Weighted Selective Aggregated Majority-OWA operator and its application in linguistic group decision-making model

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Abstract. This paper focuses on the aggregation operations in the group decision-making model based on the concept of majority opinion. The weighted-selective aggregated majority-OWA (WSAM-OWA) operator is proposed as an extension of the SAM-OWA operator, where the reliability of information sources is considered in the formulation. The WSAM-OWA operator is generalized to the quantified WSAM-OWA operator by including the concept of linguistic quantifier, mainly for the group fusion strategy. The QWSAM-IOWA operator, with an ordering step, is introduced to the individual fusion strategy. The proposed aggregation operators are then implemented for the case of alternative scheme of heterogeneous group decision analysis. The heterogeneous group includes the consensus of experts with respect to each specific criterion. The exhaustive multi-criteria group decision-making model under the linguistic domain, which consists of two-stage aggregation processes, is developed in order to fuse the experts' judgments and to aggregate the criteria. The model provides greater flexibility when analyzing the decision alternatives with a tolerance that considers the majority of experts and the attitudinal character of experts. A selection of investment problem is given to demonstrate the applicability of the developed model.

Keywords: OWA operator, majority additive-OWA, linguistic quantifiers, group decision making.

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1. Introduction

An aggregation operation is central in many applications which involve information processing, such as decision making, information retrieval, and pattern recognition. Group decision making (GDM), one of the research topics in the multi-criteria decision analysis (MCDA), relies on the aggregation operation to obtain a representative value for a group of experts. Two general frameworks or schemes that are normally used in GDM can be classified as classical and alternative schemes (Bordogna and Sterlacchini, 2014). These schemes, in general, have different approaches for aggregating the experts' judgments as the final group decision. In particular, the classical scheme refers to the consensus of experts for each alternative, while the alternative scheme deals with the consensus for each criterion. Principally, there are two main aggregation processes in GDM; they are the aggregation of criteria and the aggregation of experts. There are many aggregation functions that have been proposed as the fusion method in GDM models. One of the most commonly used aggregation operators is the ordered weighted averaging (OWA) operator introduced by Yager (1988). The OWA can be explained as a general class of aggregation functions that encompasses the operations between the min and max operators. The induced OWA (IOWA) operator, another OWA extension, has also been applied to most of the GDM models. Recent development of OWA-related aggregation operators from theoretical and application perspectives can be referred to, for instance, in Yager and Kacprzyk, (1997), Yager and Filev (1999), Merigó and Gil-Lafuente (2009), Merigó and Casanovas (2010), Merigó and Yager (2013).

Fuzzy set theory (Zadeh, 1965), on the other hand, provides MCDA models with a flexibility in the representation and/or the aggregation of information. The information used in MCDA problems, in general, is either quantitative and/or qualitative. Quantitative information may be expressed by numerical values; whereas qualitative information may be represented by linguistic assessments in order to capture the vagueness and uncertainty of the information. Human judgments, for example, involve subjective evaluations that are more suitably and conveniently modelled by the fuzzy linguistic approach. They can be represented by linguistic values using linguistic variables, i.e., the variables whose values are not numbers but words or sentences in a natural or artificial language (Zadeh, 1983). This approach is adequate for qualifying phenomena related to human perception. Many approaches have been proposed recently to model linguistic information, see for example Bordogna et al. (1997), Delgado et al. (1993), Herrera and Herrera-Viedma (2000), Merigó et al. (2010).

Fuzzy set theory is also useful in modelling the aggregation process. Soft aggregation processes can be implemented, specifically, by the inclusion of linguistic quantifiers in OWA operator (Yager, 1988; 1996). In this way, various decision strategies can be determined in order to provide a complete picture of the decision analysis. For example, considering a portion of criteria to be satisfied from "*at least one*" criterion (existential quantifier) to "*all*" criteria (universal quantifier). Analogously, with respect to the GDM, the soft majority agreement among experts can be modelled, for instance by using semantics such as "*at least 80%*" and "*most*". However, the linguistic quantifiers used to represent the majority concept as a group consensus is manipulated differently than that of the regular quantifiers in the classical OWA. For instance, instead of defining "Q of the values need to be satisfied," where the argument values are seen as truth values or degrees of satisfaction and Q represents any semantic, alternatively "Q of the relevant/similar values" is used to model the meaning of majority (Pasi and Yager, 2006; Peláez et al., 2007).

In most cases, it is difficult to achieve a unanimous decision when dealing with a group of experts. As an alternative, agreement among a majority of experts can be tolerated. In the literature, there are some approaches which have been proposed to model the majority concept using OWA operators. Pasi and Yager (2006) proposed two approaches to deal with this issue. The first is based on the use of the IOWA operator, where the support function is applied to derive a set of order-inducing, scalar-valued variables, i.e., reordered based on the most similar opinions. While the other approach is based on a fuzzy subset that represents the majority opinion under the vague concept. Correspondingly, Bordogna and Sterlacchini (2014) extended the Pasi-Yager method, specifically based on the IOWA operator, by employing the Minkowski OWA-based similarity measure to obtain the order-inducing variables. Moreover, in their method, instead of synthesizing the consensus on each alternative (classical scheme), they proposed an alternative approach where the consensus measure on each specific criterion (alternative scheme) is implemented. Furthermore, they propose to apply the importance degrees of experts to heterogeneous GDM.

In other related research, Peláez and Doña (2003a) proposed the majority additive OWA operator (MA-OWA) to aggregate the argument values that have cardinality greater than one. Particularly, this operator is an extension of the simple arithmetic mean (AM) since it is the arithmetic mean of arithmetic means. Peláez and Doña (2003a) notes that for classical aggregation operators such as the AM, the aggregated value is not representative of the majority aggregation since the result is affected by the extreme values. This results in an aggregated value that is correlated to the symmetric tendency between the values. Even though the OWA operators can be implemented as an alternative approach, they have distribution problems when aggregating arguments with cardinalities (Peláez and Doña, 2003a). Hence, the MA-OWA can be used to treat this type of problem more effectively. Furthermore, in this case, the overall value of the majority opinion is determined without elimination of the minority opinion. In other words, all the information is employed in the aggregation process. Since its inception, some extensions of the MA-OWA operator have been proposed in the literature, such as: the linguistic aggregation MA-OWA, the majority multiplicative-OWA, the quantified MA-OWA and the work committee-OWA (Peláez and Doña, 2003b; Peláez et al., 2005; Peláez et al., 2007; La Red et al., 2011). Recently, Karanik et al., (2016) has proposed the selective MA-OWA (SMA-OWA) operator to deal with the problem of fast convergence of the associated weights. More precisely, when the difference between the cardinalities of the aggregated values is huge, then, only the argument value with the highest cardinality is taken into account, while the other may be excluded. As a solution, the cardinality relevance factor (CRF) was introduced as a degree of tolerance to modify the associated weights so that all the argument values can be included. In addition, Peláez et al., (2016) has proposed the selective aggregated majority OWA (SAM-OWA) operator where the cardinality is used to calculate the individual weight for each group of argument values. Previously, in the MA-OWA and SMA-OWA operators, the individual weights were set as equally important.

