Master Thesis

Master in Advanced Mathematics

Faculty of Mathematics and Computer Science
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Shannon wavelets inverse Fourier technique for computacional finance

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Abstract

European options are financial derivatives, governed by the solution of an integral, the so-called discounted expectation of the pay-off function. For the computation of the expectation we require knowledge about the probability density function of the stochastic asset price process, which is typically available by its Fourier transform. In this project, we will explore wavelets theory to be able to construct the Shannon wavelets and use them to describe the density function. Also, a numerical method proposed by Luis Ortiz-Gracia and Cornelis W. Oosterlee to price these derivatives will be presented. This is called SWIFT (Shannon wavelet inverse Fourier technique).

Contents

1 Introduction.

Financial derivatives are contracts whose price depends on the value of an underlying asset. This underlying asset refers for example to a stock, a foreign exchange or even a derivative in itself. Derivatives are useful for traders to be prepared for a bullish or a bearish period. There are several kinds of derivatives and we will focus on one in particular that is called European derivative.

A European derivative is a contract that gives to the buyer the right, but not the obligation, to buy or sell an underlying asset for a fixed price K (strike) in a certain date T (maturity). If we decide to buy we will have a call option and if we choose to sell we will get a put option. One of the aims in financial mathematics is to price these derivatives. To be able to do this, we will need a pay-off which is the profit that the buyer of the option can gain at maturity.

The pay-off is a positive random variable because we do not know the value of the asset price at maturity S_T. Stochastic calculus is used to model the evolution of S_t. For this purpose, two models will be presented: Black-Scholes and Heston model. With the first one we have analytic formulas but because the volatility is not constant, Heston proposed another one. In this second model it is necessary to use numerical analysis because there are no analytic expressions.

That is why this work exposes a numerical method for pricing European derivatives. This method is called SWIFT (Shannon wavelet inverse Fourier technique) and it was proposed by Ortiz-Gracia and Oosterlee in [8, Ort16]. Whereas the density function f of the random variable S_t is typically not known, the characteristic function $\hat{f}$ of the log-asset price is often avaliable. The characteristic function is related with the Fourier transform of f. Also we need an orthonormal basis to describe f as a sum of some specific functions. Because the density function is smooth and defined on the real line we will use Shannon wavelets.

The master thesis is organized as follows. In Section 2 we will construct the Shannon wavelets. For that aim we need to understand concepts of Hilbert spaces and what an orthonormal basis is. Then, the Fourier transform and also results of multiresolution analysis will appear. We recall that this part is made to present the Shannon wavelets which are used to describe the density function f as a combination of basis functions. In Section 3 there are results of probability and stochastic calculus. Here we will give the definitions of European derivatives and we will give details on the aforementioned models. In Section 4, we will present the numerical method SWIFT. At the end of this section we will present some numerical results for a European call option and also for another derivate: a cash-or-nothing option. Both Black-Scholes and Heston models are used to drive the dynamics of the asset price. In Section 5, we will give some conclusions.

2 Hilbert spaces and wavelet theory.

In this chapter we will review some basic facts about Hilbert spaces, specially about orthonormal basis. This will be necessary to develope the wavelets theory. The aim of this section is to expose all the theory we need to be able to understand how to construct the Shannon wavelet.

2.1 Basic concepts on Hilbert spaces.

Definition 2.1 (**Scalar product**)**.** A scalar product in a real or complex vector space H is a $\mathbf{K}$ ($\mathbf{K} = \mathbb{R}, \mathbb{C}$) valued function on $H \times H$

$$<,>: H \times H \to \mathbf{K}$$

with the following properties,

i) For every $x \in H$, $< \cdot, x >$ is a linear map on H and $< x, \cdot >$ is a skewlinear map meaning that

$$< x, y_1 + y_2 >=< x, y_1 > + < x, y_2 >, \quad < x, \lambda y >= \bar{\lambda} < x, y >,$$

where $\bar{\lambda}$ is the complex conjugate of λ.

ii) Symmetry : $< y, x >= \overline{< x, y >}$.

iii) $< x, x > \geq 0$ and $< x, x >= 0 \iff x = 0$.

Remark 2.1. Every scalar product induces a **norm** defined as follows

$$\|x\| = \sqrt{< x, x >}.$$

It is easy to prove that $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$. It is also verified $\|\lambda x\| = |\lambda| \|x\|$, so we have to prove if the triangle inequality holds. This means

$$\|x + y\| \leq \|x\| + \|y\|.$$

First of all, let us recall the Cauchy-Schwartz inequality

$$| < x, y > | \leq \|x\| \|y\|.$$

Therefore,

$$\begin{aligned} \|x + y\|^2 =< x + y, x + y >= \|x\|^2 + \|y\|^2 + 2\Re(< x, y >) \\ \leq \|x\|^2 + \|y\|^2 + 2\|x\| \|y\| = (\|x\| + \|y\|)^2 \end{aligned}$$

and we get what we want, $\|x + y\| \leq \|x\| + \|y\|$. We recall that $\Re$ is the real part of a complex number.

Definition 2.2. Let H be a space with a scalar product. We say that a sequence $(x_n)_{n=1}^{\infty} \subset H$ converges to an element $x \in H$ if $\|x_n - x\| \to 0$ with the norm endowed by the scalar product on H.

Definition 2.3 (Cauchy sequence). $(x_n)_{n=1}^{\infty} \subset H$ is a Cauchy sequence on H endowed with a scalar product if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that for all $m, n \geq n_0$ we have that $\|x_n - x_m\| < \varepsilon$.

Definition 2.4 (Hilbert space). A vector space H with a scalar product is called a Hilbert space if every Cauchy sequence converges to an element x that belongs to H. It is important to recall that the convergence is with the topology induced by the norm of the scalar product.

Proposition 2.1. *The scalar product is a continuos function on the product space $H \times H$, with H a Hilbert space.*

Proof. To prove that the scalar product is a continuos function, we have to verify that for every sequence $(x_n, y_n)_{n=1}^{\infty} \subset H \times H$ that converges to an element (x, y) with the topology of $H \times H$, then the sequence $(< x_n, y_n >)_{n=1}^{\infty}$ converges to $< x, y >$ with the topology of $\mathbf{K}$. Let us say first which topology has each space.
The topology for $H \times H$ is defined through the norm

$$\|(x, y)\|_{H \times H} = \max\{\|x\|_H, \|y\|_H\}$$

and the norm in $\mathbf{K}$ is the absolut value for $\mathbb{R}$ and the module for $\mathbb{C}$.
Since $(x_n, y_n) \to (x, y)$ in $H \times H$, by the definition of the norm in this space we have that the sequences $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ converge to x, y respectivelly in H,

$$\|x_n - x\|_H \to 0, \quad \|y_n - y\|_H \to 0.$$

We must only check that $< x_n, y_n > \to < x, y >$. For that we will use the Cauchy-Schwartz inequality.

$$\begin{aligned} | < x_n, y_n > - < x, y > | &= | < x_n - x, y_n > + < x, y_n - y > | \\ &\leq | < x_n - x, y_n > | + | < x, y_n - y > | \\ &\leq \|x_n - x\| \|y_n\| + \|x\| \|y_n - y\| \to 0. \end{aligned}$$

Hence the scalar product is a continous function.

$\square$

Example 2.1. The spaces $\mathbb{R}^n$, $\mathbb{C}^n$ are Hilbert spaces with the usual scalar product

$$< (x, \ldots, x_n), (y_1, \ldots, y_n) >= \sum_{i=1}^{n} x_i \overline{y_i}.$$

Example 2.2. Let $\mathbb{T}$ be the unit circle, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. This set will be idetified for our purpose by the interval $[-\pi, \pi)$ (sometimes it is used the interval $[0, 1)$, or $[0, 2\pi)$), and we consider also the real line $\mathbb{R}$. Then the space $L^2(X)$ composed by all the square integrable functions on X,

$$L^2(X) = \left\{ f : \int_X |f|^2 < \infty \right\}$$

is a Hilbert space with the scalar product

$$< f, g >= \int_X f(x) \overline{g(x)} \, dx$$

This is well defined because of the Cauchy-Schwartz inequality taking the norm like

$$\|f\|_2 = \sqrt{\int_X |f|^2}$$

Example 2.3. The space ℓ^2 defined as follows

$$\ell^2 = \left\{ (x_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}$$

is a Hilbert space with the scalar product

$$< x, y >= \sum_{n=1}^{\infty} x_n \overline{y_n}$$

From now on we will denote by H a Hilbert space.

Definition 2.5 (orthogonality). We say that $x, y \in H$ are orthogonal if

$$< x, y >= 0.$$

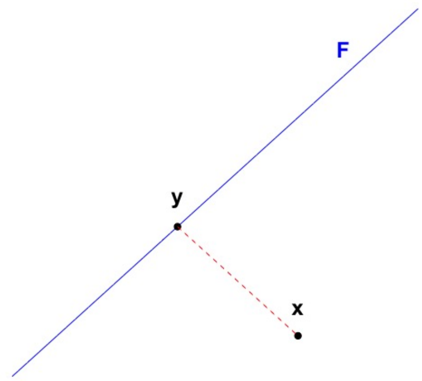

Figure 1: Orthogonal projection.

It is easy to prove the famous **Pythagoras theorem** for a Hilbert space if two elements are orthogonal only developing the definiton of the norm,

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

Another indentity that we will use is the **Parallelogram identity**,

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Definition 2.6. Given $x \in H$ and $F \subset H$ a closed subspace, we define the distance between x and F as

$$d(x, F) = \inf\{\|x-y\| : y \in F\}$$

Theorem 2.1 (**Projection theorem**)**.** *Let F be a closed subspace of H. Then for every $x \in H$ there exists a unique $y \in F$ such that $d(x, F) = \|x-y\|$. We denote this projection as $y = P_F(x)$.*

Proof. Let $d = d(x, F)$. By the definition of infimum there exists a sequence $(y_n)_{n=1}^{\infty} \subset F$ such that $d = \lim_{n\to\infty} \|x - y_n\|$. Now, by the parallelogram indentity we have

$$\begin{aligned} 2\|y_m - x\|^2 + 2\|y_n - x\|^2 &= \|y_m - y_n\|^2 + \|y_m + y_n - 2x\|^2 \\ &= \|y_m - y_n\|^2 + 4\left\|\frac{y_m + y_n}{2} - x\right\|^2. \end{aligned}$$

Since F is a subspace, $(y_m + y_n)/2 \in F$, and moreover

$$\begin{aligned}\|y_n - y_m\|^2 &= 2\|y_m - x\|^2 + 2\|y_n - x\|^2 - 4\left\|\frac{y_m + y_n}{2} - x\right\|^2 \\ &\leq 2\|y_m - x\|^2 + 2\|y_n - x\|^2 - 4d^2 \to 0.\end{aligned}$$

F is a closed subspace of a Hilbert space and therefore F is also a Hilbert space. Like $(y_n)_{n=1}^{\infty} \subset F$ is a Cauchy sequence, $y_n \to y \in F$. Hence, since the norm is a continuos function

$$d = d(x, F) = \lim_{n\to\infty} \|x - y_n\| = \|x - y\|, \quad y \in F.$$

For uniqueness: Let us assume that there exists two elements $y_1, y_2 \in F$ such that

$$d = d(x, F) = \|x - y_1\| = \|x - y_2\|.$$

Then using the previous argument with the sequence $\{y_1, y_2, y_1, y_2, \dots\}$ we obtain that this sequence has a limit and thus $y_1 = y_2$. □

Theorem 2.2. *Let F be a closed subspace of H as before. We define the **orthogonal complement** $F^{\perp}$ as follows*

$$F^{\perp} = \{x \in H : < x, y >= 0, \forall y \in F\}.$$

Then $F^{\perp}$ is a closed subspace such that

$$H = F \oplus F^{\perp}.$$

Proof. It is straightforward that $F^{\perp}$ is a subspace. Let's prove that it is closed. We take a sequence $(x_n)_{n=1}^{\infty} \subset F^{\perp}$ that converges to x in H, $\|x_n - x\| \to 0$. We have to see that $x \in F^{\perp}$. Let $y \in F$,

$$\begin{aligned}| < x, y > | &= | < x - x_n + x_n, y > | = | < x - x_n, y > + < x_n, y > | \\ &= | < x - x_n, y > | \leq \|x - x_n\|\|y\| \to 0.\end{aligned}$$

Since $< x, y >= 0$ for all $y \in F$ we have that x is in $F^{\perp}$. That proves that $F^{\perp}$ is closed.

Now, for the second part we have to verify two things. The first one is that $F \cap F^{\perp} = \{0\}$. If $x \in F \cap F^{\perp}$ then $x \in F^{\perp}$ and we have that $< x, y >= 0$ for all $y \in F$. Specially $x \in F$ and we obatin that $< x, x >= 0$ which implies that $x = 0$.

The second thing to do, is to prove that every element $x \in H$ is the sum of two elements, one that belongs to F and other that belongs to $F^\perp$.
Let $x \in H$. By the projection theorem we have that there exists an element $y \in F$ such that $y = P_F(x)$. Then it would be enough to prove that $x - y \in F^\perp$. Let t be $0 < t < 1$ and $z \in F$. We have

$$\begin{aligned}\|x-y\|^2 &\leq \|x-(1-t)y-tz\|^2 = \|x-y-t(z-y)\|^2\\ &= \|x-y\|^2 + t^2\|z-y\|^2 - 2t\Re(< z-y, x-y >)\end{aligned}$$

and hence $2\Re(< z-y, x-y >) \leq t\|z-y\|^2$. Letting t as small as possible we get that $\Re(< u, x-y >) \leq 0$ because it is for every $z \in F$ and F is a subspace. Multiplying this inequality by -1, i and $-i$ we have that $< u, x-y >= 0$, $\forall u \in F$ and we are done.

□

This theorem gives us a useful inequality. Let $x \in H$, because of the projection theorem we can put x as a sum of two elements, one is the projection on F and the other is an element on $F^\perp$,

$$x = P_F(x) + z, \quad z \in F^\perp.$$

Using the Pithagoras theorem we finally get that

$$\|P_F(x)\|^2 + \|z\|^2 = \|x\|^2 \implies \|P_F(x)\|^2 \leq \|x\|^2.$$

2.2 Orthonormal basis

Definition 2.7. We say that $\{e_i\}_{i\in\mathbb{N}} \subset H$ is an **orthonormal system** if their elements are pairwise orthogonal, and moreover, the norm of each element is equal to one, that is

$$< e_i, e_j >= \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Definition 2.8. Let $x \in H$. Given an orthonormal system $\{e_i\}_{i\in\mathbb{N}}$ we define the **Fourier coefficient** of x as

$$\hat{x}(i) =< x, e_i >, \quad i \in \mathbb{N}.$$

Then we define $\hat{x} = \{\hat{x}(i)\}_{i\in\mathbb{N}}$. The sum $\sum_{i\in\mathbb{N}} \hat{x}(i)e_i$ is called the **Fourier series** of x.

Definition 2.9. An orthonormal system $E = \{e_i\}_{i\in\mathbb{N}}$ is called complete if $E^{\perp} = \{0\}$. In other words this means that

$$\hat{x}(i) = 0, \forall\, i \in \mathbb{N} \implies x = 0.$$

An orthonormal complete system of H is also known as **orthonormal basis** of H.

