Publ. Mat. (2007), 43–106
Proceedings of the *Primeras Jornadas de Teoría de Números*.

UNIFORMIZATION OF TRIANGLE MODULAR CURVES

P. Bayer and A. Travesa

Abstract

In the present article, we determine explicit uniformizations of modular curves attached to triangle Fuchsian groups with cusps. Their *Hauptmoduln* are obtained by integration of non-linear differential equations of the third order. Series expansions involving integral coefficients are calculated around the cusps as well as around the elliptic points. The method is an updated form of a differential construction of the elliptic modular function j, first performed by Dedekind in 1877. Subtle differences between automorphic functions with respect to conjugate Fuchsian groups become apparent.

Introduction

The number of $\mathbf{PSL}(2,\mathbb{R})$-conjugacy classes of arithmetic triangle Fuchsian groups is finite and their arithmetic types are available in [**26**]. Nine of these types have cusps, and all of them can be realized by groups commensurable with the modular group $\mathbf{PSL}(2,\mathbb{Z})$. Six of the arithmetic types with cusps correspond to seven genus zero modular curves of habitual use in the current literature: the curves $X_0(N)$, for $N = 1, 2, 3, 4$, and the curves $X_0^+(N)$, for $N = 2, 3, 4$. The uniformizing functions involved (*Hauptmoduln*) are well known and can be expressed as quotients of Dedekind's eta functions (cf. [**12**], [**7**], [**4**]).

In [**15**], Klein traces the origins of the elliptic modular function j to Gauss but, in a footnote on page 116 of [**14**], he also mentions that the function j coincides with a function called *Valenz* by Dedekind: "*Was Herr Dedekind in seinem Aufsatze* Valenz *nennt, ist also nichts anderes als die absolute Invariante des Integrals*" ("What Dedekind means in his

2000 *Mathematics Subject Classification.* Primary: 11F03, 11F06; Secondary: 11F30.
Key words. Arithmetical Fuchsian group, triangle group, Fuchsian differential equation, Schwarzian derivative, modular curve.
Partially supported by MCTE BFM2003-01898 and MEC MTM2006-04895.

concept of *Valenz* is therefore none other than the absolute invariant of the integral").

Our aim is to present a new and unified computation for the uniformizing functions for any modular triangle group not using the classical representations via Dedekind's eta function, but following the main ideas contained in Dedekind's article [**8**]. For this purpose we need to explicitly describe various data including

(1) fundamental domains and presentations of the modular triangle groups;
(2) differential equations satisfied by the uniformizing functions;
(3) local uniformizing parameters.

An advantage of our presentation is that the uniformizing functions are obtained without any previous knowledge of other special functions. They can be developed as easily around the cusps as around the elliptic points and, by performing suitable choices of the local uniformizing parameters, we obtain series expansions with integral coefficients. The method has also been applied to Fermat curves and Shimura curves (cf. [**1**], [**2**], [**3**], [**5**]). Other examples and presentations can be found in [**12**] and [**18**].

1. Triangle Fuchsian groups of non-compact type

Let $\mathcal{H}$ be the upper half complex plane. We define $\overline{\mathcal{H}} = \mathcal{H} \cup \mathbf{P}^1(\mathbb{R})$ and $\mathcal{H}^* = \mathcal{H} \cup \mathbf{P}^1(\mathbb{Q})$. Let $[A_1, A_2]$ be the oriented closed geodesic segment determined by a pair of points $(A_1, A_2) \in \overline{\mathcal{H}} \times \overline{\mathcal{H}}$. More generally, let $[A_1, A_2, \dots, A_n]$ be the closed hyperbolic polygon with vertices $A_i \in \overline{\mathcal{H}}$ for $1 \leq i \leq n$.

The group $\mathbf{GL}(2, \mathbb{R})$ acts, conformally or anticonformally, on $\mathcal{H}$, $\mathbf{P}^1(\mathbb{R})$ and $\overline{\mathcal{H}}$ by linear fractional transformations:

$$M \cdot z = \frac{az + b}{cz + d}, \quad \text{for } M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{if } \det M > 0,$$

$$M \cdot z = \frac{a\overline{z} + b}{c\overline{z} + d}, \quad \text{for } M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{if } \det M < 0.$$

The actions factorize through $\mathbf{PGL}(2, \mathbb{R}) = \mathbf{GL}(2, \mathbb{R})/\mathbb{R}^* 1_2$ and $\mathbf{GL}^+(2,\mathbb{R})/\mathbb{R}^* 1_2 \simeq \mathbf{PSL}(2, \mathbb{R}) = \mathbf{SL}(2, \mathbb{R})/\{\pm 1_2\}$, where 1_2 denotes the identity matrix of degree 2.