Nevertheless, the SAM-OWA operators are limited to the case of homogeneous GDM problems. Although the SAM-OWA is associated with a set of weights that are based on cardinalities, the argument values are still considered equally important. In addition, the information to be aggregated is not associated with the reliability of information sources as in the case of heterogeneous GDM problems. In the context of GDM, each expert has an associated degree of importance that reflects his/her expertise, knowledge, skill, etc. Motivated by the heterogeneous GDM problems, the inclusion of the reliability of information sources (or degree of importance) is suggested as the extension of SAM-OWA and it is denoted as the weighted SAM-OWA operator. Furthermore, by integrating over the linguistic quantifiers, the WSAM-OWA is extended to the quantified WSAM-OWA to provide a greater flexibility in the aggregation process, specifically for the group fusion strategy. While in the individual fusion strategy, QWSAM-IOWA is introduced to deal with the ordering problem and to better represent the majority opinions of experts. Finally, based on the proposed aggregation operators, the multi-expert GDM model with respect to the alternative scheme is developed under the linguistic domain. A selection of investment problem is given as an example of the applicability of the developed model. This paper is structured as the followings. Section 2 provides some preliminaries include the definitions and basic concepts of OWA, neat OWA, IOWA and linguistic labels. In Section 3, a review of MA-OWA, SMA-OWA and SAM-OWA operators is provided. In section 4, the proposed WSAM-OWA, QWSAM-OWA and QWSAM-IOWA are presented. Section 5, the multi-criteria GDM model is developed based on the proposed aggregation operators and in section 6, a numerical example is given. Finally, the conclusion is provided in section 7.

2. Preliminaries

In this section, some definitions and basic concepts related to the OWA, neat OWA and IOWA operators and also the linguistic labels are presented.

2.1 OWA, neat OWA and IOWA operators

Definition 1 (Yager, 1988). An OWA operator of dimension n is a mapping $F_{OWA}: \mathbb{R}^n \to \mathbb{R}$ that has an associated weighting vector $W = [w_1, w_2, \dots, w_n]$ such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, defined as:

$$F_{OWA}(a_1, a_2, \dots, a_n) = \sum_{i=1}^{n} w_i a_{\sigma(i)}, \tag{1}$$

where $a_{\sigma(i)}$ is the argument value a_i being ordered in non-increasing order $a_{\sigma(i)} \geq a_{\sigma(i+1)}$.

As can be seen, the OWA is a nonlinear aggregation operator since it involve the ordering process. Moreover, it is a mean-type aggregation operator that meets all the commutative, monotonic, bounded and idempotent properties. The type of aggregation performed by OWA operator is mainly affected by the weighting vector W. It can be shown that a number of well-known aggregation operators are included in the OWA operator such as min and max operators, simple average, median, to name a few. Other families of OWA operators can be referred to Yager (1993).

Different approaches have been suggested for deriving the weights for OWA operator, such as, using the linguistic quantifiers, maximum entropy, minimal variability, and learning method. See (Xu, 2005) for a complete review of the other approaches. In particular, Yager (1988) defined the OWA operator from the proportional linguistic quantifiers Q (i.e., based on monotonic non-decreasing function) by defining the weights in the following way:

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), \qquad i = 1,2,\dots,n, \tag{2}$$

where w_i represents the increase of satisfaction in getting i with respect to $i-1$ criteria satisfied. In this case, all the criteria are associated with the identical degrees of importance, $w_i = 1/n$, as shown when $Q(x) = x$. However, in the case where each of the criteria c_i to be aggregated has an importance degree v_i associated with it, such that (v_i, c_i), the inclusion of importance degrees in OWA operators from Q can be defined as follows (Yager, 1996):

$$\omega_i = Q\left(\frac{\sum_{k=1}^{i} v_{\sigma(k)}}{T}\right) - Q\left(\frac{\sum_{k=0}^{i-1} v_{\sigma(k)}}{T}\right), \tag{3}$$

where $v_{\sigma(i)}$ are the degrees of importance associated with the criteria that has the *i*th largest satisfaction c_i such as $\left(v_{\sigma(i)}, c_{\sigma(i)}\right)$ and $T = \sum_{i=1}^{n} v_{\sigma(i)}$, the total sum of degrees of importance. The linguistic quantifiers Q can be presented in the form of (Zadeh, 1983):

$$Q(r) = \begin{cases} 0 & if \;\; r \le a, \\ \dfrac{(r-a)}{(b-a)} & if \; a < r < b, \\ 1 & if \;\; r \ge b, \end{cases} \tag{4}$$

with $a, b, r \in [0,1]$. For example, the semantic "*most*", "*almost all*" and "*at least half*" can be given as parameters (a, b) with $(0.35, 0.7), (0, 0.5)$ and $(0.5, 1)$, respectively.

Alternatively, the associated weights for the OWA operator can be obtained directly from its argument values. This method is known as the neat OWA operator and it can be defined as the following.

Definition 3 (Yager, 1993): Neat OWA or weight-dependent OWA operator is a function $F_{NOWA}: \mathbb{R}^n \to \mathbb{R}$, defined as:

$$F_{NOWA}(a_1, a_2, \dots, a_n) = \sum_{i=1}^{n} w_i\left(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}\right) a_{\sigma(i)} \tag{5}$$

where $a_{\sigma(i)}$ is the argument value a_i with any permutation and the vector valued function $w: \mathbb{R}^n \to [0,1]^n$ is normalized such that $\sum_{i=1}^{n} w_i(a_1, a_2, \dots, a_n) = 1$.

The neat OWA meets the properties of idempotency, commutativity and boundedness. However the monotonic property is generally lost. The arithmetic mean is one of the examples of neat OWA.

In addition, the induced OWA operator is another useful aggregation operator that deal with the different ordering step. Instead of ordering the arguments with respect to their magnitudes such in the OWA operator, the additional parameters called order inducing variables are used to induce the arguments. The definition of IOWA can be given as follows.

Definition 4 (Yager and Filev, 1999) An IOWA operator of dimension n is mapping $IOWA: \mathbb{R}^n \to \mathbb{R}$ that has an associated weighting vector W such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, given by the following formula:

$$I - F(\langle u_1, a_1 \rangle, \langle u_2, a_2 \rangle, \dots, \langle u_n, a_n \rangle) = \sum_{i=1}^{n} w_i a_{\sigma(i)} \tag{6}$$

where $a_{\sigma(i)}$ is the argument value of pair $\langle u_i, a_i \rangle$ of order inducing variable u_i, reordered such that $u_{\sigma(i)} \geq u_{\sigma(i+1)}$. The IOWA operators are all satisfying commutative, monotonic, bounded and idempotent properties.

2.2 Linguistic labels

The input of the decision analysis can be represented in various forms, such as in qualitative and quantitative forms. In the case of qualitative form, the linguistic labels are used to capture the information based on the subjective evaluation such as "*poor*", "*good*", "*very good*", etc. The general definition of linguistic labels can be given as follows:

Definition 4 (Herrera and Herrera-Viedma, 2000): Let a set of linguistic labels, $S = \{s_0, s_1, \dots, s_{max}\}$ be uniformly distributed on a scale, then, the ordering is defined as $(s_a, s_b) \in S, s_a < s_b \Leftrightarrow a < b$ with s_0 and s_{max} are the lowest and the highest elements, respectively. The max is given as $|S| - 1$, where $|S|$ denotes the cardinality of S.

As stated by Herrera and Herrera-Viedma (2000), the cardinality of S must be small enough so as not to impose useless precision on the experts and it must be rich enough in order to allow discrimination of the performances of each object in a limited number of grades. In the literature, there are many approaches which proposed to compute with the linguistic labels. In this paper, the method by Bordogna et al., (1997) is applied, where the linguistic labels are converted directly to the numerical values to deal with the operations in numerical environment. Finally, the results based on numerical values are reconverted to the linguistic labels as the final ranking purpose.