Theorem 2.3 (**Bessel's inequality**). *Given an orthonormal system $\{e_i\}_{i\in\mathbb{N}}$ on H, and let $x \in H$. Then*

$$\sum_{i=1}^{\infty} |\hat{x}(i)|^2 \leq \|x\|^2.$$

Proof. First of all we fix $n \in \mathbb{N}$ and let F_n be the closed subspace generated by the elements $\{e_1, \ldots, e_n\}$. Then $P_{F_n}(x) = \sum_{i=1}^{n} c_i e_i$ for some coefficients c_i. Using the Pythagoras theorem we have

$$\|P_{F_n}(x)\|^2 = \left\| \sum_{i=1}^{n} c_i e_i \right\|^2 = \sum_{i=1}^{n} |c_i|^2 \leq \|x\|^2.$$

We know that $x - P_{F_n}(x) \in F^{\perp}$ and for all $i = 1, \ldots, n$

$$< x - \sum_{i=1}^{n} c_i e_i, e_i >=< x, e_i > -c_i = 0$$

and therefore $< x, e_i >= c_i = \hat{x}(i)$. Letting n tend to infinity we obtain the Bessel's inequality. □

Remark 2.2. This inequality can be used to confirm that the function from $H \to \ell^2$ such that $x \to \hat{x}$ is well defined. Moreover, it is subjective. For all $(x_n) \in \ell^2$ the sum $\sum_{i=1}^{\infty} x_i e_i$ converges to an element $x \in H$ such that $\hat{x}(i) =< x, e_i >= x_i$. That is easy to prove.
Let us define $s_n = \sum_{i=1}^{n} x_i e_i$. If we take $p, q \in \mathbb{N}$, $q > p$ we have

$$\|s_q - s_p\|^2 = \left\| \sum_{i=p+1}^{q} x_i e_i \right\|^2 = \sum_{i=p+1}^{q} |x_i|^2 \to 0$$

Therefore $(s_n)_{n=1}^{\infty}$ is a Cauchy sequence on H and there exists an element $x \in H$ such that $\sum_{i=1}^{\infty} x_i e_i = x$. To finish, for all $i \in \mathbb{N}$ and taking into account the continuity of the scalar product we have

$$x_i = \lim_{n\to\infty} < \sum_{i=1}^{n} x_i e_i, e_i >=< \lim_{n\to\infty} \sum_{i=1}^{n} x_i e_i, e_i >=< x, e_i >= \hat{x}(i)$$

Usually this function is not injective. If $\hat{x} = 0$, the inverse by the map can not be the element $x = 0$. In case it does (the map is bijective) the space H is complete.

Now we will introduce a very important theorem that allow us to assure if some orthonormal system is an orthonormal basis. This characterization is very useful because to prove that something is an orthormal basis by means of the definition is often much more complicated that this theorem.

Theorem 2.4 (**Fischer-Riesz theorem**). *An orthonormal countable system* $\{e_n\}_{n=1}^{\infty}$ *is complete (then it is an orthonormal basis) on* H*, if and only if, for every* $x \in H$ *,*

$$x = \sum_{n=1}^{\infty} \hat{x}(n) e_n$$

and the Parseval's identity holds

$$\|x\|^2 = \sum_{n=1}^{\infty} |\hat{x}(n)|^2.$$

Proof. If we consider that x is the infinity sum of its Fourier coefficients, it is immediate that if $\hat{x}(n) = 0$, $\forall\, n \in \mathbb{N}$ we have that $x = 0$, and it is complete.
To prove the converse we know that $\hat{x} \in \ell^2$ and there existes $y \in H$ such that $\sum_{n=1}^{\infty} \hat{x}(n) e_n = y$. Thus $\hat{x}(n) = \hat{y}(n)$, and

$$0 = \hat{x}(n) - \hat{y}(n) = \widehat{x-y}(n) \implies x = y$$

and the Parseval's identity holds.

□

Remark 2.3. The previous map we have defined from H to ℓ^2 as $x \to \hat{x}$ is an isometry when the orthonormal system of H is complete due to the Fischer-Riesz theorem.

Example 2.4. The canonical basis $\{e_i,\, i = 1, \ldots, n\}$ on $\mathbb{R}^n$

$$e_i = (0, \ldots, \underbrace{1}_{\mathrm{i}}, \ldots, 0)$$

is an orthonormal basis.

Example 2.5. For the space ℓ^2 the orthonormal system $\{e_i\}_{i=1}^{\infty}$ with $e_i = (\delta_{i,n})_{n=1}^{\infty}$ is a complete orthonormal system. The delta represents the Kronecker delta,

$$\delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Example 2.6. A well known orthonomal basis on $L^2(\mathbb{T})$ with $\mathbb{T} = [-\pi, \pi)$ is

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{ikx} : k \in \mathbb{Z} \right\}.$$

Then, if $f \in L^2(\mathbb{T})$, its Fourier coefficients are

$$\hat{f}(k) =< f, e_k >= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, dx.$$

Using this basis of $L^2(\mathbb{T})$, we can encode f with the coefficients $c_k =< f, e_k >$. And then, we can reconstruct again the function f by these coefficients. This is something very used in signal theory. But functions in finance are normally defined on the real line $\mathbb{R}$ and that is why we have to use another efficient basis like the wavelets basis is. We will introduce this chapter just after a comment on a very important basis on $L^2(\mathbb{R})$.

Example 2.7. The Haar system $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ such that

$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k)$$

is an orthonormal basis on $L^2(\mathbb{R})$ where $\psi(x) = \chi_{(0,1/2)}(x) - \chi_{(1/2,1)}(x)$. It is easy to check that

$$\|\psi_{j,k}\|_2^2 = \int_{\mathbb{R}} 2^j |\psi(2^j x - k)|^2 \, dx = \int_{\mathbb{R}} |\psi(x)|^2 \, dx = \int_0^1 \, dx = 1.$$

We can observe that the support of $\psi_{j,k}$ is the interval $[k/2^j, (k+1)/2^j]$ in which the function $\psi_{j,k}$ has the values

$$\psi_{j,k}(x) = \begin{cases} 2^{j/2} & \frac{k}{2^j} < x < \frac{k+1/2}{2^j} \\ -2^{j/2} & \frac{k+1/2}{2^j} < x < \frac{k+1}{2^j} \\ 0 & \text{otherwise} \end{cases}$$

If we take two functions $\psi_{j,k}$, $\psi_{m,l}$, we can see that if $k \neq l$ we have that the intersection of the two supports is empty and thus the scalar product is zero. And if we have that $k = l$ and $j \neq m$ the intersection of both supports is the smaller one. That makes that the half of the interval is positive and the other half is negative with the same value, canceling the areas and therefore the scalar product is also zero. It can also be proved that this orthonormal system is complete on $L^2(\mathbb{R})$.

2.3 Fourier transform and orthonormal wavelets.

Definition 2.10. Let $f \in L^1(\mathbb{R})$. We define the **Fourier transform** as

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}\, dx.$$

Remark 2.4. The Fourier transform is well defined becuase $|\hat{f}(\xi)| \leq \|f\|_1$, and therefore

$$\|\hat{f}\|_\infty \leq \|f\|_1, \implies \hat{f} \in L_\infty.$$

We will often say that x is the time variable and ξ is the frequency variable.

Proposition 2.2 (**Plancherel's theorem**).

$$< f, g >= \frac{1}{2\pi} < \hat{f}, \hat{g} >, \quad f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$$

Remark 2.5. The Fourier transform can be extentended for a function $f \in L^2(\mathbb{R})$ due to Plancherel's theorem.
We know that $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, so we can find a sequence $(f_n)_{n=1}^\infty \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ than converges to f in $L^2(\mathbb{R})$. Since

$$\|\hat{f}_n - \hat{f}_m\|_2 = \|\widehat{f_n - f_m}\|_2 = \sqrt{2\pi}\,\|f_n - f_m\|_2 \to 0$$

$(\hat{f}_n)_{n=1}^\infty$ is a Cauchy sequence in $L^2(\mathbb{R})$, and thus we define the Fourier transform of $f \in L^2(\mathbb{R})$ as

$$\hat{f} = \lim_{n\to\infty} \hat{f}_n \quad \text{in} \quad L^2(\mathbb{R}).$$

Definition 2.11. The **inverse Fourier transform** is defined as

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi)e^{ix\xi}\, d\xi.$$

We will expose two examples about how to compute the inverse Fourier transform of two functions that we will need to construct the Shannon wavelet.

Example 2.8.

$$\hat{\phi}(\xi) = \chi_{[-\pi,\pi]}(\xi).$$

$$
\begin{aligned}
\phi(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\phi}(\xi) e^{ix\xi}\, d\xi = \int_{-\pi}^{\pi} e^{ix\xi}\, d\xi \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} \cos(x\xi)\, d\xi + i\int_{-\pi}^{\pi} \sin(x\xi)\, d\xi\right) \\
&= \frac{1}{2\pi}\left(\frac{\sin(x\xi)}{x}\bigg|_{-\pi}^{\pi} - i\,\frac{\cos(x\xi)}{x}\bigg|_{-\pi}^{\pi}\right) \\
&= \frac{\sin(\pi x)}{\pi x}.
\end{aligned}
$$

Example 2.9.

$$
\hat{\psi}(\xi) = e^{i\frac{\xi}{2}}\chi_I(\xi), \quad I = [-2\pi, -\pi) \cup (\pi, 2\pi].
$$

$$
\begin{aligned}
\psi(x) &= \frac{1}{2\pi}\int_I e^{i\frac{\xi}{2}} e^{i\xi x}\, d\xi = \frac{1}{2\pi}\int_I e^{i\frac{\xi}{2}(2x+1)}\, d\xi \\
&= \frac{1}{2\pi}\left[\int_I \cos\left(\frac{\xi}{2}(2x+1)\right) + i\int_I \sin\left(\frac{\xi}{2}(2x+1)\right)\right] \\
&= \frac{\sin\left(\frac{\xi}{2}(2x+1)\right)}{\pi(2x+1)}\bigg|_{-2\pi}^{-\pi} + \frac{\sin\left(\frac{\xi}{2}(2x+1)\right)}{\pi(2x+1)}\bigg|_{\pi}^{2\pi} \\
&= 2\frac{\sin(2\pi x + \pi) - \sin\left(\pi x + \frac{\pi}{2}\right)}{\pi(2x+1)} \\
&= -2\frac{\sin(2\pi x) + \cos(\pi x)}{\pi(2x+1)}.
\end{aligned}
$$

Definition 2.12. Let f be a function. Then we define the **traslation** τ_h and the **dilation** ρ_r of f as follows

$$
(\tau_h f)(x) = f(x-h), \quad (\rho_r f)(x) = f(rx).
$$

Proposition 2.3.

$$
\widehat{\tau_h f}(\xi) = e^{-ih\xi}\hat{f}(\xi)
$$

Proof.

$$
\begin{aligned}
\widehat{\tau_h f}(\xi) &= \int_{\mathbb{R}} (\tau_h f)(x) e^{-ix\xi}\, dx = \int_{\mathbb{R}} f(x-h) e^{-ix\xi}\, dx \\
&= \int_{\mathbb{R}} f(u) e^{-i(u+h)\xi}\, du = e^{-ih\xi}\int_{\mathbb{R}} f(u) e^{-iu\xi}\, du = e^{-ih\xi}\hat{f}(\xi).
\end{aligned}
$$

□

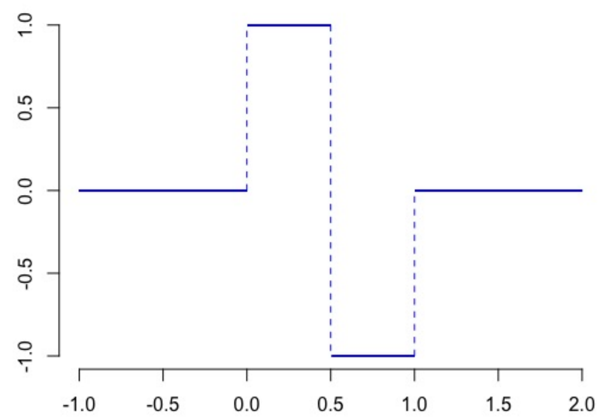

Figure 2: Haar wavelet

Definition 2.13 (Orthonormal wavelet). An orthonormal wavelet on $\mathbb{R}$ is a function $\psi \in L^2(\mathbb{R})$ such that $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$ where

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k).$$

Remark 2.6. Observe that what we are doing are translations and dilations of the function ψ. The factor $2^{j/2}$ is used to preserve orthonormality. Moreover, due to this factor all the functions $\psi_{j,k}$ have the same norm as ψ, $\|\psi_{j,k}\| = \|\psi\|$, so this system is normalized if the norm of ψ is equal to one.

Remark 2.7. Another interesting observation about using orthonormal wavelets instead of Fourier series is that we can move, stretech, or compress the wavelets to accurately represent the local properties of the functions that we will find in finance.

Example 2.10. The Haar system obtained by dilations and translations of the Haar wavelet ψ

$$\psi(x) = \chi_{(0,1/2)}(x) - \chi_{(1/2,1)}(x)$$

is an orthonormal wavelet for $L^2(\mathbb{R})$. In the section of orthonormal basis we exposed this system in one example and we checked that it is an orthonormal basis.

Proposition 2.4.

$$\widehat{\psi_{j,k}}(\xi) = e^{-i2^{-j}k\xi}2^{-j/2}\hat{\psi}(2^{-j}\xi)$$

Proof.

$$\begin{aligned}\widehat{\psi_{j,k}}(\xi) =& \int_{\mathbb{R}} \psi_{j,k}(x)e^{-ix\xi}\,dx = \int_{\mathbb{R}} 2^{j/2}\psi(2^j x - k)e^{-ix\xi}\,dx \\ =& 2^{-j/2}\int_{\mathbb{R}} \psi(u)e^{-i(u+k)2^{-j}\xi}\,du = e^{-i2^{-j}k\xi}2^{-j/2}\hat{\psi}(2^{-j}\xi).\end{aligned}$$

□

Haar wavelet is an important example in wavelets theory. But for the purpose of this thesis, we want to develope another wavelet which is called **Shannon wavelet**. The problem that the Haar basis has, is that there are discontinuities. The functions we will use are smooth and Shannon wavelet can help us for this aim. The following example is related to this wavelet.