Let $\Gamma \subseteq \mathbf{PSL}(2,\mathbb{R})$ be a Fuchsian group acting on $\overline{\mathcal{H}}$. The order e_z of a point $z \in \overline{\mathcal{H}}$ is defined as the order of its isotropy group $\Gamma_z = \{\gamma \in \Gamma : \gamma(z) = z\}$. If $e_z = \infty$, z is said to be a cusp; if $1 < e_z < \infty$, z is said to be an elliptic point. The group Γ acts on the set $\mathcal{P}_\Gamma$ of its cusps and the quotient $\Gamma \backslash (\mathcal{H} \cup \mathcal{P}_\Gamma)$ is a compact Riemann surface. We shall denote by $(g; e_1, \ldots, e_n)$ the signature of Γ, which tells us that the Riemann surface is of genus g and that any fundamental region of Γ contains exactly n inequivalent points of orders $e_i > 1$ for $1 \leq i \leq n$ (cf. [**16**]).

Henceforth, for any group $\Gamma \subseteq \mathbf{PSL}(2,\mathbb{R})$, $\widetilde{\Gamma}$ will mean its preimage in $\mathbf{SL}(2,\mathbb{R})$ under the natural projection; and, for any group $\widetilde{\Gamma} \subseteq \mathbf{SL}(2,\mathbb{R})$, $-1_2 \in \widetilde{\Gamma}$, Γ will mean its image in $\mathbf{PSL}(2,\mathbb{R})$.

Definition 1.1. Let e_1, e_2, e_3 be positive integers, or infinity, satisfying

$$\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} < 1, \quad e_1 \geq e_2 \geq e_3.$$

A Fuchsian group Γ of signature $(0; e_1, e_2, e_3)$ is said to be a triangle Fuchsian group of type (e_1, e_2, e_3).

Theorem 1.2 (cf. [**20**], [**16**]). *For any triangle Fuchsian group $\Gamma \subseteq \mathbf{PSL}(2,\mathbb{R})$ of type (e_1, e_2, e_3), the group $\widetilde{\Gamma}$ can be presented by matrices $M_1, M_2, M_3 \in \mathbf{SL}(2,\mathbb{R})$ which satisfy the following conditions:*

(i) $M_1 M_2 M_3 = -1_2$; $M_i^{e_i} = -1_2$ *if* $e_i \neq \infty$.

(ii) $\mathrm{tr}(M_i) = \pm 2\cos(\pi/e_i)$ *for* $1 \leq i \leq 3$.

(iii) *The fixed points A_i of the transformations defined by M_i are the vertices of a hyperbolic triangle $[A_1, A_2, A_3]$ of angles π/e_i.*

(iv) *The images of $[A_1, A_2, A_3]$ under successive reflections with respect to the sides of the triangle fill the hyperbolic plane without gaps or overlappings.*

(v) *Let Γ' be the subgroup of $\mathbf{PGL}(2,\mathbb{R})$ generated by Γ and the reflection with respect to the side $[A_1, A_3]$. Then $[A_1, A_2, A_3]$ is a fundamental domain for Γ'.*

(vi) *Let $[A_1, A_2', A_3]$ be the hyperbolic triangle obtained from $[A_1, A_2, A_3]$ under the reflection with respect to the side $[A_1, A_3]$. Then the quadrilateral $[A_1, A_2, A_3, A_2']$, under the identifications*

$$[A_1, A_2] \sim [A_1, A_2'], \quad [A_3, A_2] \sim [A_3, A_2'],$$

is a fundamental domain for Γ.

Definition 1.3. The hyperbolic triangle $[A_1, A_2, A_3]$ constructed in Theorem 1.2 is said to define the triangle group Γ. For the sake of brevity, we shall say that the matrices M_i, $1 \leq i \leq 3$, define a triangle presentation for the group Γ (instead of $\widetilde{\Gamma}$).

If two triangle Fuchsian groups are conjugate in $\mathbf{PSL}(2,\mathbb{R})$, then they are of the same type. Conversely, it was proved by Petersson [**20**] that triangle Fuchsian groups of the same type are conjugate in $\mathbf{PSL}(2,\mathbb{R})$.