Definition 5 (Bordogna et al., 1997): The conversion of the linguistic labels to the numbers in unit interval $[0,1]$ can be conducted by using the function $Label^{-1}$ defined as: $Label^{-1}: S \rightarrow [0,1]$, $Label^{-1}(s_i) = \frac{i}{|S|-1}$ with $i = 0,1, \dots, max$. While, the retranslation from the numerical values into the linguistic labels can be given as: $Label(x) = s_i$ for $\frac{i}{|S|} \leq x < \frac{i+1}{|S|}$, $i = 0,1, \dots, max$ and $Label\ (1) = S_{max}$.

3. Aggregation functions based on Majority-additive OWA

In this section, a review of the definitions and basic properties of MA-OWA, SMA-OWA and SAM-OWA operators are presented prior to the definition of WSAM-OWA and QWSAM-OWA operators.

Definition 4 (Peláez and Doña, 2003a): A MA-OWA operator is a function $F_{MA}: \mathbb{R}^n \times \mathbb{N}^N \rightarrow \mathbb{R}$ defined as:

$$F_{MA}(a_1, a_2, \dots, a_n) = \sum\nolimits_{i=1}^{n} w_{i,N} b_{\sigma(i)}, \quad (7)$$

where $N = \max_{1 \leq i \leq n} m_i$ and σ denotes a permutation of group of argument b_i with respect to the cardinality m_i, such that $b_{\sigma(i)} \geq b_{\sigma(i+1)}$. The weights associated to the arguments are defined by the recurrence relations:

$$w_{i,1} = \frac{1}{u_1} = \frac{1}{n}: u_1 = n, \quad (8)$$

$$w_{i,k} = \frac{\gamma_{i,k} + w_{i,k-1}}{u_k}: \forall k, 2 \leq k \leq N, \quad (9)$$

where $u_k = 1 + \sum_{j=1}^{n} \gamma_{j,k}$, and $\sum_{i=1}^{n} w_{i,k} = 1$, for $k = N$, such that:

$$\gamma_{j,k} = \begin{cases} 1 & m_{\sigma(j)} \geq k, \\ 0 & otherwise. \end{cases} \tag{10}$$

Note that k factor represents the current cardinality considered at a moment in the aggregation process. The MA-OWA operators meet all the bounded, idempotent and commutative properties. However the monotonicity is preserved if only if the cardinality vector, m is exactly the same in both aggregate sets, i.e., $F_{MA,w}(b,m) \geq F_{MA,w}(d,m), b \geq d, \forall j$. Moreover, the MA-OWA reduces to arithmetic mean, $F_{MA}(a_1, a_2, \ldots, a_n) = F_{AM}(a_1, a_2, \ldots, a_n)$, if all cardinalities, $m_i = 1$ (Peláez and Doña, 2003a).

Example 1. Assume that $A = \langle a_1, \ldots a_i, \ldots, a_n \rangle \in \mathbb{R}^n \times \mathbb{N}^n$ where $a_i = (b_i, m_i)$ represents the aggregate value b_i, and its cardinality $m_i > 0$. For $A = \{(0.6, 1), (0.2, 1), (0.1, 3)\}$, the MA-OWA can be computed as the following.

Table 1. Values of $\gamma_{i,k}$ and u_k

	$b_{\sigma(1)}$ 0.6 $m_{\sigma(1)}$	$b_{\sigma(2)}$ 0.2 $m_{\sigma(2)}$	$b_{\sigma(3)}$ 0.1 $m_{\sigma(3)}$	$\delta = 1$
$\gamma_{i,k}$	1	1	3	u_k
$\gamma_{i,1}$	1	1	1	3
$\gamma_{i,2}$	0	0	1	2
$\gamma_{i,3}$	0	0	1	2

The cardinal-dependent weights can be given as:

$$w_{1,3} = \frac{1}{2}\left(0 + \frac{1}{2}\left(0 + \frac{1}{3}1\right)\right) = \frac{1}{12},$$
$$w_{2,3} = \frac{1}{2}\left(0 + \frac{1}{2}\left(0 + \frac{1}{3}1\right)\right) = \frac{1}{12},$$
$$w_{3,3} = \frac{1}{2}\left(1 + \frac{1}{2}\left(1 + \frac{1}{3}1\right)\right) = \frac{5}{6},$$

Then, the MA-OWA operator for $\delta = 1$ can be derived as:

$$F_{MA}(\{(0.6,1),(0.2,1),(0.1,3)\}) = 0.6 \cdot \frac{1}{12} + 0.2 \cdot \frac{1}{12} + 0.1 \cdot \frac{5}{6} = 0.150.$$

While for $\delta = 0.5$, the MA-OWA operator yields: $F_{MA} = F_{AM} = 0.220$.

As can be seen, the MA-OWA indicates the better result for the majority opinion than AM, as 80% of the argument values are equal and less than 0.2 and 60% is 0.1. Hence, the representative value should be in between these two values or closer to 0.1.

As mentioned earlier, the main goal of the MA-OWA operator is to determine a synthesized value with considering all the information, i.e., the majority opinion and the minority opinion. However in certain cases, the minority opinion is excluded in the aggregation process due to the huge different between the cardinalities of arguments. In this case, the weight $w_{i,N} = 0$ is obtained for the minority opinion, whilst $w_{i,N} = 1$ is given for the majority opinion. To deal with this problem, Karanik et al., (2016) proposed the selective MA-OWA operator where the cardinality relevance factor (CRF) is introduced to weaken the $\gamma_{j,k}$ values in MA-OWA as to obtain the weight, $w_{i,N} > 0$ for the minority opinion.

Definition 5 (Karanik et al., 2016): A SMA-OWA operator is a function $F_{SMA}: \mathbb{R}^n \times \mathbb{N}^N \rightarrow \mathbb{R}$ defined as:

$$F_{SMA}(a_1, a_2, \dots, a_n) = \sum_{i=1}^{n} w_{i,N} b_{\sigma(i)}, \tag{11}$$

where $N = \max_{1 \le i \le n} m_i$ and σ denotes a permutation with respect to the cardinality m_i, such that $b_{\sigma(i)} \geq b_{\sigma(i+1)}$. Their weights are defined by the recurrence relations, such in Eq. (8) and Eq. (9), given that $u_k = 1 + \sum_{j=1}^{n} \gamma_{j,k}$ and $\sum_{i=1}^{n} w_{i,k} = 1$, for $k = N$, such that:

$$\gamma_{j,k} = \begin{cases} \delta & m_{\sigma(j)} \geq k, \\ 1 - \delta & otherwise. \end{cases} \tag{12}$$

The parameter δ is the cardinality relevance factor (CRF) with $\delta \in [0,1]$.

By assigning the appropriate value for CRF, the minority opinion can be included in the aggregation process, specifically, by increasing its associated weight, such that, $w_{i,N} > 0$. The behavior of CRF value can be explained as the following. For

$\delta \to 1$, the opinion with the largest cardinality (majority of opinion) is more emphasized than the opinion with the smallest cardinality. Hence, it is given a higher weight than the others. On the contrary, if $\delta \to 0$, the opinion with the smallest cardinality is given more priority than the largest cardinality. Meanwhile, if $\delta = 0.5$, the AM of the arguments is obtained, $F_{SMA} = F_{AM}$ such that all the cardinalities of arguments are reduced to cardinality $m_i = 1$. It can be demonstrated that the properties of idempotency, commutativity and boundedness hold for the SMA-OWA. However, the monotonicity is preserved only if the cardinality vector is exactly the same in both aggregate sets (Karanik et al., 2016).

