.$$

*Then  $\mu_\infty$  is an  $H^2$ -Carleson measure but not an  $L^2$ -Carleson measure.*



*Proof.* It is well known that  $\mu_\infty$  is an  $H^2$ -Carleson measure [11, 34]. Let us therefore prove that  $\mu_\infty$  is not an  $L^2$ -Carleson measure. Fix  $\varepsilon > 0$  and define  $w \in \mathbb{D}^\infty \cap \ell^2$  by  $w_j = p_j^{-1/2-\varepsilon}$ . We will consider the kernel of the  $d$ -dimensional Poisson transform,

$$f_d(z) = \prod_{j=1}^d \frac{1 - |w_j|^2}{|1 - \bar{z}_j w_j|^2}.$$

First observe that

$$\lim_{d \rightarrow \infty} \|f_d\|_{L^2(\mathbb{T}^\infty)}^2 = \prod_{j=1}^{\infty} \frac{1 - p_j^{-2-4\varepsilon}}{(1 - p_j^{-1-2\varepsilon})^2} = \frac{[\zeta(1+2\varepsilon)]^2}{\zeta(2+4\varepsilon)} \simeq \varepsilon^{-2}.$$

Next, we have that

$$\lim_{d \rightarrow \infty} (\mathcal{B}^{-1} \mathcal{P} f_d)(1/2 + \sigma) = \sum_{q \in \mathbb{Q}_+} q_+^{-1-\varepsilon-\sigma},$$

uniformly convergent in  $\sigma \in [0, 1]$ . Note that

$$\sum_{q \in \mathbb{Q}_+} q_+^{-1-\varepsilon-\sigma} = \sum_{n=1}^{\infty} 2^{\omega(n)} n^{-1-\varepsilon-\sigma} \simeq (\sigma + \varepsilon)^{-2},$$

since there are  $2^{\omega(n)}$  rational numbers  $q \in \mathbb{Q}_+$  such that  $q_+ = n$ . This concludes the argument, since

$$\int_0^1 \frac{d\sigma}{(\sigma + \varepsilon)^4} \simeq \varepsilon^{-3}. \quad \square$$

## REFERENCES

- [1] Mats Andersson, *Solution formulas for the  $\partial\bar{\partial}$ -equation and weighted Nevanlinna classes in the polydisc* (English, with French summary), *Bull. Sci. Math.* (2) **109** (1985), no. 2, 135–154. MR802530
- [2] N. Aronszajn, *Theory of reproducing kernels*, *Trans. Amer. Math. Soc.* **68** (1950), 337–404. MR0051437
- [3] Maxime Bailleul and Pascal Lefèvre, *Some Banach spaces of Dirichlet series*, *Studia Math.* **226** (2015), no. 1, 17–55. MR3322601
- [4] Jogia Bandyopadhyay, *Optimal concentration for  $SU(1, 1)$  coherent state transforms and an analogue of the Lieb-Wehrl conjecture for  $SU(1, 1)$* , *Comm. Math. Phys.* **285** (2009), no. 3, 1065–1086. MR2470916
- [5] Frédéric Bayart, *Hardy spaces of Dirichlet series and their composition operators*, *Monatsh. Math.* **136** (2002), no. 3, 203–236. MR1919645
- [6] William Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, *Ann. of Math.* (2) **138** (1993), no. 1, 213–242. MR1230930
- [7] Bernhard G. Bodmann, *A lower bound for the Wehrl entropy of quantum spin with sharp high-spin asymptotics*, *Comm. Math. Phys.* **250** (2004), no. 2, 287–300. MR2094518
- [8] Andriy Bondarenko, Winston Heap, and Kristian Seip, *An inequality of Hardy-Littlewood type for Dirichlet polynomials*, *J. Number Theory* **150** (2015), 191–205. MR3304613
- [9] Ole Fredrik Brevig and Karl-Mikael Perfekt, *Failure of Nehari's theorem for multiplicative Hankel forms in Schatten classes*, *Studia Math.* **228** (2015), no. 2, 101–108. MR3422087
- [10] Ole Fredrik Brevig, Karl-Mikael Perfekt, and Kristian Seip, *Volterra operators on Hardy spaces of Dirichlet series*, to appear in *J. Reine Angew. Math.* (doi:10.1515/crelle-2016-0069).
- [11] Ole Fredrik Brevig, Karl-Mikael Perfekt, Kristian Seip, Aristomenis G. Siskakis, and Dragan Vukotić, *The multiplicative Hilbert matrix*, *Adv. Math.* **302** (2016), 410–432. MR3545935
- [12] Jacob Burbea, *Sharp inequalities for holomorphic functions*, *Illinois J. Math.* **31** (1987), no. 2, 248–264. MR882113
- [13] Torsten Carleman, *Zur Theorie der Minimalflächen* (German), *Math. Z.* **9** (1921), no. 1-2, 154–160. MR1544458