Let $\mathbb{H}$ be a quaternion algebra defined over a real number field K. Let n be the reduced norm from $\mathbb{H}$ to K. Suppose that at least one real place of K is split in $\mathbb{H}$ and fix an embedding

$$\Phi \colon \mathbb{H} \hookrightarrow \mathbf{M}(2,\mathbb{R}).$$

Consider an order $\mathcal{O}$ of $\mathbb{H}$ and let

$$\mathcal{O}_1^* = \{x \in \mathcal{O} : \mathrm{n}(x) = 1\}.$$

Then $\Phi(\mathcal{O}_1^*) \subseteq \mathbf{SL}(2,\mathbb{R})$, and we denote by $\Gamma(\mathbb{H},\mathcal{O})$ the projection in $\mathbf{PSL}(2,\mathbb{R})$ of the group $\Phi(\mathcal{O}_1^*)$.

Definition 1.4. A Fuchsian group $\Gamma \subseteq \mathbf{PSL}(2,\mathbb{R})$ of the first kind is said to be arithmetic if it is commensurable with a group $\Gamma(\mathbb{H},\mathcal{O})$. A type (e_1, e_2, e_3) is said to be arithmetic if it belongs to an arithmetic triangle Fuchsian group.

The next theorem is due to Takeuchi [**26**].

Theorem 1.5. *There are exactly nine arithmetic types defined by triangle Fuchsian groups with cusps, namely:*

$$\begin{aligned} &(\infty,\infty,\infty), (\infty,\infty,3), (\infty,\infty,2),\\ &(\infty,6,6), (\infty,6,2), (\infty,4,4),\\ &(\infty,4,2), (\infty,3,3), (\infty,3,2). \end{aligned}$$

In Table 1 we give the notation of several points in $\mathcal{H}^*$ to be used throughout the article.

v	point	v	point	v	point
v_0	0	v_8	$\frac{3+i\sqrt{3}}{6}$	v_{16}	$\frac{-1}{2}$
v_1	$i\infty$	v_9	$\frac{-2+i}{5}$	v_{17}	$\frac{1}{2}$
v_2	$\frac{-1+i\sqrt{3}}{2}$	v_{10}	$\frac{-5+i\sqrt{3}}{14}$	v_{18}	1
v_3	i	v_{11}	$\frac{-3+i}{10}$	v_{19}	$\frac{3+i\sqrt{3}}{2}$
v_4	$\frac{1+i\sqrt{3}}{2}$	v_{12}	$\frac{-7+i\sqrt{3}}{26}$	v_{20}	$\frac{3+i}{2}$
v_5	$\frac{-1+i}{2}$	v_{13}	$\frac{i}{\sqrt{2}}$	v_{21}	$\frac{9+i\sqrt{3}}{6}$
v_6	$\frac{-3+i\sqrt{3}}{6}$	v_{14}	$\frac{i}{\sqrt{3}}$	v_{22}	$\frac{3}{2}$
v_7	$\frac{1+i}{2}$	v_{15}	$\frac{i}{2}$	v_{23}	$\frac{3+i}{5}$

Table 1. Notation for some points in $\mathcal{H}^*$

Our first example of a triangle Fuchsian group will be the modular group $\Gamma_0(1) = \mathbf{PSL}(2,\mathbb{Z})$. Recall that the full modular group $\widetilde{\Gamma}_0(1) = \mathbf{SL}(2,\mathbb{Z})$ is generated by the matrices

$$T := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Proposition 1.6. *The following holds:*

(i) *The group $\Gamma_0(1)$ is the triangle group defined by $[v_1, v_2, v_3]$.*

(ii) *Its arithmetic type is $(\infty, 3, 2)$.*

(iii) *The point v_1 is fixed by T, the point v_2 is fixed by $-T^{-1}S$ and the point v_3 is fixed by $-S$.*

(iv) *The polygon* $[v_1, v_2, v_3, v_4]$, *under the identifications*

$$a : [v_3, v_2] \sim [v_3, v_4], \quad z \mapsto S \cdot z,$$

$$b : [v_2, v_1] \sim [v_4, v_1], \quad z \mapsto T \cdot z,$$

is a fundamental domain for $\Gamma_0(1)$ *acting on* $\mathcal{H}^*$. *This fundamental domain is illustrated in Figure 1.*