In other related work, Peláez et al., (2016) proposed the SAM-OWA operator as the generalization of the SMA-OWA where weights are assigned to different group of arguments based on their cardinalities. The definition of SAM-OWA operator can be given as the following.

Definition 6 (Peláez et al., 2016): A SAM-OWA operator is a function $F_{SAM}: \mathbb{R}^n \times \mathbb{N}^N \to \mathbb{R}$ defined as:

$$F_{SAM}(a_1, a_2, \dots, a_n) = \sum_{i=1}^{n} w_{i,N} b_{\sigma(i)}, \tag{13}$$

where $N = \max_{1 \le i \le n} m_i$ and σ denotes a permutation with respect to the cardinality m_i, such that $b_{\sigma(i)} \ge b_{\sigma(i+1)}$. The associated weights are defined by the recurrent relations:

$$w_{i,1} = w_i = \frac{m_i}{\sum_{j=1}^{n} m_j}, \tag{14}$$

$$w_{i,k} = \frac{w_i \gamma_{i,k} y_k + w_{i,k-1}}{z_k}, \tag{15}$$

$$y_1 = 1, y_k = \begin{cases} 1, & if \sum_{j=1}^{n} w_j \gamma_{j,k} = 0, \\ \dfrac{\sum_{j=1}^{n} \gamma_{j,k}}{\sum_{j=1}^{n} w_j \gamma_{j,k}}, & otherwise, \end{cases} \tag{16}$$

$$z_1 = 1, z_k = \begin{cases} 1, & if \sum_{j=1}^{n} w_j \gamma_{j,k} = 0, \\ 1 + \sum_{j=1}^{n} \gamma_{j,k}, & otherwise, \end{cases} \tag{17}$$

where $\gamma_{j,k}$ is defined in the similar way as Eq. (12), δ is the cardinality relevance factor (CRF) such that $\delta \in [0,1]$ and $1 \le i \le n$, $2 \le k \le N$.

Example 2. Consider again the previous example where a set of aggregated values is given as $A = \{(0.6, 1), (0.2, 1), (0.1, 3)\}$. The weights $w_{i,1} = w_i$ then can be obtained as: $w_{1,1} = 1/5$, $w_{2,1} = 1/5$ and $w_{3,1} = 3/5$.

The final cardinal-dependent weights are derived as:

$$w_{1,3} = 0.050, w_{2,3} = 0.050, w_{3,3} = 0.900,$$

and the SAM-OWA operator for $\delta = 1$ yields:

$$F_{SAM}(\{(0.6, 1), (0.2, 1), (0.1, 3)\}) = 0.130.$$

However, as can be noticed, the individual weights in the MA-OWA and SMA-OWA operators are distributed uniformly to each group of arguments, i.e., $w_{i,1} = 1/u_1 = 1/n$. Thus, for each aggregated value a_i in (b_i, m_i), the weight can be given as $1/u_1 m_i = 1/nm_i$. On the contrary, for the SAM-OWA operator, the individual weights are distributed proportionally to each group of opinions, i.e., $w_{i,1} = w_i = m_i / \sum_{j=1}^{n} m_j$, such that, the weights are uniformly distributed to each argument a_i.

In general, the aggregated values in MA-OWA, SMA-OWA and SAM-OWA are independent of the degrees of importance or the reliability of information sources. In the context of group decision making, they can be considered as the homogenous GDM problems. However, under the heterogeneous GDM problems, each argument value is associated with the degree of importance as to reflect the knowledge, expertise or experience of each expert. Hence, in the next section, the weighted SAM-OWA operator is proposed as an extension of the SAM-OWA operator to deal with the mentioned problem. In addition, the quantified WSAM-OWA operator for the group fusion strategy and the QWSAM-Induced OWA for the individual fusion strategy are presented.

4. Weighted SAM-OWA aggregation functions

In this section the WSAM-OWA operator is presented and the QWSAM-OWA and QWSAM-IOWA operators are proposed as its generalization and extension.

4.1 Weighted SAM-OWA operator

Definition 7: A WSAM-OWA operator is a function $F_{WSAM}: \mathbb{R}^n \times \mathbb{N}^N \to \mathbb{R}$ that has an associated weighting vector V of dimension n such that $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, defined as:

$$F_{WSAM}(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} w_{i,N} b_{\sigma(i)}, \tag{18}$$

where $N = \max_{1 \le i \le n} m_i$ and σ denotes a permutation with respect to the cardinality m_i. The associated weights are defined by the recurrent relations:

$$w_{i,1} = \omega_i = \begin{cases} v_i, & if \;\; m_i = 1, \\ \sum_{i=1}^{m_i} v_i\,, & if \;\; m_i > 1, \end{cases} \tag{19}$$

and the cardinal-dependent weights are given as,

$$w_{i,k} = \frac{\omega_i \gamma_{i,k} y_k + w_{i,k-1}}{z_k}, \tag{20}$$

$$y_1 = 1, y_k = \begin{cases} 1, & if \sum_{j=1}^{n} \omega_j \gamma_{j,k} = 0, \\ \dfrac{\sum_{j=1}^{n} \gamma_{j,k}}{\sum_{j=1}^{n} \omega_j \gamma_{j,k}}, & otherwise, \end{cases} \tag{21}$$

$$z_1 = 1, z_k = \begin{cases} 1, & if \sum_{j=1}^{n} \omega_j \gamma_{j,k} = 0, \\ 1 + \sum_{j=1}^{n} \gamma_{j,k}\,, & otherwise, \end{cases} \tag{22}$$

where $\gamma_{j,k}$ is defined in the similar way as Eq. (12), the parameter δ is the cardinality relevance factor (CRF) and $1 \le i \le n$, $2 \le k \le N$.

Similarly, it can be demonstrated that the WSAM-OWA operator meets the bounded, idempotent and monotonic properties. However, they are not commutative as involve the importance degrees or weighted arithmetic mean (WAM).

Property 1: Boundedness

Let m_k is the cardinality of the lowest argument value of vector A, if $m_k \to \infty$ and $\delta \to 1$, then $F_{WSAM}\big((b_i, m_i)\big) = b_k, Min[a_i]$.

Let m_k is the cardinality of the highest argument value of vector A, if $m_k \to \infty$ and $\delta \to 1$, then $F_{WSAM}\big((b_i, m_i)\big) = b_k, Max[a_i]$.

Hence, it is bounded by $Min[a_i] \leq F_{WSAM}\big((b_i, m_i)\big) \leq Max[a_i]$.

Property 2: Idempotency

An aggregation function F_{WSAM} is idempotent if, $F_{WSAM}\big((b, m)\big) = b$ for any δ and m.

Property 3: Monotonicity

The monotonicity is preserved if and only if the cardinality vector is exactly the same in both aggregate sets, i.e., $F_{WSAM}(b_i, m) \geq F_{WSAM}(d_i, m), b_i \geq d_i$ for all $i = 1,2, \dots, n$.

Property 4: Commutativity

An aggregation function F_{WSAM} is commutative if and only if $v_i = 1/n$ for all $i = 1,2, \dots, n$.

