Deriving the Hard Thermal Loops of QCD from Classical Transport Theory

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Classical transport theory is employed to analyze the hot quark-gluon plasma at the leading order in the coupling constant. A condition on the (covariantly conserved) color current is obtained. From this condition, the generating functional of hard thermal loops with an arbitrary number of soft external bosonic legs can be derived. Our approach, besides being more direct than alternative ones, shows that hard thermal loops are essentially classical.

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It is the purpose of the present Letter to derive the hard thermal loops (HTL's) of QCD from classical transport theory. The generating functional of HTL's (with an arbitrary number of soft external bosonic legs) arises as a leading order effect in this formalism.

In the diagrammatic approach to high-temperature QCD it was realized that a resummation procedure is necessary in order to take into account consistently all contributions at leading order in the coupling constant [1]. Such contributions arise from one-loop diagrams with external soft legs and hard internal momenta, the so-called "hard thermal loops." The HTL approach provides gauge invariant results for physical quantities. An effective action for HTL's was given in [2] by exploring the gauge invariance condition for the generating functional. This condition has been identified with the equations of motion for the topological Chern-Simons theory at zero temperature [3]. Furthermore, the eikonal for a Chern-Simons theory has been used to obtain a non-Abelian generalization of the Kubo formula [4], which governs, through the current induced by HTL's, the response of a hot quark-gluon plasma.

Another successful description of the hard thermal loops in QCD is based on (a truncation of) the hierarchy of Schwinger-Dyson equations [5]; the generating functional for HTL's is obtained by performing a consistent expansion in the coupling constant. Alternatively, one can start from the composite effective action [6] by requiring its stationarity [7], and by using the approximation scheme developed in [5]. The gauge invariance condition for the generating functional can then be obtained as a consequence of symmetry properties due to the form of the momentum integration measure (see [7]).

The diagrammatic techniques of [1], as well as the consistent expansion in the coupling constant presented in [5], although remarkably insightful, are technically very involved and necessitate lengthy computations. Furthermore, they are puzzling in respect to the very nature of hard thermal loops. One wonders to what extent these are quantum effects: on the one hand, HTL's emerge in loop diagrams and are described by the Schwinger-Dyson equations of quantum field theory. On the other hand, it is generally believed that hard thermal effects are UV finite since they are due exclusively to *thermal* scattering processes.

This motivates us to find an alternative, *classical* formalism for hard thermal loops in QCD. The obvious setting for such a search is the classical transport theory of plasmas (see, for instance, [8]). Our effort was encouraged by the fact that for an Abelian plasma of electrons and ions, the dielectric tensor that can be computed [9] from classical transport theory is the same as the one that can be extracted from the hard thermal corrections to the vacuum polarization tensor [1,4]. Moreover, the same situation is encountered for non-Abelian plasmas [10,11]. The transport theory for colored plasmas has been established in [10]. There has not been, to the best of our knowledge, any attempt at deriving the complete set of HTL's from this formalism. It is our purpose to do so, for the HTL's with an arbitrary number of external soft bosonic legs.

Consider a particle bearing a non-Abelian SU(N) color charge Q^a, $a = 1,\ldots,N^2-1$, traversing a world line $x^\alpha(\tau)$, where τ denotes the proper time. The dynamical effects of the spin of the particle shall be ignored, as they are typically small. The Wong equations [12] describe the dynamical evolution of x^μ, p^μ, and Q^a:

$$m\,\frac{dx^\mu}{d\tau} = p^\mu \ , \tag{1a}$$

$$m\,\frac{dp^\mu}{d\tau} = g\,Q^a F_a^{\mu\nu} p_\nu \ , \tag{1b}$$

$$m\,\frac{dQ^a}{d\tau} = -g\,f^{abc} p^\mu A_\mu^b Q^c \ . \tag{1c}$$

The f^{abc} are the structure constants of the group, $F_a^{\mu\nu}$ denotes the field strength, g is the coupling constant, and we set $c = \hbar = k_B = 1$ henceforth. Equation (1b) is the non-Abelian generalization of the Lorentz force law, and Eq. (1c) describes the precession in color space of the charge in an external color field A_μ^a. It is noteworthy that the color charge is itself subject to dynamical evolution, a feature which distinguishes the non-Abelian theory from electromagnetism.