Proof: The general statement is well known. We only mention that by setting $M_1 = T$, $M_2 = -T^{-1}S$, $M_3 = -S$, we have

$$M_1 M_2 M_3 = M_2^3 = M_3^2 = -1_2,$$

and the matrices M_i generate $\widetilde{\Gamma}_0(1)$ (cf. Table 4). □

Remark 1.7. Not all presentations of a Fuchsian group obtained through matrices fulfilling conditions (i), (ii) and (iii) in Theorem 1.2 can be used to infer a structure of triangle group. In this way, if we define

$$N_1 = T, \quad N_2 = -U^{-1}, \quad N_3 = UT^{-1},$$

where $U := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, then these matrices satisfy

$$N_1 N_2 N_3 = N_3^3 = -1_2, \quad \operatorname{tr} N_1 = -\operatorname{tr} N_2 = 2, \quad \operatorname{tr} N_3 = 1,$$

and generate $\widetilde{\Gamma}_0(1)$, but the triple $(\infty, \infty, 3)$ does not correspond to the type of $\Gamma_0(1)$ as a triangle group.

Lemma 1.8. *Let* $F = \langle m_1, m_2, m_3 \rangle$ *be the free group on three generators. Take* $\chi \colon F \to \{\pm 1\}$ *to be the character defined by*

$$\chi(m_2) = 1, \quad \chi(m_1) = \chi(m_3) = -1.$$

Then $\ker \chi$ *is the normal subgroup of* F *generated by* $\{m_1^2, m_2, m_3 m_1^{-1}\}$.

Proof: Consider the following subgroups of F:

$$F_1 = \langle m_1, m_3 \rangle = \langle m_1, m_3 m_1^{-1} \rangle, \quad F_2 = \langle m_1 \rangle,$$

and denote by χ_i the restriction of the character χ to F_i. For $i = 1, 2$, let $p_1 \colon F \to F_1$, $p_2 \colon F_1 \to F_2$ be the projections defined by

$$p_1(m_2) = 1, \qquad p_1(m_3) = m_3, \quad p_1(m_1) = m_1,$$

$$p_2(m_3 m_1^{-1}) = 1, \quad p_2(m_1) = m_1.$$

Then $\chi = \chi_1 \circ p_1$ and $\chi_1 = \chi_2 \circ p_2$. The claim follows from the fact that $\ker \chi$ is the normal subgroup of F generated by $\ker \chi_1$ and m_2; $\ker \chi_1$ is the normal subgroup of F_1 generated by $\ker \chi_2$ and $m_3 m_1^{-1}$; and $\ker \chi_2$ is the normal subgroup of F_2 generated by m_1^2. □

Remark 1.9. Let Γ be any group on three generators written as a quotient of F by means of an epimorphism $p\colon F \to \Gamma$. Suppose that $\ker p \subseteq \ker \chi$, χ being the quadratic character in Lemma 1.8. Let $\chi^*\colon \Gamma \to \{\pm 1\}$ be the quadratic character such that $\chi^* \circ p = \chi$ and define $\Gamma^* = \ker \chi^*$. The subgroup Γ^* is an index two subgroup of Γ. A set of representatives for Γ modulo Γ^* is given by $\{1, T\}$, where here T is any element in Γ such that $\chi^*(T) = -1$.

The following result is due to Petersson [**20**].

Theorem 1.10. *Let Γ be a triangle Fuchsian group of type $(\infty, e, 2)$. Then $\Gamma^* := \ker \chi^*$ is a triangle Fuchsian group of type (∞, e, e).*

To illustrate Theorem 1.10, consider the group epimorphism $p\colon F \to \Gamma_0(1)$ defined by $p(m_1) = T$, $p(m_2) = -T^{-1}S$, $p(m_3) = -S$ (cf. Proposition 1.6). Since $\ker p \subseteq \ker \chi$, we obtain an index two subgroup $\Gamma_0(1)^* = \ker \chi^*$ of $\Gamma_0(1)$. Proposition 1.11 gives a presentation of this group as a triangle group.

Proposition 1.11. *The following holds:*

(i) *The group $\Gamma_0(1)^*$ is the triangle group defined by $[v_1, v_2, v_4]$.*

(ii) *Its arithmetic type is $(\infty, 3, 3)$.*

(iii) *The point v_1 is fixed by T^2, the point v_2 is fixed by $-T^{-1}S$ and the point v_4 is fixed by $-ST^{-1}$ (cf. Table 4).*

(iv) *The polygon $[v_1, v_2, v_4, v_{19}]$, under the identifications*

$$a : [v_4, v_2] \sim [v_4, v_{19}], \quad z \sim TS \cdot z,$$

$$b : [v_2, v_1] \sim [v_{19}, v_1], \quad z \sim T^2 \cdot z,$$

is a fundamental domain for $\Gamma_0(1)^*$ acting on $\mathcal{H}^*$. This fundamental domain is illustrated in Figure 14.