Example 2.11. Let ψ be a function such that

$$\hat{\psi}(\xi) = \chi_I(\xi), \quad I = [-2\pi, -\pi] \cup [\pi, 2\pi].$$

Then ψ is an orthonormal wavelet. Let's prove it. First of all we will check if $\|\psi_{j,k}\| = 1$. Recall that it is only necessary to see if $\|\psi\| = 1$,

$$\|\psi\|_2^2 = \frac{1}{2\pi}\|\hat{\psi}\|_2^2 = \frac{1}{2\pi}2\pi = 1.$$

On the other hand, if $j \neq l$ the intersection of the supports of the functions $\hat{\psi}_{j,k}, \hat{\psi}_{l,m}$ is empty and by Plancherel's theorem we have

$$< \psi_{j,k}, \psi_{l,m} >= \frac{1}{2\pi} < \hat{\psi}_{j,k}, \hat{\psi}_{l,m} >= 0 \quad j \neq l$$

Now, if $j = l$,

$$\begin{aligned} < \psi_{j,k}, \psi_{j,m} > &= \frac{1}{2\pi} < \hat{\psi}_{j,k}, \hat{\psi}_{j,m} > \\ &= \frac{1}{2\pi}2^{-j}\int |\hat{\psi}(2^{-j}\xi)|^2 e^{-i2^{-j}(k-m)\xi}\,d\xi \\ &= \frac{1}{2\pi}\int |\hat{\psi}(u)|^2 e^{-i(k-m)u}\,du \\ &= \frac{1}{2\pi}\left(\int_{-2\pi}^{-\pi} e^{-i(k-m)u}\,du + \int_{\pi}^{2\pi} e^{-i(k-m)u}\,du\right) \\ &= \delta_{k,m}.\end{aligned}$$

This proves that the system $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ is an orthonormal system. Let us check that it is exactly an orthonormal basis.

$$
\begin{aligned}
\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} | < f, \psi_{j,k} > |^2 =& \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \frac{1}{4\pi^2} | < \hat{f}, \hat{\psi}_{j,k} > |^2 \\
=& \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \frac{2^{-j}}{4\pi^2} \left| \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{\psi}(2^{-j}\xi)} e^{i2^{-j}k\xi}\, d\xi \right|^2 \\
=& \sum_{j\in\mathbb{Z}} \frac{2^j}{2\pi} \sum_{k\in\mathbb{Z}} \left| \int_I \hat{f}(2^j u) \frac{1}{\sqrt{2\pi}} e^{iku}\, du \right|^2 \\
=& \sum_{j\in\mathbb{Z}} \frac{2^j}{2\pi} \int_I |\hat{f}(2^j u)|^2\, du \\
=& \frac{1}{2\pi} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} \chi_I(2^{-j}\xi) |\hat{f}(\xi)|^2\, d\xi \\
=& \frac{1}{2\pi} \int_{\mathbb{R}} \left(\sum_{j\in\mathbb{Z}} \chi_I(2^{-j}\xi) \right) |\hat{f}(\xi)|^2\, d\xi \\
=& \frac{1}{2\pi} \|\hat{f}\|_2^2 = \|f\|_2^2.
\end{aligned}
$$

Since the Parseval's identity holds, the function ψ is an orthonormal wavelet on $L^2(\mathbb{R})$.

2.4 Multiresolution analysis

Definition 2.14. A **multiresolution analysis** (MRA) consists of a sequence of closed subspaces $V_j, j \in \mathbb{Z}$ of $L^2(\mathbb{R})$ satisfying the following properties.

1. $V_j \subset V_{j+1}, \quad \forall j \in \mathbb{Z}$
2. $f(x) \in V_j \iff f(2x) \in V_{j+1}, \quad \forall j \in \mathbb{Z}$
3. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$
4. $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$
5. There exists a function $\phi \in V_0$, such that $\{\tau_k\phi : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0. This function ϕ is called the **scaling function** of the given MRA.

Remark 2.8. Typically given a function ϕ we can define the closed subspace V_0 as

$$V_0 = \overline{\operatorname{span}\{\tau_k \phi : k \in \mathbb{Z}\}}$$

and by Property 5 every function $f \in V_0$ can be written as the following sum

$$f(x) = \sum_{k \in \mathbb{Z}} c_k \phi(x-k), \quad c_k =< f, \tau_k \phi > .$$

Clearly through Property 2 since $\phi(x) \in V_0$ then $\phi(2^j x) \in V_j$ and the system

$$\{2^{j/2}\phi(2^j \cdot -k) : k \in \mathbb{Z}\}$$

is an orthonormal basis for V_j, chosen V_j as

$$V_j = \overline{\operatorname{span}\{2^{j/2}\phi(2^j \cdot -k : k \in \mathbb{Z}\}}.$$

Another interesting result is deduced from Property 3 which says that $\bigcup_{j \in \mathbb{Z}} V_j$ is dense on $L^2(\mathbb{R})$. It means that given a function $f \in L^2(\mathbb{R})$ and for every $\varepsilon > 0$ there exists a function $g \in V_j$ for some $j \in \mathbb{Z}$ such that $\|f - g\|_2 < \varepsilon$.

Definition 2.15. Let ϕ the scaling function of an MRA and let $\phi_{j,k}$ the function such that $\phi_{j,k} = 2^{j/2}\phi(2^j x - k)$. For each $j \in \mathbb{Z}$ we define the **projection operator** $P_j : L^2(\mathbb{R}) \to V_j$ by

$$P_j f(x) = \sum_{k \in \mathbb{Z}} c_{j,k}(x) \phi_{j,k}(x)$$

where $c_k =< f(x), \phi_{j,k}(x) >$.

Proposition 2.5. *With the same previous assumptions, for all $f \in L^2(\mathbb{R})$ we have that*

$$\lim_{j \to \infty} \|P_j f - f\|_2 = 0$$

Proof. Because of the Property 4 of MRA, we can assure that there exists a $J \in \mathbb{Z}$ such that $\|f - g\|_2 \leq \varepsilon/2$ for some $g \in V_J$. By Property 1 we have also that $g \in V_j$, $\forall j \geq J$ and therefore $P_j g = g$, $\forall j \geq J$. Finally, for all $j \geq J$

$$\begin{aligned}\|f - P_j f\|_2 =& \|f - g + g - P_j f\|_2 \leq \|f - g\|_2 + \|P_j(g - f)\|_2 \\ \leq& 2\|f - g\|_2 < 2\,\frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

and that proves that $\|f - P_j f\|_2 \to 0$ when $j \to \infty$. $\square$

The next proposition is a characterization about orthonormal systems that we will use several times. It can be used to prove that $\{\tau_k \phi : k \in \mathbb{Z}\}$ is an orthonormal system for V_0.

Proposition 2.6. *If $\phi \in L^2(\mathbb{R})$ then the system $\{\tau_k \phi : k \in \mathbb{Z}\}$ is an orthonormal system if and only if*

$$\Phi(\xi) = \sum_{k \in \mathbb{Z}} |\hat{\phi}(\xi + 2k\pi)| = 1, \quad a.e.\, \xi$$

Naturally, the function Φ is 2π-periodic.

Proof. We have that for every $k \in \mathbb{Z}$

$$\begin{aligned}
\delta_{0,k} =< \phi, \tau_k \phi >= & \frac{1}{2\pi} < \hat{\phi}, \widehat{\tau_k \phi} >= \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{\phi}(\xi)|^2 e^{ik\xi}\, d\xi \\
= & \frac{1}{2\pi} \sum_{j \in \mathbb{Z}} \int_{2j\pi}^{2(j+1)\pi} |\hat{\phi}(\xi)|^2 e^{ik\xi}\, d\xi \\
= & \frac{1}{2\pi} \sum_{j \in \mathbb{Z}} \int_0^{2\pi} |\hat{\phi}(u + 2j\pi)|^2 e^{iku}\, du \\
= & \frac{1}{2\pi} \int_0^{2\pi} \sum_{j \in \mathbb{Z}} |\hat{\phi}(u + 2j\pi)|^2 e^{iku}\, du
\end{aligned}$$

and the proposition holds. □

2.5 Construction of wavelets from a multiresolution analysis

Now we will show how to construct an orthonormal wavelet from a mutiresolution analysis MRA. We suppose that we have a collection of closed subspaces $\{V_j : j \in \mathbb{Z}\} \subset L^2(\mathbb{R})$ forming an MRA with a scaling function ϕ. Let W_j be the orthogonal complement of V_j in V_{j+1}, it means

$$V_{j+1} = V_j \oplus W_j$$

Since $V_j \to \{0\}$ we have that

$$V_{j+1} = \bigoplus_{i=-\infty}^{j} W_i$$

and because of $V_j \to L^2(\mathbb{R})$ when $j \to \infty$ we see that

$$\bigoplus_{j=-\infty}^{\infty} W_j = L^2(\mathbb{R})$$

Then, to find an orthonormal wavelet what we need is to find a function $\psi \in W_0$ such that the system $\{\tau_k \psi : k \in \mathbb{Z}\}$ is an orthonormal basis on W_0. Then by the second property of an MRA we have that $\{2^{j/2}\psi(2^j \cdot -k) : k \in \mathbb{Z}\}$ is an orthonormal basis on W_j. Thus the system $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$ and finally we can conclude that ψ is an orthonomal wavelet.

Remark 2.9. It is clear that we can write the projection operator $P_j : L^2(\mathbb{R}) \to V_j$ in another way using the functions $\psi_{j,k}$ like

$$P_j f(x) = \sum_{i=-\infty}^{j-1} \sum_{k=-\infty}^{\infty} d_{i,k} \psi_{i,k}(x)$$

where $d_{i,k} =< f, \psi_{i,k} >$. By Proposition 2.5 we can see also that when $j \to \infty$

$$\|f - P_j f\|_2^2 = \sum_{i=j}^{\infty} \sum_{k=-\infty}^{\infty} |d_{i,k}|^2 \to 0.$$

Let us consider the scalling function $\phi \in V_0$. Then the function $\frac{1}{2}\phi(\frac{x}{2}) \in V_{-1} \subset V_0$. With this information, we can write $\frac{1}{2}\phi(\frac{x}{2})$ as a sum by the orthonormal basis $\{\tau_{-k}\phi : k \in \mathbb{Z}\}$,

$$\frac{1}{2}\phi(\frac{x}{2}) = \sum_{k \in \mathbb{Z}} c_k \phi(x + k)$$

where $c_k =< \frac{1}{2}\phi(\frac{x}{2}), \phi(x+k) >$, and we know that the sequence $(c_k)_{k=1}^{\infty} \in \ell^2$. Taking the Fourier transform on both sides of the equation we have

$$\hat{\phi}(2\xi) = \sum_{k \in \mathbb{Z}} c_k \widehat{\tau_{-k}\phi}(\xi) = \hat{\phi}(\xi) \sum_{k \in \mathbb{Z}} c_k e^{ik\xi} = \hat{\phi}(\xi) m_0(\xi)$$

where $m_0(\xi) = \sum_{k \in \mathbb{Z}} c_k e^{ik\xi}$. This function is $2\pi - periodic$ in $L^2(\mathbb{T}) = L^2([-\pi, \pi))$. The function m_0 is called the **low pass filter** associated to the scaling function ϕ.

Proposition 2.7. *Let m_0 be the low pass filter of a scaling function ϕ. Then it is verified the following equality*

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1, \quad a.e. \, \xi$$

We can deduce from here that m_0 is bounded.

Proof. By Propotion 2.6 we know that

$$\sum_{k\in\mathbb{Z}} |\hat{\phi}(2\xi + 2k\pi)|^2 = 1, \quad a.e.\, \xi$$

and we have also that

$$\sum_{k\in\mathbb{Z}} |\hat{\phi}(2(\xi + k\pi))|^2 = \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi + k\pi)|^2 |m_0(\xi + k\pi)|^2 = 1, \quad a.e.\, \xi$$

Now we separate the sum into two sums, one with the even integers and the other with the odd integers. Taking into account the Proposition 2.6 and the $2\pi - periodicity$ of m_0 we finally get

$$\begin{aligned}
1 =& \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi + 2k\pi)|^2 |m_0(\xi + 2k\pi)|^2 + \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi + (2k+1)\pi)|^2 |m_0(\xi + (2k+1)\pi)|^2 \\
=& |m_0(\xi)|^2 \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi + 2k\pi)|^2 + |m_0(\xi + \pi)|^2 \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi + (2k+1)\pi)|^2 \\
=& |m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi + \pi + 2k\pi)|^2 \\
=& |m_0(\xi)|^2 + |m_0(\xi + \pi)|^2.
\end{aligned}$$

□

Proposition 2.8. *Let ϕ the scaling function of an MRA and m_0 the low pass filter associated. Then we can characterize the spaces V_{-1} and V_0 as follows*

$$V_{-1} = \{f : \hat{f}(\xi) = m(2\xi) m_0(\xi) \hat{\phi}(\xi),\, m \in L^2(\mathbb{T})\}$$

$$V_0 = \{f : \hat{f}(\xi) = l(\xi) \hat{\phi}(\xi),\, l \in L^2(\mathbb{T})\}$$

for some m, l functions that are 2π-periodic.

Proof. Let's begin to prove that

$$V_{-1} \subseteq \{f : \hat{f}(\xi) = m(2\xi) m_0(\xi) \hat{\phi}(\xi),\, m \in L^2(\mathbb{T})\}.$$

Let $f \in V_{-1}$. Then $\{\frac{1}{\sqrt{2}}\phi(\frac{\cdot}{2} - k) : k \in \mathbb{Z}\}$ is an orthonormal basis on V_{-1} and we can write f like

$$f(x) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} c_k \phi\left(\frac{x}{2} - k\right)$$

Then we take the Fourier transform to obtain

$$\begin{aligned}\hat{f}(\xi) &= \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} c_k\widehat{\phi_{-1,k}}(\xi) = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} c_k e^{-i2k\xi}\sqrt{2}\hat{\phi}(2\xi) \\ &= \hat{\phi}(2\xi)\sum_{k\in\mathbb{Z}} c_k e^{-i2k\xi} = m(2\xi)m_0(\xi)\hat{\phi}(\xi)\end{aligned}$$

where $m(\xi) = \sum_{k\in\mathbb{Z}} c_k e^{-ik\xi}$ is a 2π-periodic function in $L^2(\mathbb{T})$.
Conversely, let $m \in L^2(\mathbb{T})$ a function that is 2π-periodic. We want to check if $f \in V_{-1}$ has the form

$$\hat{f}(\xi) = m(2\xi)m_0(\xi)\hat{\phi}(\xi)$$

Thus, we have to verify that $f \in L^2(\mathbb{R})$. We define $h(\xi) = m(2\xi)m_0(\xi)$. We know that m_0 is bounded and since $m \in L^2(\mathbb{T})$ we get that $h \in L^2(\mathbb{T})$. Since m, m_0 are 2π-periodic, h is also 2π-periodic. And thus,

$$\begin{aligned}\|\hat{f}\|_2^2 =& \|h(\xi)m(\xi)\|_2^2 = \int_{\mathbb{R}} |h(\xi)|^2|\hat{\phi}(\xi)|^2\, d\xi = \sum_{k\in\mathbb{Z}}\int_0^{2\pi} |h(\xi)|^2|\hat{\phi}(\xi+2k\pi)|^2\, d\xi \\ &= \int_0^{2\pi} |h(\xi)|^2\left(\sum_{k\in\mathbb{Z}}|\hat{\phi}(\xi+2k\pi)|^2\right) d\xi = \int_0^{2\pi} |h(\xi)|^2 = \|h\|^2_{L^2(\mathbb{T})} < \infty\end{aligned}$$

Since $\hat{f} \in L^2(\mathbb{R})$ the function f is in $L^2(\mathbb{R})$.