- [14] Eric A. Carlen, *Some integral identities and inequalities for entire functions and their application to the coherent state transform*, J. Funct. Anal. **97** (1991), no. 1, 231–249. MR1105661
- [15] Sun-Yung A. Chang, *Carleson measure on the bi-disc*, Ann. of Math. (2) **109** (1979), no. 3, 613–620. MR534766
- [16] Brian J. Cole and T. W. Gamelin, *Representing measures and Hardy spaces for the infinite polydisk algebra*, Proc. London Math. Soc. (3) **53** (1986), no. 1, 112–142. MR842158
- [17] Olivia Constantin, *Weak product decompositions and Hankel operators on vector-valued Bergman spaces*, J. Operator Theory **59** (2008), no. 1, 157–178. MR2404468
- [18] Peter Duren and Alexander Schuster, *Bergman spaces*, Mathematical Surveys and Monographs, vol. 100, American Mathematical Society, Providence, RI, 2004. MR2033762
- [19] P. L. Duren and A. L. Shields, *Restrictions of  $H^p$  functions to the diagonal of the polydisc*, Duke Math. J. **42** (1975), no. 4, 751–753. MR0402101
- [20] Sarah H. Ferguson and Michael T. Lacey, *A characterization of product BMO by commutators*, Acta Math. **189** (2002), no. 2, 143–160. MR1961195
- [21] Leonard Gross, *Strong hypercontractivity and relative subharmonicity*, J. Funct. Anal. **190** (2002), no. 1, 38–92. Special issue dedicated to the memory of I. E. Segal. MR1895529
- [22] Haakan Hedenmalm, Boris Korenblum, and Kehe Zhu, *Theory of Bergman spaces*, Graduate Texts in Mathematics, vol. 199, Springer-Verlag, New York, 2000. MR1758653
- [23] Håkan Hedenmalm, Peter Lindqvist, and Kristian Seip, *A Hilbert space of Dirichlet series and systems of dilated functions in  $L^2(0, 1)$* , Duke Math. J. **86** (1997), no. 1, 1–37. MR1427844
- [24] Henry Helson, *Dirichlet series*, Henry Helson, Berkeley, CA, 2005. MR2116531
- [25] Henry Helson, *Hankel forms and sums of random variables*, Studia Math. **176** (2006), no. 1, 85–92. MR2263964
- [26] Henry Helson, *Hankel forms*, Studia Math. **198** (2010), no. 1, 79–84. MR2640082
- [27] Charles Horowitz, *Factorization theorems for functions in the Bergman spaces*, Duke Math. J. **44** (1977), no. 1, 201–213. MR0427650
- [28] Svante Janson, *On hypercontractivity for multipliers on orthogonal polynomials*, Ark. Mat. **21** (1983), no. 1, 97–110. MR706641
- [29] Michael Lacey and Erin Terwilleger, *Hankel operators in several complex variables and product BMO*, Houston J. Math. **35** (2009), no. 1, 159–183. MR2491875
- [30] Giovanni Leoni, *A first course in Sobolev spaces*, Graduate Studies in Mathematics, vol. 105, American Mathematical Society, Providence, RI, 2009. MR2527916
- [31] Gianni Mancini and Kunnath Sandeep, *On a semilinear elliptic equation in  $\mathbb{H}^n$* , Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **7** (2008), no. 4, 635–671. MR2483639
- [32] Zeev Nehari, *On bounded bilinear forms*, Ann. of Math. (2) **65** (1957), 153–162. MR0082945
- [33] Jan-Fredrik Olsen, *Local properties of Hilbert spaces of Dirichlet series*, J. Funct. Anal. **261** (2011), no. 9, 2669–2696. MR2826411
- [34] Jan-Fredrik Olsen and Eero Saksman, *On the boundary behaviour of the Hardy spaces of Dirichlet series and a frame bound estimate*, J. Reine Angew. Math. **663** (2012), 33–66. MR2889706
- [35] Joaquim Ortega-Cerdà and Kristian Seip, *A lower bound in Nehari’s theorem on the polydisc*, J. Anal. Math. **118** (2012), no. 1, 339–342. MR2993031
- [36] Karl-Mikael Perfekt and Alexander Pushnitski, *On Helson matrices: moment problems, non-negativity, boundedness, and finite rank*, Proc. Lond. Math. Soc. (3) **116** (2018), no. 1, 101–134, DOI 10.1112/plms.12068. MR3747045
- [37] Hervé Queffélec, *Espaces de séries de Dirichlet et leurs opérateurs de composition* (French, with French summary), Ann. Math. Blaise Pascal **22** (2015), no. S2, 267–344. MR3453287
- [38] Hervé Queffélec and Martine Queffélec, *Diophantine approximation and Dirichlet series*, Harish-Chandra Research Institute Lecture Notes, vol. 2, Hindustan Book Agency, New Delhi, 2013. MR3099268
- [39] Walter Rudin, *Function theory in polydiscs*, W. A. Benjamin, Inc., New York-Amsterdam, 1969. MR0255841
- [40] Walter Rudin, *Function theory in the unit ball of  $\mathbb{C}^n$* , Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 241, Springer-Verlag, New York-Berlin, 1980. MR601594
- [41] Eero Saksman and Kristian Seip, *Integral means and boundary limits of Dirichlet series*, Bull. Lond. Math. Soc. **41** (2009), no. 3, 411–422. MR2506825