Remark 1. It can be demonstrated that for $\delta = 0.5$, then WSAM-OWA is reduced to WAM, $F_{WSAM} = F_{WAM}$. In addition, for $\delta \to 1$, a higher weight is given to the argument with greater cardinality (majority opinion) and if $\delta \to 0$, then a higher weight is given to the argument with lower cardinality (minority opinion).

Remark 2. Conversely, when $\omega_i = w_i$ (or $v_i = 1/n$), then WSAM-OWA is reduced to SAM-OWA, $F_{WSAM} = F_{SAM}$.

The issue that may arise in WSAM-OWA operator is how to aggregate the argument values based on cardinality with respect to the inclusion of the degrees of importance. In WAM, the degrees of importance reflect the reliability of information sources, for example, given more priority to the most skilled or experience person. Nevertheless, the majority of information which represents the highest degree of importance is not directly emphasized in the WAM. Here, the WSAM-OWA can be used to include both characteristics, i.e., the degree of importance and the majority concept. Note that in the SAM-OWA operators, the emphasis is directly given on cardinality or majority opinion since the degrees of importance are uniform. In WSAM-OWA, the CRF is suggested as a tolerant factor in considering the majority and the degrees of importance simultaneously. This value can be derived as the following formula (Karanik, et al., 2016):

$$\delta = 1 - \left(2 + s^2\left(m_{\sigma(i)}\right)\right)^{-1} \tag{23}$$

where $s^2\left(m_{\sigma(i)}\right)$ is the variance of cardinality values, such that $\delta \in [0,1]$. Notice that in Karanik et al., (2016) the expected value is calculated as $E\left(m_{\sigma(i)}\right) = \sum_{i=1}^{n} w_{i,1}\, m_{\sigma(i)}$, where $w_{i,1} = 1/n$. For the case of WSAM-OWA, the degrees of importance, $w_{i,1} = \omega_i$ are used such in Eq. (19), then the variance can be given as $s^2\left(m_{\sigma(i)}\right) = \sum_{i=1}^{n} \omega_i \left(m_{\sigma(i)} - E\left(m_{\sigma(i)}\right)\right)^2$. Hence, by formulating in this way, the influence of the degrees of importance is taken into account in deriving the CRF value for the overall aggregation process. Should be noted that, in the case of WSAM-OWA, the CRF is applied to provide a compensation between the degrees of importance and the cardinalities of aggregated values instead of the obtaining the $w_{i,1} > 0$ for the minority opinion.

Remark 3. It can be shown that for $\omega_k = 1$ and $\omega_i = 0$ for all $i \neq k$, then $F_{WSAM}\left((b_i, m_i)\right) = b_k$ for any $\delta = (0,1]$.

Example 3: Given that $A = \langle 0.6, 0.2, 0.1, 0.1, 0.1 \rangle$ and their associated weights are provided as $V = \langle 0.1,0.1,0.3,0.3,0.2 \rangle$. For simplicity it can be represented as $A = \{(0.6, 1, 0.1), (0.2, 1, 0.1), (0.1, 3, 0.8)\}$, where $a_i = (b_i, m_i, \omega_i)$. Based on the cardinalities and degrees of importance, the CRF can determined as follows:

$$E\left(m_{\sigma(i)}\right) = (0.1 \cdot 1) + (0.1 \cdot 1) + (0.8 \cdot 3) = 2.6,$$

$$s^2\left(m_{\sigma(i)}\right) = 0.1(1 - 2.6)^2 + 0.1(1 - 2.6)^2 + 0.8(3 - 2.6)^2 = 0.64.$$

$$\delta = 1 - (2 + 0.64)^{-1} = 0.621$$

Table 2. Values of $\gamma_{i,k}$, u_k and y_k

	$b_{\sigma(1)}$	$b_{\sigma(2)}$	$b_{\sigma(3)}$		
	0.6	0.2	0.1		
	$m_{\sigma(1)}$	$m_{\sigma(2)}$	$m_{\sigma(3)}$	$\delta = 0.621$	
$\gamma_{i,k}$	1	1	3	u_k	y_k
$\gamma_{i,1}$	1	1	1	3	
$\gamma_{i,2}$	0.379	0.379	0.621	2.379	2.407
$\gamma_{i,3}$	0.379	0.379	0.621	2.379	2.368

The cardinal-dependent weights are:

$$w_{1,3} = 0.064, w_{2,3} = 0.064, w_{3,3} = 0.872,$$

and the WSAM-OWA operator yields:

$$F_{WSAM}(\{(0.6, 1, 0.1), (0.2, 1, 0.1), (0.1, 3, 0.8)\}) = 0.138.$$

In this example, the WAM is given as, $F_{WAM} = 0.160$. Similarly to the MA-OWA, in this case, the representative value is expected to be closer to 0.1 as the highest weight (the total sum of individual weights) is belong to the group of arguments $b_3 = 0.1$, which is the majority opinion.

Example 4: Assume that $A = \{(0.6, 1, 0.6), (0.2, 1, 0.1), (0.1, 3, 0.3)\}$ where $V = \langle 0.6,0.1,0.1,0.1,0.1 \rangle$. In this example, the highest weight is associated with the minority opinion. Based on the cardinalities and the degrees of importance, the CRF is obtained as 0.648.

The cardinal-dependent weights are derived as:

$$w_{1,3} = 0.457, w_{2,3} = 0.076, w_{3,3} = 0.467,$$

and the WSAM-OWA operator yields:

$$F_{WSAM}(\{(0.6, 1, 0.6), (0.2, 1, 0.1), (0.1, 3, 0.3)\}) = 0.336.$$

In this example, the WAM is given as, $F_{WAM} = 0.410$. As can be seen, this value is lower than WAM which reflects the majority opinion with the relevancy of the degrees of importance.

4.2 Quantified Weighted SAM-OWA operators

In the previous section, all the majority operators take into account not only the majority opinion but also the minority opinion in deriving the aggregated value. As mentioned by Peláez et al., (2007), this definition in general uses the majority semantics which consider "*all*" of the arguments, but it is not able to model the majority concepts like "*most*" or "*at least 80%*" of arguments. Hence, Peláez et al., (2007) proposed the inclusion of linguistic quantifiers as to generalize the MA-OWA operator. Two quantified weights in MA-OWA operators were introduced, namely the individual fusion strategy and the group fusion strategy. The individual fusion strategy can be explained as applying the semantics of the quantifier on each individual weight of the aggregation process. While the group fusion strategy applies the semantics of quantifier to each group of arguments with respect to their cardinalities. Analogously, in this paper, both decision strategies can be extended to the case of WSAM-OWA operator. The method for the group fusion strategy can be applied directly to the case of WSAM-OWA since the ordering of the group of cardinalities is not affecting the overall result. The definition of the group fusion strategy of WSAM-OWA is given as the following.

Definition 8: A QWSAM-OWA operator under the group fusion strategy is a function $F_{WSAM}: \mathbb{R}^n \times \mathbb{N}^N \rightarrow \mathbb{R}$ that has an associated weighting vector V of dimension n such that $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, defined as:

$$F_{QWSAM}(a_1, a_2, \dots, a_n) = \sum_{i=1}^{n} w_i^{Q-G} b_{\sigma(i)}, \tag{24}$$

where $N = \max_{1 \le i \le n} m_i$ and the weights are defined by the recurrent relations such in F_{WSAM}. The weights for the group fusion strategy can be presented as in the following expression (Peláez et al., 2007):

$$w_i^{Q-G} = \frac{\omega_i}{m_i} \cdot \sum_{j=1}^{m_i} Q\left(\frac{j}{m_i}\right) + \left[\sum_{j=1}^{m_i} Q\left(\frac{j}{m_i}\right) \cdot \frac{1 - \sum_{i=1}^{n} \frac{\omega_i}{m_i} \cdot \sum_{j=1}^{m_i} Q\left(\frac{j}{m_i}\right)}{\sum_{i=1}^{n} \sum_{j=1}^{m_i} Q\left(\frac{j}{m_i}\right)}\right], \tag{25}$$

where Q is the quantifier, n is the number of majority groups and m_i is the cardinality of the group i. This fusion strategy avoids the exclusion of any group in the aggregation process. Moreover, in this way it is possible to eliminate the distribution problems in the group decision-making problems.

