Master Thesis

MASTER IN ADVANCED MATHEMATICS

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Stochastic Volatility Models: Present, Past and Future

Author: Claudia Cristina Reyes Montes de Oca

Advisors:	**Dra. Elisa Alòs Alcalde (UPF)**
	Dr. Josep Vives (UB)
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Introduction

From ancient times to the present, the need to protect ourselves from the fluctuations of market prices forced to create ways to minimize the risk and its consequences, thus creating the options.

In 1900, Louis Bachelier first proposed to use the Brownian motion to model the dynamics of stock price in his dissertation *Théorie de la spéculation.* He was considered to be a pioneer in the study of financial mathematics. Also too, Robert Brown, Albert Einstein and Kiyosi Itô made contributions to the foundations of financial mathematics through the study of Brownian motion and stochastic integration theory. But it was not until in 1973 when Fisher Black and Myron Scholes published a fundamental paper on option pricing, where they introduced the Black-Scholes model [5]. In the same year, Robert Merton published a paper on the same topic independently [18]. The differences between these works are that the model introduced by Black and Scholes had a major effect on the world of finance since it gave an answer to the problem of pricing options.

The Black-Scholes model has important assumptions, these are that the underlying asset price process is continuous and that the volatility is constant. However, as we know, in the world of options, the second assumption cannot be considered realistic. The implied volatility is an important concept associated with an option price theory on an underlying asset, it is calculated by inverting an option price via the Black–Scholes price formula and it is simple to show that the implied volatility of an underlying asset is not constant but varies with the maturity time and with respect to the strike price. Associated with this concept we have the volatility smile, which is considered as one of the main problems of quantitative finance and the main tools involved with it can be, depending on the approach used, stochastic calculus and mathematical finance, partial differential equations, numerical analysis.

The Black-Scholes model does not fit real market data. To have a more realistic approach to the problem of option pricing, alternative models have been proposed, popular ones include, Merton's jump stochastic model [19] and stochastic volatility models. In the model proposed by Merton, he allowed the underlying assets to have random jumps to have more realistic behavior. This approach suggested by Merton gave rise to the development of what is now known as jump-diffusion models.

In the stochastic volatility models the volatility is described by a stochastic process. These models are used in order to price options where volatility varies over time and they are useful because they explain why options with different strikes and maturities have different Black-Scholes implied volatilities. One of the most interesting stochastic volatility models is Heston's model introduced in 1993, this model allows the spot and the volatility processes to have positive, negative or zero correlation. We have too, the fractional stochastic volatility models, in which the volatility may exhibit a long-range dependent or a antipersistent behavior.

In Chapter 1, we will introduce the Black-Scholes model and a brief introduction to quantitative finance concepts related to this model. In Chapter 2, we will talk about implied volatility and how to calculate it by numerical methods. In Chapter 3 we will introduce the stochastic volatility models and the jump volatility models studied by Hull and White in [12], Fouque, Papanicolau and Sircar in [8] and by Merton in [19]. In Chapter 4, we will introduce the statics and dynamics of implied volatility based on Lee's paper [16]. In addition, we will plot the volatility smile and volatility skew based on models introduced in Chapter 3.

In Chapter 5 we will introduce fractional Brownian motion, which has an important role in many fields, as meteorology, finance, telecommunications and hydrology, the last is because Hurst observed that Nile river water had a consistent cyclical behavior, which for seven consecutive years the water level increased and was greater than in the following seven years, which in turn created a cycle of seven years of abundance and seven years of scarcity. Until then, it was thought that there was no depending on the behavior of the increase in water between one year and another. In addition, we will introduce some concepts on Malliavin calculus to introduce the fractional volatility model studied by Alòs, León and Vives in [2].

Chapter 1

Black-Scholes Model

With Itô's stochastic calculus as the main tool we obtain the formula that is a solution to the Black-Scholes model [5], and we will introduce this model to solve problems of valuation and hedge of European options considering the prices in a continuous market, this model is given by a stochastic differential equation.

This model assesses the price of a European option through the values of five variables: the price of the underlying asset at the current date, the maturity time, the strike price, the risk free interest rate and the volatility. Almost all these values can be taken from the prices of the options observed in the market data, an exception is the volatility, that turns out to be the indicator that gave us an idea of the behavior that will have the value of the option in the future.

1.1 Black-Scholes model

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, let $(\mathscr{F}_t)_{t\geq 0}$ a filtration on $(\Omega, \mathscr{F})$ and let $[0, T]$ a time interval. We consider a financial market with two stocks.

First, we consider the *asset price* or a *bank account* at time t given by:

$$S_t^0 = e^{rt},\ t \geq 0, r \geq 0,$$

where r is the instantaneous interest rate, and we can note that the process $(S_t^0)_{t\geq 0}$ can be writing in differential form

$$\begin{cases} dS_t^0 = rS_t^0 dt \\ S_0^0 = 1 \end{cases}$$

since the unique solution to this differential equation is $S_t^0 = e^{rt}$.

Now, we consider a stochastic process, named *stock price* $(S_t)_{t\geq 0}$ given by the following stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1.1}$$

where $\mu \in \mathbb{R}$ is the drift, $\sigma > 0$ is the volatility (we assume that σ is constant) and $(W_t)_{t\geq 0}$ is a standard Brownian motion adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$. The unique solution to this stochastic differential equation is given by

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right), \quad t \geq 0. \tag{1.2}$$

In this case we say that $(S_t)_{t\geq 0}$ follows a *geometric Brownian motion*, as we can see in the Figure 1.1.

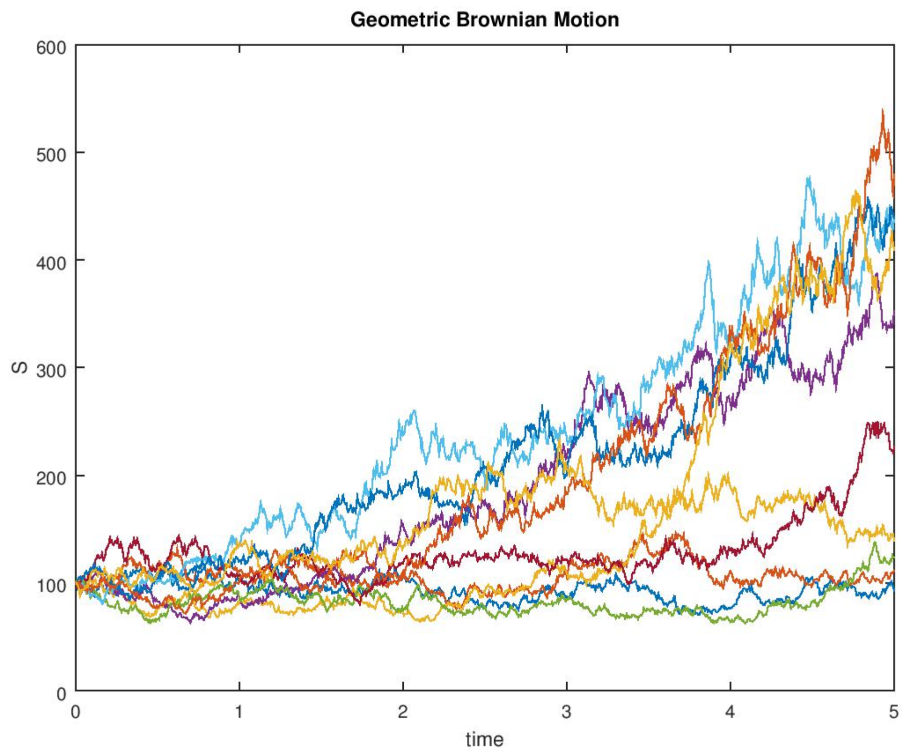


Figure 1.1: We plot 10 sample paths of GBM with $S_0 = 100$, $\mu = 0.2$ and $\sigma = 0.3$ on interval $[0, 5]$.

We rewrite the equation (1.2) in integral form, and we obtain

$$S_t = S_0 + \int_0^t \mu S_u du + \int_0^t \mu S_u dB_u,$$

then we can solve it applying Ito's formula to the Itô's process $(S_t)_{t\geq 0}$ with $v_t = \mu S_t$ and $u_t = \sigma S_t$, as follows.

First, we consider a function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ defined by $f(t, S_t) = \log(S_t)$ with $f(t,x) = \log(x)$, then we have the following derivatives:

$$\begin{aligned}
\partial_t f(t, S_t) &= 0 \\
\partial_x f(t, S_t) &= \frac{1}{S_t} \\
\partial^2_{xx} f(t, S_t) &= -\frac{1}{S_t^2}
\end{aligned}$$

Now, we apply Ito's formula to obtain

$$\begin{aligned}
\log(S_t) &= f(0, S_0) + \int_0^t \partial_x f(u, S_u) u_u dW_u + \int_0^t \partial_x f(u, S_u) v_u du + \frac{1}{2}\int_0^t \partial^2_{xx} f(u, S_u) u_u^2 du \\
&= \int_0^t \frac{1}{S_u} \sigma S_u dW_u + \int_0^t \frac{1}{S_u} \mu S_u du - \frac{1}{2}\int_0^t \frac{1}{S_u^2} \sigma^2 S_u^2 du \\
&= \int_0^t \sigma dW_u + \int_0^t \mu du - \frac{1}{2}\int_0^t \sigma^2 du.
\end{aligned}$$

Then,

$$d\log(S_t) = \sigma dW_t + \mu dt - \frac{1}{2}\sigma^2 dt = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t.$$

Equivalently, we have

$$\int_0^t d\log(S_t) = \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) du + \int_0^t \sigma dW_u = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma(W_t - W_0),$$

hence

$$\log(S_t) - \log(S_0) = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t,$$

since $W_0 = 0$. Finally, we apply the exponential function in both sides to obtain

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right), \quad t \geq 0.$$

We have the following properties:

Theorem 1.1. *Let $\mu \in \mathbb{R}, \sigma > 0$ and $(S_t)_{t\geq 0}$ be a stochastic process such that*

$$dS_t = S_t(\mu dt + \sigma dW_t), \qquad S_0 = S_t^0$$

then S_t is a log-normal random variable.

Proof. Since the unique solution to

$$dS_t = S_t(\mu dt + \sigma dW_t), \qquad S_0 = S_t^0$$

is given by the equation (1.2), then if we if we take logarithm in both sides of this equation, we obtain that

$$\log(S_t) = \log(S_0) + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t,$$

where $W_t \sim \mathcal{N}(0, t)$. Then, we can say that

$$\log(S_t) \sim \mathcal{N}\left(\log(S_0) + \left(\mu - \frac{\sigma^2}{2}\right) t, \sigma^2 t\right).$$

This means that S_t has a log-normal distribution with mean $\log(S_0) + \left(\mu - \frac{\sigma^2}{2}\right) t$ and variance $\sigma^2 t$. ■

From the properties of log-normal distribution, we have that

$$\mathbb{E}\left[S_t\right] = S_0 \exp(\mu t)$$

and that

$$\mathrm{Var}\left(S_t\right) = (S_0)^2 \exp(2\mu t)\left(\exp(\sigma^2 t) - 1\right).$$

Proposition 1.2. *Let $\mu \in \mathbb{R}, \sigma > 0$ and $(S_t)_{t \geq 0}$ a stochastic process such that*

$$dS_t = S_t(\mu dt + \sigma dW_t) \quad S_0 = S_t^0$$

then, $\log(S_t)$ is a Brownian motion, no necessarily standard.

Proof. It is sufficient to prove that $(\log(S_t))_{t \geq 0}$ satisfies the following properties:

- Continuity of sample paths:
  Since $(W_t)_{t \geq 0}$ is continuous with respect t, we have that $\log(S_t) = \log(S_0) + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t$ is continuous with respect t.

- Independent increments:
  We want to prove that, if $u \leq t$ then $\log(S_t) - \log(S_u) = \log\left(\frac{S_t}{S_u}\right)$ is independent of $\sigma\left(\log(S_v), v \leq u\right)$. This is equivalent to prove that if $u \leq t$, then $\frac{S_t}{S_u}$ or the relative increments $\frac{S_t - S_u}{S_u}$ are independent of $\sigma\left(S_v, v \leq u\right)$. Since,

  $$\frac{S_t}{S_u} = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(t - u) + \sigma(W_t - W_u)\right)$$

  and since we know that $(W_t - W_u)$ is independent of $\sigma\left(W_v, v \leq u\right)$, then $\frac{S_t}{S_u}$ is independent of $\sigma\left(S_v, v \leq u\right)$.

- Stationary increments:

  We want to prove that for $u \leq t$, the law of $\ln\left(\frac{S_t}{S_u}\right)$ is the same that the law of $\ln\left(\frac{S_{t-u}}{S_0}\right)$. Or, equivalently to prove that the relative increments of $(S_t)_{t\geq 0}$ are stationary, that is, if $u \leq t$, then the distribution of $\frac{S_t - S_u}{S_u}$ is identically equal to $\frac{S_{t-u} - S_0}{S_0}$. For this, let $z \in \mathbb{R}_+$, then

$$\begin{aligned}\mathbb{P}\left(\frac{S_t}{S_u} < z\right) &= \mathbb{P}\left(\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(t-u) + \sigma(W_t - W_u)\right) < z\right)\\ &= \mathbb{P}\left(\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(t-u) + \sigma W_{t-u}\right) < z\right)\\ &= \mathbb{P}\left(\frac{S_{t-u}}{S_0} < z\right).\end{aligned}$$

  Therefore, the law of $\frac{S_t}{S_u}$ and $\frac{S_{t-u}}{S_0}$ are the same.

■

1.2 Self-financing strategies

Let ϕ_t^0 and ϕ_t^1 be the quantities invested at time t, respectively in the assets S_t^0 and S_t. A *strategy* is a stochastic process

$$\phi = (\phi_t)_{t\geq 0} = \left(\phi_t^0, \phi_t^1\right)_{t\geq 0}$$

whit real values and such that is adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$ of Brownian motion.

Definition 1.3. *The value of the portfolio at time t for the strategy ϕ is given by*

$$V_t(\phi) = \phi_t^0 S_t^0 + \phi_t^1 S_t, \quad 0 \leq t \leq T. \tag{1.3}$$

Definition 1.4. *A self-financing strategy ϕ is a pair of adapted processes $(\phi_t^0)_{t\geq 0}$ and $(\phi_t^1)_{t\geq 0}$ such that*

1. $\int_0^T |\phi_t^0| dt + \int_0^T (\phi_t^1)^2 dt < \infty \quad \mathbb{P}-a.s.$
2. $V_t(\phi) = V_0(\phi) + \int_0^t \phi_u^0 dS_u^0 + \int_0^t \phi_u^1 dS_u, \quad 0 \leq t \leq T.$

The condition 1 guarantee that the integrals in condition 2 are well-defined. That is,

$$\int_0^T |\phi_u^0| du < \infty \text{ implies that } \int_0^T \phi_u^0 dS_u^0 = \int_0^T \phi_u^0 r e^{ru} du < \infty$$

and

$$\int_0^T (\phi_t^1)^2 dt < \infty \text{ implies that } \int_0^T \phi_u^1 dS_u = \int_0^T \phi_u^1 \mu S_u du + \int_0^T \phi_u^1 \sigma S_u dW_u < \infty.$$

Proposition 1.5. *Let ϕ be a strategy with value of the portfolio given as equation (1.3). The strategy ϕ is a self-financing strategy if and only if*

$$\tilde{V}_t(\phi) = \tilde{V}_0(\phi) + \int_0^t \phi_u^1 d\tilde{S}_u, \quad t \geq 0, \tag{1.4}$$

where $\tilde{V}_t(\phi) = e^{-rt}V_t(\phi)$ and $\tilde{S}_t = e^{-rt}S_t$ are the respective discounted values.