Let us prove another characterization of V_0. Let $g \in V_0$. We know that the system $\{\tau_k\phi : k \in \mathbb{Z}\}$ is an orthonormal basis on V_0. Then g can be written as

$$g(x) = \sum_{k\in\mathbb{Z}} c_k\phi(x-k).$$

Taking the Fourier transform on both sides

$$\hat{g}(\xi) = \sum_{k\in\mathbb{Z}} c_k\widehat{\tau_k\phi}(\xi) = \hat{\phi}(\xi)\sum_{k\in\mathbb{Z}} c_k e^{-ik\xi}$$

where $l(\xi) = \sum_{k\in\mathbb{Z}} c_k e^{-ik\xi}$ is a function 2π-periodic that belongs to $L^2(\mathbb{R})$. Now conversely we apply the same as we did for V_{-1} taking $h(\xi) = l(\xi)$. □

We will continue with the construction of the wavelet ψ. We recall that the elements of W_{-1} are functions such that $f \in V_0$ and are orthogonal to V_{-1}, $< f, g >=$

0 for all $g \in V_{-1}$. We define the function $U : V_0 \to L^2(\mathbb{T})$ by $U(f) = l$ taking l the function that we have constructed on the space V_0 by Proposition 1.8. U satisfies

$$\|U(f)\|^2_{L^2(\mathbb{T})} = \|l\|^2_{L^2(\mathbb{T})} = 2\pi \sum_{k\in\mathbb{Z}} |c_k|^2 = 2\pi \sum_{k\in\mathbb{Z}} | < f, \tau_k\phi > |^2 = 2\pi\|f\|^2_2.$$

By the polarization identity and this last equality we have

$$< f, g >_{L^2(\mathbb{R})} = \frac{1}{2\pi} < Uf, Ug >_{L^2(\mathbb{T})}, \quad f, g \in V_0.$$

If $g \in V_{-1} \subset V_0$, we take $Ug(\xi) = m(2\xi)m_0(\xi)$ due to the characterization of V_{-1}. It shows us that

$$0 =< f, g >_{L^2(\mathbb{R})} = \frac{1}{2\pi} < l(\xi), m(2\xi)m_0(\xi) >_{L^2(\mathbb{T})}$$

which implies that l is orthogonal to $m(2\xi)m_0(\xi)$ for all 2π-period function m in $L^2(\mathbb{T})$.

$$0 = \int_0^{2\pi} l(\xi)\overline{m(2\xi)m_0(\xi)}\, d\xi = \int_0^{\pi} \overline{m(2\xi)} \left(l(\xi)\overline{m_0(\xi)} + l(\xi+\pi)\overline{m_0(\xi+\pi)} \right) d\xi$$

This tell us that the π-periodic function inside the parentheses is orthogonal for all π-periodic square integrable functions and thus

$$l(\xi)\overline{m_0(\xi)} + l(\xi+\pi)\overline{m_0(\xi+\pi)} = 0, \quad a.e.\, \xi \in \mathbb{T}$$

Therefore for some function λ we can put the pair $l(\xi), l(\xi+\pi)$ as

$$\begin{cases} l(\xi) = -\lambda(\xi+\pi)\overline{m_0(\xi+\pi)} \\ l(\xi+\pi) = \lambda(\xi+\pi)\overline{m_0(\xi)} \end{cases}$$

and taking now $\xi = \mu + \pi$ and using that l and m_0 are 2π-periodic

$$\begin{cases} l(\mu+\pi) = -\lambda(\mu+2\pi)\overline{m_0(\mu)} \\ l(\mu) = \lambda(\mu+2\pi)\overline{m_0(\mu+\pi)} \end{cases}$$

Because of that we can observe

$$\begin{cases} l(\xi) = -\lambda(\xi+\pi)\overline{m_0(\xi+\pi)} = \lambda(\xi+2\pi)\overline{m_0(\xi+\pi)} \\ l(\xi+\pi) = \lambda(\xi+\pi)\overline{m_0(\xi)} = -\lambda(\xi+2\pi)\overline{m_0(\xi)} \end{cases}$$

Due to Proposition 2.7 we obtain that $\lambda(\xi+\pi) = -\lambda(\xi+2\pi)$, and substracting π on both sides we have $\lambda(\xi) = -\lambda(\xi+\pi)$, for almost every point ξ. Clearly λ is a 2π-periodic function in $L^2(\mathbb{T})$ and for some function s, that is also 2π-periodic and in $L^2(\mathbb{T})$, we can write λ like $\lambda(\xi) = e^{i\xi}s(2\xi)$. With this form the function λ verifies the property we have obtained

$$\lambda(\xi+\pi) = e^{i\xi}e^{-i\pi}s(2\xi+2\pi) = -e^{-i\xi}s(2\xi) = -\lambda(\xi)$$

Finally, if we take again the function l and we substitute the value λ in one of the previous equalities we get

$$l(\xi) = -\lambda(\xi+\pi)\overline{m_0(\xi+\pi)} = e^{i\xi}s(2\xi)\overline{m_0(\xi+\pi)}$$

In conclusion, we have found a characterization for the space W_{-1}

$$W_{-1} = \left\{ f : \hat{f}(\xi) = e^{i\xi}s(2\xi)\overline{m_0(\xi+\pi)}\hat{\phi}(\xi) : \; s \in L^2(\mathbb{T}) \right\}$$

with s a function that is 2π-periodic. By the property of dilations for an MRA we can establish the characterization of W_0

Lemma 2.1. *If ϕ is a scaling function of an MRA, and m_0 the low pass filter associated, then*

$$W_0 = \{ f : \hat{f}(2\xi) = e^{i\xi}s(2\xi)\overline{m_0(\xi+\pi)}\hat{\phi}(\xi) : \; s \in L^2(\mathbb{T}) \}$$

for some function s that is 2π-periodic.

With all this theory we can define our orthonormal wavelet like the function ψ that verifies

$$\hat{\psi}(2\xi) = e^{i\xi}\overline{m_0(\xi+\pi)}\hat{\phi}(\xi)$$

taking $s \equiv 1$. This is the principal idea to begin to construct wavelets.

We will end wavelets theory with a proposition that shows how to obtain $|\hat{\phi}|$ from $|\hat{\psi}|$. It will help to find a candidate scaling fuction given an orthonormal wavelet.

Proposition 2.9. *Let ϕ, ψ, the scaling function and the wavelet of an MRA respectively. Therefore the following equality holds*

$$|\hat{\phi}(\xi)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j\xi)|.$$

Proof. We recall that

$$\hat{\phi}(2\xi) = \hat{\phi}(\xi)m_0(\xi), \quad \hat{\psi}(2\xi) = e^{i\xi}\overline{m_0(\xi+\pi)}\hat{\phi}(\xi)$$

and considering these equalities we get

$$|\hat{\phi}(2\xi)|^2 + |\hat{\psi}(2\xi)|^2 = |\hat{\phi}(\xi)|^2(|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2) = |\hat{\phi}(\xi)|^2$$

Iterating this result we have

$$|\hat{\phi}(\xi)|^2 = |\hat{\phi}(2^n\xi)|^2 + \sum_{i=1}^{n} |\hat{\psi}(2^i\xi)|^2.$$

The function $\hat{\phi}$ is bounded by one due to Propostition 2.6, $|\hat{\phi}(\xi)|^2 \leq 1$, and thus the sequence

$$a_n = \sum_{i=1}^{n} |\hat{\psi}(2^i\xi)|^2, \quad n = 1, 2, \dots$$

is increasing and bounded by 1. Since the limit of a_n exists, the limit of $|\hat{\phi}(2^n\xi)|$ also exists. Moreover we know that

$$\int_{\mathbb{R}} |\hat{\phi}(2^n\xi)|^2 \, d\xi = \frac{1}{2^n}\int_{\mathbb{R}} |\hat{\phi}(u)|^2 \, du = \frac{\|\hat{\phi}\|_2^2}{2^n} \xrightarrow{n\to\infty} 0$$

and by Fatou's lemma

$$\int_{\mathbb{R}} \lim_{n\to\infty} |\hat{\phi}(2^n\xi)|^2 \, d\xi \leq \lim_{n\to\infty} \int_{\mathbb{R}} |\hat{\phi}(2^n\xi)|^2 = 0$$

Hence $|\hat{\phi}(2^n\xi)|^2 = 0$ and the equality of the proposition holds.

□

Finally, we arrive to the aim of this section. We will construct the Shannon wavelet defining a specific function and then we will give the scaling function associated to some multiresolution analysis.

Example 2.12. The Shannon wavelet is the function ψ whose Fourier transform is

$$\hat{\psi}(\xi) = e^{i\frac{\xi}{2}}\chi_I(\xi), \quad I = [-2\pi, -\pi) \cup (\pi, 2\pi].$$

To prove that ψ is a wavelet we use the Proposition 2.9 to be able to find a scaling function ϕ for some MRA generated by this wavelet. We have that

$$\hat{\psi}(2^j\xi) = e^{i2^{j-1}\xi}\chi_{I_j}(\xi), \quad I_j = [-2^{-j+1}\pi, -2^{-j}\pi) \cup (2^{-j}\pi, 2^{-j+1}\pi].$$

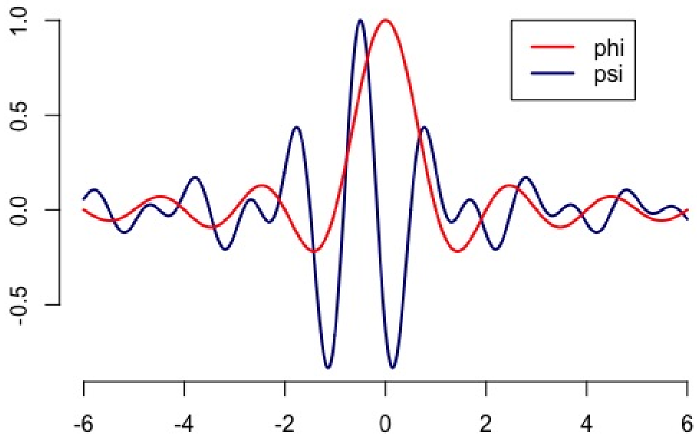

Figure 3: Shannon wavelet ψ is drawn by the blue line. The red line is the scaling function ϕ.

The sets I_j are disjointed for all $j = 1, 2, \ldots$ and the union of all of them is the interval $[-\pi, \pi]$. Since

$$\sum_{j=1}^{\infty} |\hat{\psi}(2^j \xi)| = 1, \quad \forall \xi \in [-\pi, \pi]$$

a possible candidate ϕ for the scaling function we are finding could be a function ϕ such that

$$\hat{\phi}(\xi) = \chi_{[-\pi,\pi]}(\xi).$$

Due to Proposition 2.6 and because of the lenght of the interval $[-\pi, \pi]$ is 2π,

$$\sum_{k \in \mathbb{Z}} |\hat{\phi}(\xi + 2k\pi)|^2 = \sum_{k \in \mathbb{Z}} |\chi_{[\pi,\pi]}(\xi + 2k\pi)|^2 = 1$$

the system $\{\tau_k \phi : k \in \mathbb{Z}\}$ is an orthonormal system.
We choose for V_j the closed subspace generated by the functions $\phi_{j,k}$

$$V_j = \overline{\text{span}\,(\{\phi_{j,k} = 2^{j/2}(2^j \cdot -k) : k \in \mathbb{Z}\})}, \, \forall j \in \mathbb{Z}.$$

If we see that $\frac{1}{2}\phi(\frac{x}{2})$ is an element of V_0 we will have that $\{V_j : j \in \mathbb{Z}\}$ is an MRA. This is equivalent to find the low pass filter m_0 verifying $\hat{\phi}(2\xi) = \hat{\phi}(\xi)m_0(\xi)$ for some $m_0 \in L^2(\mathbb{T})$ and 2π-periodic. As we have said m_0 must satisfy

$$\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\xi) = \chi_{[-\pi,\pi]}(\xi)m_0(\xi)$$

We can define m_0 on the interval $[-\pi, \pi]$ as

$$m_0(\xi) = \begin{cases} 1 & \xi \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ 0 & \text{otherwise} \end{cases}$$

and we extend periodically this function to $\mathbb{R}$.
Finally we recall that a function ψ given a scaling function ϕ and a low pass filter m_0 is characterized by Lemma 1.1 and we obtain

$$\hat{\psi}(2\xi) = e^{i\xi}\overline{m_0(\xi + \pi)}\hat{\phi}(\xi) = e^{i\xi}m_0(\xi + \pi)\chi_{[-\pi,\pi]}(\xi) = e^{i\xi}X_J(\xi)$$

with $J = [-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ and thus we have proved that the function ψ we have defined by its Fourier transform is exactly a wavelet.
The analytic functions of ϕ and ψ were computed in the examples of the inverse Fourier transform, and they are

$$\phi(x) = \frac{\sin(\pi x)}{\pi x}, \quad \psi(x) = -2\frac{\sin(2\pi x) + \cos(\pi x)}{2x + 1}.$$

3 Black-Scholes and Heston model in finance.

3.1 European options.

This section is designed for all those who are not familiar with financial concepts, specially with European options.

Let us see one example on Figure 4. Here we have the price of NETFLIX asset from the 3rd July 2017 until 1st August 2018. Imagine that you have some assets of this company and you are worried because you wait a bearish period. That means that the price will decrease in the following days and you want to be prepared for the worst case. Then you have the option to prepare yourself for that event with financial options. We will only explain the European options because we will try to compute them in Section 4.

Definition 3.1. A **European option** is a contract that gives to the buyer the right (but NOT the obligation) to buy/sell (Call / Put) an underlying asset for a certain price K (Strike) in a concrete date T (maturity). The seller of the European option has the obligation to sell (the buyer has a call) or buy (the buyer has a put) in case the buyer of the option uses his right.

We continue with our example of NETFLIX. Because we expect that there will be a bearish period we buy a put option. This option gives us the right to sell the asset for a certain price K at maturity time T. On the date 1st August 2018 the price of the NETFLIX asset is $S_0 = 338.38\$$. We want to sell this asset at least by the strike $K = 300\$$ at maturity $T = 3/12$ (3 months). With these assumptions, what can happen at T?

We have two possibilites. If the price at maturity S_T is greater than the strike we will not exercise our right and will sell the asset for the price S_T on the market. However, if the price S_T is less than K we will exercise our right and we will get a profit of $K - S_T$. That makes that if I finally decide to sell the asset on the market I will earn S_T, plus the profit of $K - S_T$, I will exactly have the strike K.

To do this we need to price the option. To buy this right you have to pay a premium that the seller will use to get the profit for his worst case (if the buyer finally applies his right). Mathematical tools have been developed to get this purpose. The value of the option at maturity V_T will be called the pay-off, and it will be the possible profit that the buyer could get. In the following section we will explain the

Figure 4: Price of NETFLIX asset.

way to compute European options at any time t between today and the maturity date.