Example 5: By extending the previous example (Example 3), the group fusion strategy using the QWSAM-OWA operator can be implemented. Firstly, the cardinal-dependent weight vector is obtained, $W = [0.064, 0.064, 0.872]$ as in F_{WSAM}. After that, the value of the quantifier with semantics "*most*" such in Eq. (4) can be calculated for each group. The Q vectors for each majority group are obtained as:

Group with cardinality, $m = 1$: $[1]$,
Group with cardinality, $m = 1$: $[1]$,
Group with cardinality, $m = 3$: $[0, 0.633, 1]$.

Then, the quantified weight vector for the group fusion strategy is obtained as $W^{Q-G} = [0.173, 0.173, 0.653]$, where:

$$w_1^{Q-G} = \frac{0.064}{1} \cdot 1 + 1 \cdot \frac{1-0.603}{3.633} = 0.173,$$

$$w_2^{Q-G} = \frac{0.064}{1} \cdot 1 + 1 \cdot \frac{1-0.603}{3.633} = 0.173,$$

$$w_2^{Q-G} = \frac{0.872}{3} \cdot 1.633 + 1.633 \cdot \frac{1-0.603}{3.633} = 0.653.$$

Finally, the QWSAM-OWA operator for the group fusion strategy yields:

$$F_{QWSAM}(\{(0.6, 1, 0.3), (0.2, 1, 0.3), (0.1, 3, 0.4)\}) = 0.211.$$

4.3 Quantified Weighted SAM-IOWA operators

For the individual fusion strategy, an extension of QMA-OWA to the QWSAM-IOWA is proposed as to deal with the issue of reordering process. As can be noticed, in this case each weight, $w_{i,N}$ is multiplied by the linguistic quantifier, $Q(i/n)$ of monotonically non-decreasing function. Peláez et al., (2007) suggests the reordering of arguments with respect to their cardinalities, i.e., in non-decreasing order such that, the greater the cardinality of argument, then the higher weight is associated to that argument. However, the problem may arise in the case where there are two or more arguments with identical cardinality, i.e., different order of these arguments may produce different results of the aggregation processes. For example, let say $(b_i, m_i) = (\langle 0.2,1\rangle, \langle 0.3,1\rangle, \langle 0,3\rangle)$. The ordering of $(0.2,0.3,0,0,0)$ and $(0.3,0.2,0,0,0)$ then producing distinct results if the quantified weight vector is given as $(0,0.1,0.2,0.3,0.4)$. Hence, in this paper, the extension of the individual fusion strategy to the case of IOWA operator is suggested, where the order inducing variable reflects the similarity between arguments. Note that, in this case, both majority opinions and similarity between arguments are considered, but more emphasis is given to the most similar values. As can be seen, in this case, the order of $(0.3,0.2,0,0,0)$ is better represent the similarity between arguments. In the following, the definition of QWSAM-IOWA operator is presented.

Definition 8: A QWSAM-IOWA operator of the individual fusion strategy is a function $I - F_{QWSAM}: \mathbb{R}^n \to \mathbb{R}$ that has an associated weighting vector V of dimension n such that $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, defined as:

$$I - F_{QWSAM}(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \dots, \langle u_n, a_n\rangle) = \sum\nolimits_{i=1}^{n} w_i^{Q-I} a_{\sigma(i)}, \tag{26}$$

where $a_{\sigma(i)}$ is the argument value of pair $\langle u_i, a_i\rangle$ of order inducing variable u_i, with $u_{\sigma(i)} \leq u_{\sigma(i+1)}$, such that:

$$u_i = \left(\sum_{i=1}^{j} v_i s_i^{\lambda}\right)^{1/\lambda}, i = 1,2, \dots, n, \tag{27}$$

and $s_i = s(a_i, a_j) = 1 - |a_i - a_j|$ is a similarity measure between each argument a_i with respect to arguments a_j, $(j = 1,2, \dots, n)$, $i \in j$ and λ is a parameter in a range $\lambda \in (-\infty, \infty) \setminus \{0\}$. The individual fusion weight w_i^{Q-I} is obtained from the following equation:

$$w_i^{Q-I} = \frac{w_{i,N} \cdot v_i}{\omega_i} \cdot Q\left(\frac{i}{n}\right) + \left[Q\left(\frac{i}{n}\right) \cdot \frac{1 - \sum_{i=1}^{n}\left(\frac{w_{i,N} \cdot v_i}{\omega_i} \cdot Q\left(\frac{i}{n}\right)\right)}{\sum_{i=1}^{n} Q\left(\frac{i}{n}\right)}\right], \tag{28}$$

where $w_{i,N}$ is the weight determined by the recurrent relations such in F_{WSAM} for $N = \max_{1 \le i \le n} m_i$, Q is the linguistic quantifier and the expression in the bracket is the $Q-$ normalization. It can be demonstrated that, the QWSAM-IOWA satisfies bounded, idempotent, monotonic properties. However, it is not commutative as it involves the WAM.

As can be noticed, in this expression, some modifications have been made to the original QMA-OWA of individual fusion strategy where the weight, $w_{i,N}$ is multiplied by v_i/ω_i to decompose its individual weights proportionally with respect to their degrees of importance. In the original form (Peláez et. al., 2007), the weight $w_{i,N}$ is divided equally with respect to its cardinality. Moreover, the order inducing variable is introduced to order the arguments with respect to their degrees of similarity and also resolve the issue of ordering problem.

Example 6: Let $A = \langle 0.6, 0.2, 0.1, 0.1, 0.1 \rangle$ and its weight vector is provided as $V = \langle 0.1,0.1,0.3,0.3,0.2 \rangle$. The individual fusion strategy using the QWSAM-IOWA operator with semantics "*most*" can be computed as the following. Firstly, the order inducing variable is computed for each argument:

$$u_1 = \sum_{i=1}^{5} v_i s_i = 0.1(1 - |0.6 - 0.6|) + 0.1(1 - |0.6 - 0.2|) + 0.3(1 - |0.6 - 0.1|) + 0.3(1 - |0.6 - 0.1|) + 0.2(1 - |0.6 - 0.1|) = 0.56.$$

Similarly, the rest of order inducing variables can be determined, such that:

$$A = \langle (0.56,0.6), (0.88,0.2), (0.94,0.1), (0.94,0.1), (0.94,0.1) \rangle.$$