Proof. We suppose that ϕ is a self-financing strategy. We note that in the previous definition, condition 2 can be written in differential form and in the case of the Black-Scholes model, we have

$$dV_t(\phi) = \phi_t^0 dS_t^0 + \phi_t^1 S_t(\mu dt + \sigma dW_t),$$

so, if we substitute the value of dS_t^0, we obtain

$$\begin{aligned} dV_t(\phi) &= \phi_t^0 r S_t^0 dt + \mu\phi_t^1 S_t dt + \sigma\phi_t^1 S_t dW_t \\ &= r(\phi_t^0 S_t^0 + \phi_t^1 S_t)dt + (\mu - r)\phi_t^1 S_t dt + \sigma\phi_t^1 S_t dW_t \\ &= rV_t dt + (\mu - r)\phi_t^1 S_t dt + \sigma\phi_t^1 S_t dW_t, \end{aligned}$$

now, we multiply by e^{-rt} in both sides

$$e^{-rt}dV_t(\phi) = re^{-rt}V_t dt + (\mu - r)e^{-rt}\phi_t^1 S_t dt + \sigma e^{-rt}\phi_t^1 S_t dW_t$$

and since $d\tilde{V}_t = -re^{-rt}V_t dt + e^{-rt}dV_t$, then

$$\begin{aligned} d\tilde{V}_t &= -re^{-rt}V_t dt + re^{-rt}V_t dt + (\mu - r)e^{-rt}\phi_t^1 S_t dt + \sigma e^{-rt}\phi_t^1 S_t dW_t \\ &= (\mu - r)\phi_t^1 \tilde{S}_t dt + \phi_t^1 \sigma \tilde{S}_t dW_t \\ &= \phi_t^1 \Big((\mu - r)\tilde{S}_t dt + \sigma\tilde{S}_t dW_t\Big) \\ &= \phi_t^1 d\tilde{S}_t. \end{aligned}$$

Finally, we integrate in both sides to obtain

$$\tilde{V}_t - \tilde{V}_0 = \int_0^t \phi_u^1 d\tilde{S}_u, \quad 0 \leq t \leq T.$$

For the converse, we suppose that $d\tilde{V}_t = \phi_t^1 d\tilde{S}_t$, and since $V_t = e^{rt}\tilde{V}_t$, then

$$\begin{aligned}
dV_t(\phi) &= re^{rt}\tilde{V}_t dt + e^{rt} d\tilde{V}_t \\
&= re^{rt}\tilde{V}_t dt + e^{rt}\phi_t^1 d\tilde{S}_t \\
&= rV_t dt + e^{rt}\phi_t^1 d\tilde{S}_t \\
&= rV_t dt + e^{rt}\phi_t^1 \left((\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dW_t\right) \\
&= rV_t dt + \phi_t^1(\mu - r)S_t dt + \sigma\phi_t^1 S_t dW_t \\
&= rV_t dt - r\phi_t^1 S_t dt + \phi_t^1 \left(\mu S_t dt + \sigma S_t dW_t\right) \\
&= \phi_t^0 dS_t^0 + \phi_t^1 dS_t.
\end{aligned}$$

If we integrate both sides, we obtain

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_u^0 dS_u^0 + \int_0^t \phi_u^1 dS_u, \quad t \geq 0,$$

which satisfies the definition of self-financing strategy. ■

We note that according to (1.4), the self-financing portfolio price V_t can be written as

$$V_t = e^{rt}V_0 + (\mu - r)\int_0^t e^{r(t-u)}\phi_u^1 S_u du + \sigma \int_0^t e^{r(t-u)}\phi_u^1 S_u dW_u, \quad t \geq 0.$$

1.3 Arbitrage and risk neutral probability measure

If we put additional restrictions on self-financing strategies, we obtain the admissible strategies whose total value V_t remains nonnegative for all times $t \in [0, T]$.

Definition 1.6. *A strategy $\phi = \left(\phi_t^0, \phi_t^1\right)_{t\geq 0}$ is admissible if it is self-financing and its discounted value*

$$\tilde{V}_t(\phi) = \phi_t^0 + \phi_t^1 S_t \geq 0,$$

for all t.

Definition 1.7. *An arbitrage is an admissible strategy ϕ such that satisfies*

(i) $V_0(\phi) = 0$,

(ii) $V_t(\phi) \geq 0$ *for all* $0 \leq t \leq T$,

(iii) $\mathbb{P}(V_T(\phi) > 0) > 0$.

The condition (i) means that the investor starts with zero capital or even with a debt, the condition (ii) means that he wants no loss and (iii) means that he wishes to sometimes make a strictly positive gain.

1.3.1 Existence of a risk neutral measure

We want to find a risk neutral probability measure under which the discounted price process

$$(\tilde{S}_t)_{t\geq 0} = (e^{-rt}S_t)_{t\geq 0}$$

is a martingale.

For this, we need to introduce the Girsanov theorem and some topics related with it, to try to find this probability measure.

Change of probability measure and the Girsanov theorem

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. A probability measure $\mathbb{Q}$ on a measure space $(\Omega, \mathscr{F})$ is *absolutely continuous* with respect to $\mathbb{P}$ if

$$\mathbb{P}(A) = 0 \text{ implies } \mathbb{Q}(A) = 0. \tag{1.5}$$

for all $A \in \mathscr{F}$.

Theorem 1.8. *A probability measure $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ if and only if there exits a non-negative random variable Z on $(\Omega, \mathscr{F})$ with $\mathbb{E}[Z] = 1$ such that*

$$\mathbb{Q}(A) = \int_A Z(\omega) d\mathbb{P}(\omega),$$

for all $A \in \mathscr{F}$. Z is called the density of $\mathbb{Q}$ with respect to $\mathbb{P}$ and denoted by $d\mathbb{Q}/d\mathbb{P}$.

Now, let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space and $(W_t)_{t\geq 0}$ be a standard Brownian motion $\mathscr{F}_t$-measurable. The Girsanov theorem describes how we can change a probability measure by an equivalent probability measure. For the proof of the theorem, see [14].

Theorem 1.9. *Let $(\theta_t)_{t\geq 0}$ be an adapted process satisfying $\int_0^T \theta_t^2 dt < \infty$ almost surely and such that the process $(L_t)_{t\geq 0}$ defined by*

$$L_t = \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds \right)$$

is martingale. Then, under the probability $\mathbb{P}^L$ with density L_T with respect to $\mathbb{P}$, the process $(B_t)_{t\geq 0}$ defined by

$$B_t = W_t + \int_0^t \theta_s ds$$

is a standard Brownian motion $\mathscr{F}_t$-measurable.

Risk neutral probability measure

Proposition 1.10. *In the Black-Scholes model there exists a probability measure equivalent to $\mathbb{P}$, under which $(\tilde{S}_t)_{t\geq 0}$ is a martingale.*

Proof. First, from the stochastic differential equation (2.1) satisfied by S_t, we have

$$\begin{aligned} d\tilde{S}_t &= d(e^{-rt}S_t) \\ &= -re^{-rt}S_t dt + e^{-rt}dS_t \\ &= -r\tilde{S}_t dt + e^{-rt}\left(S_t(\mu dt + \sigma dW_t)\right) \\ &= -r\tilde{S}_t dt + e^{-rt}S_t\mu dt + e^{-rt}S_t\sigma dW_t \\ &= -r\tilde{S}_t dt + \tilde{S}_t\mu dt + \tilde{S}_t\sigma dW_t \\ &= \tilde{S}_t\left((\mu - r)dt + \sigma dW_t\right) \\ &= \sigma\tilde{S}_t\left(\left(\frac{\mu - r}{\sigma}\right)dt + dW_t\right), \end{aligned}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion. Let

$$B_t = W_t + \int_0^t \left(\frac{\mu - r}{\sigma}\right) ds,$$

then

$$dB_t = dW_t + \left(\frac{\mu - r}{\sigma}\right) dt.$$

Therefore, we have that $d\tilde{S}_t = \sigma\tilde{S}_t dB_t$. By Girsanov's theorem, we have that under probability measure $\mathbb{P}^*$, the stochastic process $(B_t)_{t\geq 0}$ is a Brownian motion. For this, let $\theta_t = \dfrac{\mu - r}{\sigma}$, i.e., $(\theta_t)_{t\geq 0}$ is a constant process, then $\displaystyle\int_0^T \left(\frac{\mu - r}{\sigma}\right)^2 ds < \infty$, and if we define

$$\begin{aligned} L_t &= \exp\left(-\int_0^t \left(\frac{\mu - r}{\sigma}\right) dW_s - \frac{1}{2}\int_0^t \left(\frac{\mu - r}{\sigma}\right)^2 ds\right) \\ &= \exp\left(-\left(\frac{\mu - r}{\sigma}\right) W_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right). \end{aligned}$$

Then, L_t is a martingale under $\mathbb{P}$. By Girsanov's theorem , we have that under $\mathbb{P}^{(L_T)}$ the process $(B_t)_{t\geq 0}$ is a Brownian motion.

Then, the stochastic differential equation $d\tilde{S}_t = \sigma\tilde{S}_t dB_t$ has a unique solution given by

$$\tilde{S}_t = S_0 \exp\left(-\frac{\sigma^2}{2}t + \sigma B_t\right)$$

which is a martingale under $\mathbb{P}^{L_T}$.

Finally, we need to prove that $\mathbb{P}$ is equivalent to $\mathbb{P}^{L_T}$, i.e., we need to prove the following:

- $\mathbb{P}^{L_T} \ll \mathbb{P}$: Let $A \in \mathscr{F}$ such that $\mathbb{P}(A) = 0$, then

$$\mathbb{P}^{L_t}(A) = \int_A L_T d\mathbb{P} = 0.$$

- $\mathbb{P} \ll \mathbb{P}^{L_T}$: Let $A \in \mathscr{F}$ such that $\mathbb{P}^{L_T}(A) = 0$, then

$$\mathbb{P}(A) = \int_A d\mathbb{P} = \int_A \frac{L_T}{L_T} d\mathbb{P} = \int_A \frac{1}{L_T} L_T d\mathbb{P} = \int_A \frac{1}{L_T} d\mathbb{P}^{L_T} = 0.$$

■

In the following, we will denote by $\mathbb{P}^*$ the probability measure equivalent to $\mathbb{P}$. Then, under $\mathbb{P}^*$ we can write

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t\right),$$

where $(W_t)_{t\geq 0}$ is a $\mathbb{P}^*$ Brownian motion.

1.3.2 Absence of arbitrage and completeness

For the notion of absence of arbitrage, it is sufficient the existence of a risk neutral probability measure $\mathbb{P}^*$ to have a model without arbitrage and complete. Then, we have the following result.

Theorem 1.11. *The Black-Scholes model is free of arbitrage.*

Proof. We know that under $\mathbb{P}^*$, $(\tilde{S}_t)$ is a martingale. If we consider an admissible strategy ϕ with zero initial value, we have

$$\begin{aligned} \tilde{V}(\phi) &= \int_0^t \phi_u^1 d\tilde{S}_u \\ &= \int_0^t \phi_u^1 \sigma \tilde{S}_u dW_t > 0. \end{aligned}$$

So, $\tilde{V}(\phi)$ is a local martingale, that is, there exists an increasing sequence of stopping times $(\tau_n)_{n\geq 0}$ with respect to $(\mathscr{F}_t)_{t\geq 0}$ with $\tau_n \uparrow \infty$ such that for n fixed, $(\tilde{V}_{t\wedge\tau_n})_{t\geq 0}$ is a $\mathbb{P}^*$ martingale for all $n \geq 0$. Then we have

$$\mathbb{E}_{\mathbb{P}^*}\left[\tilde{V}_{t\wedge\tau_n}(\phi)\right] = 0.$$

Since $\tilde{V}_{t\wedge\tau_n}(\phi) \geq 0$, we obtain that $\tilde{V}_{t\wedge\tau_n}(\phi) = 0$ $\mathbb{P}^*$-almost surely for all $n \geq 0$. Consequently,

$$\tilde{V}_T(\phi) = \lim_{n\to\infty} \tilde{V}_{t\wedge\tau_n}(\phi) = 0 \quad \mathbb{P}^* - a.s.$$

Finally, since $\mathbb{P}^*$ is equivalent to $\mathbb{P}$, we obtain that $\tilde{V}_T(\phi) = 0$ $\mathbb{P}$-a.s. So, no arbitrage is possible. ■

The next result is the second fundamental theorem of asset pricing in continuous time.

Theorem 1.12. *A market model free of arbitrage is complete if and only it admits a unique risk neutral probability measure* $\mathbb{P}^*$.

For the proof, see [27].

In Black Scholes model, we proved the existence of a unique risk neutral probability measure, hence the model is complete.

Chapter 2

Option Pricing and Implied Volatility

In Black-Scholes model we assume that the stock price $(S_t)_{t\geq 0}$ follows a geometric Brownian motion, which has constant volatility. But, this model ignores possible behaviors, one interesting case is when we have changes in volatility. However the financial market still uses the Black-Scholes formula in order to price an option.

This leads us to ask the following question: which value of volatility we should include in Black-Scholes formula in order to obtain the right option price?. To try to answer this, we will introduce the concept of *implied volatility*, since volatility is the most important parameter in the Black-Scholes model. The implied volatility of options of different maturities has an interesting characteristic. There is a pattern that implied volatility is not constant for different strike prices, and this is called volatility smile or volatility skew.

2.1 Option pricing in Black-Scholes model

We can use $(S_t)_{t\geq 0}$ to price an option with maturity time T and initial time t, strike price K, risk free interest rate r, current stock price S_0 and volatility σ. We will focus only on European options.

Definition 2.1. *A European option is a contract that gives you the right but not the obligation, to get a payoff X at maturity T, where X is a non-negative F_T-measurable random variable. Then, we say that:*

- *A call option is an European option that gives the right but not the obligation, to buy one unit of an underlying asset for a predetermined strike price K and maturity time T. If S_T is the price of the* *underlying asset at maturity time T, then the value of this contract at maturity is:*

$$h(S_T) = (S_T - K)_+ = \begin{cases} S_T - K & \textit{if} \;\; S_T > K, \\ \quad 0 & \textit{if} \;\; S_T \leq K. \end{cases}$$

- *A put option is a European option that gives the right, but not the obligation, to sell a unit of an underlying asset for a strike price K at the maturity date T. This payoff is:*

$$h(S_T) = (K - S_T)_+ = \begin{cases} K - S_T & \textit{if} \;\; S_T < K, \\ \quad 0 & \textit{if} \;\; S_T \geq K. \end{cases}$$

The price of a call option at time t under the risk neutral probability measure $\mathbb{P}^*$ is the discounted expected value to the initial time,

$$C_t = \mathbb{E}_{\mathbb{P}^*}\left[e^{-r(T-t)}(S_T - K)_+ | \mathscr{F}_t\right],$$

by definition of a call option and properties of expectation, we have

$$\begin{aligned} C_t &= e^{-r(T-t)}\mathbb{E}_{\mathbb{P}^*}\left[S_T \mathbb{1}_{\{S_T > K\}} | \mathscr{F}_t\right] - K e^{-r(T-t)}\mathbb{E}_{\mathbb{P}^*}\left[\mathbb{1}_{\{S_T > K\}} | \mathscr{F}_t\right] \\ &= e^{-r(T-t)} S_t \mathbb{E}_{\mathbb{P}^*}\left[\frac{S_T}{S_t}\mathbb{1}_{\left\{\frac{S_T}{S_t} > \frac{K}{S_t}\right\}} | \mathscr{F}_t\right] - K e^{-r(T-t)}\mathbb{E}_{\mathbb{P}^*}\left[\mathbb{1}_{\left\{\frac{S_T}{S_t} > \frac{K}{S_t}\right\}} | \mathscr{F}_t\right] \\ &= e^{-r(T-t)} S_t \mathbb{E}_{\mathbb{P}^*}\left[\frac{S_T}{S_t}\mathbb{1}_{\left\{\frac{S_T}{S_t} > \frac{K}{x}\right\}}\right]_{x=S_t} - K e^{-r(T-t)}\mathbb{E}_{\mathbb{P}^*}\left[\mathbb{1}_{\left\{\frac{S_T}{S_t} > \frac{K}{x}\right\}}\right]_{x=S_t}. \end{aligned}$$

First, we know that

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right),$$

and that

$$\frac{S_T}{S_t} = \exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W_T - W_t)\right),$$

since $W_T - W_t = W_{T-t}$ in law, for the first term, we have

$$\begin{aligned} & e^{-r(T-t)} S_t \mathbb{E}_{\mathbb{P}^*}\left[\frac{S_T}{S_t}\mathbb{1}_{\left\{\frac{S_T}{S_t} > \frac{K}{x}\right\}}\right]_{x=S_t} \\ &\quad = e^{-r(T-t)} S_t \mathbb{E}_{\mathbb{P}^*}\left[\exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma W_{T-t}\right)\mathbb{1}_{\left\{\exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma W_{T-t}\right) > \frac{K}{x}\right\}}\right] \\ &\quad = e^{-r(T-t)} S_t \mathbb{E}_{\mathbb{P}^*}\left[e^{r(T-t)}\exp\left(\left(-\frac{\sigma^2}{2}\right)(T-t) + \sigma W_{T-t}\right)\mathbb{1}_{\left\{\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma W_{T-t} > \log\left(\frac{K}{x}\right)\right\}}\right] \\ &\quad = S_t \mathbb{E}_{\mathbb{P}^*}\left[\exp\left(\left(-\frac{\sigma^2}{2}\right)(T-t) + \sigma W_{T-t}\right)\mathbb{1}_{\left\{\sigma W_{T-t} > \log\left(\frac{K}{x}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right\}}\right] \end{aligned}$$

$$
\begin{aligned}
&= S_t \mathbb{E}_{\mathbb{P}^*}\left[\exp\left(\left(-\frac{\sigma^2}{2}\right)(T-t)+\sigma\sqrt{T-t}Y\right)\mathbb{1}_{\left\{\sigma\sqrt{T-t}Y>\log\left(\frac{K}{x}\right)-\left(r-\frac{\sigma^2}{2}\right)(T-t)\right\}}\right] \\
&= S_t \mathbb{E}_{\mathbb{P}^*}\left[\exp\left(\left(-\frac{\sigma^2}{2}\right)(T-t)+\sigma\sqrt{T-t}Y\right)\mathbb{1}_{\left\{Y>\frac{\log\left(\frac{x}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right\}}\right]
\end{aligned}
$$

where Y is a standard Gaussian random variable, and since S_t follows a standard Gaussian law, we have

$$
\begin{aligned}
&e^{-r(T-t)} S_t \mathbb{E}_{\mathbb{P}^*}\left[\frac{S_T}{S_t}\mathbb{1}_{\left\{\frac{S_T}{S_t}>\frac{K}{x}\right\}}\right]_{x=S_t} \\
&\quad = S_t \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\log\left(\frac{x}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}} \exp\left(-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{T-t}y\right)\exp\left(-\frac{y^2}{2}\right)dy \\
&\quad = S_t \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\log\left(\frac{x}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}} \exp\left(-\frac{1}{2}\left(\sigma\sqrt{T-t}+y\right)^2\right)dy \\
&\quad = S_t \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\log\left(\frac{x}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}} \exp\left(-\frac{u^2}{2}\right)du \\
&\quad = S_t N(d_1)
\end{aligned}
$$

For the second summand, we have

$$
\mathbb{E}_{\mathbb{P}^*}\left[\mathbb{1}_{\left\{\frac{S_T}{S_t}>\frac{K}{x}\right\}}\right]_{x=S_t} = \mathbb{P}^*\left(\frac{S_T}{S_t}>\frac{K}{x}\right),
$$

and if we take logarithm, we obtain

$$
\begin{aligned}
\mathbb{P}^*\left(\log\frac{S_T}{S_t}>\log\left(\frac{K}{x}\right)\right) &= \mathbb{P}^*\left(\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma W_{T-t}>\log\left(\frac{K}{x}\right)\right) \\
&= \mathbb{P}^*\left(\sigma W_{T-t}>\log\left(\frac{K}{x}\right)-\left(r-\frac{\sigma^2}{2}\right)(T-t)\right) \\
&= \mathbb{P}^*\left(\frac{W_{T-t}}{\sqrt{T-t}}>\frac{\log\left(\frac{K}{x}\right)-\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= N(d_2).
\end{aligned}
$$

Therefore,

$$
C_t = S_t N(d_1) - Ke^{-r(T-t)}N(d_2).
$$

In other words, the price of a European call option at time t and for an observed risky asset price S_0 is given by the *Black-Scholes formula*:

$$
\mathrm{C}_{\mathrm{BS}}(S_0, K, r, T, t, \sigma) = S_0 N(d_1) - Ke^{-r(T-t)}N(d_2), \tag{2.1}
$$

where

$$d_1 = \frac{\log(S_0/K) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t} = \frac{\log(S_0/K) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy$$

is the standard Gaussian cumulative distribution function.