2. Modular triangle groups

In this section we identify the arithmetic types with cusps by means of triangle groups commensurable with $\Gamma_0(1)$.

Let N be a positive integer and let $\widetilde{\Gamma}_0(N)$ be the congruence subgroup of level N whose elements are the matrices $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \widetilde{\Gamma}_0(1)$ with N dividing c. Let $\Gamma_0(N)$ be its image in $\mathbf{PSL}(2, \mathbb{Z})$. As usual, we denote by $X_0(N)$ the modular curve defined by $\Gamma_0(N)$.

N	ν_2	ν_3	ν_∞	$\Gamma_0(N)$-type
1	1	1	1	$(\infty, 3, 2)$
2	1	0	2	$(\infty, \infty, 2)$
3	0	1	2	$(\infty, \infty, 3)$
4	0	0	3	(∞, ∞, ∞)

Table 2. Constants for $X_0(N)$

Proposition 2.1. *The groups $\Gamma_0(N)$ are triangle Fuchsian groups if and only if $N \leq 4$. Table 2 lists the arithmetic type of each of these groups.*

Proof: The Riemann-Hurwitz formula for the genus tells us that

$$g(X_0(N)) = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}, \tag{2.1}$$

where $\mu = [\Gamma(1) : \Gamma_0(N)]$, ν_e is the number of inequivalent elliptic points of order e, and ν_∞ is the number of inequivalent cusps, under the action of $\Gamma_0(N)$. From formula (2.1), one can show that the genus zero curves $X_0(N)$ are exactly those with $1 \leq N \leq 10$ or $N = 12, 13, 16, 18, 25$. From well known formulas for the constants $\mu(N)$ and $\nu_i(N)$ (cf. [**25**]), it follows that only for $N \leq 4$ holds that

$$\nu_2(N) + \nu_3(N) + \nu_\infty(N) = 3.$$

The values of the constants are specified in Table 2. □

The matrix $W_N := \frac{1}{\sqrt{N}} \left[\begin{smallmatrix} 0 & 1 \\ -N & 0 \end{smallmatrix}\right]$ defines a transformation which belongs to the normalizer of $\widetilde{\Gamma}_0(N)$ in $\mathbf{SL}(2, \mathbb{R})$. It gives rise to what is known as Fricke involution w_N of $X_0(N)$. We let $\widetilde{\Gamma}_0^+(N)$ be the subgroup of $\mathbf{SL}(2, \mathbb{R})$ generated by $\widetilde{\Gamma}_0(N)$ and W_N. We also consider the quotient curve $X_0^+(N) := X_0(N)/\langle w_N \rangle$.

Remark 2.2. Since $W_1 = -S \in \widetilde{\Gamma}_0(1)$, we have $\Gamma_0^+(1) = \Gamma_0(1)$ and, therefore, $X_0^+(1) = X_0(1)$.

N	ν_2^+	ν_3^+	ν_4^+	ν_6^+	ν_∞^+	$\Gamma_0^+(N)$-type	ν_3^*	ν_4^*	ν_6^*	ν_∞^*	$\Gamma_0^+(N)^*$-type
1	1	1	0	0	1	$(\infty,3,2)$	2	0	0	1	$(\infty,3,3)$
2	1	0	1	0	1	$(\infty,4,2)$	0	2	0	1	$(\infty,4,4)$
3	1	0	0	1	1	$(\infty,6,2)$	0	0	2	1	$(\infty,6,6)$
4	1	0	0	0	2	$(\infty,\infty,2)$	0	0	0	3	(∞,∞,∞)

Table 3. Constants for $X_0^+(N)$ and for $X_0^+(N)^*$

Proposition 2.3. *The groups $\Gamma_0^+(N)$ are triangle Fuchsian groups if and only if $N \leq 4$. Table 3 lists the arithmetic type of each of these groups.*