Secondly, the F_{WSAM} aggregation operator is applied to obtain the cardinal-dependent weighting vector, $W_N = [0.064, 0.064, 0.872]$. The individual weighting vector W is given as:

$$W = [0.064, 0.064, 0.327, 0.327, 0.218],$$

where $w_{3,N} = 0.872$ can be decomposed to: $w_{3,N}^1 = w_{3,N}^2 = (0.872 \times 0.3)/0.8 = 0.327$, $w_{3,N}^3 = (0.872 \times 0.2)/0.8 = 0.218$. Then, the individual weights w_i^{Q-I} are calculated using the above expression:

$$W^{Q-I} = [0, 0.017, 0.162, 0.389, 0.432]$$

Finally, the WSAM-OWA operator for individual fusion strategy yields:

$$I - F_{QWSAM}(\{(u_1, 0.6, 1, 0.1), (u_2, 0.2, 1, 0.1), (u_3, 0.1, 0.3), (u_4, 0.94, 0.1), (u_5, 0.94, 0.1)\}) = 0.102.$$

5. Multi-criteria group decision making under linguistic domain

In this section, a multi-criteria group decision-making model under the linguistic domain is developed. Two-stage aggregation processes are involve, in particular, the proposed WSAM-OWA operator and its extensions are used as group aggregators. While the classical OWA operator with the inclusion of degrees of importance is applied to aggregate the criteria as the final ranking. The proposed model is based on the extension of Bordogna-Fedrizzi-Pasi model (Bordogna et al., 1997), specifically it is extended to the case of alternative scheme. The input provided by the experts are based on the linguistic labels. These input are then directly converted to the numeric values in unit interval $[0,1]$ to simplify the aggregation process. The algorithm of the proposed model is explained step by step as the following.

Stage 1: Majority aggregation for experts' judgments

Step 1: Construct a decision matrix of dimension $M \times N$ for each expert, $D^h, (h = 1,2, \dots, k)$ as follows:

$$D^h = \begin{array}{c c} & \begin{array}{ccc} C_1 & \dots & C_n \end{array} \\ \begin{array}{c} A_1 \\ \vdots \\ A_m \end{array} & \begin{pmatrix} a_{11}^h & \cdots & a_{1n}^h \\ \vdots & \ddots & \vdots \\ a_{m1}^h & \cdots & a_{mn}^h \end{pmatrix} \end{array}, \tag{29}$$

where A_i indicates the alternative i $(i = 1,2, \dots, m)$, C_j denotes the criterion j $(j = 1,2, \dots, n)$, and a_{ij}^h denotes the preferences for alternative

A_i with respect to criterion C_j. The input value a_{ij}^h is the linguistic label provided by each expert based on the predefined linguistic scale, S.

Step 2: Determine the degree of importance (or trust) of each expert with respect to each criterion, such that, $T = \{t_1, t_2, \dots, t_k\}$. The degree of importance, t_h is drawn from the same linguistic scale, S.

Step 3: Transform the performance labels and the importance labels of all experts into the numeric values by applying the function $Label^{-1}: S \to [0,1]$. Then, the numeric value $t_h \in T$ is normalized to form $\hat{T} = \{\hat{t}_1, \hat{t}_2, \dots, \hat{t}_k\}$, where $\hat{t}_h = t_l / \sum_{h=1}^{k} t_h$, such that $\sum_{h=1}^{k} \hat{t}_h = 1$. With respect to each criterion, the transformed values (performance and importance labels of each expert) are used to determine the cardinality relevance factor (CRF), δ such in Eq. (23).

Step 4: Aggregate the experts' preferences using the WSAM-OWA operator to form a group decision matrix: Eq. (18) – Eq. (22). Note that, at this stage, the decision strategy (consensus on experts) can also be implemented by specifying the semantics "*most*" and manipulated either using the group fusion strategy: Eq. (24) – Eq. (25) or the individual fusion strategy: Eq. (26) – Eq. (28).

Stage 2: Aggregation of criteria and ranking process

Step 5: Determine the importance degrees of criteria, $V = (v_1, v_2, \dots, v_n)$, such that v_j are drawn from the linguistic scale, S. Then, these weights are transformed to the numerical values using the function $Label^{-1}: S \to [0,1]$. At this stage, the OWA weights can be computed using the Eq. (3).

Step 6: Aggregate the judgment matrix of the majority of experts using the OWA operator such in Eq. (1) with respect to the weighting vector obtained in Step 5. Finally, rank the alternatives based on their values. Note that here, the proportion of criteria is subject to the attitudinal character of the majority of experts. Specifically, by assigning any semantics to the linguistic quantifiers, specifically in Eq. (3), various decision strategies can be obtained.

6. Numerical example

In this section, an investment selection problem is studied where a group of experts are assigned for the judgment and selection of an optimal strategy. Assume that a company plans to invest some money in one or several available options (allocated proportionally based on their rankings). Primarily, five possible investment options are considered as follows: $A_1 =$ hedge funds, $A_2 =$ investment funds, $A_3 =$ bonds, $A_4 =$ stocks and $A_5 =$ equity derivatives. These investment options are described with respect to the following characteristics: $C_1 =$ benefits in the short term, $C_2 =$ benefits in the long term, $C_3 =$ risk of the investment, $C_4 =$ social responsible investment and $C_5 =$ difficulty of the investment. In order to evaluate these options, the investor has brought together a group of experts which consist of five persons; with different backgrounds or areas of expertise. To enable the experts to formulate their judgments in a natural way, a set S of linguistic labels is supplied. For example, S can be defined so as its elements are uniformly distributed on a scale on which a total order is defined as:

$$S = \begin{Bmatrix} s_0 = none,\ s_1 = very\ low,\ s_2 = low,\ s_3 = medium, \\ s_4 = high,\ s_5 = very\ high,\ s_6 = perfect \end{Bmatrix}$$

in which $s_a < s_b$ if and only if $a < b$. Based on this linguistic scale S, a decision matrix for each expert can be constructed for options A_i with respect to the characteristics C_j as shown in Table 4 and the reliability of each expert on specific criterion is given in Table 5.

Table 4. Available investment strategies of each expert, E_h

E_1				
C_1	C_2	C_3	C_4	C_5
s_3	s_2	s_3	s_2	s_5
s_4	s_6	s_1	s_6	s_2
s_2	s_3	s_2	s_2	s_1
s_5	s_2	s_4	s_6	s_5
s_1	s_3	s_3	s_4	s_5

E_2				
C_1	C_2	C_3	C_4	C_5
s_2	s_5	s_6	s_5	s_5
s_6	s_3	s_1	s_6	s_4
s_1	s_5	s_4	s_3	s_2
s_5	s_1	s_3	s_6	s_5
s_3	s_3	s_5	s_5	s_5

E_3				
C_1	C_2	C_3	C_4	C_5
s_1	s_3	s_5	s_4	s_5
s_5	s_5	s_1	s_6	s_3
s_4	s_4	s_3	s_3	s_2
s_5	s_1	s_4	s_6	s_3
s_4	s_3	s_4	s_5	s_4

E_4				
C_1	C_2	C_3	C_4	C_5
s_1	s_3	s_5	s_4	s_4
s_5	s_3	s_2	s_5	s_2
s_2	s_2	s_1	s_4	s_1
s_3	s_1	s_3	s_3	s_5
s_2	s_2	s_3	s_4	s_5

E_5				
C_1	C_2	C_3	C_4	C_5
s_1	s_2	s_3	s_2	s_4
s_5	s_4	s_1	s_5	s_1
s_2	s_2	s_1	s_4	s_3
s_4	s_3	s_2	s_5	s_4
s_1	s_2	s_4	s_2	s_4