2.2 Implied Volatility

In Black-Scholes model the unique parameter which cannot be observed in the financial market is the volatility, since the values of the parameters S_0, K, r, T, t used to price an option via the Black-Scholes formula can be observed. To estimate the volatility coefficient σ can be a more difficult task, and several estimation methods are considered. Almost always, the inversion of the Black-Scholes formula to get the implied volatility is done with some sort of solver method, for example, the Newton-Raphson method.

We assume that stock price S follows a geometric Brownian motion and that interest rate is constant, then given a constant volatility, and put $\tau = T - t$, we know that Black-Scholes formula is given by

$$\mathrm{C_{BS}}(S, K, r, \tau, \sigma) := SN(d_1) - Ke^{-r\tau}N(d_2),$$

where

$$d_1 = \frac{\log(S/K) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad d_2 = \frac{\log(S/K) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

Since

$$\text{vega} := \frac{\partial\, \mathrm{C_{BS}}(S, K, r, \tau, \sigma)}{\partial \sigma} = \frac{S\exp(-d_1^2/2)\sqrt{\tau}}{\sqrt{2\pi}} = S_0\sqrt{\tau}N(d_1)$$

is positive, then $\mathrm{C_{BS}}(S, K, r, \tau, \sigma)$ is strictly increasing on σ. In addition, we have

$$\lim_{\sigma\to 0} \mathrm{C_{BS}}(S, K, r, \tau, \sigma) = (S - Ke^{-r\tau})_+,$$

$$\lim_{\sigma\to\infty} \mathrm{C_{BS}}(S, K, r, \tau, \sigma) = S.$$

We want to mention that the *vega* of an option is the sensitive of the option price to a change in volatility.

From the properties of European option, we have that the price of a call option satisfies

$$(S - Ke^{-r\tau})_+ \leq C(K, \tau, S) \leq S.$$

independent of the model. Therefore, for any price $C^*(K, \tau, S)$ observed in the market for a European call option with maturity time τ and strike price K, there exists a unique solution $\sigma^{imp}(K, \tau, S)$ such that

$$\mathrm{C_{BS}}(S, K, r, \tau, \sigma^{imp}) = C^*(K, \tau, S),$$

where $\sigma^{imp}(K, \tau, S)$ is called the *implied volatility.*

Knowing S, the implied volatility σ^{imp} is a function of τ and K. If Black-Scholes model is true, σ^{imp} must be a constant function, however this is not compatible with the data.

The Black-Scholes model implies that σ^{imp} of all options on the same S must be the same. However, when calculating σ^{imp} from prices of different options observed in the market, we find that

- The implied volatility is always higher than the volatility of S.
- The implied volatility depends on the strike and maturity.

To obtain the value of implied volatility σ^{imp} we must to find the volatility that equals the theoretical price of B-S model with the real market price C^*, since the other parameters are given. As this volatility can not be obtained directly from the Black-Scholes formula we must apply numerical methods to find roots, the method that we will present below is an iterative method that allows us to approximate the solution of an equation of type:

$$f(\sigma) = \mathrm{C_{BS}}(S, K, r, \tau, \sigma) - C^* = 0.$$

There exist a lot of methods to approximate the roots of a function. For calculate the volatility, we can use Newton-Raphson method, Bisection method, Secant method, etc. In our case, we used only the first to estimate the implied volatility.

2.2.1 Newton-Raphson method

The Newton-Raphson method is a powerful technique for solving numerically equations of the form

$$f(x) = 0.$$

That is, this method approximate roots of a function. It is an iterative algorithm, which converges (usually) rapidly.

In the Newton–Raphson method, we take a tangent line to the curve $y = f(x)$ under the assumption that f is a differentiable function which derivatives are non zero. It can be derived from the Taylor series expansion as well.

In the Taylor series approach, at point x the equation for the tangent line to $y = f(x)$ is given by

$$f(x) \approx f(x_n) + f'(x_n)(x - x_n) + \frac{f''(x_n)}{2!}(x - x_n)^2 + \cdots$$

If x in near to x_n, we can ignore the terms $\frac{f''(x_n)}{2!}(x-x_n)^2+\cdots$ to obtain

$$f(x) \approx f(x_n) + f'(x_n)(x-x_n).$$

Now, we take $x = x_{n+1}$ and since the function value at x_{n+1} (at the intersection with the x-axis) has a root, i.e. $f(x_{n+1}) = 0$, then we can write the previous equation as

$$f(x_n) + f'(x_n)(x_{n+1}-x_n) = 0$$

which simplifies as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This method is very efficient since we only need to give an initial value, in addition to which it converges quickly, but it has an inconvenient as it is necessary to know the derivative and if it takes values close to zero, the method may not converge, also if $f'(x_n) = 0$ the method cannot be applied.

2.3 Implied volatility in real market

In theory, the implied volatility should be equal for any choice of stock prices S and strike prices K, but in practice the graph of implied volatility versus moneyness S/K is convex. The most quoted phenomenon testifying to the limitations of the Black–Scholes model is established by the relationship between volatility and the exercise price, which should give rise to a form of a *smile*. So that, we have two types of smile depending of the type of distribution followed by the stock price, then will talk about implied volatility smile and implied volatility skew.

There is a conflict between the implied volatility and Black-Scholes model, since according to this model, we should obtain a horizontal straight line which implied that any options for buying or selling the same underlying stock with the same expiration date, but with different exercise prices, should have the same implied volatility, see Figure 2.1 (b). But one of the limitations of Black-Scholes models is the *smile effect*, which say that implied volatilities are not constant for options with the same maturity time, but different strike prices as we can see in Figure 2.1 (a).

2.3.1 Volatility Smile

As described by Emanuel Derman in *Laughter in the Dark-The Problem of the Volatility Smile* in [7], the volatility smile is the empirical relationship observed between the implied volatility and the stock price of an option. The problem is that currently there is no known model that adequately adjusts the complexity of this relationship.

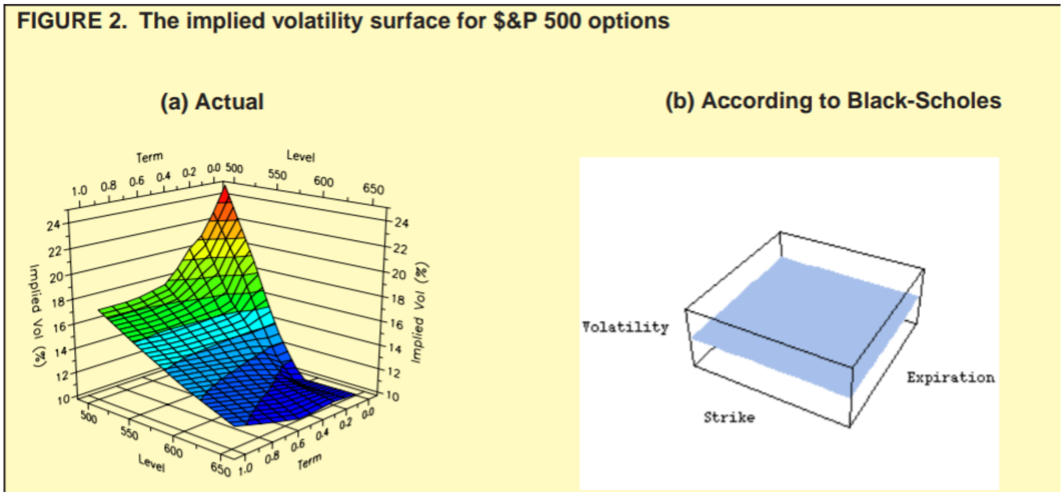


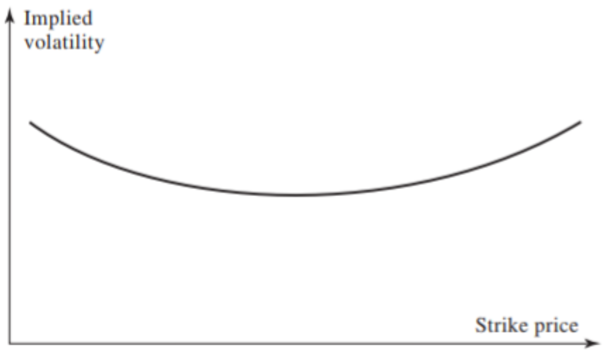
Figure 2.1: Figure taken from Derman [7].

What is known as the volatility smile appears by first time in the financial markets after the crisis stock market in October 1987, but the investors afterwards began to reassess the probabilities of rare events such as financial disasters, and caused higher evaluations of some options. This reflects the fact that the standard Black–Scholes assumes log-normal distributions of asset price so that for these options there has been a very marked and rapid deterioration. Although it was until December 1990 that said problem attracted the attention of Derman, one of the main researchers of the subject.

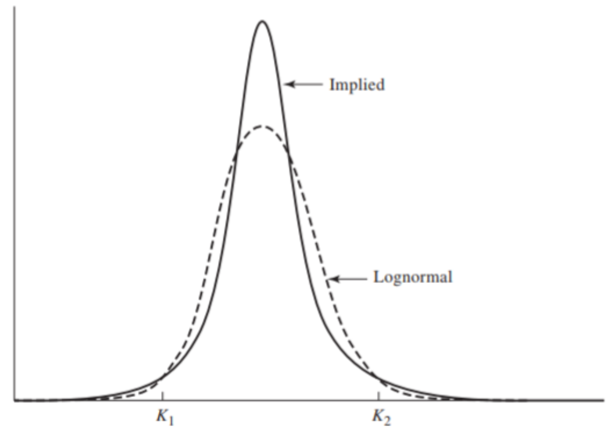
A volatility smile is a common graph shape that results from plotting the strike price and implied volatility of a group of options with the same maturity date, see Figure 2.2 (a).

In Figure 2.2 (b) we can observe that both tails of the implied distribution are heavier than the log-normal distribution, so this distribution tells us that small and big movements are more probably than with the log-normal distribution. Therefore, if the implied distribution has heavier tails than the log-normal, the market prices of the options will be higher than those that would be obtained with the Black-Scholes formula since there is a greater probability than the one assumed by the Black-Scholes model. That is, as explained by Hull, if we assume a call option with strike price K_2, this option only obtains benefits if at the time of expiration the price is above K_2. The same happens with a put option with strike price K_1 to obtain the benefits of a price lower than K_1.

If the correlation between the volatility and the stock price is not allowed, we get a volatility smile, which is symmetric as shown below, where the implied volatility increases if the strike moves away from the price of the stock price.



(a) Volatility smile



(b) Distribution of implied volatility

Figure 2.2: Figures taken from Hull [11].

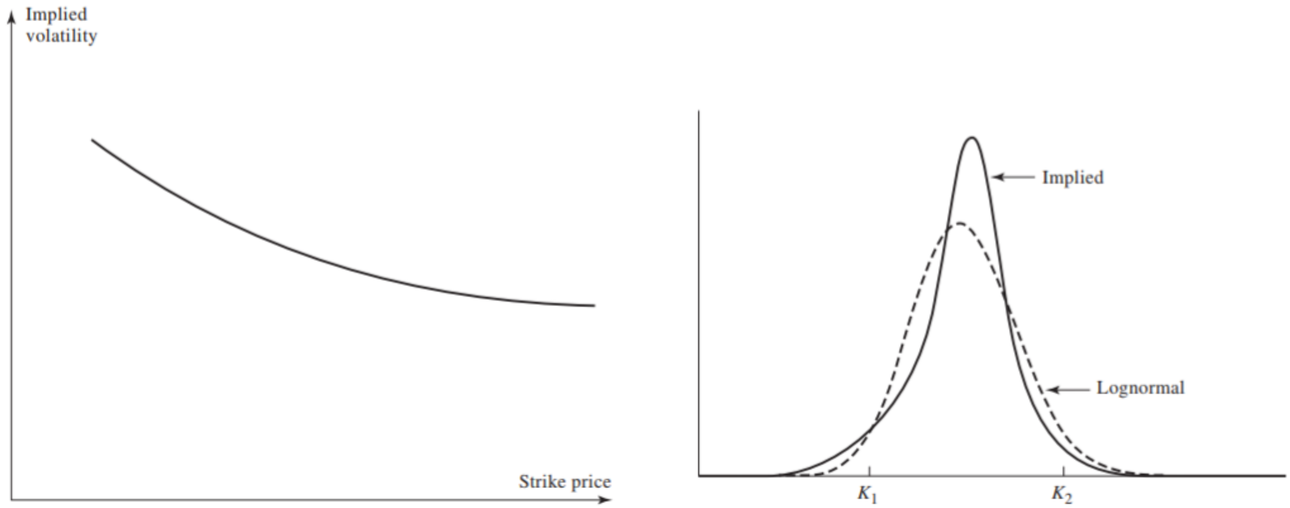
2.3.2 Volatility Skew

The *skew* is important as it reflects the market's perception of *risk* or *expected volatility* for different potential share price outcomes. The skew looks at implied volatilities across different strike prices for a particular maturity time.

An implied volatility skew tells us that the volatility decreases as the strike price increases. There is a premium charged for out of the-money put options above their Black–Scholes formula computed with at-the-money implied volatility. The market prices as though the log-normal model fails to capture probabilities of large downward stock price movements and supplements the Black–Scholes prices to account for this.

In the case of the volatility skew, see Figure 2.3, the implicit distribution is only heavier in the left tail, since the right tail is lighter than the log-normal distribution. The implicit distribution would indicate that both the probability of small movements and large falls is greater than the probability of the log-normal distribution. This implicit distribution means that if we plot the volatility as a function of the strike, it will be decreasing.

Following the reasoning of Hull, if we assume a call option with an exercise price of K_2, this option only obtains benefits if at the time of expiration the price is above K_2. If we observe Figure 2.3 (b) the probability of this happening is lower in the implicit distribution than in the log-normal, therefore the price will be lower than that calculated by Black-Scholes and, a lower price implies a lower volatility. Now, if we assume a put with an exercise price K_1, benefits will be obtained if at the expiration the price is less than K_1. It is more likeless that this happens with the implicit distribution than with the log-normal



(a) Volatility skew (b) Distribution of implied volatility

Figure 2.3: Figures taken from Hull [11].

distribution, that's why the price of the option will be higher and, consequently, the volatility too.

If the correlation between the volatility and the stock price is allowed, a linear term is added that will distort the smile, producing asymmetric smiles that can also match the smiles of the real price index.

Why is there a skew? For stock prices the graph of the implied volatility is always changing. There is generally a skew, however, so that for any fixed maturity T, the implied volatility decreases with the strike K. It is most pronounced at shorter expirations. Now, we explain one of the principal explanations for the skew: Risk aversion which can appear in many guises:

(a) Stock prices do not follow a geometric Brownian motion with a fixed volatility. Markets often jump and jumps to the downside tend to be larger and more frequent than jumps to the upside.