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The usual (x,p) phase space is enlarged by including color degrees of freedom for colored particles. Physical constraints are enforced by the choice of the volume element $dx\,dP\,dQ$. The momentum measure

$$dP = \frac{d^4p}{(2\pi)^3}\, 2\,\theta(p_0)\, \delta(p^2 - m^2) \tag{2}$$

guarantees on-shell evolution and positivity of the energy. The color charge measure enforces the conservation of the group invariants, e.g., for SU(3),

$$dQ = d^8Q\, \delta(Q_aQ^a - q_2)\, \delta(d_{abc}Q^aQ^bQ^c - q_3)\ , \tag{3}$$

where the constants q_2 and q_3 fix the values of the Casimirs.

The one-particle distribution function $f(x,p,Q)$ is the classical probability density for finding the particle in the state $\{x,p,Q\}$. This probability density evolves in time via a transport equation,

$$m\,\frac{df(x,p,Q)}{d\tau} = C[f](x,p,Q)\ , \tag{4}$$

where $C[f](x,p,Q)$ denotes the collision integral, which we henceforth set to zero. Using the equations of motion (1), in the collisionless case, (4) becomes the Boltzmann equation,

$$p^\mu \left[\frac{\partial}{\partial x^\mu} - g\,Q_aF^a_{\mu\nu}\frac{\partial}{\partial p_\nu} - g\,f_{abc}A^b_\mu Q^c\frac{\partial}{\partial Q_a}\right] f(x,p,Q) = 0\ . \tag{5}$$

A self-consistent set of (non-Abelian Vlasov) equations for the distribution function and the mean color field is obtained by augmenting the Boltzmann equation with the Yang-Mills equations,

$$[D_\nu F^{\nu\mu}]^a(x) = J^{\mu\, a}(x)\ . \tag{6}$$

The covariant derivative is defined as $D^{ac}_\mu = \partial_\mu\delta^{ac} + g\,f^{abc}A^b_\mu$. The total color current $J^{\mu\, a}(x)$ is given by the sum of all contributions from particle species and helicities,

$$J^{\mu\, a}(x) = \sum_{\text{species}}\sum_{\text{helicities}} j^{\mu\, a}(x)\ . \tag{7}$$

Each $j^{\mu\, a}(x)$ (spin and species indices are implicit) is computed from the corresponding distribution function,

$$j^{\mu\, a}(x) = g\int dPdQ\; p^\mu Q^a f(x,p,Q)\ , \tag{8}$$

and it is covariantly conserved,

$$\left(D_\mu j^\mu\right)^a(x) = 0\ . \tag{9}$$

This can be checked by using the Boltzmann equation with the aid of a Jacobi-like identity relating f_{abc} and d_{def}. For later convenience, we define the total and individual current momentum densities,

$$J^{\mu\, a}(x,p) = \sum_{\text{species}}\sum_{\text{helicities}} j^{\mu\, a}(x,p)\ ,$$

$$j^{\mu\, a}(x,p) = g\int dQ\; p^\mu Q^a f(x,p,Q)\ . \tag{10}$$

The above framework is now employed to study the soft excitations in a hot, color-neutral quark-gluon plasma. In the high-temperature limit, the masses of the particles can be neglected and shall henceforth be taken to vanish. The wavelength of the soft excitations is of order $1/g\,|A|$, and the coupling constant g is assumed to be small. We then expand the distribution function $f(x,p,Q)$ in powers of g,

$$f = f^{(0)} + gf^{(1)} + g^2f^{(2)} + \cdots\ , \tag{11}$$

where $f^{(0)}$ is the equilibrium distribution function in the absence of a net color field, and is given by

$$f^{(0)}(p_0) = Cn_{B,F}(p_0)\ . \tag{12}$$

Here C is a normalization constant and $n_{B,F}(p_0) = 1/(e^{\beta|p_0|} \mp 1)$ is the bosonic (fermionic) probability distribution.