In Figure 5 there are two plots with the payoff of a call option C_T, and a put option P_T with strike $K = 20$. The analytic expressions are the following

$$C_T = (S_T - K)^+ = \begin{cases} S_T - K & S_T \geq K \\ 0 & S_T < K \end{cases}$$

$$P_T = (K - S_T)^+ = \begin{cases} K - S_T & S_T \leq K \\ 0 & S_T > K \end{cases}$$

There is a famous relation between the pay-off of a European call and a European put which is called the Put-Call parity. It implies that if we know the price of one of these options we can calculate the other. It can be easy to check through the formulas or looking at Figure 5 that this relation is

$$C_T - P_T = S_T - K.$$

So the Put-Call Parity at any time t between 0 and T will be

$$C_t - P_t = S_t - Ke^{-r(T-t)}$$

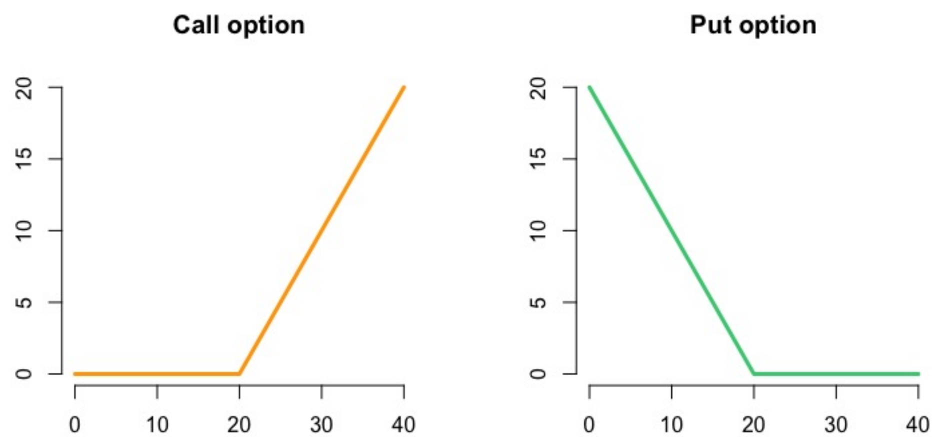

Figure 5: Pay-off for the call and put european options.

where $e^{-r(T-t)}$ is the discount factor on the interval $[t, T]$. It means that the amount of $e^{-r(T-t)}$ at time t will become a unity at time T.

Anyway we can create any option given a pay-off. The pay-off will be a positive random variable X. Another famous option that we will compute is the cash-or-nothing option. This option has the pay-off

$$X = \mathbb{1}_{\{S_T > K\}}$$

that gives to the buyer the profit of one amount of money if the price at maturity is greater than the strike, or 0 otherwise.

The goal for the next section is to construct a probabilistic model to model the evolution of the underlying price asset S_t.

3.2 Important results of stochastic calculus.

In this section we will expose some basic concepts and results of probability theory and stochastic calculus. Two imporant theorems, Itô's lemma and Girsanov's theorem, will be mentioned to develope the Black-Scholes model.

Let us begin with a probability space $(\Omega, \mathcal{F}, P)$ such that

i) Ω is the sample space. This set is the collection of all events ω of our experiment.

ii) $\mathcal{F}$ is a σ-field.

iii) P is a probability.

Definition 3.2 (Conditional expectation). Let X be an integrable random variable ($E|X| < \infty$) and $\mathcal{G}$ a σ-field such that $\mathcal{G} \subset \mathcal{F}$. Then the conditional expectation of X given $\mathcal{G}$ is the unique random variable

$$E[X|\mathcal{G}] : \Omega \to \mathbb{R}$$

such that

i) $E[X|\mathcal{G}]$ is $\mathcal{G}$-measurable.

ii) $E[E[X|\mathcal{G}]\mathbb{1}_G] = E[X\mathbb{1}_G], \quad G \in \mathcal{G}$.

Remark 3.1. A simple observation gives us the first property of conditional expectation. Taking $G = \Omega$

$$E[E[X|\mathcal{G}]] = E[X].$$

There is another interesting thing we can observe if we take the σ-field $\mathcal{G} = \{\emptyset, \Omega\}$. A possible candidate for the conditional expectation in this case should be a constant $a = E[X|\mathcal{G}]$ (a constant is $\{\emptyset, \Omega\}$-measurable), and by the previous observation

$$E[E[X|\mathcal{G}]] = E[X] = E[a] = a$$

and the value of the constant a is $E[X]$.

There are two important properties that are very useful throughout this section. The first one is that if Y is a random variable $\mathcal{G}$-measurable then

$$E[XY|\mathcal{G}] = YE[X|\mathcal{G}].$$

And the other one is that if X and $\mathcal{G}$ are independent then

$$E[X|\mathcal{G}] = E[X].$$

Definition 3.3 (Stochastic process). A stochastic process $\{X_t : t \in I\}$ is a collection of random variables indexed by a set I. In finance, the set I usually is a bounded interval of $\mathbb{R}$ of the form $[0, T]$, representing that the time goes from an initial time 0 until a final date T called maturity.

Definition 3.4 (**Sample path**). If we have a stochastic process $\{X_t : t \in [0,T]\}$, then for every $\omega \in \Omega$ the map

$$X(\cdot\,,\omega) : [0,T] \to \mathbb{R}$$

is called the sample path or also the trajectory.

Definition 3.5. We say that a stochastic process $\{B_t : t \in [0,T]\}$ is a **Brownian motion** if it verifies the following properties

i) $B_0 = 0, \quad$ a.s.

ii) The trajectories B_t are continuos almost surely.

iii) For every partition of the interval $[0,T]$,

$$t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n = T$$

the random variables of the increments $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent.

iv) The increments $B_t - B_s$ with $s < t$, follow a normal distribution of mean 0 and variance the length of the interval $[s,t]$,

$$B_t - B_s \sim N(0, t-s).$$

Definition 3.6. A family of σ-algebras $\{\mathcal{F}_t : t \in [0,T]\}$ is a filtration if $\mathcal{F}_s \subset \mathcal{F}_t$, $\forall s \leq t$, and $\mathcal{F}_t \subset \mathcal{F}$, $\forall t \in [0,T]$.

Definition 3.7 (**Martingale**). We consider a stochastic process $\{M_t : t \in [0,T]\}$ and a filtration $\{\mathcal{F}_t : t \in [0,T]\}$. We say that M is a martingale with respect this filtration if

i) M_t is $\mathcal{F}_t$-measurable (M is adapted).

ii) $E|M_t| < \infty$.

iii) $E[M_t|\mathcal{F}_s] = M_s$, for all $s < t$.

Remark 3.2. Obviously applying the properties of the definition we have that

$$E[M_t] = E[E[M_t|\mathcal{F}_s]] = E[M_s],\ s < t.$$

In particular,

$$E[M_t] = E[M_0], \quad \forall t \in [0,T].$$

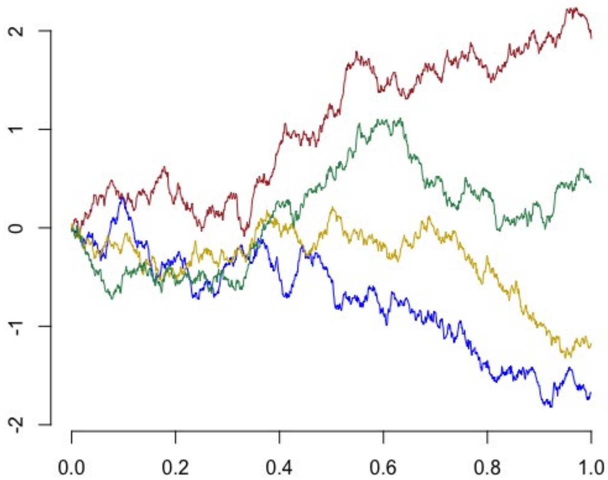

Figure 6: Different sample paths of a Brownian motion.

Example 3.1. The Brownian motion $\{B_t : t \in [0,T]\}$ is a martingale with the natural filtration defined as

$$\mathcal{F}_t = \sigma\{B_s : 0 \leq s \leq t\} = \{X_s^{-1}(B) : 0 \leq s \leq t, B \in B(\mathbb{R})\}.$$

It is clear thath the first and second property are verified. Let us check the third one.

$$\begin{aligned} E[B_t|\mathcal{F}_s] &= E[B_t - B_s + B_s|\mathcal{F}_s] = E[B_t - B_s|\mathcal{F}_s] + E[B_s|\mathcal{F}_s] = \\ &= E[B_t - B_s] + B_s = B_s \end{aligned}$$

Example 3.2. Fix $a \in \mathbb{R}$. The stochastic process

$$M_t = e^{\left(aB_t - \frac{a^2}{2}t\right)}$$

is a martingale with respect to the natural filtration generated by the Brownian motion. Obviously the first property holds. In order to prove the other properties we check that

$$E\left[e^{aZ}\right] = e^{\frac{a^2}{2}}, \quad Z \sim N(0,1).$$

$$E\left[e^{aZ}\right] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{ax} e^{-\frac{x^2}{2}} = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{ax-\frac{x^2}{2}}$$
$$= e^{\frac{a^2}{2}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2}} = e^{\frac{a^2}{2}}.$$

First we compute the expectation of M_t

$$E[M_t] = e^{-\frac{a^2}{2}t} E[a^{aB_t}] = e^{-\frac{a^2}{2}t} E[e^{a\sqrt{t}Z}] = e^{-\frac{a^2}{2}t} e^{\frac{a^2}{2}t} = 1$$

And finally let us see if the third property holds

$$E[M_t|\mathcal{F}_s] = E\left[\frac{M_t}{M_s} M_s \middle| \mathcal{F}_s\right] = M_s E\left[e^{a(B_t-B_s)-\frac{a^2}{2}(t-s)} \middle| \mathcal{F}_s\right]$$
$$= M_s E\left[e^{a(B_t-B_s)-\frac{a^2}{2}(t-s)}\right] = M_s e^{-\frac{a^2}{2}(t-s)} E\left[e^{a\sqrt{t-s}Z}\right]$$
$$= M_s e^{-\frac{a^2}{2}(t-s)} e^{\frac{a^2}{2}(t-s)} = M_s$$

We will continue the **Itô's integral**. The goal is to define and compute the integral

$$\int_0^T X_t \, dB_t$$

where $\{B_t : t \in [0,T]\}$ is a Brownian motion and $\{X_t : t \in [0,T]\}$ is another stochastic process. It can be proved that the trajectories of a Brownian motion have infinite variation on every finite interval and that makes impossible to define this integral with the definition of Riemann-Stieltjes. However, the trajectories of a Brownian motion have finite quadratic variation on every finite interval with value exactly the length of the interval. Therefore we give the following definition

Definition 3.8. Let $\{X_t : t \in [0,T]\}$ be a stochastic process such that

i) X_t is adapted.

ii) $\int_0^T E[X_t^2] \, dt < \infty$.

Then the Itô's integral is defined as follows

$$\int_0^T X_t \, dB_t = \lim_{n\to\infty} \sum_{i=1}^{n} X_{t_{i-1}} (B_{t_i} - B_{t_{i-1}})$$

where the limit in probability is taken over a sequence of partitions P_n of $[0,T]$, where $P_n = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ with the mesh tending to 0.

Remark 3.3. Let $\{X_t : t \in [0,T]\}$ be an adapted stochastic process and such that $\int_0^T E[X_t^2]\, dt < \infty$. Then for every $t \in [0,T]$, the new process $\{M_t : t \in [0,T]\}$,

$$M_t = \int_0^t X_t \, dB_t = \int_0^T X_t \mathbb{1}_{[0,t]} \, dB_t$$

is a martingale.

Definition 3.9. Let $\{\mu_t : t \in [0,T]\}$, $\{\sigma_t : t \in [0,T]\}$ be two adapted stochastic processes such that

$$\int_0^T |\mu_t| \, dt < \infty \text{ a.s.} \qquad \int_0^T \sigma_t^2 \, dt < \infty \text{ a.s.}$$

Then, the stochastic process of the form

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dB_s, \quad t \in [0,T]$$

is called an **Itô's process**. It is also used the differential notation

$$dX_t = \mu_t dt + \sigma_t dB_t.$$

Theorem 3.1 (**Itô's formula**)**.** *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a function in $C^{1,2}$, and let X_t be an Itô's process*

$$dX_t = \mu_t dt + \sigma_t dB_t.$$

Thus, the following formula holds

$$df(t, X_t) = f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2} f_{xx}(t, X_t)(dX_t)^2.$$

We compute $(dX_t)^2$ taking into account the equalities

$$\begin{cases} dt \cdot dt = 0 \\ dt \cdot dB_t = dB_t \cdot dt = 0 \\ dB_t \cdot dB_t = dt \end{cases}$$

Example 3.3 (**GMB: Geometric Brownian Motion**). The Itô's process S_t

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

where μ, σ are constants is called a Geometric Brownian motion. To compute the value of S_t knowing an initial value S_0 we will use the Itô's formula. Let $f(x) = \log(x)$. This function is twice differentiable and its derivatives are

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}.$$

To be able to compute $d(\log(S_t))$ we see first that $(dS_t)^2 = \sigma^2 S_t^2 dt$, and now we apply the previous theorem.

$$d(\log(S_t)) = \frac{1}{S_t} S_t(\mu dt + \sigma dB_t) - \frac{1}{S_t^2}\sigma^2 S_t^2 dt = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t$$

which implies that

$$\log(S_t) = \log(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t$$

and finally we obtain the famous formula for a geometric brownian motion

$$S_t = S_0 e^{\left(\mu - \sigma^2/2\right)t + \sigma B_t}.$$

A GBM is not usually a martingale. But a new process envolving the evolution of a GBM can be a martingale respect a change of probability for continuos random variables. We will give a simplified version of the Girsanov theorem wich we will apply to explain the Black-Scholes model.

Theorem 3.2 (Girsanov theorem). *Let $\{B_t, t \in [0,T]\}$ be a Brownian motion. Therefore there exists a unique probability P^* equivalent to P $(P(A) > 0 \iff P(A^*) > 0, \forall A \in \mathcal{F})$ such that the stochastic proces $\{B_t^* : t \in [0,T]\}$ defined as*

$$B_t^* = \lambda t + B_t, \quad \lambda \in \mathbb{R}$$

is a P^*-Brownian motion.

3.3 Black-Scholes model.

We consider a market in continuos time $t \in [0,T]$. There is an asset without risk S_t^0 (bond, bank account), that follows the following differential equation

$$\begin{cases} dS_t^0 = rS_t^0 dt \\ S_0^0 = 1 \end{cases} \implies S_t^0 = e^{rt}$$

where r is the risk-free rate that we suppose is a positive number. There is also a risky asset with price S_t at time t which we model with a Geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$

Both μ and σ are constant. μ is the drift of S_t and σ is the standard desviation of the stock's return that we call volatility. We know the value S_0 at time 0, so S_0 is a constant value. We take the natural filtration generated by the Brownian motion $\{B_t : t \in [0, T]\}$.