(b) Supply and demand. Investors like to protect their portfolio by purchasing out-of-the-money puts and so there is more demand for options with lower strikes.

Chapter 3

Alternatives to Black-Scholes Model

Extensions to Black Scholes model for option pricing appeared in finance literature after the publication of the paper of Black and Scholes [5] in 1973. For example, in 1976 Merton [19] generalized the Black Scholes formula, incorporating jump diffusion models for the underlying asset, and, in 1987 stochastic volatility models were first studied by Hull and White in [12], which the underlying price is modeled as a stochastic process driven by a random volatility that may or may not be independent.

3.1 Stochastic volatility models

In stochastic volatility models, the stock price $(S_t)_{t\geq 0}$ satisfies the stochastic differential equation

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t \tag{3.1}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion, $(\mu_t)_{t\geq 0}$ is the instantaneous drift of stock price return process, and $(\sigma_t)_{t\geq 0}$ is called the volatility process.

Let us consider the volatility as a function of a one-dimensional Itô's process $\sigma_t = f(Y_t)$ where f is some smooth, positive, increasing function and Y_t is a one-dimensional processes satisfying the stochastic differential equation

$$dY_t = \alpha(Y_t)dt + \beta(Y_t)dZ_t \tag{3.2}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$, $\beta : \mathbb{R} \to \mathbb{R}_+$ and $(Z_t)_{t\geq 0}$ is a standard Brownian motion.

We want to incorporate the concept of *correlation* with stock price changes by correlating Brownian motions. The Brownian motions Z_t and W_t have constant correlation $\rho \in [-1, 1]$, where the correlation coefficient is defined by

$$\rho dt = d\langle W_t, Z_t\rangle_t.$$

3.1.1 Pricing with equivalent martingale measures

We want to derive the pricing partial differential equation assuming the following stochastic volatility model:

$$\begin{aligned} dS_t &= \mu(Y_t)S_t dt + \sigma_t S_t dW_t, \\ \sigma_t &= f(Y_t), \\ dY_t &= \alpha(Y_t)dt + \beta(Y_t)dZ_t, \end{aligned} \tag{3.3}$$

where μ only depends of Y. It is also convenient to decompose Z_t in terms of W_t and a independent standard Brownian motion B_t as follows

$$Z_t = \rho W_t + \sqrt{1-\rho^2} B_t.$$

We suppose that there is an equivalent probability measure $\mathbb{P}^*$ under the discounted stock price $\tilde{S}$ is a martingale, and we know that the price of an option with maturity time T is given by the formula

$$C_t = \mathbb{E}_{\mathbb{P}^*}\left[e^{-r(T-t)}h(S_T)|\mathscr{F}_t\right], \tag{3.4}$$

then for each $\mathbb{P}^*$ we can find a reasonable option price C_t. Now, we want to find a family of equivalent risk neutral probability measures as in Section 1.3. Let $\theta_t := \dfrac{\mu(Y_s)-r}{f(Y_s)}$, and we define

$$W_t^* = W_t + \int_0^t \theta_s ds,$$

and for an arbitrary adapted and square integrable process $(\gamma_t)_{t\geq 0}$,

$$B_t^* = B_t + \int_0^t \gamma_s ds.$$

By Girsanov's theorem, W_t^* and B_t^* are independent standard Brownian motions under the probability measure $\mathbb{P}^{*(\gamma)}$ defined by

$$\frac{d\mathbb{P}^{*(\gamma)}}{d\mathbb{P}} = \exp\left(-\int_0^T \theta_s dW_s - \int_0^T \gamma_s dB_s - \frac{1}{2}\int_0^T \left(\theta_s^2 + \gamma_s^2\right) ds\right).$$

Then, under $\mathbb{P}^{*(\gamma)}$, the stochastic differential equation (3.3) become

$$dS_t = rS_t dt + f(Y_t)S_t dW_t^*, \tag{3.5}$$

$$dY_t = \left(\alpha(Y_t) - \beta(Y_t)\Lambda_t\right)dt + \beta(Y_t)dZ_t^*, \tag{3.6}$$

$$Z_t^* = \rho W_t^* + \sqrt{1-\rho^2} B_t^*,$$

where

$$\Lambda_t = \rho\theta_t + \gamma_t\sqrt{1-\rho^2}$$

is the total risk premium process. Then, we can price a payoff $h(S_T)$ under $\mathbb{P}^{*(\gamma)}$ as

$$C_t = \mathbb{E}_{\mathbb{P}^{*(\gamma)}}\left[e^{-r(T-t)}h(S_T)|\mathscr{F}_t\right]. \tag{3.7}$$

The process $(\gamma_t)_{t\geq 0}$ is called the volatility risk premium or the market price of volatility risk, which parametrizes the space of risk neutral probabilities measures.

3.1.2 Uncorrelated Stochastic Volatility Models

We consider stochastic volatility models where the stock price satisfies the following stochastic differential equation:

$$dS_t = rS_t dt + \sigma_t S_t dW_t^*.$$

We suppose that the volatility process $(\sigma_t)_{t\geq 0}$ is independent of the Brownian motion W_t^* under the probability measure $\mathbb{P}^*$. We refer to such models as *uncorrelated stochastic volatility models.*

In the case of diffusion stochastic volatility models of the form (3.5) and (3.6), this corresponds to the case $\rho = 0$ and γ_t independent of the Brownian motion W_t^*.

Hull-White Formula

The pricing formula given in (3.7) for a European option $h(S_T)$ can be simplified under some assumptions. If we condition on the path of volatility process $(\sigma_t)_{t\geq 0}$ and by iterated expectations then

$$C_t = \mathbb{E}_{\mathbb{P}^{*(\gamma)}}\left[\mathbb{E}_{\mathbb{P}^{*(\gamma)}}\left[e^{-r(T-t)}h(S_T)|\mathscr{F}_t, \sigma_s, t\leq s\leq T\right]\Big|\mathscr{F}_t\right].$$

We can note that the inner conditional expectation is the Black-Scholes formula with a time dependent volatility. Therefore, we can write the previously equation as

$$C_t = \mathbb{E}_{\mathbb{P}^{*(\gamma)}}\left[\mathrm{C}_{\mathrm{BS}}(S_t, K, r, T, , t, \sqrt{\overline{\sigma^2}})|\mathscr{F}_t\right],$$

where

$$\overline{\sigma^2} := \frac{1}{T-t}\int_t^T \sigma_s^2 ds.$$

We observe that the call option price is the average over all possible volatility paths.

The previous results appears in Renault and Touzi [23] and in [8].

3.1.3 Correlated Stochastic Volatility Models

We consider stochastic volatility models where we suppose that the volatility process $(\sigma_t)_{t\geq 0}$ is correlated with the Brownian motion W_t. Under an equivalent martingale measure $\mathbb{P}^{*(\gamma)}$, we can rewrite the model given by (3.5)-(3.6) as the stock price satisfying the stochastic differential equation:

$$\frac{dS_t}{S_t} = rdt + \sigma_t\left(\sqrt{1-\rho^2}d\hat{W}_t^* + \rho dZ_t^*\right),$$
$$dY_t = (\alpha(Y_t) - \beta(Y_t)\Lambda_t)\,dt + \beta(Y_t)dZ_t^*,$$

where W_t^* is decomposed in terms of Z_t^* and an independent standard Brownian motion $\hat{W}_t^*$. Then, we can apply Itô's formula to obtain

$$d\log(S_t) = \left(r - \frac{\sigma_t^2}{2}\right)dt + \sigma_t\left(\sqrt{1-\rho^2}d\hat{W}_t^* + \rho dZ_t^*\right).$$

Since in Chapter 1, we proved that $(S_t)_{t\geq 0}$ is a log-normal random variable, if we take conditional expectation on the path of the Brownian motion Z_t^*, we have that the mean is

$$\mathbb{E}_{\mathbb{P}^{*(\gamma)}}\left[\log(S_t)|\mathscr{F}_t, Z_{[t,T]}^*\right] = \log(S_0) + \rho\int_t^T \sigma_s Z_t^* - \frac{\rho^2}{2}\int_t^T \sigma_s^2 ds + \left(r - \frac{(1-\rho^2)\overline{\sigma^2}}{2}\right)(T-t),$$
$$\mathrm{Var}_{\mathbb{P}^{*(\gamma)}}\left[\log(S_t)|\mathscr{F}_t, Z_{[t,T]}^*\right] = (1-\rho^2)\overline{\sigma^2}(T-t),$$

where $\overline{\sigma^2} = \dfrac{1}{T-t}\displaystyle\int_t^T \sigma_s^2 ds$. We can reformulate the Hull-White formula for correlated stochastic volatility models as follows

$$C_t = \mathbb{E}_{\mathbb{P}^{*(\gamma)}}\left[\mathrm{C_{BS}}(S_t\xi_t, K, r, T, , t, \sqrt{(1-\rho^2)\overline{\sigma^2}})\Big|\mathscr{F}_t\right], \tag{3.8}$$

where

$$\xi_t = \exp\left(\rho\int_t^T \sigma_s Z_t^* - \frac{\rho^2}{2}\int_t^T \sigma_s^2 ds\right).$$

In this way, the price C_t is a mixture of Black-Scholes formula with different volatilities and different stock prices.

The correlated Hull-White formula given in equation (3.8) is given by Willard in [28].

3.2 Jump diffusion models

The two basic building blocks of every jump diffusion model are the Brownian motion (the diffusion part) and the Poisson process (the jump part).

In this section we will introduce the construction of processes with jumps and independent increments, including the Poisson and compound Poisson processes. The first application of jump processes in option pricing was introduced by Robert Merton in [19].

3.2.1 Poisson Process

The most elementary jump process is the Poisson process $N_t : \Omega \to \mathbb{Z}_+$ which is a counting process. The process $(N_t)_{t\geq 0}$ has jumps of size $+1$ only, and its paths are constant in between two jumps. In other words, the value N_t at time t is given by

$$N_t = \sum_{n\geq 1} \mathbb{1}_{\{t\geq T_n\}}(t), \quad t \geq 0$$

where, $T_n = \sum_{i=1}^{n} \tau_i$ with $(\tau_i)_{i\geq 1}$ a sequence of independent exponential random variables with parameter λ, that is, with distribution $\mathbb{P}\left(\tau_i \geq y\right) = e^{-\lambda y}$.

Then, we can say that N_t counts the number of random times T_n which occur between 0 and t. The Poisson process has the following properties:

- *Independent increments*: for all $0 \leq t_0 < t_1 < \cdots < t_n$ and for all $n \geq 1$ the increments

  $$N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}},$$

  are independent random variables.

- *Stationary increments*: $N_{t+h} - N_{s+h}$ has the same distribution as $N_t - N_s$ for all $h > 0$ and $0 \leq s \leq t$.

The property of stationary increments means that for all $n \in \mathbb{N}$, we have

$$\mathbb{P}(N_{t+h} - N_{s+h} = n) = \mathbb{P}(N_t - N_s = n), \quad h > 0$$

that is, does not depend on $h > 0$, for all $0 \leq s \leq t$ fix. With these two properties, we have the following property:

- *Poisson distribution*: The Poisson process N_t follows a Poisson distribution with intensity $\lambda > 0$. For all $n \in \mathbb{N}$, and for any $0 \leq s \leq t$ we have

  $$\mathbb{P}(N_t - N_s = n) = e^{-\lambda(t-s)} \frac{(\lambda(t-s))^n}{n!}.$$

3.2.2 Compound Poisson Process

The Poisson process itself appears to be too limited to develop realistic price models as its jumps are of constant size. Therefore, for financial applications, it is interesting to consider jump processes that can have random jump sizes. The compound Poisson process is a generalization where the waiting times between jumps are exponential, but the jump size can have an arbitrary distribution.

Let $(Y_k)_{k>1}$ be an independent and identically distributed sequence of square-integrable random variables distributed as the common random variable Y with probability distribution $\gamma(dx)$ on $\mathbb{R}$, independent of the Poisson process $(N_t)_{t\geq 0}$. We have

$$\mathbb{P}(Z \in [a,b]) = \gamma([a,b]) = \int_a^b \gamma(dx), \quad -\infty < a \leq b < \infty.$$

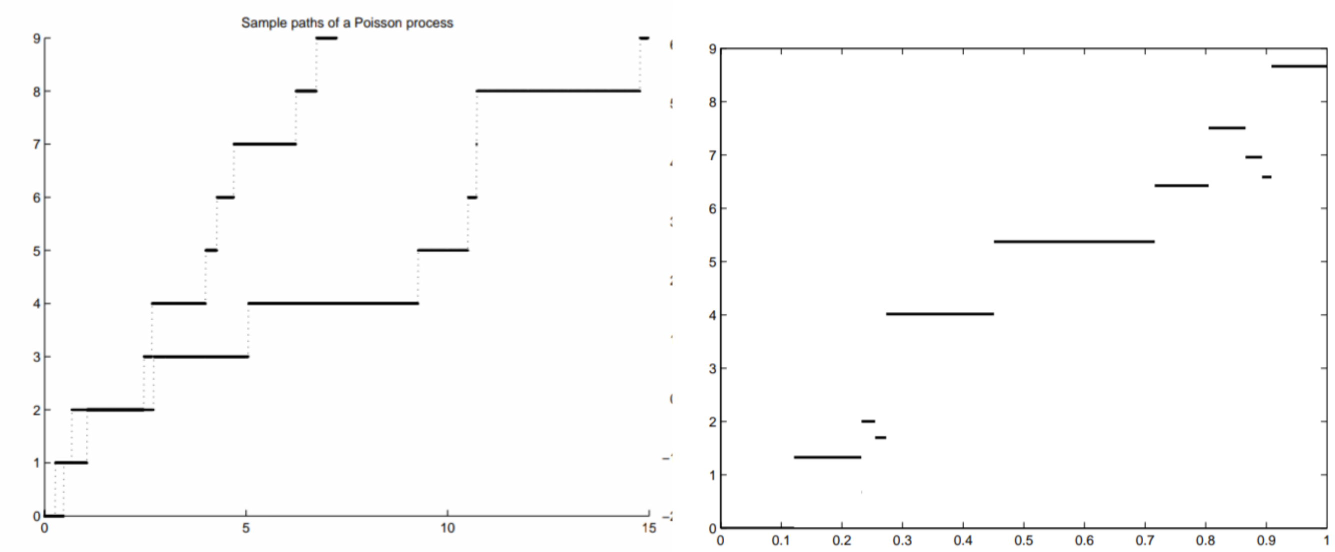
Definition 3.1. *The stochastic process* $(X_t)_{t\geq 0}$ *defined by*

$$X_t = Y_1 + Y_2 + \cdots + Y_{N_t} = \sum_{k=1}^{N_t} Y_k, \quad t \geq 0,$$

is called a compound Poisson process *with intensity* $\lambda > 0$.
When $Y_k = 1$ *we obtain that* X_t *is the Poisson process.*

We can assume that the jump sizes follows a Gaussian law with mean m and variance δ^2, in this case $\gamma(dx)$ is given by

$$\gamma(dx) = \frac{1}{\sqrt{2\pi\delta^2}} \exp\left(\frac{-(x-m)^2}{2\delta^2}\right) dx.$$



(a) Poisson Process with jump size +1 (b) compound Poisson process with Gaussian jump sizes

Figure 3.1: Simulation of sample paths of Poisson and compound Poisson process, the images were obtained from Cont & Tankov [6].

3.2.3 Merton's approach

If we combine the Brownian motion with drift and a compound Poisson process, we obtain the simple case of jump diffusion given by

$$X_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} Y_k.$$

Merton consider the jump-diffusion model given by:

$$S_t = S_0 \exp(X_t) = S_0 \exp\left(\mu t + \sigma W_t + \sum_{k=1}^{N_t} Y_k\right), \tag{3.9}$$

where W_t is a standard Brownian motion, N_t is a Poisson process with intensity λ independent of W_t and $(Y_k)_{k\geq 1}$ are independent and identically distributed Gaussian random variables with mean m and variance δ^2 and independent from W_t and N_t.

The purpose of Merton was change only the drift of Brownian motion in Black-Scholes model, then under the risk-free probability measure $\mathbb{P}^*$, we have:

$$S_t = S_0 \exp\left(\mu^M t + \sigma W_t + \sum_{k=1}^{N_t} Y_k\right), \tag{3.10}$$

where W_t is a standard Brownian motion, N_t and Y_k are independent from W_t and we choose μ^M such that $\widetilde{S}_t = S_t e^{-rt}$ are a $\mathbb{P}^*-$martingale:

$$\begin{aligned}\mu^M &= r - \frac{\sigma^2}{2} - \lambda \mathbb{E}\left[e^{Y_t} - 1\right] \\ &= r - \frac{\sigma^2}{2} - \lambda\left(\exp\left(m + \frac{\delta^2}{2}\right) - 1\right).\end{aligned}$$

Merton justified the choice (3.10) by assuming that *jump risk* is diversified, therefore, no risk premium is attached to it. In particular, the distribution of jump times and jump sizes is unchanged.

