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ON FINITE GROUPS ACTING ON ACYCLIC COMPLEXES OF DIMENSION TWO

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Abstract

We conjecture that every finite group G acting on a contractible CW-complex X of dimension 2 has at least one fixed point. We prove this in the case where G is solvable, and under this additional hypothesis, the result holds for X acyclic.

0. Introduction

Let G be a group and A an abelian group. Dicks and Dunwoody ([**4**, Chapter IV]) proved that for each element ζ of $H^1(G;AG)$ there exists a G-tree T with finite edge stabilizers, with the property that for each subgroup H of G, the restriction of ζ to H is zero if and only if H fixes a point of T. It is natural to look for analogous geometric explanations of elements of higher cohomology groups; thus, for example, one can ask if for each element ζ of $H^2(G;AG)$ there exists a contractible 2-dimensional CW-complex X admitting an action of G with finite stabilizers for 2-cells, with the property that for each subgroup H of G, the restriction of ζ to H is zero if and only if H acts trivially on X in some sense, perhaps leaving invariant a subtree of the 1-skeleton of X. The restriction of ζ to any finite subgroup of G is zero, but if a finite group leaves a subtree invariant then it fixes a point. With this motivation, we optimistically conjecture that every finite group G acting on a contractible 2-dimensional CW-complex X has at least one fixed point.

In this note we prove this conjecture in the case where G is solvable. Our argument is based on a classical result of P.A. Smith ([**8**], [**9**]), stating that every action of a finite p-group on a finite dimensional $\mathbf{Z}/p$-acyclic CW-complex has a $\mathbf{Z}/p$-acyclic fixed-point set (see [**2**, Chapter III] and further developments e.g. in [**1**], [**3**], [**7**]).

In our context, the hypothesis that X has no cells above dimension 2 is essential. It is known that any finite nilpotent group whose order is not a prime power acts on some contractible 3-dimensional CW-complex without fixed points ([**1**]).

On the other hand, we shall prove that for a finite solvable group G acting on a 2-dimensional CW-complex X, in order to ensure the existence of a fixed point it suffices to assume that X is *acyclic*. For X acyclic, however, the condition that G be solvable cannot be removed, because the alternating group A_5 acts on the 2-skeleton of the Poincaré sphere –which is acyclic– without fixed points ([**6**]). Recall that the 1-skeleton of the Poincaré sphere is the complete graph on 5 vertices, and the 2-skeleton is obtained by adding 6 pentagonal faces so as to extend the natural action of A_5 on the set of vertices. The fundamental domain of the action is a triangle with angles $\pi/2$, $\pi/5$, $3\pi/10$, and the 60 copies of this fundamental domain triangulate the 2-skeleton, from which it follows that there are no fixed points. The fundamental group of this space is isomorphic to $SL_2(\mathbf{F}_5)$.

Since X being contractible is equivalent to X being simply-connected and acyclic, the question that remains open is: If we add the condition that X be simply-connected, can we delete the condition that G be solvable?

1. Statement and proof of the result

Let G be a finite group acting on a CW-complex X of dimension 2, and denote by X^G the set of fixed points under the action of G. We shall assume that the action is *cellular* ([**5**]); that is, each translation of an open cell is an open cell, and, if a cell is invariant, then it is pointwise fixed. Thus X^G is a subcomplex of X. For a subcomplex $Y \subseteq X$, we denote by $C_n(X, Y)$ the group of relative cellular n-chains of the pair (X, Y).

Given a nonzero abelian group A, a space X is said to be A-*acyclic* if $\widetilde{H}_k(X; A) = 0$ for all k, where $\widetilde{H}$ denotes reduced homology. Recall that the condition $\widetilde{H}_{-1}(X; A) = 0$ is equivalent to the augmentation homomorphism $C_0(X) \otimes A \to A$ being surjective, and hence equivalent to X being nonempty.

We prove

Theorem 1.1. *Let G be a finite solvable group acting on a CW-complex X of dimension 2. If $\widetilde{H}_*(X; \mathbf{Z})$ is finite, and the orders of G, $H_1(X; \mathbf{Z})$ are coprime, then the natural map $\widetilde{H}_*(X^G; \mathbf{Z}) \to \widetilde{H}_*(X; \mathbf{Z})$ is injective.*

Proof: Under our assumptions, the graded group $\widetilde{H}_*(X;\mathbf{Z})$ is necessarily concentrated in degree 1, since it is free abelian in all other degrees. Moreover, $H_1(X;\mathbf{Z}/p)=0$ (and hence X is $\mathbf{Z}/p$-acyclic) for every prime p dividing the order of G.

Since G is solvable, we can find a series of subgroups

$$\{1\}=G_0\subset G_1\subset G_2\subset\cdots\subset G_k=G \tag{1.1}$$

in which each G_{i-1} is normal in G_i and $G_i/G_{i-1}\cong\mathbf{Z}/p_i$, where p_i is a prime. Then the action of G on X induces an action of G_i/G_{i-1} on $X^{G_{i-1}}$ and

$$X^{G_i}=\left(X^{G_{i-1}}\right)^{G_i/G_{i-1}}. \tag{1.2}$$

We prove inductively that the map $\widetilde{H}_*(X^{G_i};\mathbf{Z})\to\widetilde{H}_*(X;\mathbf{Z})$ is a monomorphism for all $i=0,\dots,k$. This is trivial for G_0. Thus suppose that it has been established for G_{i-1}. Then $X^{G_{i-1}}$ is $\mathbf{Z}/p$-acyclic for every prime p dividing the order of G. Since the order of G_i/G_{i-1} is a prime p_i, applying Smith's Theorem ([**9**]) to the action of G_i/G_{i-1} on $X^{G_{i-1}}$ we obtain, by (1.2), that X^{G_i} is $\mathbf{Z}/p_i$-acyclic. This tells us in particular that X^{G_i} is nonempty and connected. Further, for every abelian group A we have an exact sequence

$$\begin{aligned}(1.3)\quad 0&\longrightarrow H_2(X^{G_i};A)\longrightarrow H_2(X;A)\longrightarrow H_2(X,X^{G_i};A)\longrightarrow\\ &\longrightarrow H_1(X^{G_i};A)\longrightarrow H_1(X;A)\longrightarrow H_1(X,X^{G_i};A)\longrightarrow 0,\end{aligned}$$

from which we infer that $H_2(X,X^{G_i};\mathbf{Z}/p_i)=0$. But, since X has no cells above dimension 2, the group $H_2(X,X^{G_i};\mathbf{Z})$ embeds in the free abelian group $C_2(X,X^{G_i})$ and hence it is free abelian itself. This forces $H_2(X,X^{G_i};\mathbf{Z})=0$, showing that $H_1(X^{G_i};\mathbf{Z})$ embeds in $H_1(X;\mathbf{Z})$. ■

Corollary 1.2. *Every action of a finite solvable group G on a $\mathbf{Z}$-acyclic CW-complex X of dimension 2 has at least one fixed point.*

Proof: It follows from Theorem 1.1 that the fixed-point set X^G is $\mathbf{Z}$-acyclic, so in particular it is nonempty. ■

Note. Robert Oliver has kindly informed us that an, as yet unpublished, paper by Yoav Segev contains a different proof of Corollary 1.2, with the additional assumption that X be finite.

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