At leading order in g, the induced current (10) is

$$j^{\mu\, a}(x,p) = g^2\int dQ\; p^\mu Q^a f^{(1)}(x,p,Q)\ , \tag{13}$$

while the Boltzmann equation (5) reduces to

$$p^\mu\left(\frac{\partial}{\partial x^\mu} - g\,f^{abc}A^b_\mu Q_c\frac{\partial}{\partial Q^a}\right) f^{(1)}(x,p,Q) = p^\mu Q_aF^a_{\mu\nu}\frac{\partial}{\partial p_\nu}f^{(0)}(p_0)\ . \tag{14}$$

Because of the softness of the excitation, the ∂_x term in the above equation is of order $g\,|A|$, so we are taking into account consistently all contributions in order g.

Equations (13) and (14) yield the following constraint on the induced current:

$$[p\cdot D\; j^\mu(x,p)]^a = g^2\,p^\mu p^\nu F^b_{\nu\rho}\frac{\partial}{\partial p_\rho} \times \left(\int dQ\; Q^aQ_bf^{(0)}(p_0)\right)\ , \tag{15}$$

where, from color symmetry, we expect that $\int dQ\,Q^aQ_bf^{(0)}(p_0) = C_{B,F}\; n_{B,F}(p_0)\,\delta^a_b$ with $C_B = N$ and $C_F = \frac{1}{2}$ for gluons and fermions, respectively. Thus, upon summation over all species (N_F quarks, N_F antiquarks, and one $[(N^2-1)$-plet$]$ gluon) and helicities (2 for quarks-antiquarks and massless gluon), (15) yields

$$[p \cdot D\, J^\mu(x,p)]^a = 2\,g^2\, p^\mu p^\nu F^a_{\nu 0}(x) \times \frac{d}{dp_0}[N\, n_B(p_0) + N_F\, n_F(p_0)] \; . \quad (16)$$

This equation is equivalent to the condition (3.13) of [7] for the induced current. [This condition has been previously obtained in [5]. Our notational conventions differ from those of [7]: the $\delta(p^2)$ function appearing in (3.13) there is here contained in the (massless limit of the) momentum measure dP (2).]

The subsequent steps which lead to the gauge invariance condition for HTL's can be found in [7]. Assuming that the induced current can be expressed as the functional derivative of an effective action allows one to derive, from (16), a condition on the latter action. The result can be presented as [Eq. (3.26) in [7]]

$$\partial_+ \frac{\delta W(A_+)}{\delta A_+} + g\left[A_+, \frac{\delta W(A_+)}{\delta A_+}\right] = 4\sqrt{2}\,\pi^3\, m_D^2\, \partial_0 A_+ \; , \quad (17)$$

where $\partial_+ \equiv Q^\mu_+\, \partial_\mu$, $A_+ \equiv Q^\mu_+\, A_\mu$, with $Q^\mu_+ = \frac{1}{\sqrt{2}}(1, \frac{\mathbf{p}}{p_0})$, and $m_D = gT\sqrt{\frac{N+N_F/2}{3}}$ is the Debye screening mass. The effective action generating the HTL's has the form (first derived in [1])

$$\Gamma_{\rm HTL} = \frac{m_D^2}{2}\int d^4x\, A_0^a(x) A_0^a(x) - \int \frac{d\Omega}{(2\pi)^4}\, W(A_+) \; , \quad (18)$$

where $d\Omega$ denotes integration over all angular directions of the unit vector $\frac{\mathbf{p}}{p_0}$. The gauge invariance condition (17) has been solved for W [2,3], thereby obtaining a closed expression for the generating functional $\Gamma_{\rm HTL}$. This concludes our derivation of hard thermal loops from classical transport theory.

Let us now summarize and discuss our results. The classical transport theory for a hot collisionless plasma of colored particles is a well-suited formalism for studying the hot quark-gluon plasma. The recently proposed HTL's arise naturally in this formalism. The spin-statistics theorem is the only "quantum mechanical" input we used in our analysis. Its use is entirely justified by the conditions of temperature and density which prevail in a hot quark-gluon plasma. With respect to alternative descriptions our approach is much simpler. It also demonstrates explicitly that the HTL's of QCD are a classical effect.

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