A European option with payoff $h(S_T)$ can then be priced according to the following formula:

$$C_t^M = e^{-r(T-t)} \mathbb{E}_{\mathbb{P}^*}\left[h(S_T)|\mathscr{F}_t\right].$$

Since S_t is a Markov process under $\mathbb{P}^*$, the option price C_t^M can be expressed as a deterministic function of t and S_t, as follows:

$$\begin{aligned}C^M(t, S_t) &= e^{-r(T-t)} \mathbb{E}_{\mathbb{P}^*}\left[(S_T - K)_+|\mathscr{F}_t\right] \\ &= e^{-r(T-t)} \mathbb{E}\left[h\left(x \exp\left\{\mu^M (T-t) + \sigma W_{T-t} + \sum_{k=1}^{N_{T-t}} Y_k\right\}\right)\right]_{x=S_t}\end{aligned}$$

By conditioning on the number of jumps N_t, we can express $C(t, S_t)$ as a weighted sum of Black-Scholes prices:

$$C^M(t, S_t) = e^{-r(T-t)} \sum_{n\geq 0} \mathbb{P}^*(N_t = n)\mathbb{E}_{\mathbb{P}^*}\left[h\left(x \exp\left\{\mu^M(T-t) + \sigma W_{T-t} + \sum_{k=1}^{N_{T-t}} Y_k\right\}\right)\right],$$

but we know that $\sum_{k=1}^{N_{T-t}} Y_k \sim \mathcal{N}\left(nm, n\delta^2\right)$, and if we substitute the value of μ^M, we obtain

$$\begin{aligned}
&C^M(t, S_t) = \\
&= e^{-r(T-t)} \sum_{n\geq 0} \mathbb{P}^*(N_{T-t} = n)\mathbb{E}_{\mathbb{P}^*}\left[h\left(x \exp\left(r - \frac{\sigma^2}{2} - \lambda\left(\exp\left(m + \frac{\delta^2}{2}\right) - 1\right)(T-t) + \sigma W_{T-t} + nm + \frac{n\delta^2}{2}\right)\right)\right] \\
&= e^{-r(T-t)} \sum_{n\geq 0} \frac{e^{-\lambda(T-t)}(\lambda(T-t))^n}{n!} \mathrm{C}_{\mathrm{BS}}(t, T, S_n, \sigma_n),
\end{aligned}$$

where $\sigma_n^2 = \sigma^2 + \dfrac{n\delta^2}{T-t}$,

$$S_n = x \exp\left(nm + \frac{n\delta^2}{2} - \lambda(T-t)\exp\left(m + \frac{\delta^2}{2}\right) + \lambda(T-t)\right)$$

and

$$\mathrm{C}_{\mathrm{BS}}(t, T, x, \sigma) = e^{-r(T-t)}\mathbb{E}\left[h\left(x \exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma W_{T-t}\right)\right)\right]$$

is the value of a European option wit maturity time $T-t$ and payoff $h(S_T)$ in Black-Scholes model with volatility σ.

Chapter 4

Implied Volatility: Statics and Dynamics

In [16], Roger Lee intends to apart from the Black-Scholes model and look for volatility values that come from alternative methods, such as stochastic volatility models. First, he begins with the basic analysis of implied volatility, for which the following assumptions are made:

- The price of the underlying asset S_t is strictly positive.
- We take a call option with maturity time T and strike price K.
- We have a self-financing portfolio for the maturity time, i.e, its value is $(S_T - K)^+$ at time T.
- The option price is a function $C(K,T)$ for S_t with t the actual date.
- The risk free interest rate will be a constant r.
- We write the log-moneyness of an option at time t as:

$$x := \log\left(\frac{K}{S_t e^{r(T-t)}}\right)$$

We assume a frictionless market, the Black-Scholes model suppose that S_t follows a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

then the no-arbitrage call price satisfies

$$C = \mathrm{C}_{\mathrm{BS}}(\sigma),$$

where the Black-Scholes formula is defined by

$$\mathrm{C}_{\mathrm{BS}}(\sigma) := \mathrm{C}_{\mathrm{BS}}(S_t, K, r, t, T, \sigma) := S_t N(d_1) - K e^{-r(T-t)} N(d_2),$$

with

$$d_{1,2} = \frac{\log(S_t e^{r(T-t)}/K)}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2},$$

where N is the cumulative Gaussian distribution function.

On the other hand, given market prices $C^*(K,T)$, the implied volatility of the strike price K and maturity time T is defined as the $I(K,T)$ that solves

$$C^*(K,T) = \mathrm{C}_{\mathrm{BS}}(K,T,I(K,T)).$$

In Chapter 2 we say that the solution is unique since C_{BS} is strictly increasing in σ, and since $\sigma \to 0$ (resp. $\sigma \to \infty$) the function $\mathrm{C}_{\mathrm{BS}}(\sigma)$ approaches the lower (resp. upper) bounds on a call.

We can write the implied volatility as a function $\tilde{I}$ of log-moneyness and time, so

$$\tilde{I}(x,T) := I(S_t e^{x+r(T-t)}, T).$$

In the subsequent we write I to refer us I as a function of K or x.

The derivation of Black-Scholes formula can proceed by a hedging argument that yields a PDE for $C(S,t)$:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0$$

with condition $C(S,T) = (S-K)_+$.

4.1 Statics

The term *static* refers to the analysis of $I(x,T)$ or $I(K,T)$ for t fixed. As we saw in Chapter 2, the plot of I is not constant with respect to K (or x), it can take the shape of a smile, in which $I(K)$ is greater for K out-of-money than it is for K near-the-money, however, is a skew in which at-the-money I slopes downward, and the smile is far more pronounced for small K than for large K. Empirically smile or skew flattens as T increases.

4.1.1 Statics under absence of arbitrage

Assuming only the absence of arbitrage, one obtains bounds on the slope of the implied volatility surface, as well as a characterization of how fast I grows at extreme strikes.

Slope bounds

Hodges [10] gives bounds on implied volatility based on the non-negativity of call spreads and put spreads. If the strike prices satisfies $K_1 < K_2$ then

$$C(K_1) \geq C(K_2) \quad \text{and} \quad P(K_1) \leq P(K_2). \tag{4.1}$$

Jim Gatheral [9] observed that

$$C(K_1) \geq C(K_2) \quad \text{and} \quad \frac{P(K_1)}{K_1} \leq \frac{P(K_2)}{K_2} \tag{4.2}$$

which is evident from a comparison of the respective payoff functions. Assuming the differentiability of option prices in K, we have

$$\frac{\partial C}{\partial K} \leq 0 \quad \text{and} \quad \frac{\partial}{\partial K}\left(\frac{P}{K}\right) \geq 0.$$

Substituting $C = \mathrm{C_{BS}}(I)$ and $P = \mathrm{P_{BS}}(I)$ and simplifying, we have

$$-\frac{N(-d_1)}{\sqrt{T}N'(d_1)} \leq \frac{\partial I}{\partial x} \leq \frac{N(d_2)}{\sqrt{T}N'(d_2)},$$

where the upper and lower bounds come from the call and put constraints, respectively. Now, using the Mill's Ratio $R(d) := (1 - N(d))/N'(d)$, we rewrite the last inequality as

$$-\frac{R(d_1)}{\sqrt{T}} \leq \frac{\partial I}{\partial x} \leq \frac{R(-d_2)}{\sqrt{T}}.$$

From (4.1) without Gatheral's observation (4.2) yields the significantly weaker lower bound $-R(d_2)/\sqrt{T}$.

Of particular interest is the behavior at-the-money, when $x = 0$. In the short-dated limit, as $T \to 0$, we assume that $I(0,T)$ is bounded above. Then

$$d_{1,2}(x = 0) = \pm I(0,T)\sqrt{T}/2 \to 0.$$

Since $R(0)$ is a positive constant, the at-the-money skew slope must have the short-dated behavior

$$\left|\frac{\partial I}{\partial x}(0,T)\right| = O\left(\frac{1}{\sqrt{T}}\right) \quad \textbf{as} \quad T \to 0. \tag{4.3}$$

In the long-dated limit, as $T \to \infty$, we assume that $I(0,T)$ is bounded away from 0. Then

$$d_{1,2}(x = 0) = \pm I(0,T)\sqrt{T}/2 \to \pm\infty.$$

Since $R(d) \sim d^{-1}$ as $d \to \infty$, the at-the-money skew slope must have the long-dated behavior

$$\left|\frac{\partial I}{\partial x}(0,T)\right| = O\left(\frac{1}{T}\right) \quad \textbf{as} \quad T \to \infty. \tag{4.4}$$

According to (4.4), the rule of thumb that approximates the skew slope decay rate as $T^{-1/2}$ cannot maintain validity into long-dated expiries.

4.1.2 Statics under stochastic volatility

We assume that the stock prices (S_t) follows a stochastic differential equation of the form

$$dS_t = rS_t dt + \sqrt{Y_t} S_t dW_t$$
$$dY_t = \alpha(Y_t) + \beta(Y_t) dZ_t$$

where W_t and Z_t are Brownian motions with correlation ρ. From here one obtains, via perturbation methods, approximations to the implied volatility skew I.

Zero Correlation

Renault and Touzi [23] proved that in the case when the correlation is zero, the implied volatility is a symmetric smile, symmetric in the sense that

$$I(x, T) = I(-x, T)$$

and a smile in the sense that I is increasing in x for $x > 0$.

According Hull and White in [12], we may illustrate the qualitative difference between Black-Scholes prices and option prices under stochastic volatility in the following intuitive way. We expand the function $C(v) := \mathrm{C_{BS}}(\sqrt{v})$ where $v = \mathbb{E}[\overline{Y}]$ and $\overline{Y}$ is the average variance defined as $\overline{Y} = \dfrac{1}{T}\displaystyle\int_0^T \sigma_s^2 ds$, we have

$$C = \mathrm{C_{BS}}(I) \approx \mathrm{C_{BS}}(\mathbb{E}[\overline{Y}]) + (I^2 - \mathbb{E}[\overline{Y}])\frac{\partial\, \mathrm{C_{BS}}}{\partial Y},$$

and using Taylor power series expansion,

$$C = \mathbb{E}[\mathrm{C_{BS}}(I)] \approx \mathrm{C_{BS}}(\mathbb{E}[\overline{Y}]) + \frac{1}{2}\,\mathrm{Var}(\overline{Y})\frac{\partial^2\, \mathrm{C_{BS}}}{\partial Y^2}$$

yields the approximation

$$I^2 \approx \mathbb{E}[\overline{Y}] + \frac{1}{4}\frac{\mathrm{Var}(\overline{Y})}{\mathbb{E}[\overline{Y}]^2}\left(\frac{x^2}{T} - \mathbb{E}[\overline{Y}] - \frac{1}{4}\mathbb{E}[\overline{Y}]^2 T\right),$$

which is quadratic in x, with minimum at $x = 0$.

To the extent that implied volatility skews are empirically not symmetric in equity markets, stochastic volatility models with zero correlation will not be consistent with market data.

Fast mean reversion

In [8] Fouque, Papanicolaou and Sircar model stochastic volatility as a function f of a state variable Y_t that follows a rapidly mean-reverting diffusion process, as we saw in Section 3.1.1.

The fast-mean-reversion approximation is particularly suited for pricing long-dated options, in that long time horizon, volatility has time to undergo much activity, so relative to the time scale of the option's lifetime, volatility can indeed be considered to mean-revert rapidly.

In the case of Ornstein-Uhlenbeck process Y, this means that for some large α,

$$\begin{aligned} dS_t &= \mu_t S_t dt + f(Y_t) S_t d\hat{W}_t, \\ dY_t &= \alpha(\theta - Y_t)dt + \beta d\hat{Z}_t, \end{aligned}$$

where the Brownian motions $\hat{W}_t$ and $\hat{Z}_t$ have correlation ρ. Under the pricing probability measure, we have

$$\begin{aligned} dS_t &= rS_t dt + f(Y_t) S_t dW_t, \\ dY_t &= [\alpha(\theta - Y_t) - \beta\Lambda(Y_t)]dt + \beta dZ_t, \end{aligned}$$

where the volatility risk premium Λ is assumed to depend only on Y. Let p_Y denote the invariant density of Y, which is normal with mean θ and variance $\beta^2/(2\alpha)$. As in the Chapter 3, we denote the quadratic average of volatility with respect to p_Y by $\bar{\sigma}_\infty^2 := \langle f^2 \rangle$. By a singular perturbation analysis of the PDE for call price, the implied volatility has an expansion with leading terms

$$I(x, T) = A\frac{x}{T} + B + O(1/\alpha),$$

where

$$\begin{aligned} A &:= -\frac{V_3}{\bar{\sigma}_\infty^3} \\ B &:= \bar{\sigma}_\infty + \frac{3V_3/2 - V_2}{\bar{\sigma}_\infty}, \end{aligned}$$

and

$$\begin{aligned} V_2 &:= \frac{\beta}{2\alpha} \left\langle (2\rho f - \Lambda)\phi \right\rangle, \\ V_3 &:= \frac{\beta}{2\alpha} \left\langle (\rho f \phi \right\rangle, \\ \phi(y) &:= \frac{2\alpha}{\beta^2 p_Y(y)} \int_{-\infty}^{\infty} (f^2(z) - \langle f^2 \rangle) p_Y(z) dz. \end{aligned}$$

Today's volatility plays no role in the leading-order coefficients A and B. Intuitively, the assumption of large mean-reversion rapidly erodes the influence of today's volatility, leaving the long-run averages to determine A and B. Furthermore, the slope of the long-dated implied volatility skew satisfies

$$\left|\frac{\partial I}{\partial x}(0, T)\right| \approx \frac{1}{T} \quad \textbf{as} \quad T \to \infty.$$

As a consistency check, note that the long-dated asymptotics are consistent with the no-arbitrage constraint (4.4). Specifically, the $T \to \infty$ skew slope decay of these stochastic volatility models achieves the $O(T^{-1})$ bound.

Slow mean reversion

The slow-mean-reversion approximation is particularly suited for pricing short-dated options, in that short time horizon, volatility has little time in which to vary, so relative to the time scale of the option's lifetime, volatility can indeed be considered to mean-revert slowly.

Assuming that for a constant parameter ε,

$$d\sigma_t = \varepsilon\alpha(V_t)dt + \sqrt{\varepsilon}\beta(V_t)dW_t,$$

Sircar and Papanicolaou [26] develop, and Lee [17] extends, a regular perturbation analysis of the PDE

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \sqrt{\varepsilon} S\sigma\beta \frac{\partial^2 C}{\partial S \partial \sigma} + \frac{1}{2}\varepsilon\beta^2 \frac{\partial^2 C}{\partial \sigma^2} + \varepsilon\alpha \frac{\partial C}{\partial \sigma} + rS\frac{\partial C}{\partial S} = rC.$$

satisfied by the call price under stochastic volatility. This leads to an expansion for C in powers of ε, which in turn leads to the implied volatility expansion

$$\begin{aligned} I \approx\ & \sigma_0 + \sqrt{\varepsilon}\left(\frac{\rho\beta}{2\sigma_0}x + \frac{\rho\sigma_0\beta}{4}T\right) \\ & + \varepsilon\Bigg[\left(\left(\frac{\beta\beta'}{6\sigma^2} - \frac{5\beta^2}{12\sigma^3}\right)\rho^2 + \frac{\beta^2}{6\sigma^3}\right)x^2 + \left(\left(\frac{\sigma\beta^2}{12} + \frac{\sigma^2\beta\beta'}{24}\right)\rho^2 - \frac{\sigma\beta^2}{24}\right)T^2 \\ & \left(\frac{\beta^2}{24\sigma} - \frac{\beta\beta'}{6}\right)\rho^2 Tx + \left(\left(\frac{\beta^2}{24\sigma} - \frac{\beta\beta'}{6}\right)\rho^2 + \frac{\alpha}{2} - \frac{\beta^2}{12\sigma}\right)T\Bigg], \quad \text{where } \beta' = \frac{\partial\beta}{\partial\sigma}. \end{aligned}$$

In particular, short-dated implied volatility satisfies

$$I(x,0) \approx \sigma_0 + \sqrt{\varepsilon}\frac{\rho\beta}{2\sigma_0}x.$$

In contrast to the case of rapid mean-reversion, the level to which volatility reverts here plays no role in the leading-order coefficients. With a small rate of mean-reversion, today's volatility will have the dominant effect.

For $\rho \neq 0$, the at-the-money skew exhibits a slope whose sign agrees with ρ. For $\rho = 0$ the skew has a parabolic shape.

4.2 Dynamics

Models for the dynamics of implied volatility surfaces treat it as a random process and try to model it based on option prices quoted in the market. We will present a new class of models to specify directly the dynamics of one or more implied volatilities.

We assume t the current date that is not fixed at 0, because we are now concerned with the time evolution of I.