Remark 3.4. We have computed the solution of a GBM which is

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma B_t}$$

and from here we can deduce some properties. The first one is that the expectation of the stock price depends only on the drift.

$$E[S_t] = S_0 e^{(\mu - \sigma^2/2)t} E[e^{\sigma B_t}] = S_0 e^{\mu t} e^{-\frac{\sigma^2}{2}t} e^{\frac{\sigma^2}{2}t} = S_0 e^{\mu t}$$

The quocient S_t/S_s for any $0 \leq s \leq t \leq T$ is equal to

$$\frac{S_t}{S_s} = e^{(\mu - \sigma^2/2)(t-s) + \sigma(B_t - B_s)}$$

which is independent of $\mathcal{F}_s$. This fact will be used to compute the value of European options. And the last property due to this last equality is that the law of the random variable $\log(S_t/S_s)$ follows a normal distribution with mean $(\mu - \sigma^2/2)(t-s)$ and variance $\sigma^2(t-s)$,

$$\log\left(\frac{S_t}{S_s}\right) \sim N((\mu - \sigma^2/2)(t-s), \sigma^2(t-s)).$$

Now we consider a new process called the discounted price process $\{\widetilde{S}_t : t \in [0, T]\}$,

$$\widetilde{S}_t = e^{-rt} S_t.$$

Note $\widetilde{S}_t$ is the quantity that we need to invest today with the risk-free rate r to get S_t at time t. The risk-free rate is the rate of return of an investment with no risk of financial loss.

Let f be the function $f(t, x) = e^{-rt}x$. Its partial derivatives are

$$f_t(t, x) = -re^{-rt}x, \quad f_x(t, x) = e^{-rt}, \quad f_{xx} = 0$$

and we are able to apply the Itô's formula to compute $d\widetilde{S}_t$

$$\begin{aligned} d\widetilde{S}_t &= df(t, S_t) = -re^{-rt}S_t dt + e^{-rt}dS_t \\ &= -re^{-rt}S_t dt + \mu e^{-rt}S_t dt + \sigma e^{-rt}S_t dB_t \\ &= \widetilde{S}_t((\mu - r)dt + \sigma dB_t) \\ &= \sigma\widetilde{S}_t\left(\left(\frac{\mu - r}{\sigma}\right) + dB_t\right) \\ &= \sigma\widetilde{S}_t dB_t^* \end{aligned}$$

where B_t^* is

$$B_t^* = \frac{\mu - r}{\sigma} + B_t.$$

Note B_t^* is a Brownian motion with respect to a new probability P^* equivalent to P due to Girsanov theorem. Finally the main result is that the stochastic process

$$\widetilde{S}_t = S_0 + \sigma \int_0^t \widetilde{S}_s \, dB_s^*$$

is a P^*-martingale by Remark 3.3.

Definition 3.10. A continous time **portfolio** with the risky asset S_t and the riskless asset S_t^0 is a stochastic process

$$\Phi = \{\Phi_t = (D_t, H_t) : t \in [0, T]\}$$

where D_t is the amount of riskless asset and H_t is the amount of the risky asset at time t. We assume that both are continuos adapted processes, such that

$$\int_0^T |D_t| \, dt < \infty \text{ a.s,} \quad \int_0^T H_t^2 \, dt < \infty \text{ a.s.}$$

Naturally, the value of the portfolio at time t is

$$V_t = D_t e^{rt} + H_t S_t.$$

Definition 3.11. Let Φ be a portfolio. If it holds that $V_t \geq 0$, for all $t \in [0, T]$, we say that the portfolio is **admissible**. And it is said that the portfolio is self-financing if for each $t \in [0, T]$ we have

$$dV_t = D_t dS_t^0 + H_t dS_t.$$

We follow defining the discounted value of the portfolio $\widetilde{V}_t$ in the natural way,

$$\widetilde{V}_t = e^{-rt} V_t.$$

If the portfolio Φ is self-financing we have that $\widetilde{V}_t$ is a P^*-martingale. To check it, we will use the Itô's formula again and we will see that dV_t only depends on some Brownian motion.

$$\begin{aligned}
d\widetilde{V}_t &= -re^{-rt}V_t dt + e^{-rt}(D_t dS_t^0 + H_t DS_t) \\
&= -re^{-rt}V_t dt + re^{-rt}D_t S_t^0 dt + e^{-rt}H_t dS_t \\
&= -re^{-rt}(V_t - D_t e^{rt})dt + e^{-rt}H_t dS_t \\
&= -re^{-rt}H_t S_t dt + e^{-rt}H_t dS_t \\
&= H_t(-re^{-rt}S_t dt + e^{-rt}dS_t) \\
&= H_t d\widetilde{S}_t = \sigma H_t \widetilde{S}_t dB_t^*.
\end{aligned}$$

Thus $\widetilde{V}_t$ is a P^*-martingale.

Definition 3.12. We say that a positive random variable X is **replicable** if there exists and admissible portfolio $\{\Phi_t : t \in [0,T]\}$ such that the value of the portfolio at time T is equal to the random variable X, $V_T = X$.

Definition 3.13. It is said that a model for a market is **complete** if any positive random variable X is replicable.

The aim of this part is the following theorem. We will not prove it because the principal idea is to compute the value of some financial derivatives.

Theorem 3.3. *The Black-Scholes model is complete. It means, under de probability P^* and for every non negative random variable X that is $\mathcal{F}_T$-measurable, there exists an admissible self-financing portfolio Φ such that $V_T = X$.*

Because of the stochastic process $\widetilde{V}_t$ is a P^*-martingale we can compute the value of the portfolio at any time t as

$$\widetilde{V}_t = E^*[\widetilde{V}_T|\mathcal{F}_t] = e^{-rT}E^*[V_T|\mathcal{F}_t] = e^{-rT}E^*[X|\mathcal{F}_t]$$

and therefore we get finally

$$V_t = e^{-r(T-t)}E^*[X|\mathcal{F}_t].$$

In particular, the value of the portfolio at time 0 does not need the condicional expectation (we know the stock price at time 0 and $\mathcal{F}_0 = \{\emptyset, \Omega\}$) and its value is interesting because it will be the premium we will have to pay for an option

$$V_0 = e^{-rT}E^*[X].$$

3.4 Pricing options with the Black-Scholes model.

Now it is time to compute the value of a European call and a European put option at any time $t \in [0,T]$ if we suppose the Black-Scholes model. We have to apply Theorem 3.3. and we are done.
We recall that the pay-off for a European call is

$$X = C_T = (S_T - K)^+ = (S_T - K)\mathbb{1}_{\{S_T>K\}} = S_T\mathbb{1}_{\{S_T>K\}} - K\mathbb{1}_{\{S_T>K\}}$$

and thus

$$C_t = e^{-r(T-t)}\left(E[S_T\mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] - KE[\mathbb{1}_{S_T>K}|\mathcal{F}_t]\right)$$

The price of the asset follows a Geometric Brownian motion and we can modify the expression $S_T > K$ to get an inequality with a standard normal distribution Z. Let t be such that $0 \leq t \leq T$ and we know all the information until time t what it means we know the value of S_t. Therefore

$$\begin{aligned}
\frac{S_T}{S_t} > \frac{K}{S_t} &\iff e^{(r-\sigma^2/2)(T-t)+\sigma(B_T-B_T)} > \frac{K}{S_t} \\
&\iff (r-\sigma^2/2)(T-t) + \sigma(B_T - B_t) > \log(K/S_t) \\
&\iff Z > \frac{\log(K/S_t) - (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\
&\iff Z < \frac{\log(S_t/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}$$

To make the computations easier we denote this quantity like d_2

$$d_2 = \frac{\log(S_t/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

We call another quantity as d_1 with the value

$$d_1 = d_2 + \sigma\sqrt{T-t} = \frac{\log(S_t/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

To get the value of a European call option, we will compute both expectations separately. We begin with the easiest one.

$$E[\mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] = E[\mathbb{1}_{\{S_T>K\}}] = P(S_T > K) = P(Z < d_2) = \Phi(d_2)$$

where Φ is the cumulative standard normal distribution. And here we have the computations on the other expectation.

$$\begin{aligned}
E[S_T \mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] &= S_t E\left[\frac{S_T}{S_t}\mathbb{1}_{\{S_T>K\}}\middle|\mathcal{F}_t\right] \\
&= S_t E[e^{(r-\sigma^2/2)(T-t)+\sigma(B_T-B_t)}\mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] \\
&= S_t e^{(r-\sigma^2/2)(T-t)} E[e^{\sigma(B_T-B_t)}\mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] \\
&= S_t e^{(r-\sigma^2/2)(T-t)} E[e^{\sigma(B_T-B_t)}\mathbb{1}_{\{S_T>K\}}] \\
&= S_t e^{(r-\sigma^2/2)(T-t)} E[e^{\sigma\sqrt{T-t}Z}\mathbb{1}_{\{Z<d_2\}}]
\end{aligned}$$

For this expectation we will use directily the definition,

$$\begin{aligned}
E[e^{\sigma\sqrt{T-t}Z}\mathbb{1}_{\{Z<d_2\}}] &= \int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} e^{\sigma\sqrt{T-t}\,x}\,dx \\
&= e^{\frac{\sigma^2}{2}(T-t)} \int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\sigma\sqrt{T-t})^2}{2}}\,dx \\
&= e^{\frac{\sigma^2}{2}(T-t)} \int_{-\infty}^{d_2+\sigma\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx \\
&= e^{\frac{\sigma^2}{2}(T-t)}\Phi(d_1).
\end{aligned}$$

what it finally does

$$E[S_T \mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] = S_t e^{r(T-t)}\Phi(d_1).$$

With all these calculations the value of the European call option at time t, $0 \leq t \leq T$ is

$$C_t = e^{-r(T-t)}(S_t e^{r(T-t)}\Phi(d_1) - K\Phi(d_2)) = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2).$$

The price of a European call option has been obtained. The price for a European put option can be computed with the Put-Call parity and thus its value is

$$\begin{aligned}
P_t &= C_t - S_t + Ke^{-r(T-t)} \\
&= S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) - S_t + Ke^{-r(T-t)} \\
&= S_t(\Phi(d_1) - 1) + Ke^{-r(T-t)}(1 - \Phi(d_2)) \\
&= -S_t\Phi(-d_1) + Ke^{-r(T-t)}\Phi(-d_2).
\end{aligned}$$

The other financial option we mentioned is the cash-or-nothing option. Its payoff is the random variable $X = \mathbb{1}_{\{S_T>K\}}$ and hence the value for this option is

$$V_t = e^{-r(T-t)}E[\mathbb{1}_{\{S_T>K\}}|\mathcal{F}_t] = e^{-r(T-t)}\Phi(d_2).$$

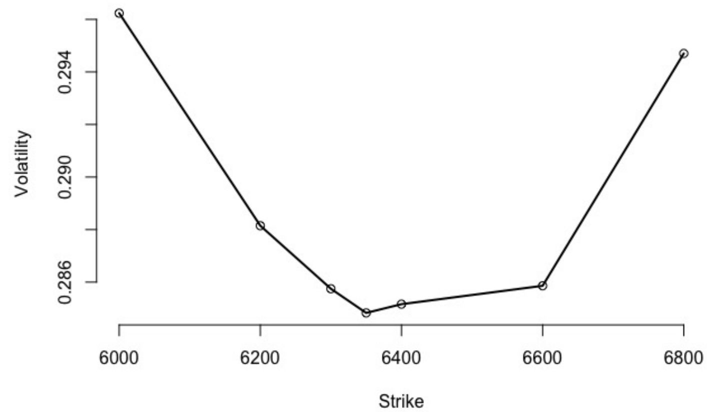

Figure 7: Implied volatility.

3.5 Heston model.

The evolution of the price follows a geometric Brownian motion according to Black-Scholes model. For this model we assume that the volatily is constant. However, it can be proved that this is not true. It seems that the volatility changes over time. This is simple to check due to the implied volatility. To know more about this matter, you can see it on [5, HULL].

Since the price of the derivatives are known on the market we can construct an implicit equation fixing the parameters S_t, T, r, K. In another way, if we can observe the price of a derivative, we are able to give the implied volatility for certain parameters. Figure 7 shows the implied volatility according to the Strike and we can see that the volatility is not constant. It is moved between 0.28 to 0.3. This shows that the assumption of a constant volatility in Black-Scholes model is wrong.

In 1993 Heston introduced another model. Here we assume that the volatility is not constant, but follows a random process. The volatility is denoted by u_t and x_t will be the log-asset price $x_t = \log(S_t/K)$. Here we have the stochastic differential equation

$$\begin{cases} dx_t &= (\mu - \frac{1}{2}u_t)x_t dt + \sqrt{u_t} x_t dB_{1t} \\ du_t &= \lambda(\bar{u} - u_t)dt + \eta\sqrt{u_t} dB_{2t} \end{cases}$$

The parameters $\lambda \geq 0, \bar{u} \geq 0, \eta \geq 0$ are called the speed of mean reversion, the mean level of variance, and the volatility of volatility, repectively. Moreover we assume the two Brownian motions B_{1t}, B_{2t} are correlated with correlation coefficient ρ.

As we have said this model has no analytic formulas to compute the value of an option as the Back-Scholes model has. This thing motivates the use of numerical analysis to obtain a value for a specific derivative. The following section introduce a new numerical method for this aim.

4 A numerical method for option pricing. SWIFT.

4.1 Exposition of the SWIFT method.

We have arrived to the final section. Here we will explain a numerical method to compute the value of European options which is called SWIFT (Shannon wavelet inverse Fourier technique). This method was introduced by Ortiz-Gracia and Oosterlee in [8, Ort16]. We remember that in the first section we have developed the theory of Hilbert spaces and wavelets theory to be able to understand the Shannon wavelet.

The main idea in the SWIFT method is to expand a density function f in terms of the Shannon scaling function ϕ

$$\phi(x) = \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x},$$

and therefore we will use

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k) = 2^{j/2}\frac{\sin(\pi(2^j x - k))}{\pi(2^j x - k)}.$$

In Section 2 we have seen the value of a European option is computed as the expectation of a pay-off multiplied by the discounted factor. The formula is

$$v(x,t) = e^{-r(T-t)}\int_{\mathbb{R}} v(y,T)f(y|x)\,dy,$$

where v is the option value, T is the maturity, t is the initial date and $f(y|x)$ is the probability function of y given x. The variables x, y denote the log-asset prices at time t and T respectively

$$x = \log(S_t/K), \quad y = \log(S_T/K).$$

To show an example of the function v at maturity if we consider a European call option we have that the pay-off function is

$$v(y,T) = (S_T - K)^+ = (K(e^y - 1))^+.$$

It is important to recall that whereas f is typically unknown, the characteristic function of the log-asset price is usually avaliable as the Fourier transform of f.