Proof: The Riemann-Hurwitz formula for the genus tells us that

$$g(X_0^+(N)) = \frac{1}{2}(1 + g(X_0(N))) - \frac{1}{4}\nu^+(N), \tag{2.2}$$

where $\nu^+(N)$ is the number of fixed points of w_N. From formula (2.2), one can show that the genus zero curves $X_0^+(N)$ are exactly those with $N \leq 21$ or

$$N = 23, 24, 25, 27, 29, 31, 32, 35, 36, 41, 47, 49, 50, 59, 71.$$

If we denote by $\nu_e^+(N)$ the number of non-equivalent points of order e under the action of $\Gamma_0(N)^+$, then only for $N \leq 4$ we shall have

$$\sum_{e>1} \nu_e^+(N) = 3.$$

The values of the constants are specified in Table 3. □

Now, by the procedure explained in Theorem 1.10, we obtain further triangle groups $\Gamma_0^+(N)^*$ for $N \leq 4$. Note that $\Gamma_0^+(1)^* = \Gamma_0(1)^*$. The arithmetic type of the triangle groups $\Gamma_0^+(N)^*$, for $N \leq 4$, is also specified in Table 3. We shall denote by $X_0^+(N)^*$ the corresponding modular curves.

Corollary 2.4. *Any arithmetic type with cusps (∞, e_2, e_3) admits a modular realization. That is, it can be obtained from a triangle group commensurable with $\Gamma_0(1)$.*

3. Fundamental domains

In Section 2, we have identified all the arithmetic types with cusps by means of modular triangle groups. We are now interested in obtaining fundamental domains for each of these groups. All the fundamental domains computed in this section will be adapted to Theorem 1.2. Note that this was already the case of the fundamental domains obtained in Propositions 1.6 and 1.11.

Lemma 3.1. *For $2 \leq N \leq 4$, we have*

(i) $\widetilde{\Gamma}_0(N) = \langle T, U^{-N}, -U^N T^{-1}\rangle$, *where* $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

(ii) $\widetilde{\Gamma}_0^+(N) = \langle T, T^{-1}W_N, W_N\rangle$, *where* $W_N = \dfrac{1}{\sqrt{N}} \begin{bmatrix} 0 & 1 \\ -N & 0 \end{bmatrix}$.

Proof: Let $\widetilde{\Gamma}' := \langle T, U^{-N}, -U^N T^{-1}\rangle$. Clearly, $\widetilde{\Gamma}'$ is a subgroup of $\widetilde{\Gamma}_0(N)$. To obtain the equality asserted, we shall apply a reduction process. Let $M = \begin{bmatrix} a & b \\ cN & d \end{bmatrix}$ be an arbitrary element in $\widetilde{\Gamma}_0(N)$. If $c = 0$, then $a = d = \pm 1$ and, therefore, $M = \pm T^{\pm b}$ lies in $\widetilde{\Gamma}'$. If $c \neq 0$, then the division of a by cN yields an integer k such that $|a + kcN| \leq |c|N/2$. Therefore, replacing M by $T^k M$ we can assume that $|a| \leq |c|N/2$. Since $N > 1$, $a \neq 0$. The division of c by a yields an integer k' such that $|c - k'a| \leq |a|/2$. Thus,

$$|c - k'a| \leq \frac{|a|}{2} \leq |c|\frac{N}{4}.$$

If $N = 2$ or $N = 3$, then $|c - k'a| < |c|$. If $N = 4$, taking into account that $\gcd(a, cN) = 1$, we have $2|c - k'a| < |a|$ and, also, $|c - k'a| < |c|$. Therefore, replacing M by $U^{-k'N}M$ we can assume that $|c'| < |c|$. After a finite number of steps, we shall obtain a matrix with the entry $c = 0$, hence $M \in \widetilde{\Gamma}'$.

To prove the second assertion let $\widetilde{\Gamma}'' := \langle T, T^{-1}W_N, W_N\rangle$. Since $W_N^2 = -1_2$ and $W_N T W_N^{-1} = U^{-N}$, it is clear that

$$\widetilde{\Gamma}'' = \langle T, W_N\rangle = \langle T, W_N, -1_2\rangle = \langle T, U^{-N}, -1_2, W_N\rangle = \widetilde{\Gamma}_0^+(N). \qquad \square$$

The general principles recalled in the following lemmas will be used in our construction of fundamental domains.