4.2.1 No-arbitrage approach

One implied volatility

We consider the time evolution of a single implied volatility I at some fixed strike price K and maturity time T. Schönbucher in [25] presented a market for the implied volatility and the main issues to achieve the absence of arbitrage in market models. Further, an advantage is that in this model there is no need to specify the market price of risk process since it is implied in the observed option prices. Schönbucher proposes the following dynamics in order to model one implied volatility

$$dI_t = u_t dt + \gamma_t dW_t^{(0)} + v_t dW_t,$$

where W_t and $W_t^{(0)}$ are independent Brownian motions. The spot price has dynamics

$$dS_t = rS_t dt + \sigma_t S_t dW_t^{(0)},$$

where σ_t is yet to be specified.

Since the discounted call price $e^{-r(T-t)}\,\mathrm{C_{BS}}(t, S_t, I_t)$ must be a martingale under the risk free probability measure, then we have for all $I > 0$ the following drift restriction on the call price:

$$\frac{\partial\,\mathrm{C_{BS}}}{\partial t} + rS\frac{\partial\,\mathrm{C_{BS}}}{\partial S} + u\frac{\partial\,\mathrm{C_{BS}}}{\partial I} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2\,\mathrm{C_{BS}}}{\partial S^2} + \gamma\sigma S\frac{\partial^2\,\mathrm{C_{BS}}}{\partial I\partial S} + \frac{1}{2}v^2\frac{\partial^2\,\mathrm{C_{BS}}}{\partial I^2} = r\,\mathrm{C_{BS}}\,.$$

This reduces to a joint restriction on the diffusion coefficients of I, the drift of I, and the instantaneous volatility σ:

$$Iu = \frac{I^2 - \sigma^2}{2(T-t)} - \frac{1}{2}d_1 d_2 v^2 + \frac{d_2}{\sqrt{T-t}}\sigma\gamma \tag{4.5}$$

Since S, t, and T are observable in the financial market, we have that the volatility of I, together with the drift of I, determines the spot volatility.

Schönbucher imposes a further constraint to ensure that I does not *blow up* as t tends to T. He requires that

$$(I^2 - \sigma^2) - d_1 d_2 (T-t) v^2 + 2d_2\sqrt{T-t}\sigma\gamma = O(T-t) \tag{4.6}$$

which simplifies to

$$I^2\sigma^2 + 2\gamma x I\sigma - I^4 + x^2 v^2 = 0.$$

This can be solved to get expiration-date implied volatility in terms of expiration date spot volatility. The solution is particularly simple in the zero-correlation case, where $\gamma = 0$. Then, suppressing subscripts T,

$$I^2 = \frac{1}{2}\sigma^2 + \sqrt{\frac{\sigma^2}{4} + x^2 v^2}.$$

Under condition 4.6, therefore, implied volatility behaves as $\sigma + O(x^2)$ for x small, but $O(|x|^{1/2})$ for x large.

4.2.2 Term structure of implied volatility

Schönbucher extends this model for the forward volatility to handle a set of option with M maturity times. The implied volatilities to be modelled are $I_t(K_m, T_m)$ for $m = 1, \ldots, M$, where $T_1 < T_2 < \cdots < T_M$. Let

$$V_t^{(m)} := I_t^2(K_m, T_m)$$

be the implied variance. One specifies the dynamics for the shortest-dated variance $V^{(1)}$, as well as all *forward* variances

$$V^{(m,m+1)} := \frac{(T_{m+1} - t)V^{(m+1)} - (T_m - t)V^{(m)}}{T_{m+1} - T_m}$$

The spot volatility σ_t and the drift and diffusion coefficients of $V_t^{(1)}$ are jointly subject to the drift restriction (4.5) and the no-explosion condition (4.6). Then, given the σ_t and $V_t^{(1)}$ dynamics, specifying each $V^{(m,m+1)}$ diffusion coefficient determines the corresponding drift coefficient, by applying (4.5) to $V^{(m+1)}$.

4.3 Plots of implied volatility for correlated and uncorrelated cases

In the following, we explain Ornstein–Uhlenbeck process, in order to show the plots of implied volatilities, in correlated and uncorrelated cases.

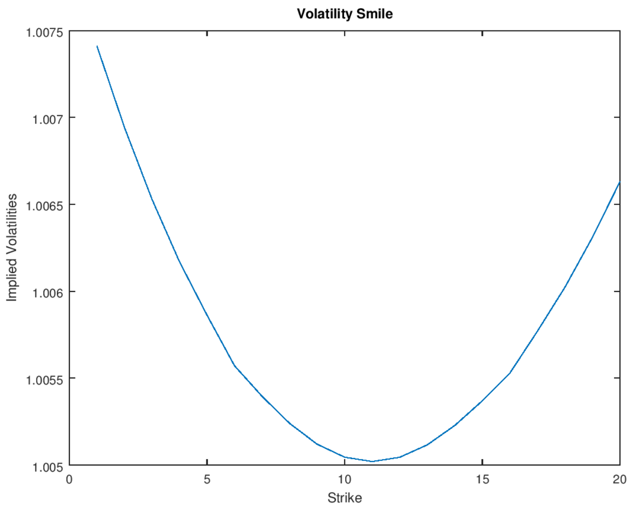
An Ornstein–Uhlenbeck process is a stationary Gaussian and Markov process, which means that it is a Gaussian process, a Markov process, and is temporally homogeneous. Overtime is a mean reverting process and satisfies the following stochastic differential equation

$$dX_t = -aX_t dt + b dW_t, \quad X_0 = x$$

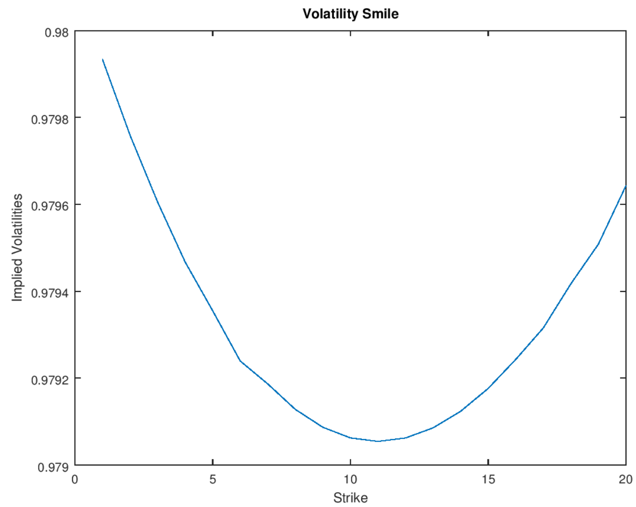
where a and b are constants, X_0 is the initial condition and W_t is a Brownian motion.

In Chapter 3, we introduce the stochastic volatility models. In uncorrelated case $\rho = 0$, we obtain the *smile* when we use the Hull-White formula and where the implied volatility is calculated for different strike prices and the same maturity time T, as we can see un Figure 4.1. For correlated case $\rho \in [-1, 1]$, we extended the Hull-White formula when the stock price is random as we mentioned and if we plot the implied volatilities for different strikes prices we obtain the *skew* as we can see un Figure 4.2.

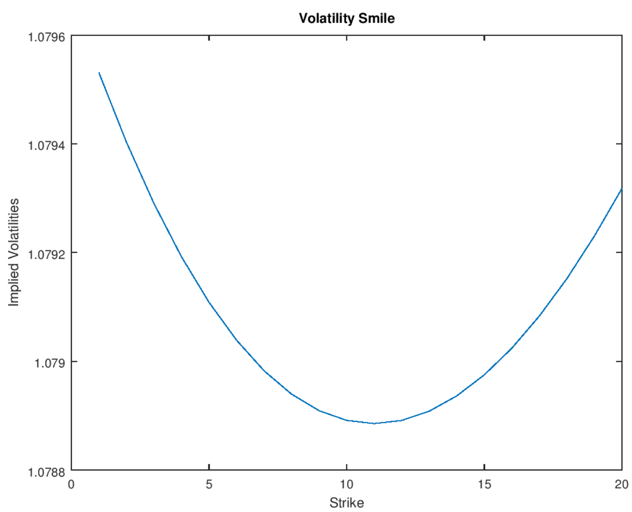
Uncorrelated case



(a) At maturity time $T = 0.5$



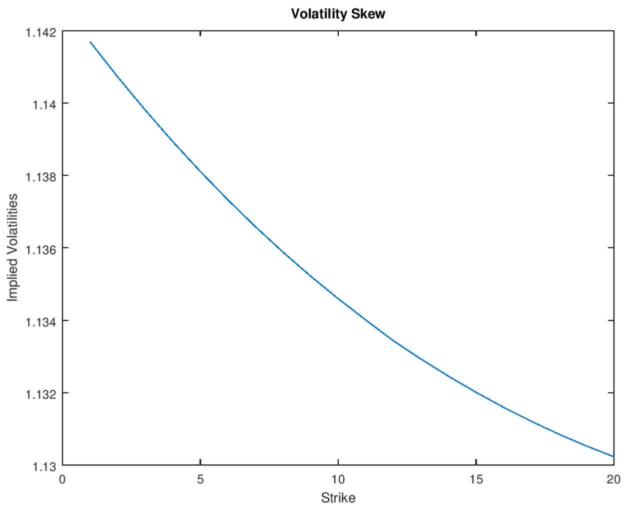
(b) At maturity time $T = 1$



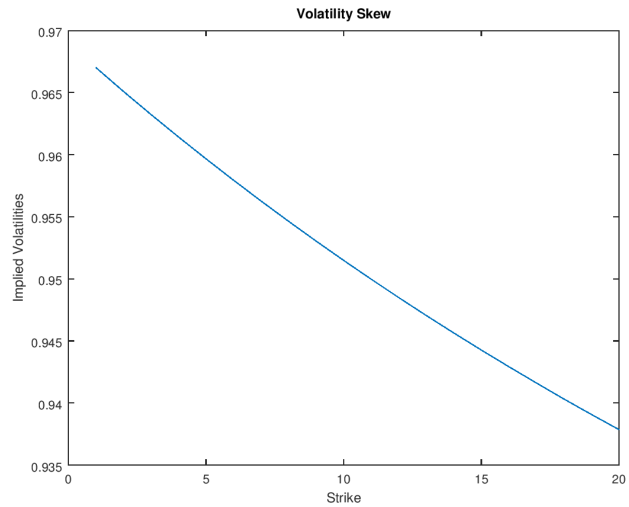
(c) At maturity time $T = 2$

Figure 4.1: We plot the implied volatility for different strikes prices and maturity times

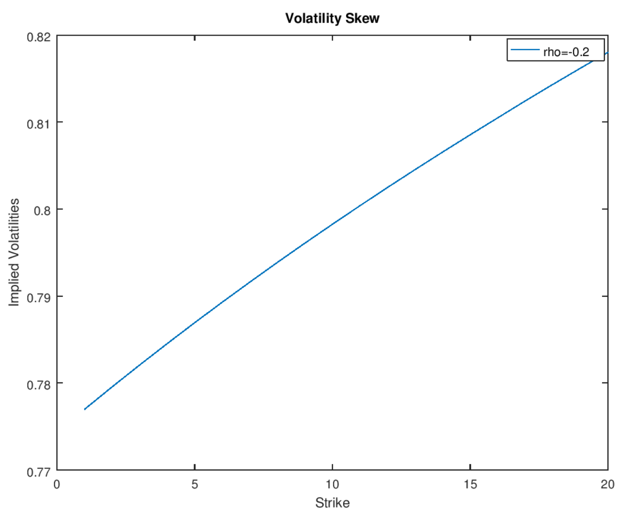
Correlated case



(a) At maturity time $T = 1$ and correlation $\rho = 0.2$



(b) At maturity time $T = 2$ and correlation $\rho = 0.2$



(c) At maturity time $T = 1$ and correlation $\rho = -0.2$

Figure 4.2: We plot the implied volatility for different strikes prices and maturity times

Chapter 5

Fractional Brownian Motion

Fractional Brownian motion was introduced in 1940 by Andrey Kolmogorov [15]. But was Benoît Mandelbrot who recognized the importance of this random process, and jointly with John Van Ness [20], gives the first mathematical definition and the first properties. They established a representation for fractional Brownian motion as an integral with respect to standard Brownian motion which involves a fractional parameter $H \in (0,1)$, this parameter is called *Hurst parameter* from the statical analysis developed by Harold Edwin Hurst [13], who studied the water level in the Nile River.

5.1 Fractional Brownian motion

Definition 5.1. *A Gaussian process $B^H = (B_t^H)_{t\geq 0}$ is called a fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$, if it has zero mean and covariance function given by*

$$R_H(s,t) = \mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \quad s,t \geq 0.$$

Usually is assumed that $B_0^H = 0$.

When $H = 1/2$, the covariance function of fBm is:

$$R_{1/2}(s,t) = \frac{1}{2}\left(s + t - |t-s|\right) = \min(s,t),$$

which is the covariance function of Brownian motion.

The fractional Brownian motion has the following properties:

- *Self-similarity.* For all $a > 0$, the process $\left(a^{-H} B_{at}^H\right)_{t\geq 0}$ is a fractional Brownian motion with Hurst parameter H;

- *Stationary increments.* For all $0 \leq s < t$, the increments of the process $\left(B_t^H - B_s^H\right)_{t\geq 0}$ has a Gaussian distribution with zero mean and variance
$$\mathbb{E}[\left(B_t^H - B_s^H\right)^2] = |t-s|^{2H}.$$
- *No independent increments.* Unlike to Brownian motion, we want to mention that fractional Brownian motion has not independent increments.

Hence, for any integer $n \geq 1$, we have
$$\mathbb{E}[\left(B_t^H - B_s^H\right)^{2n}] = \frac{(2n)!}{n!2^n}|t-s|^{2Hn}$$
and we can apply Kolmogorov's continuity criterion to affirm the following property:

- *$\alpha-$Hölder continuous sample paths.* Fractional Brownian motion $(B_t^H)_{t\geq 0}$ has continuous trajectories, i.e., there exists $C > 0$ such that
$$\sup_{s\leq t} |B_t^H(\omega) - B_s^H(\omega)| \leq C|t-s|^{\alpha},$$
for all $\alpha \in (0, H)$. But fBm does not have $\alpha-$Hölder continuous sample paths for $\alpha \geq H$.

By a result given by Mandelbrot and Van Ness [20] we have that the sample paths of fBm are almost surely nowhere differentiable at any point.

- *No differentiable sample paths.* The sample paths of $(B_t^H)_{t\geq 0}$ are not differentiable. In fact, at any point $t_0 \in [0, \infty)$ it satisfies
$$\mathbb{P}\left(\limsup_{t\to t_0}\left|\frac{B_t - B_{t_0}}{t-t_0}\right| = \infty\right) = 1.$$

5.1.1 Fractional Brownian motion is not a semimartingale

We will introduce another notion of regularity of the sample paths, called $p-$variation. The definition of Itô's integral is a direct consequence of the martingale property of the Brownian motion. But fBm does not exhibit this property, in fact, it is not even a semimartingale, except when $H = 1/2$, which is an impediment to defining the stochastic integral in the Itô sense, reason why other techniques are required to define an integral with respect to fBm.

First, we want to study the asymptotic behavior of the $p-$variation of fBm, in order to find what is it $p-$variation.

We consider $T > 0$ and we fix an interval $[0, T]$. Let $X = (X_t)_{t\geq 0}$ be a stochastic process and we consider a sequence of partitions $(\pi_n)_{n\in\mathbb{N}}$ given by $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ such that
$$\limsup_{n\to\infty, k\leq n} (t_k - t_{k-1}) = 0.$$

Definition 5.2. *We define the p-variation of a stochastic process X as*

$$V_p(X,[0,T]) = \sup_{\pi} \sum_{k=1}^{n} |X_{t_k} - X_{t_{k-1}}|^p,$$

for $1 \leq p < \infty$. If $V_p(X,[0,T]) < \infty$, then $(X_t)_{t\geq 0}$ has bounded p-variation.

Then, we have the following result: let B^H be a fBm with Hurst parameter $H \in (0,1)$, $p \in [1,\infty)$ and $\mathrm{N} \sim \mathcal{N}(0,1)$. Then, when $n \to \infty$, we have the following limit in $\mathrm{L}^2(\Omega)$

$$\sum_{k=1}^{n} |B_{k/n} - B_{(k-1)/n}|^p \longrightarrow \begin{cases} 0 & \text{si } p > 1/H, \\ \mathbb{E}[|\,\mathrm{N}\,|^p] & \text{si } p = 1/H, \\ +\infty & \text{si } p < 1/H. \end{cases}$$

Rogers proved in [24] that fBm has finite $p-$variation when $p = 1/H$ and in consequence that fBm is not a semimartingale, except in the case $H = 1/2$ since if $H < 1/2$, the 2-variation is infinite, and if $H > 1/2$ the 2-variation is zero and for all $H \in (0,1)$ we have that 1-variation is infinite. We recall that semimartingales are processes for which a stochastic calculus can be developed, and they can be expressed as the sum of a bounded variation process and a local martingale which has finite $2-$variation.

To define an integral of the form

$$\int_0^T X_t \, \mathrm{d}B_t^H,$$

we cannot apply Itô's calculus because B^H is not a semimartingale and we cannot apply Lebesgue-Stieltjes integral because the sample paths of B^H are not of bounded variation. Therefore, other techniques are required so that an integral with respect to the fractional Brownian motion is well defined.