- [42] Gérald Tenenbaum, *Introduction à la théorie analytique et probabiliste des nombres* (French), 2nd ed., Cours Spécialisés [Specialized Courses], vol. 1, Société Mathématique de France, Paris, 1995. MR1366197
- [43] Dragan Vukotić, *The isoperimetric inequality and a theorem of Hardy and Littlewood*, Amer. Math. Monthly **110** (2003), no. 6, 532–536. MR1984405
- [44] Fred B. Weissler, *Logarithmic Sobolev inequalities and hypercontractive estimates on the circle*, J. Funct. Anal. **37** (1980), no. 2, 218–234. MR578933
- [45] Kehe Zhu, *Spaces of holomorphic functions in the unit ball*, Graduate Texts in Mathematics, vol. 226, Springer-Verlag, New York, 2005. MR2115155

UNIVERSITÉ CLERMONT AUVERGNE, CNRS, LMBP, F-63000 CLERMONT-FERRAND, FRANCE  
*Email address:* bayart@math.univ-bpclermont.fr

DEPARTMENT OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY (NTNU), NO-7491 TRONDHEIM, NORWAY  
*Email address:* ole.brevig@math.ntnu.no

FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, OSKAR-MORGENSTERN-PLATZ 1, 1090 WIEN, AUSTRIA  
*Email address:* antti.haimi@univie.ac.at

DEPARTAMENT DE MATEMÀTIQUES I INFORMÀTICA, UNIVERSITAT DE BARCELONA & BARCELONA GRADUATE SCHOOL IN MATHEMATICS, GRAN VIA 585, 08007 BARCELONA, SPAIN  
*Email address:* jortega@ub.edu

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37996  
*Email address:* karlmikael.perfekt@gmail.com