Let $f \in L^2(\mathbb{R})$ be the density function of a certain random variable X. We have defined the Fourier transform as

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx.$$

The characteristic function of f is defined in probability as

$$\varphi(t) = E[e^{itX}]$$

which differs from a sign with the definition of the Fourier transform,

$$\varphi(t) = E[e^{itX}] = \int_{\mathbb{R}} f(x) e^{itx}\, dx = \hat{f}(-t).$$

Following the wavelets theory, the density function f can be approximated at a level of resolution j by the Projection operator

$$f(x) \approx P_j f(x) = \sum_{k \in \mathbb{Z}} c_{j,k} \phi_{j,k}(x)$$

where $P_j f$ converges to f in $L^2(\mathbb{R})$ as j tends to infinity :

$$\|f - P_j f\|_2 \xrightarrow{j \to \infty} 0.$$

Following the multiresolution analysis theory we have that

$$\|f - P_j f\|_2^2 = \sum_{i \geq j} \sum_{k \in \mathbb{Z}} |d_{i,k}|^2$$

where $d_{i,k} =< f, \psi_{i,k} >$. By Parseval's identity we can develope this scalar product. Here we will use the Fourier transform of the Shannon wavelet

$$\hat{\psi}(\xi) = e^{-i\frac{\xi}{2}} \chi_I(\xi), \quad I = [-2\pi, -\pi) \cup (\pi, 2\pi]$$

and by Proposition 2.4. we have

$$\hat{\psi}_{j,k}(\xi) = e^{-i2^{-j}k\xi} 2^{-j/2} \hat{\psi}(2^{-j}\xi) = e^{-i\xi(k+1/2)2^{-j}} 2^{-j/2} \chi_{I_{j,k}}(\xi)$$

where $I_{j,k} = [-2^{j+1}\pi, -2^j\pi) \cup (2^j\pi, 2^{j+1}\pi]$. Thus we finally get

$$\begin{aligned} < f, \psi_{j,k} > &= \frac{1}{2\pi} \left\langle \hat{f}, \hat{\psi}_{j,k} \right\rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) \hat{\psi}_{j,k}(\xi)\, d\xi \\ &= \frac{2^{-j/2}}{2\pi} \left(\int_{-2^{j+1}\pi}^{-2^j\pi} \hat{f}(\xi) e^{i\xi(k+1/2)2^{-j}}\, d\xi + \int_{2^j\pi}^{2^{j+1}\pi} \hat{f}(\xi) e^{i\xi(k+1/2)2^{-j}}\, d\xi \right). \end{aligned}$$

Since

$$|d_{i,k}| \leq \frac{2^{-i/2}}{2} 2^i \left(\max_{\xi \in [-2^{i+1}\pi, -2^i\pi)} |\hat{f}(\xi)| + \max_{\xi \in [2^i\pi, 2^{i+1}\pi)} |\hat{f}(\xi)| \right)$$

we deduce that the approximation of f by the projection operator is highly dependent on the rate of decay of the Fourier transform $\hat{f}$ of f. Anyway, this approximation is defined as an infinite sum of some functions $\phi_{j,k}$ and this does not interest us. We would like to get some interval for which we could sum finite values without loss of considerable density mass. The following lemma guarantees we can do that.

Lemma 4.1. *Let f be a function that tends to zero at plus and minus infinite. Then $f\left(\frac{h}{2^j}\right) \approx 2^{j/2} c_{j,h}$, $h \in \mathbb{Z}$, and it is verified that*

$$\lim_{h \to \pm\infty} c_{j,h} = 0.$$

Proof. We take $h \in \mathbb{Z}$, then

$$f\left(\frac{h}{2^j}\right) \approx P_j f\left(\frac{h}{2^j}\right) = \sum_{k \in \mathbb{Z}} c_{j,k} \phi_{j,k} \left(\frac{h}{2^j}\right) = 2^{j/2} \sum_{k \in \mathbb{Z}} c_{j,k} \phi(h-k)$$

and the value of the Shannon scalling function for integer numbers are

$$\phi(h-k) = \frac{\sin(\pi(h-k))}{\pi(h-k)} = \delta_{h,k}.$$

Therefore

$$f\left(\frac{h}{2^j}\right) \approx 2^{j/2} \sum_{k \in \mathbb{Z}} c_{j,k} \delta_{h,k} = 2^{j/2} c_{j,h}.$$

Since we assume that the function f tends to zero at infinite we obtain that

$$\lim_{h \to \pm\infty} c_{j,h} = 0.$$

□

As we have said, the Lemma 4.1 assures that we can approximate well the function f by a finite sum without loss of considerable density mass,

$$f(x) \approx P_j f(x) \approx f_j(x) = \sum_{k=k_1}^{k_2} c_{j,k} \phi_{j,k}(x).$$

Once the sum is limited between two integer numbers k_1, k_2 we have to compute the scaling coefficients $c_{j,k}$. We recall that these coefficients are computed by a scalar product

$$c_{j,k} =< f, \phi_{j,k} >= \int_{\mathbb{R}} f(x)\phi_{j,k}(x)\, dx = 2^{j/2} \int_{\mathbb{R}} f(x)\phi(2^j x - k)\, dx.$$

Also we define the integers k_1 and k_2 as the smallest integers such that

$$\frac{k_1}{2^j} \leq a \leq b \leq \frac{k_2}{2^j}$$

where a, b are numbers which conserve enough mass of the density function in the sense

$$\int_{\mathbb{R}} f(x)\, dx \approx \int_a^b f(x)\, dx.$$

These numbers were calculated in the article [3, Fan08] using the nth cumulant of $\log(S_T/K)$, and S_T will depend on the financial model we are working on. Therefore the interval $[a, b]$ is

$$[a, b] = \left[c_1 - L\sqrt{c_2 + \sqrt{c_4}}, c_1 + L\sqrt{c_2 + \sqrt{c_4}} \right], \quad L = 10.$$

Remark 4.1. Let X be a random variable. The cumulant generating function $K(t)$ of X is defined as

$$K(t) = \log E\left[e^{tX}\right],$$

and the nth cumulant denoted by c_n is

$$c_n = K^{n)}(0).$$

Finally with these assumptions the main idea of this method is the following. We set the interval $I_j = [\frac{k_1}{2^j}, \frac{k_2}{2^j}]$ as we have said. Then the price of the option at time t can be approximated as

$$v(x, t) = e^{-r(T-t)} \int_{\mathbb{R}} v(y, T) f(y|x)\, dy \approx e^{-r(T-t)} \sum_{k=k_1}^{k_2} c_{j,k}(x) V_{m,k},$$

where $V_{j,k}$ is called the pay-off coefficient

$$V_{j,k} = \int_{I_j} v(y, T)\phi_{j,k}(y)\, dy = 2^{j/2} \int_{I_j} v(y, T)\phi(2^j y - k)\, dy.$$

To be able to continue and compute the scaling coefficients $c_{j,k}(x)$, we will apply the Vieta's formula. This formula can be used to write the cardinal sine as an infinite product of cosines and as we did to approximate the function f we limit this product by a finite product,

$$\operatorname{sinc}(x) = \prod_{l=1}^{\infty} \cos\left(\frac{\pi x}{2^l}\right) \approx \prod_{l=1}^{L} \cos\left(\frac{\pi x}{2^l}\right).$$

Thanks to the cosine product-to-sum identity

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$$

we transform the product into a sum

$$\prod_{l=1}^{L} \cos\left(\frac{\pi x}{2^l}\right) = \frac{1}{2^{L-1}} \sum_{l=1}^{2^{L-1}} \cos\left(\frac{2l-1}{2^L}\pi x\right).$$

In conclusion, we can approximate the cardinal sinus function as the following sum

$$\operatorname{sinc}(x) \approx \operatorname{sinc}(x)^* = \frac{1}{2^{L-1}} \sum_{l=1}^{2^{L-1}} \cos\left(\frac{2l-1}{2^L}\pi x\right).$$

Then we will give a lemma to be able to estimate the error for this approximation of the cardinal sinus function.

Lemma 4.2. *We define the following error $\varepsilon(x) = sinc(x) - sinc^*(x)$. Then*

$$|\varepsilon(x)| \le \frac{(\pi c)^2}{2^{2(L+1)} - (\pi c)^2}$$

for $x \in [-c, c]$, where $c \in \mathbb{R}$, $c > 0$, and $L \ge \log_2(\pi c)$.

Proof. Taking into account that

$$\operatorname{sinc}\left(\frac{x}{2^L}\right) = \prod_{l=1}^{\infty} \cos\left(\frac{\pi x}{2^{L+l}}\right) = \prod_{l=L+1}^{\infty} \cos\left(\frac{\pi x}{2^l}\right)$$

we obtain the expression

$$\operatorname{sinc}(x) = \operatorname{sinc}\left(\frac{x}{2^L}\right) \prod_{l=1}^{L} \cos\left(\frac{\pi x}{2^l}\right) = \operatorname{sinc}\left(\frac{\pi x}{2^L}\right) \operatorname{sinc}^*(x).$$

Therefore
$$|\varepsilon(x)| = \left|\operatorname{sinc}\left(\frac{\pi x}{2^L}\right) - 1\right| \left|\operatorname{sinc}^*(x)\right| \leq \left|\operatorname{sinc}\left(\frac{\pi x}{2^L}\right) - 1\right|.$$
The Taylor serie expansion for the cardinal sinus is

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x} = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n+1)!},$$

and we can use this serie to get

$$|\varepsilon(x)| \leq \sum_{n=1}^{\infty} \frac{\pi^{2n}}{(2n+1)2^{2Ln}} |x|^{2n} \leq \sum_{n=1}^{\infty} \left(\frac{\pi c}{2^{L+1}}\right)^{2n},$$

if we consider $|x| < c$, with $c > 0$, and observe that $(2n+1)! > 2n$. The error $\varepsilon(x)$ is narrowed by a geometric serie and since $L \geq \log_2(\pi c)$ we have that $\frac{\pi c}{2^{J+1}} < 1$ and thus the serie converges to the value

$$|\varepsilon(x)| \leq \frac{\frac{(\pi c)^2}{2^{2(L+1)}}}{1 - \frac{(\pi c)^2}{2^{2(L+1)}}} = \frac{(\pi c)^2}{2^{2(L+1)} - (\pi c)^2}.$$

□

This lemma can be applied for choosing the value of L. We define $M_{j,k} = \max\{|2^j a - k|, |2^j b + k|\}$ and then we take $M_j = \max_{k_1 \leq k \leq k_2} M_{j,k}$. The value of L will be $L = \ell = \lceil \log_2(\pi M_j) \rceil$ where $\lceil x \rceil$ denotes the smallest integer number greater or equal than x. The proof can be seen by Theorem 1 in [8, Ort16].

With all of this information, we can replace the function $\phi(x) = \operatorname{sinc}(x)$ in the integral of the scaling coefficient $c_{j,k}$ to get another approximation

$$c_{j,k} \approx c^*_{j,k} = \frac{2^{j/2}}{2^{L-1}} \sum_{l=1}^{2^{L-1}} \int_{\mathbb{R}} f(x) \cos\left(\frac{2l-1}{2^L}\pi(2^j x - k)\right) dx$$

Taking into account that $\Re(\hat{f}(\xi)) = \int_{\mathbb{R}} f(x) cos(x\xi)\, dx$ we have on the one hand that

$$\hat{f}(\xi) e^{ik\pi \frac{2l-1}{2^L}} = \int_{\mathbb{R}} f(x) e^{-i\left(x\xi - k\pi\frac{2l-1}{2^L}\right)} dx$$

and therefore

$$\hat{f}\left(\frac{2l-1}{2^L}\pi 2^j\right) e^{ik\pi\frac{2l-1}{2^L}} = \int_{\mathbb{R}} f(x) e^{-i\left(\frac{2l-1}{2^L}\pi(2^j x - k)\right)} dx.$$

On the ohter hand the real part of this Fourier transform is used to compute the approximation of the scaling coefficients,

$$c_{j,k} \approx c^*_{j,k} = \frac{2^{j/2}}{2^{L-1}} \sum_{l=1}^{2^{L-1}} \Re \left(\hat{f} \left(\frac{2l-1}{2^L} \pi 2^j \right) e^{\frac{ik\pi(2l-1)}{2^L}} \right)$$

For the rest of this work we will give some computations for different options, in particular for a cash-or-nothing option and for a European call option. We have shown how to proceed with the scaling coefficients. Now, for each option we will have an integral which depends on the value of the pay-off. This allows to find the value of the pay-off coefficient $V_{j,k}$.

4.2 Cash-or-nothing option pricing.

This option has the following payoff

$$X = \mathbb{1}_{\{S_T > K\}}.$$

This function can be written by the pay-off $v(y, T)$ in the log-asset space as

$$v(y, T) = \begin{cases} 1 & y > 0, \\ 0 & y < 0. \end{cases}$$

So if we replace the function f by its approximation f_j for a certain j we have that the value for this option for every time t

$$\begin{aligned} v(x, t) &= e^{-r(T-t)} \int_{\mathbb{R}} v(y, T) f(y|x) \, dy = e^{-r(T-t)} \int_0^\infty f(y|x) \, dy \\ &\approx e^{-r(T-t)} \int_0^\infty f_j(y|x) \, dy = e^{-r(T-t)} \sum_{k=k_1}^{k_2} \left(c_{j,k}(x) \int_0^\infty \phi_{j,k}(y) \, dy \right) \\ &= e^{-r(T-t)} \sum_{k=k_1}^{k_2} c_{j,k}(x) V_{j,k} \end{aligned}$$

The value $V_{j,k} = \int_0^\infty \phi_{j,k} \, dy$ is the pay-off coefficient. The following lemma can be applied to calculate these coefficients.

Lemma 4.3. *The pay-off coefficient $V_{j,k}$ for a cash-or-nothing option is*

$$V_{j,k} = 2^{-j/2} \left(sign(k) Si(|k|) + \frac{1}{2} \right)$$

where sign is the sign function

$$sign(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

and Si is the sine integral function

$$Si(x) = \int_0^x sinc(t)\, dt.$$

Proof.

$$\begin{aligned} V_{j,k} &= 2^{j/2} \int_0^\infty \phi_{j,k}(y)\, dy = 2^{j/2} \int_0^\infty \phi(2^j y - k)\, dy = 2^{-j/2} \int_{-k}^\infty \phi(x)\, dx = 2^{-j/2} \int_{-k}^\infty \frac{\sin(\pi x)}{\pi x}\, dx \\ &= 2^{-j/2} \left(\int_{-k}^0 \frac{\sin(\pi x)}{\pi x}\, dx + \int_0^\infty \frac{\sin(\pi x)}{\pi x}\, dx \right) = 2^{-j/2} \left(\int_{-k}^0 \mathrm{sinc}(x)\, dx + \frac{1}{2} \right) \\ &= 2^{-j/2} \left(\mathrm{sign}(k) \mathrm{Si}(|k|) + \frac{1}{2} \right) \end{aligned}$$

□

Anyway, this integral should be computed by a finite sum, so we can replace the value in the sinc integral function Si by the approximation of the cardinal sinus we have done with the Vieta's formula. Finally the value of the function Si will be calculated as follows

$$\begin{aligned} \mathrm{Si}(x) \approx \mathrm{Si}^*(x) &= \int_0^x \mathrm{sinc}(x)\, dx = \frac{1}{2^{L-1}} \sum_{l=1}^{2^{L-1}} \left(\int_0^x \cos\left(\frac{2l-1}{2^L} \pi x \right) dx \right) \\ &= \frac{2}{\pi} \sum_{l=1}^{2^{L-1}} \frac{1}{2l-1} \sin\left(\frac{2l-1}{2^L} \pi x \right). \end{aligned}$$

As we did with the density coefficients, we consider now a new value L for the approximation of the sine integral function Si, which is different of the constant L of the approximation of the scaling coefficient. Then, we define this new value as $\bar{\ell} = \lceil \log_2(\pi M) \rceil$, where $M = \max\{|k_1|, |k_2|\}$. This proof is done in Proposition 1 in [8, Ort16].