5.1.2 Fractional Brownian motion is not a Markov process

In addition, fractional Brownian motion loses the property of being a Markov process when the Hurst parameter $H \neq 1/2$.

Let $X = (X_t)_{t\geq 0}$ be a real-valued stochastic process. We say that $(X_t)_{t\geq 0}$ is a *Markov process* if for all Borel set $A \subset \mathbb{R}$ and all real numbers $t > s > 0$,

$$\mathbb{P}(X_t \in A \mid X_u,\ u \leq s) = \mathbb{P}(X_t \in A \mid X_s).$$

That is, $(X_t)_{t\geq 0}$ is a process without memory, which signifies that the conditional probability of the future time of a stochastic process uniquely depends on present time, being independent of the history of that process. For fractional Brownian motion, we have the following result (for the proof see [21, Theorem 2.3].

Theorem 5.3. *Let $B^H = (B_t^H)_{t\geq 0}$ be a fractional Brownian motion of Hurst index $H \in (0,1)$. Then B^H is not a Markov process for $H \neq 1/2$.*

5.1.3 Long-range dependence

Definition 5.4. *A stationary sequence of random variables $(X_n)_{n\in\mathbb{N}}$, we say that has long-range dependence if the covariance sequence*

$$\rho(n) = \mathrm{Cov}(X_k, X_{k+1}), \quad k, n \in \mathbb{N}$$

satisfies

$$\lim_{n\to\infty} \frac{\rho(n)}{cn^{-\alpha}} = 1.$$

for some constants c and $\alpha \in (0,1)$. In this case, the dependence between X_k and X_{k-1} falls slowly when n tends to infinity, since $\rho(n) = O(cn^{-\alpha})$, that is,

$$\limsup_{n\to\infty} \left| \frac{\rho(n)}{cn^{-\alpha}} \right| < \infty.$$

For fractional Brownian motion, we have that has long-range dependence only when $H > 1/2$, since

$$\rho_H(n) \approx n^{2H-2} H(2H-1) \to 0 \ \text{ as } \ n \to \infty$$

for all $H \in (0,1)$. From here we can observe that covariance of the increments tends to zero in the same order than n^{2H-2}, so fBm is a long-range process. Furthermore,

$$\lim_{n\to\infty} \frac{\rho_H(n)}{H(2H-1)n^{2H-2}} = 1,$$

hence, taking $c = H(2H-1)$ and $\alpha = 2-2H$ in Definition 5.4, we have that only when $H \in (1/2, 1)$ it satisfies that $\alpha \in (0,1)$, consequently, when $H \in (0, 1/2)$ fBm has not long-range dependence.

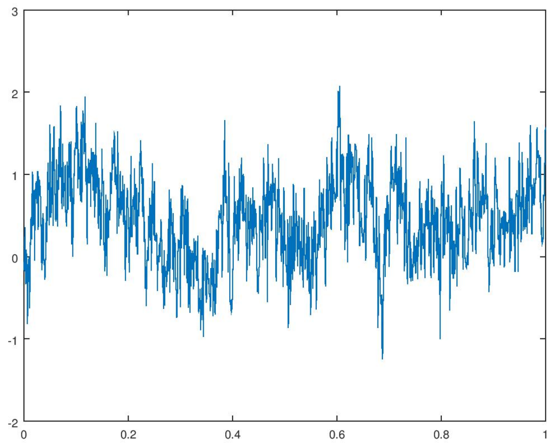
Now, to examine the characteristics that the fBm has in the case $H \in (1/2, 1)$, we analyze the covariance of his increments and for it, we will introduce the following definitions to try to understand in a different context, how is the behavior of the sample path of fBm.

Definition 5.5. *We say that a stochastic process is:*

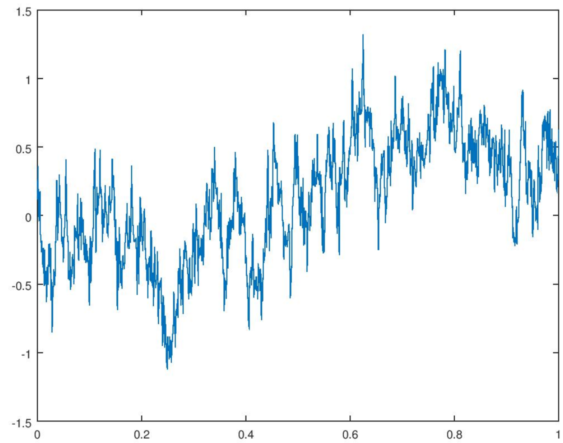
- *Persistent: when the sample paths of the process tend to go in the same direction.*
- *Anti-persistent: when the sample paths of the process tend to back on itself.*

As we can see from the following figures 5.1 and 5.2, the trajectories of fractional Brownian motion behave differently for different values of Hurst index $H \in (0,1)$.

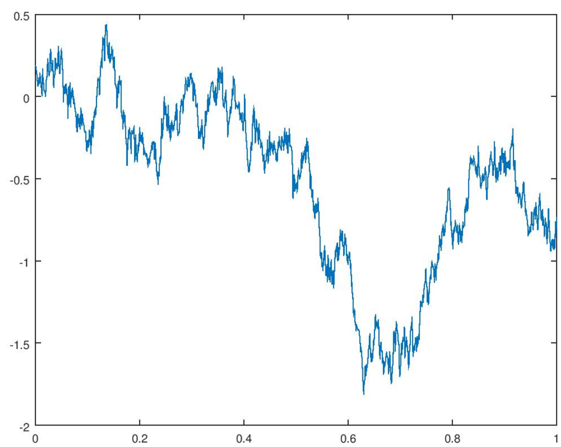
When $H < 1/2$, the increments of fBm tends in opposite directions, that is, it is anti-persistent. In other words, the behavior of trajectories is very irregular, when H is very close to zero the trajectories are



(a) $H = 0.1$



(b) $H = 0.25$

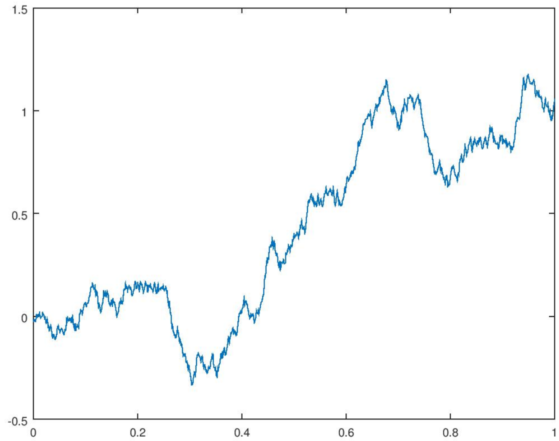


(c) $H = 0.4$

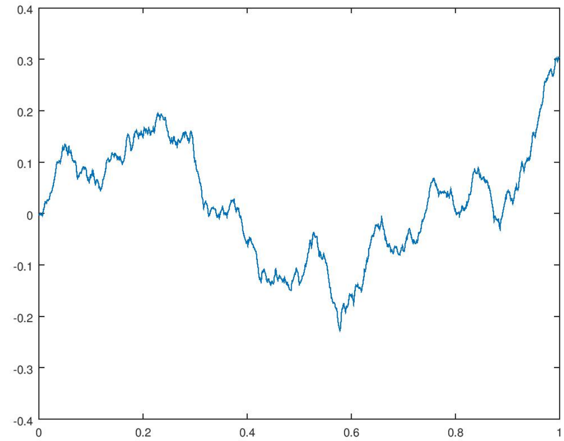
Figure 5.1: Simulation of sample paths of fBm on interval $[0, 1]$ for different values of Hurst parameter $H < 1/2$

very erratic, while the value of H is closer to $1/2$ the trajectories are similar to trajectories of Brownian motion.

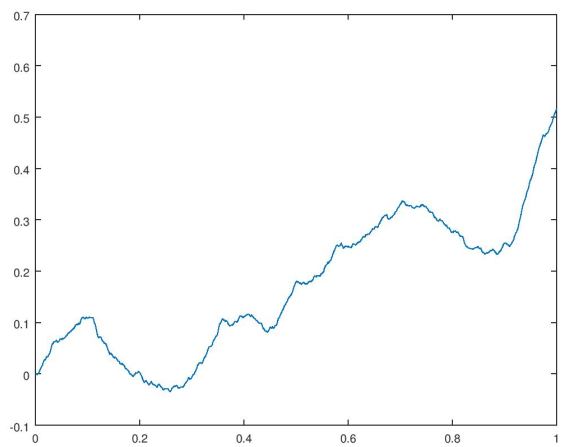
When $H > 1/2$, the increments of fBm tends in the same direction, that is, it is persistent. The trajectories of B^H are essentially $\alpha-$Hölder continuous with $0 < \alpha < H$, so we have a better management on these, since there is a kind of continuity in the Hölder sense.



(a) $H = 0.6$



(b) $H = 0.75$



(c) $H = 0.9$

Figure 5.2: Simulation of sample paths of fBm on interval $[0, 1]$ for different values of Hurst parameter $H > 1/2$

5.2 Preliminaries on Malliavin calculus

The *Malliavin calculus* is an infinite dimensional differential calculus introduced by Paul Malliavin to provide a probabilistic proof of the Hörmander hypoellipticity theorem. Malliavin calculus is called too, *anticipating stochastic calculus*, which is a powerful extension of the classical Itô calculus that allows us to work with non-adapted processes.

The basic operators of Malliavin calculus are the derivative operator and its adjoint the divergence operator. For a complete introduction to this subject, we refer [22].

We fix a time interval $[0,T]$. We consider $W=(W_t)_{t\geq 0}$ a standard Brownian motion defined in a filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},\mathbb{P})$. Let $\mathscr{H}$ be the Hilbert space $\mathrm{L}^2([0,T])$. For any $h\in\mathscr{H}$ we denote by $W(h)$ the Wiener integral

$$W(h)=\int_0^T h(t)dW_t.$$

We define by $\mathcal{S}$ the set of smooth and cylindrical random variables on $(\Omega,\mathscr{F},\mathbb{P})$, of the form

$$F=f\left(W\left(h_1\right),\ldots,W\left(h_n\right)\right)$$

where $n\geq 1$, $h_1,\ldots,h_n\in\mathscr{H}$ and $f\in\mathcal{C}_p^\infty(\mathbb{R}^n)$ (i.e., $f:\mathbb{R}^n\to\mathbb{R}$ is infinitely differentiable such that f and its partial derivatives have polynomial growth order).

Definition 5.6 (Derivative operator)**.** *Let $F\in\mathcal{S}$. The derivative of a smooth random variable F, is the stochastic process $D=(D_tF)_{t\geq 0}$ given by*

$$D_tF=\sum_{i=1}^n\frac{\partial f}{\partial x_i}\left(W\left(h_1\right),\ldots,W\left(h_n\right)\right)h_i(t),\quad t\in[0,T].$$

The derivative DF is an element of the space $\mathrm{L}^2([0,T]\times\Omega)$. More generally, we can define in a general form, the *iterated derivatives* of a smooth random variable F as

$$D^n_{t_1,\ldots,t_n}F=D_{t_1}\cdots D_{t_1}F.$$

The iterative derivative operator D^n is a closable unbounded operator from $\mathrm{L}^2(\Omega)$ into $\mathrm{L}^2([0,T]^n\times\Omega)$ for each $n\geq 1$.

We have the following integration by parts formula: Let $F\in\mathcal{S}$, then for all $h\in\mathscr{H}$, we have

$$\mathbb{E}\left[\langle DF,h\rangle_{\mathscr{H}}\right]=\mathbb{E}\left[FW(h)\right]$$

and as a consequence, if F and G are in $\mathcal{S}$, and $h\in\mathscr{H}$, then

$$\mathbb{E}\left[G\langle DF,h\rangle_{\mathscr{H}}\right]=\mathbb{E}\left[-F\langle DG,h\rangle_{\mathscr{H}}+FGW(h)\right]$$

We denote by $\mathbb{D}^{n,2}$ the closure of $\mathcal{S}$ with respect to the norm

$$||F||^2_{n,2}=||F||^2_{\mathrm{L}^2(\Omega)}+\sum_{k=1}^n||D^kF||^2_{\mathrm{L}^2([0,T]^k\times\Omega)}.$$

The divergence operator δ is the adjoint of the derivative operator D that is also called *Skorohod integral* with respect to Brownian motion $(W_t)_{t\geq 0}$.

We say that a random variable $u \in \mathrm{L}^2(\Omega, \mathscr{H})$ belongs to the domain of the divergence operator, denoted by Domδ, if there is a constant C such that

$$|\mathbb{E}\left[\langle DF, u\rangle_{\mathscr{H}}\right]| \leq C||F||_{\mathrm{L}^2(\Omega)}$$

for all $F \in \mathcal{S}$. If $u \in$ Domδ, then $\delta(u)$ is an element of $\mathrm{L}^2(\Omega)$ defined by the *duality relationship*:

$$E(F\delta(u)) = \mathbb{E}\left[\langle DF, u\rangle_{\mathscr{H}}\right]$$

for all $F \in \mathcal{S}$ and where $\delta(u) := \displaystyle\int_0^T u_t dW_t$.

For all $n \geq 1$, let $\mathbb{L}^{n,2} := \mathrm{L}^2([0,T]; \mathbb{D}^{n,2})$ be equipped with the norm

$$||v||_{n,2}^2 = ||v||_{\mathrm{L}^2([0,T]\times\Omega)}^2 + \sum_{k=1}^{n} ||D^k v||_{\mathrm{L}^2([0,T]^{k+1}\times\Omega)}^2.$$

We recall that $\mathbb{L}^{1,2}$ is included in Domδ, and for a process $u \in \mathbb{L}^{1,2}$ we can compute the variance of the Skorohod integral of u as follows:

$$\mathbb{E}[\delta(u)^2] = \mathbb{E}\left[\int_0^T u_t^2 dt\right] + \mathbb{E}\left[\int_0^T \int_0^T D_s u_t D_t u_s ds dt\right].$$

5.2.1 Itô's formula

Alòs [1] and Alòs and Nualart [4], proved the following version of Itô's formula for anticipating process.

Theorem 5.7. *Let us consider a process of the form*

$$X_t = X_0 + \int_0^t u_s dW_s + \int_0^t v_s ds,$$

where X_0 is an $\mathscr{F}_0-$measurable random variable and $u, v \in \mathrm{L}_a^2([0,T]\times\Omega)$. Consider also a process $Y_t = \displaystyle\int_t^T \theta_s ds$ for some $\theta \in \mathbb{L}^{1,2}$. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a twice continuously differentiable function such that there exists a positive constant C such that, for all $t \in [0,T]$, F and its derivatives evaluated in (t, X_t, Y_t) are bounded by C. Then it follows that

$$F(t, X_t, Y_t) = F(0, X_0, Y_0) + \int_0^t \frac{\partial F}{\partial s}(s, X_s, Y_s)ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s, Y_s)dX_s + \int_0^t \frac{\partial F}{\partial y}(s, X_s, Y_s)dY_s$$
$$+ \int_0^t \frac{\partial^2 F}{\partial x \partial y}(s, X_s, Y_s)\left(\int_s^T D_s\theta_r dr\right)u_s ds + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s, Y_s)u_s^2 ds.$$

5.3 Fractional stochastic volatility models

Alòs, León and Vives in [2], used the Malliavin calculus techniques to obtain an expression for the short-dated behavior of the implied volatility skew for general jump-diffusion stochastic volatility models.

5.3.1 Statement of the model, notation and main results

The authors consider the following model for the log-price of a stock under the risk neutral probability measure $\mathbb{P}^*$:

$$X_t = x + (r - \lambda k) - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s \left(\rho dW_s + \sqrt{1-\rho^2} dB_s\right) + Z_t, \quad t \in [0,T], \tag{5.1}$$

where x is the current log-price, r is the instantaneous interest rate, W and B are independent standard Brownian motions, $\rho \in (-1,1)$ is the correlation coefficient, Z_t is a compound Poisson process with intensity, Levy measure ν, independent of W and B, $k = \dfrac{1}{\lambda}\displaystyle\int_{\mathbb{R}} (e^y - 1)\nu(dy) < \infty$, and the volatility process σ is a squared-integrable stochastic process adapted to the filtration generated by W.

We denote by $\mathscr{F}^W, \mathscr{F}^B$ and $\mathscr{F}^Z$ the filtrations generated by W, B and Z respectively. Moreover, we define $\mathscr{F} := \mathscr{F}^W \vee \mathscr{F}^B \vee \mathscr{F}^Z$.

As we saw in the previous chapters, the price of a European call with strike price K is given by

$$C_t = e^{-r(T-t)} \mathbb{E}_{\mathbb{P}^*}\left[(e^{X_T} - K)_+ | \mathscr{F}_t\right].$$

We denote the future average volatility by $v_t := (\overline{Y_t})^{1/2}$, with $\overline{Y_t} := \dfrac{1}{T-t}\displaystyle\int_t^T \sigma_s^2 ds$. $\mathrm{C_{BS}}(t,x,\sigma)$ denote the price of a European call under Black-Scholes model with constant volatility σ, current log stock price x, maturity time $T-t$, strike price k and interest rate r:

$$\mathrm{C_{BS}}(t,x,\sigma) = e^x N(d_1) - K e^{-r(T-t)} N(d_2),$$

where

$$d_{1,2} = \frac{x - x_t^*}{\sigma\sqrt{T-t}} \pm \frac{\sigma\sqrt{T-t}}{2},$$

with $x_t^* := \log K - r(T-t)$ and N is the cumulative Gaussian distribution function. And finally we define $G(t,x,\sigma) := (\partial_{xx}^2 - \partial_x)\,\mathrm{C_{BS}}(t,x,\sigma)$.