Table 5. Reliability of experts on each criterion

	E_1	E_2	E_3	E_4	E_5
C_1	s_5	s_4	s_5	s_3	s_3
C_2	s_4	s_5	s_3	s_4	s_4
C_3	s_3	s_3	s_5	s_4	s_5
C_4	s_4	s_4	s_5	s_4	s_3
C_5	s_3	s_4	s_4	s_5	s_4

At this stage, after transforming the preference labels and the importance labels into numbers in [0,1], the group aggregation based on majority concept can be implemented. For example, the computation for the majority aggregation of option A_1 with respect to characteristic C_1 can be shown as follows:

$$A_1 = \{E_1 = s_3, E_2 = s_2, E_3 = s_1, E_4 = s_1, E_5 = s_1\},$$

$$A_1 = \{Label^{-1}(high), Label^{-1}(medium),\ Label^{-1}(low), Label^{-1}(low), Label^{-1}(low)\},$$

$$A_1 = \{0.667, 0.5, 0.333, 0.333, 0.333\} = \{(0.667,1), (0.5, 1), (0.333,3)\}.$$

Similarly, weights are transformed to the numerical values:

$$T = \{E_1 = s_5, E_2 = s_4, E_3 = s_5, E_4 = s_3, E_5 = s_3\}$$

$$T = \{Label^{-1}(very\ high), Label^{-1}(high), Label^{-1}(very\ high), Label^{-1}(medium), Label^{-1}(medium)\},$$

then $T = \{0.833, 0.667, 0.833, 0.5, 0.5\}$ and they are normalized so that the sum of all weights is one, $\hat{T} = \{0.25, 0.2, 0.25, 0.15, 0.15\}$.

Based on the cardinalities and the normalized degrees of importance, the CRF can determined and is given as $\delta = 0.666$. Then the resulted cardinal-dependent weights are:

$$w_{1,3} = 0.155, w_{2,3} = 0.124, w_{3,3} = 0.720,$$

and the WSAM-OWA operator on C_1 yields:

$$F_{WSAM}(\{(0.667, 1), (0.5, 1), (0.333, 3)\}) = 0.406,$$

The overall aggregated results of majority opinions based on WSAM-OWA are given in Table 6.

Table 6. Majority opinion based on WSAM-OWA

			E_{maj}		
	C_1	C_2	C_3	C_4	C_5
A_1	0.406	0.466	0.722	0.545	0.779
A_2	0.828	0.681	0.177	0.938	0.403
A_3	0.349	0.514	0.354	0.528	0.290
A_4	0.761	0.239	0.552	0.919	0.772
A_5	0.391	0.430	0.623	0.664	0.779

Having the decision matrix which represent the majority opinion of experts on each criteria, then the aggregation process to aggregate the final judgment or ranking of alternatives are conducted, where the weight of each criterion is provided as s_4, s_5, s_5, s_3, s_3, for each criterion C_1, C_2, C_3, C_4 and C_5, respectively. For example, the computation for A_1 can be given as the following:

$$I_{numeric} = [C_1 = I_4, C_2 = I_5, C_3 = I_5, C_4 = I_3, C_5 = I_3]$$

$$I_{numeric} = \{Label^{-1}(high), Label^{-1}(very\ high),\ Label^{-1}(very\ high), Label^{-1}(medium), Label^{-1}(medium)\},$$
$$I_{numeric} = \{I_4 = 0.667, I_5 = 0.833, I_5 = 0.833, I_3 = 0.5, I_3 = 0.5\}$$

The weight vector W_{most} is then obtained by applying the Eq. (3): $W_{most} = [0,0.2,0.3,0.5,0]$. The overall aggregation process can be determined using classical OWA operator, Eq. (1):

$$F_{OWA-W_{most}}(0.406, 0.466, 0.722, 0.545, 0.779) = 0.5411.$$

Finally, the linguistic overall performance value is obtained as: $Label(0.5411) = s_3 = medium$.

The aggregated results for all the alternative are presented in Table 7. In addition, the aggregated results based on SMA-OWA and SAM-OWA are also given as to see the results of the majority aggregation processes without the inclusion of the degrees of importance.

Table 7. Overall aggregated results based on SMA-OWA, SAM-OWA and WSAM-OWA

	SMA-OWA	Rank	SAM-OWA	Rank	WSAM-OWA	Rank
A1	S_3, 0.5372	4	S_3, 0.5544	4	S_3, 0.5411	3
A2	S_4, 0.5738	2	S_4, 0.6034	2	S_3, 0.5619	2
A3	S_2, 0.3557	5	S_2, 0.3975	5	S_2, 0.3842	5
A4	S_4, 0.6837	1	S_4, 0.6646	1	S_4, 0.6391	1
A5	S_3, 0.5708	3	S_3, 0.5672	3	S_3, 0.5069	4

In the case where only "*most*" of the experts are needed for the overall decision, then, the individual fusion strategy or the group fusion strategy can be implemented as given in the Table 8 and Table 9. Note that, the results of the individual fusion strategy are derived based on QWSAM-IOWA operator, while the group fusion strategy is mainly based on QWSAM-OWA.

Table 8. Majority opinion and overall aggregated results based on QWSAM-IOWA

			E_{maj}			Overall	
	C_1	C_2	C_3	C_4	C_5	Aggregation	Ranking
A_1	0.3384	0.4566	0.7523	0.5762	0.8273	S_3, 0.5102	2

A_2	0.8384	0.5991	0.1667	0.9935	0.3990	S_3, 0.4526	4
A_3	0.3435	0.4324	0.2756	0.6214	0.2952	S_2, 0.3243	5
A_4	0.8283	0.1770	0.6261	0.9952	0.8289	S_4, 0.5969	1
A_5	0.2955	0.4928	0.6261	0.7881	0.8273	S_3, 0.4672	3

Table 9. Majority opinion and overall aggregated results based on QWSAM-OWA

			E_{maj}			Overall	
	C_1	C_2	C_3	C_4	C_5	Aggregation	Ranking
A_1	0.4510	0.5056	0.7470	0.5774	0.7688	S_3, 0.5458	1
A_2	0.8282	0.7160	0.2011	0.9287	0.4166	S_3, 0.5232	3
A_3	0.3639	0.5494	0.3931	0.5115	0.3089	S_2, 0.3971	5
A_4	0.7157	0.2843	0.5299	0.8553	0.7248	S_3, 0.5365	2
A_5	0.4176	0.4238	0.6422	0.6339	0.7668	S_3, 0.5072	4

7. Conclusion

In this paper, the aggregation operators based on the majority concept are discussed, specifically, the majority additive-OWA, the selective MA-OWA and the selective aggregated majority-OWA operators. Those aggregation operators are applicable only in the case of homogeneous GDM problems. The weighted SAM-OWA (WSAM-OWA) operator then is proposed as the extension of the SAM-OWA to deal with heterogeneous GDM problems. In particular, it is formulated with the inclusion of the reliability of information sources. By integrating with the linguistic quantifiers, the WSAM-OWA is extended to the quantified WSAM-OWA operator, mainly for the group fusion strategy. Moreover, the QWSAM-IOWA operator is introduced for the individual fusion strategy. The similarity between experts' opinions as order inducing variables is included to present the majority under the semantics given for the linguistic quantifier. The multi-criteria GDM model under the linguistic domain then is developed where the proposed aggregation operators can be implemented as the group aggregator and the weighted OWA operator is applied to derive the final ranking of alternatives. The selection of investment problem is provided to demonstrate the applicability of the developed model. In general, the proposed model has offered greater flexibility in analyzing the decision alternatives with a tolerance in the aggregation processes.

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