4.3 European call option pricing.

In this part we will price a European call option. This option has as pay-off the random variable

$$X = (S_T - K)^+$$

and the function $v(y,T)$ in the log-asset space will be

$$v(y,T) = \begin{cases} K(e^y - 1) & y > 0 \\ 0 & y < 0 \end{cases}$$

We truncate the real line to a finite domain $I_j = [\frac{k_1}{2^j}, \frac{k_2}{2^j}]$ as we have justified before due to Lemma 4.1. Thus, the value of the call option at time t is approximated by

$$\begin{aligned} v(x,t) &\approx e^{-r(T-t)} \int_{I_j} v(y,T) f_j(y|x)\, dy = e^{-r(T-t)} \int_{I_j \cap [0,\infty)} K(e^y - 1) f_j(y|x)\, dy \\ &= e^{-r(T-t)} \sum_{k=k_1}^{k_2} c_{j,k}(x) V_{j,k} \end{aligned}$$

with $V_{j,k}$ the pay-off coefficient which is

$$V_{j,k} = K 2^{j/2} \left(\int_{I_j \cap [0,\infty)} e^y \phi(2^j y - k)\, dy - \int_{I_j \cap [0,\infty)} \phi(2^j y - k)\, dy \right).$$

Let us define the following integrals

$$I_1(a,b) = \int_a^b e^y \cos(C_l(2^j y - k))\, dy,$$

$$I_2(a,b) = \int_a^b \cos(C_l(2^j y - k))\, dy$$

where $C_l = \frac{2l-1}{2^L}\pi$. Therefore, using the approximation of the sinus cardinal we obtain now the approximation for the pay-off coefficient by a finite sum

$$V_{j,k} \approx V_{j,k}^* = \frac{K 2^{j/2}}{2^{L-1}} \sum_{l=1}^{2^{L-1}} \left(I_1\left(\frac{\bar{k}_1}{2^j}, \frac{k_2}{2^j}\right) - I_2\left(\frac{\bar{k}_1}{2^j}, \frac{k_2}{2^j}\right) \right)$$

where $\bar{k_1} = \max\{k_1, 0\}$. Both integrals, $I_1(a,b), I_2(a,b)$ are easy to compute. Of course the second one can be calculated directly and the first integral $I_1(a,b)$ has to be computed by integration by parts. The results of both integrals are

$$I_1(a,b) = \frac{1}{1+(C_l 2^j)^2}\Big(C_l 2^j \Big(e^b \sin(C_l(2^j b - k)) - e^a \sin(C_l(2^j a - k))\Big) \\ + e^b \cos(C_l(2^j b - k)) - e^a \cos(C_l(2^j a - k))\Big)$$

$$I_2(a,b) = \frac{1}{C_l 2^j}(\sin(C_l(2^j b - k)) - \sin(C_l(2^j a - k)))$$

Finally, as we did in the case for the cash-or-nothing option we have to narrow now the value L for the pay-off coefficient. Then we define $\bar{\ell} = \lceil \log_2(\pi N) \rceil$, where $N = \max_{k_1 \leq k \leq k_2} N_k$, and $N_k = \max\{|\bar{k_1} - k|, |k_2 - k|\}$. This proof can be seen in Proposition 3 in [8, Ort16].

4.4 Numerical results.

As we have said in the presentation of the SWIFT method, whereas the density function f is not known, the function of the log-asset price is available as the Fourier transform of f. The two models that have been treated are the Black-Scholes and Heston model.

The characteristic function for the Black-Scholes model (we recall that the stock price follows a Geometric Brownian motion GBM) is

$$\hat{f}_{GBM}(\omega, x) = \exp(-i\omega x) \exp\left(-i\omega\left(r - \frac{1}{2}\sigma^2\right)(T-t) - \frac{1}{2}\sigma^2\omega^2(T-t)\right)$$

where r is the risk-free rate, σ the volatility, T the maturity, and $x = \log(S_t/K)$ is the log-asset price at time t.

The Heston model has a characteristic function the following function,

$$\hat{f}(\omega, x, u_o) = \exp(-i\omega x)\exp\left(-i\omega\mu(T-t) + \frac{u_0}{\eta^2}\left(\frac{1-e^{-D(T-t)}}{1-Ge^{-D(T-t)}}\right)(\lambda + i\rho\eta\omega - D)\right) \\ \cdot \exp\left(\frac{\lambda\bar{u}}{\eta^2}\left((\lambda + i\rho\eta\omega - D)(T-t) - 2\log\left(\frac{1-Ge^{-D(T-t)}}{1-G}\right)\right)\right)$$

with the values D, G

$$D = \sqrt{(\lambda + i\rho\eta\omega) + (\omega^2 - i\omega)\eta^2}, \quad G = \frac{\lambda + i\rho\eta\omega - D}{\lambda + i\rho\eta\omega + D}.$$

The other parameters are T the maturity, x the log-asset price at time t, λ the speed of mean reversion, $\bar{u}$ the mean level of variance, η the volatility of volatility and u_0 the volatility of the underlying asset at initial time. For this case we have called μ the risk-free rate.

In this section we will use the SWIFT method to calculate the price of a cash-or-nothing and a European call option. Both options will be computed with both models. We remember that the SWIFT method needs some integers k_1 and k_2 for a specific j level of multiresolutional analysis. These parameters are useful to obtain the most quantity of mass of the density function. Another two important parameters are ℓ and $\bar{\ell}$ which are the truncation values for the sum on the scaling coefficients and on the pay-off coefficients respectively. There will be two values for $\bar{\ell}$, one for the cash-or-nothing and other for the European call.

The next step is to change the parameter j from 1 until another number (5 or 6) for a certain K, and with these previous parameters $k_1, k_2, \ell, \bar{\ell}$, we will obtain the price for the option. Let us begin with the Black-Scholes model.

We recall that to get the integer values k_1 and k_2 we need to compute first of all an interval $[a, b]$ and these values depend on the nth cumulants. This was explained on page 46. For the Black-Scholes we know the behaviour of S_t,

$$S_t = S_0 e^{(r-\sigma^2/2)t+\sigma B_t}, \quad B_t \sim N(0,t).$$

But for this model we are using the log-asset price $X = \log(S_t/K)$. To begin with this aim, we give the cumulant generating function which is

$$\begin{aligned} K(s) &= \log E\left[e^{sX}\right] = \log E\left[\left(\frac{S_t}{K}\right)^s\right] = \log E\left[\left(\frac{S_0}{K}\right)^s e^{(r-\sigma^2/2)ts+\sigma B_t s}\right] \\ &= sx_0 + \left(r - \frac{\sigma^2}{2}\right)ts + \frac{\sigma^2 t}{2}s^2 \end{aligned}$$

where $x_0 = \log(S_0/K)$. The derivatives of $K(s)$ are

$$\begin{cases} K'(s) & = x_0 + \left(r - \frac{\sigma^2}{2}\right)t + \sigma^2 ts \\ K''(s) & = \sigma^2 t \\ K^{n)}(s) & = 0, \quad \forall s \geq 3 \end{cases}$$

and therefore the cumulants we will use are

$$c_1 = K'(0) = x_0 + \left(r - \frac{\sigma^2}{2}\right)t, \quad c_2 = K''(0) = \sigma^2 t.$$

The interval $[a, b]$ is computed as

$$[a, b] = \left[x_0 + \left(r + \frac{\sigma^2}{2}\right)t + 10\sigma\sqrt{t}, x_0 + \left(r + \frac{\sigma^2}{2}\right)t - 10\sigma\sqrt{t}\right].$$

This allows to get the parameters k_1, k_2, and moreover $\ell, \bar{\ell}$.

We set now the following parameters for the Black-Scholes model,

$$r = 0.1, \quad T = 0.1, \quad \sigma = 0.25, \quad S_0 = 100, \quad K = 80.$$

Like for this model we have analytic formulas we will give the results first and will try to compare them with the values obtained with the SWIFT method. With the previous paramenters we get at time 0 for a European call C_0, and for a cash-or-nothing option CN_0 the numbers

$$C_0 = 20.79923, \quad CN_0 = 0.988258.$$

Let us see what happens with the numerical results for the cash-or-nothing.

j	k_1	k_2	ℓ	$\bar{\ell}$	SWIFT	error
1	-2	3	4	4	0.917466	0.070792
2	-3	5	5	4	1.026831	0.038573
3	-5	9	6	5	0.986008	0.00225
4	-9	17	7	6	0.988259	1.11778e-06
5	-18	33	8	7	0.988258	1.11022e-16

By theory we know that as we increase the value of j, the operator projection converges to the exact value. It seems that with $j = 5$ we obtain practically the value for a cash-or-nothing. In addition the absolut error of SWIFT and CN_0 has increased a lot between 4 and 5 which proves that the method converges very fast to the exact price.

Finally we will show what happens for a European call. In the following table we will not put the values for k_1, k_2, ℓ because they are the same as before. But we have to write the value for $\bar{\ell}$ because it depends on the new option. Here are the results,

j	$\bar{\ell}$	SWIFT	error
1	4	16.05752	4.74171
2	5	17.01339	3.78584
3	6	21.25494	0.455713
4	7	20.79991	0.000682
5	8	20.79923	2.13163e-14

The result obtained with $j = 4$ for a European call option is practically the exact value obtained with the Black-Scholes formula. As we can see the convergence of the method is very fast and we can conclude that the value for this concret option is equal to 20.79923.

To end with this numerical section, we will present some results for the Heston model. For this concrete model we do not have analytic formulas unlike we had in the previous model. Thus it would be necessary to compare with another numerical method to assure that the values we obtain with SWIFT are the correct ones. One simple method is called Euler-Maruyama for stochastic differential equations.

We suppose we have the following stochastic differential system,

$$\begin{cases} dX_{1t} &= \mu_{1t}dt + \sigma_{1t}dB_{1t} \\ dX_{2t} &= \mu_{2t}dt + \sigma_{2t}(\rho dB_{1t} + \sqrt{1-\rho^2}dB_{2t}) \end{cases}$$

whith ρ the correlation coefficient between the two Brownian motions for each equation. That is why we can wirte the second one as a relation between two independent Brownian motions B_{1t} and B_{2t} using the correlation ρ. Then we discretize the time interval $[0, T]$ in n equal parts and instead of having a continous time dt we will have a discrete time $\Delta t = \frac{T}{n}$. We suppose also that at time 0 we know the value of $X_{1,0}$ and $X_{2,0}$. Therefore the Euler-Maruyama method is to get all the values X_{1,t_i}, X_{2,t_i} with

$$t_i = t_0 + i\Delta t, \quad i = 1, \dots, n.$$

doing the following iterations from i equal to 1 until n,

$$\begin{cases} X_{1,t_{i+1}} &= X_{1,t_i} + \mu_{1t}\Delta t + \sigma_{1t}\Delta B_{1t} \\ X_{2,t_{i+1}} &= X_{2,t_i} + \mu_{2t}\Delta t + \sigma_{2t}(\rho\Delta B_{1t} + \sqrt{1-\rho^2}\Delta B_{2t}) \end{cases}$$

where $\Delta B_{1,t}$ and $\Delta B_{2,t}$ are two independent Brownian motions which follow a normal distribution with mean 0 and variance Δt.

Explained the Euler-Maruyama we go back to Heston model. The parameters we will use for this case will be

$$\mu = 0, \quad T = 0.1, \quad \lambda = 1.5768, \quad \eta = 0.5751, \quad \bar{u} = 0.0398,$$

$$S_0 = 100, \quad u_0 = 0.0175, \quad K = 100.$$

First of all we apply the Euler-Maruyama method for the stochastic differential system of Heston model exposed on page 41. We will repeat this method several times to get different prices at time T. Then we will use the concrete pay-off and with the discounted factor we will obtain a price for the option. This will make possible to give a confidence interval with confindence level of 95%. We recall that we are doing this to compare these results with the SWIFT method.

The confidence interval at 95% for the cash-or-nothing and the European call with the previous parameters are

$$\text{Cash-or-nohting:} \quad [0.514,\ 0.612],$$

$$\text{European call:} \quad [1.4781,\ 1.817].$$

Let us expose the numerical results with the SWIFT method. We will check if the result obtained is within the interval to know if we are proceeding well. Also, we will give one more value for j to see better if there is convergence.

For Heston model the computations of the cumulants are much more difficult than for the Black-Scholes model and because of this we will not write them. Anyway they have been programmed and it allows to get the integers $k_1, k_2, \ell, \bar{\ell}$. Do not forget that for $\bar{\ell}$ we will have two different values depending on the option. We begin with the cash-or-nothing. Here are the results

j	k_1	k_2	ℓ	$\bar{\ell}$	SWIFT	within
1	-1	1	3	2	0.499455	no
2	-2	2	4	3	0.496365	no
3	-4	4	5	4	0.503436	no
4	-7	7	6	5	0.543152	yes
5	-14	14	7	6	0.564314	yes
6	-28	28	8	7	0.566585	yes

In fact there is convergence and from the value $j = 4$ we can assure that the SWIFT method gives a correct price for the cash-or-nothing because these results are within $[0.514, 0.612]$. We could price this derivative as the amount of 0.566585 using the Heston model. Finally, here are the results for the European call,

j	$\bar{\ell}$	SWIFT	within
1	3	11.588814	no
2	4	0.0021	no
3	5	1.028843	no
4	6	1.672078	yes
5	7	1.663541	yes
6	8	1.636663	yes

We conclude the same as before. The method seems to converge to an element and from $j = 4$ the values are within the interval [1.4781, 1.817] with a 95% confidence level. The price for a call option with the Heston model could be the amount 1.636663.

5 Conclusions.

In this master thesis we have studied a new numerical method for computacional finance. This method is called SWIFT and to apply it, it is important to understand wavelets theory.
Wavelets are orthonormal basis that can be extended on the real line, what improves the fact to use Fourier series. There are several orthonormal wavelets but SWIFT uses one in particular: Shannon wavelets. These functions are smooth and that is why this wavelet has been chosen instead of the Haar basis.
The derivatives shown in this project are European derivatives. Also the cash-or-nothing option has been explained . The use of financial derivatives by traders makes important the tool of numerical analysis in mathematical finance. A lot of models try to model the evolution of the asset price, but we have exposed two very well-known models: Black-Scholes and Heston.
Black-Scholes is very simple but gives us fast results with easy and analytic formulas. Heston model tries to solve the problem with the constant volatility that Black-Scholes assumes, and thus we complicate the model and we do not obtain simple formulas. Like Heston is being used more and more by financial companies is important to develope strong numerical techniques for pricing derivatives.
Explained the SWIFT method we finally get some numerical results for some specific parameters depending on the model. By wavelet theory we know that the projection operator converges to the density function in $L^2(\mathbb{R})$. We have proved with our results that it is absolutely true. Moreover the convergence is very fast and only with value $j = 6$ we can obtain a price for the option. With the Black-Scholes model we were able to compare the numerical results, but for the Heston model we gave a confidence interval for which the exact price is within a 95% confidence level.
This SWIFT method is very useful in the sense that a lot of numerical methods use simulations. Monte Carlo is one of them. This method is very slow when we increase the quantity of simulations. Also with Monte Carlo we obtain different results because we generate random numbers. Here with SWIFT we get the same value with the same inputs, it never changes. And also the time for the computer is reduced because with a low level of multiresolution analysis we can price financial derivatives.
With all this knowledge some improvements could be done to continue studying another complex derviates like Asian options. Also it could check if there are other wavelets that improve the time of the CPU.

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