In [2], Alòs, León and Vives studied the short-time behavior of the implied volatility. Let $I_t(X_t)$ denote the implied volatility process which is a $\mathscr{F}-$adapted process such that

$$C_t = \mathrm{C_{BS}}(t, X_t, I_t(X_t)).$$

Furthermore, in [2, Proposition 4] they given an expression for the derivative of the implied volatility:

$$\frac{\partial I_t}{\partial X_t}(x_t^*) = \left.\frac{\mathbb{E}\left[\int_t^T \left(\partial_x F(s,X_s,v_s) - \frac{1}{2}F(s,X_s,v_s)\right) ds | \mathscr{F}_t\right]}{\partial_\sigma \mathrm{C_{BS}}(t, x_t^*, I_t(x_t^*))}\right|_{X_t = x_t^*}, \quad \text{a.s.}$$

where

$$\begin{aligned}F(s,X_s,v_s) &:= \frac{\rho}{2}e^{-r(s-t)}\partial_x G(s,X_s,v_s)\left(\int_s^T D_s\sigma_r^2 dr\right)\sigma_s \\ &+ \int_{\mathbb{R}} e^{-r(t-s)}\left(\mathrm{C}_{\mathrm{BS}}(s,X_s+y,v_s) - \mathrm{C}_{\mathrm{BS}}(s,X_s,v_s)\right)\nu(dy) \\ &- \lambda k e^{-r(t-s)}\partial_x \mathrm{C}_{\mathrm{BS}}(s,X_s,v_s).\end{aligned}$$

In [2, Section 6], the limit of $\frac{\partial I_t}{\partial X_t}(x_t^*)$ is studied when $T \to t$ under the following hypothesis:

(H1) $\sigma \in \mathbb{L}^{2,4}$.

(H2) There exists a constant $a > 0$ such that $\sigma > a > 0$.

(H3) There exists a constant $\delta > -\frac{1}{2}$ such that for all $0 < t < s < r < T$,

$$\mathbb{E}\left[(D_s\sigma_r)^2|\mathscr{F}_t\right] \le C(r-s)^{2\delta},$$

$$\mathbb{E}\left[(D_\theta D_s\sigma_r)^2|\mathscr{F}_t\right] \le C(r-s)^{2\delta}(r-\theta)^{-2\delta}.$$

(H4) σ has right-continuous trajectories.

(H5) For every $t > 0$,

$$\sup_{s,t,\theta\in[t,T]} \mathbb{E}\left[\left(\sigma_s\sigma_r - \sigma_\theta^2\right)^2\right] \to 0, \quad \text{as} \quad T \to t.$$

Theorem 5.8. *([2, Theorem 7]) Under conditions (H1)-(H5) and considering the model in (5.1):*

1. *Assume that δ in (H3) is nonnegative and that there exists a $\mathscr{F}_t$-measurable random variable $D_t^+\sigma_t$ such that, for every $t > 0$,*

$$\sup_{s,r\in[t,T]} \left|\mathbb{E}\left[\left(\sigma_s\sigma_r - \sigma_\theta^2\right)^2\right]\right| \to 0, \quad \textit{as} \quad T \to t.$$

Then

$$\lim_{T\to t}\frac{\partial I_t}{\partial X_t}(x_t^*) = -\frac{1}{\sigma_t}\left(\lambda k + \rho\frac{D_t^+\sigma_t}{2}\right).$$

2. *Assume that δ in (H3) is negative and that there exists a $\mathscr{F}_t-$measurable random variable $L_t^{\delta,+}\sigma_t$, where $L := (\partial_{xx}^2 - \frac{1}{2}\partial_x)G$, such that, for every $t > 0$,*

$$\frac{1}{(T-t)^{2+\delta}}\int_t^T\int_s^T \mathbb{E}\left[D_s\sigma_r|\mathscr{F}_t\right]drds - L_t^{\delta,+}\sigma_t \to 0, \quad \textit{as} \quad T \to t.$$

Then

$$\lim_{T\to t}(T-t)^{-\delta}\frac{\partial I_t}{\partial X_t}(x_t^*) = -\frac{\rho}{\sigma_t}L_t^{\delta,+}.$$

5.3.2 Fractional volatility models

Assume that the volatility σ can be written as $\sigma_r = f(Y_r)$, where $f \in \mathcal{C}_b^1(\mathbb{R})$ and Y_t is a stochastic process of the form

$$Y_r = m + (Y_t - m)e^{-\alpha(r-t)} + c\sqrt{2\alpha}\int_t^r e^{-\alpha(r-s)} dW_s^H, \tag{5.2}$$

where $W_s^H := \displaystyle\int_0^s (s-u)^{H-\frac{1}{2}} dW_u$.

Case $H > 1/2$

Assume the volatility model in equation (5.2) for some $H > 1/2$. From Alòs, Mazet and Nualart [3], $\displaystyle\int_t^r e^{-\alpha(r-s)} dW_s^H$ can be written as

$$\left(H - \frac{1}{2}\right)\int_0^r \left(\int_s^r \mathbb{1}_{[t,r]}(u) e^{-\alpha(r-u)}(u-s)^{H-\frac{1}{2}} du\right) dW_s,$$

from where it follows that

$$\sup_{s,r\in[t,T]} |\mathbb{E}[D_s\sigma_r|F_t]| \to 0 \quad \text{as} \quad T \to t.$$

Then, Theorem 5.8 gives us that

$$\lim_{T\to t} \frac{\partial I_t}{\partial X_t}(x_t^*) = -\frac{\lambda k}{\sigma_t}.$$

That is, the at-the-money short-dated skew slope of the implied volatility is not affected by the correlation in this case.

Case $H < 1/2$

Assume the volatility model in equation (5.2) for some $0 < H < 1/2$. From Alòs, Mazet and Nualart [3], $\displaystyle\int_t^r e^{-\alpha(r-s)} dW_s^H$ can be written as

$$\begin{aligned}\left(\frac{1}{2} - H\right)\int_0^r \left(\int_s^r \left[\mathbb{1}_{[t,r]}(u) e^{-\alpha(r-u)} - \mathbb{1}_{[t,r]}(s) e^{-\alpha(r-s)}\right](u-s)^{H-\frac{1}{2}} du\right) dW_s \\ + \int_t^r e^{-\alpha(r-s)}(r-s)^{H-\frac{1}{2}} dW_s\end{aligned}$$

Then hypothesis (H3) holds for every $\delta = H - \dfrac{1}{2}$ and we have that

$$\mathbb{E}\left[\frac{1}{(T-t)^{2+H-\frac{1}{2}}}\int_t^T \int_s^T D_s\sigma_r dr ds - c\sqrt{2\alpha} f'(Y_t)|\mathscr{F}_t\right] \to 0 \quad \text{as } T \to t.$$

Then, Theorem 5.8 gives us that

$$\lim_{T\to t}(T-t)^{\frac{1}{2}-H}\frac{\partial I_t}{\partial X_t}(x_t^*) = -c\sqrt{2\alpha}\frac{\rho}{\sigma_t}f'(Y_t).$$

That is, the introduction of fractional components with Hurst parameter $H < 1/2$ in the definition of the volatility process allows us to reproduce a skew slope of order $\mathrm{O}(T-t)^{\delta}$, for every $\delta > -1/2$.

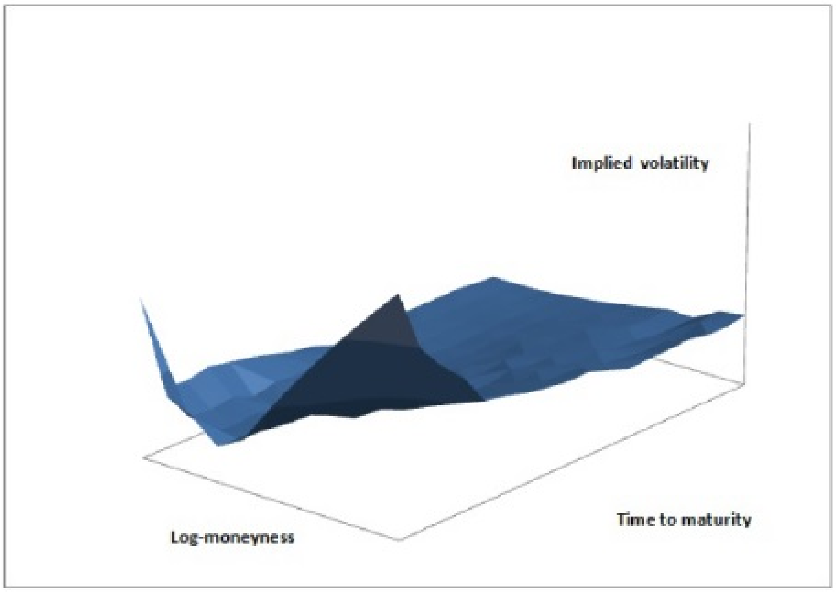


Figure 5.3: Volatility surface. (Courtesy of Rafael de Santiago).

From this, we can say that the results obtained in fractional volatility models whit Hurst index $H < 1/2$ allow to describe the blow up observed for the short-term slope, in Section 4.2.1. According to Roger Lee in [16] also comment this blow up. These results also show that the compound Poisson process does not allow to describe the blow up.

Appendix A

Octave codes

Geometric Brownian motion

Listing A.1: Geometric Brownian motion sample paths in Octave

```
function GBM(S0,mu,sigma,T,N,M)
% This function plot M sample paths of Geometric Brownian motion
% where N is the number of subintervals
X  = zeros(M,N+1);
X(:,1) = S0;
dt = T/N;
t=0:dt:T; %Time
drift = (mu-0.5*sigma^2)*dt; % Calculation of the drift term.
diff = sigma*sqrt(dt); % Calculation of the diffusion term.
for i=1:M
   for j=1:N
      X(i,j+1) = X(i,j)*exp(drift+diff*randn);
   end
end
%Plot Sample Paths
plot(t,X);
title('Geometric Brownian Motion')
xlabel('time')
ylabel('S')
```

Implied volatility

Listing A.2: Black-Scholes formula in Octave

```
function [Call] = BS_price(S0,K,r,T,sigma)
%% This function compute the value of a European option
if T > 0
d1 = (log(S0./abs(K)) + (r+sigma.^2./2).*T)./(sigma.*sqrt(T));
d2 = d1-sigma.*sqrt(T);
N1 = 0.5.*(1+erf(d1./sqrt(2)));
N2 = 0.5.*(1+erf(d2./sqrt(2)));
Call = S0.*N1-K.*exp(-r.*T).*N2;
else
Call = max(S0-K,0);
end
```

Listing A.3: Function vega in Octave

```
function [vega] = Vega(S0,K,r,T,sigma)
%% This function compute the vega in BS
d1 = (log(S0./abs(K)) + (r+sigma.^2/2)*T)./(sigma.*sqrt(T));
N1 = exp(-0.5*d1.^2)./sqrt(2*pi);
vega = S0*sqrt(T).*N1;
```

Listing A.4: Implied volatility calculated by means Newton-Raphson method in Octave

```
function [ImpVol] = BS_ImpVol(S0,K,r,T,C)
%% This function compute the implied volatility (Newton-Raphson method)
% I N P U T S:
% S0 : stock price
% K : strike price
% r : risk free interest rate
% T : time to maturity
% C : value of a European Call Option
% sigma0 : initial value for the implied volatility
n = 20; %number of iterations
tol=0.001; %tolerance
```

```
sigma0 = 0.2; % initial iteration
f = 'BS_price';
df = 'Vega';
for i=1:n-1
  sigma1=sigma0-((((feval(f,S0,K,r,T,sigma0))-C)/(feval(df,S0,K,r,T,sigma0)));
  if(abs((feval(f,S0,K,r,T,sigma0))-C)<tol)
      break
  end
sigma0=sigma1;
end
ImpVol = sigma0;
```

Volatilty smile

Listing A.5: Function to plot volatility smile by Hull-White formula in uncorrelated case in Octave

```
clc
clear all
%%%%%%%%%% INPUTS OU %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
a = 0.5; b = 1.5; X0 = 1;
%%%%%%%%%% INPUTS BS %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
S0 = 100; K = 20; r = 0;  T = 0.5; t0 = 0; %initial time
%%%%%%%%%% INPUTS HW %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
N = 1000; %% Number of time steps per path
M = 50; %% Number of paths that we simulate
% Time step
dt = ((T-t0)/N); t = t0:dt:T;
%% Generate random numbers
X = zeros(M,N);           X(:,1) = X0; %initial condition
sigma = zeros(M,N);     sigma(:,1) = 0; %initial condition
dW = sqrt(dt)*randn(M,N);
% Simulation of N-step trajectories for the OU process
for i=1:N
    X(:,i+1) = X(:,i) - a*X(:,i)*dt + b*dW(:,i);
    X_square(:,i) = X(:,i).^2;
```

```
end
% We use numerical integration (trapezoidal rule) to compute the integral
sigma = (1/2*X_square(:,1) + 1/2*X_square(:,end)
                              + sum(X_square(:,2:end-1),2))*dt/T;
sigma_bar = sqrt(sigma);
% We create the strikes prices vector
for l=1:K
  Strike(l) = 90+((l-1));
end
% We apply B-S formula
for l=1:K
  price(l)=0;
  for j=1:M
    Call(j,l) = BS_price(S0,Strike(l),r,T,sigma_bar(j));
    price(l) = price(l)+Call(j,l)./M;
  end
ImpVol(l)=BS_ImpVol(S0,Strike(l),r,T,price(l));
end
for l=1:K
  ImpVol(l);
end
plot(ImpVol)
title('Volatility Smile')
xlabel('Strike');
ylabel('Implied Volatilities');
```

Volatilty skew

Listing A.6: Function to plot volatility skew by extended Hull-White formula in correlated case in Octave

```
clc
clear all
%% % % % % % % % % INPUTS HW WITH CORRELATION % % % % % % % % % % % %
a = 0.5; b = 1.5; X0 = 1; S0 = 100; K = 20; r = 0;  T = 1; t0 = 0;
N = 1000; % % Number of time steps per path
```

```
M = 10; % % Number of paths that we simulate
rho = 0.2; %Correlation between [-1,1]
% Time step
dt = ((T-t0)/N); t = t0:dt:T;
%% Generate random numbers
X = zeros(M,N);          X(:,1) = X0; %initial condition
sigma = zeros(M,N);      sigma(:,1) = 0; %initial condition
dW = sqrt(dt)*randn(M,N);
%% Simulation of N-step trajectories for the OU process
for i=1:N
  X(:,i+1) = X(:,i) - a*X(:,i)*dt + b*dW(:,i);
  X_square(:,i) = X(:,i).^2;
end
%% We use numerical integration (trapezoidal rule) to compute the integral
sigma = (1/2*X_square(:,1) + 1/2*X_square(:,end)
                              + sum(X_square(:,2:end-1),2))*dt/T;
sigma_bar = sqrt(sigma*(1-rho^2)); %we multiply by (1-rho^2)
%% We compute the stochastic integral in extension H-W formula
Stoch_Int = sum(X(:,2:end).*dW,2); %Stochastic Integral
S = S0*exp((rho*Stoch_Int)-(0.5*(rho^2)*sigma)); %New Stock Price
%% We create the vector strikes
for l=1:K
  Strike(l) = 90+((l-1));
end
%% We apply B-S formula
for l=1:K
  Price(l) = 0;
  for j=1:M
    Call(j,l) = BS_price(S(j),Strike(l),r,T,sigma_bar(j));
    Price(l) = Price(l) + Call(j,l)./M;
    ImpVol(l) = BS_ImpVol(S(j),Strike(l),r,T,Price(l));
  end
end
for l=1:K
  ImpVol(l);
```

```
end
plot(ImpVol)
title('Volatility Skew')
ylabel('Implied Volatilities');
xlabel('Strike');
```

Fractional Brownian motion

Listing A.7: Fractional Brownian motion sample paths in Octave

```
function FBM(H)
%We plot a fractional Brownian motion on interval [0,1]
% with Hurst index H in (0,1)
n = 2^10; % number of point
r = nan(n+1,1); r(1) = 1;
for k=1:n
r(k+1) = 0.5*((k+1)^(2*H) - 2*k^(2*H) + (k-1)^(2*H));
end
r = [r; r(end-1:-1:2)]; % First row of a circular matrix
lambda = real(fft(r))/(2*n); % Eigenvalues
B = fft(sqrt(lambda).*complex(randn(2*n,1),randn(2*n,1)));
B = n^(-H)*cumsum(real(B(1:n+1))); % Rescaling
plot((0:n)/n,B);
```

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