Lemma 3.2. *Let $\mathcal{D}$ be a fundamental domain for a group $\Gamma \subseteq \mathbf{PSL}(2,\mathbb{R})$ acting on $\mathcal{H}^*$. Given a subgroup $\Gamma' \subseteq \Gamma$ of finite index, let $\{\gamma_j\}_j$ be a set of right coset representatives for Γ modulo Γ', i.e. $\Gamma = \bigcup_j \Gamma'\gamma_j$), as a disjoint union. Then $\bigcup_j \gamma_j\mathcal{D}$ is a fundamental region for the group Γ'.*

Lemma 3.3. *Let $\mathcal{D}_0$ be a fundamental domain for a discrete group Γ. Suppose that $\mathcal{D}_0 = \mathcal{D}' \cup \mathcal{D}''$, where $\mathcal{D}'$ and $\mathcal{D}''$ are regions whose interiors are disjoint. Let $\gamma \in \Gamma$ be such that the interiors of $\mathcal{D}'$ and $\gamma \cdot \mathcal{D}''$ are also disjoint. Then $\mathcal{D} := \mathcal{D}' \cup \gamma \cdot \mathcal{D}''$ is a fundamental region for Γ. The fundamental region $\mathcal{D}$ is said to be obtained from $\mathcal{D}_0$ by cutting $\mathcal{D}''$ and pasting it throughout γ.*

Proposition 3.4. *Figures 1, 2, 3 and 4 illustrate fundamental domains for the groups $\Gamma_0(N)$, for $1 \leq N \leq 4$.*

Proof: Let $\mathcal{D}$ be the quadrilateral $[v_1, v_2, v_3, v_4]$. By Proposition 1.6, we know that $\mathcal{D}$ is a fundamental domain for $\Gamma_0(1)$. The sets

$$\{I, S, ST\}, \quad \{I, S, ST, ST^2\}, \quad \{I, S, ST, ST^2, ST^3, ST^2S\}$$

provide right coset representatives for the groups $\widetilde{\Gamma}_0(1)$ modulo $\widetilde{\Gamma}_0(2)$, $\widetilde{\Gamma}_0(3)$, $\widetilde{\Gamma}_0(4)$, respectively. The assertions of the proposition follow from Lemma 3.2 applied to $\mathcal{D}$. □

Proposition 3.5. *Figures 1, 5, 6 and 7 illustrate fundamental domains for the groups $\Gamma_0(N)$, for $1 \leq N \leq 4$, which are symmetrical with respect to the imaginary axis.*

Proof: We now use Lemma 3.3. To obtain the fundamental domain for $\Gamma_0(2)$, we cut the hyperbolic triangle $[v_0, v_5, v_6]$ and paste it throughout the transformation $U^2 \in \Gamma_0(2)$. To obtain the fundamental domain for $\Gamma_0(3)$, we cut the hyperbolic triangle $[v_0, v_6, v_{10}]$ and paste it throughout the transformation $U^3 \in \Gamma_0(3)$. To obtain the fundamental domain for $\Gamma_0(4)$, we cut the hyperbolic triangle $[v_0, v_{16}, v_{12}]$ and paste it throughout the transformation $U^4 \in \Gamma_0(4)$. □

The fundamental domains obtained in Proposition 3.5 will be used with a twofold purpose in Theorem 3.6. On the one hand, they will provide the domains for the groups $\Gamma_0^+(N)$ and, on the other, for the groups $\Gamma_0^+(N)^*$.

Theorem 3.6. *For the groups* $\Gamma_0(N)$, $\Gamma_0^+(N)$, $\Gamma_0^+(N)^*$, $1 \le N \le 4$, *consider the entries* M_i, A_i, e_i, *defined in accordance with Table 4.*

(i) *Each group* $\widetilde{\Gamma}_0(N)$, $\widetilde{\Gamma}_0^+(N)$, $\widetilde{\Gamma}_0^+(N)^*$ *is generated by the corresponding matrices* M_1, M_2, M_3.

(ii) *Each point* A_i *is fixed under the action of* M_i *on* $\mathcal{H}^*$. *Each hyperbolic triangle* $[A_1, A_2, A_3]$ *has interior angles* π/e_i *at* A_i.

(iii) *Let* A_2' *be the point obtained from* A_2 *under the reflection in the side* $[A_1, A_3]$. *Then each quadrilateral* $[A_1, A_2, A_3, A_2']$, *with the identifications* $[A_2, A_1] \sim [A_2', A_1]$ *and* $[A_2, A_3] \sim [A_2', A_3]$, *is a fundamental domain for the corresponding triangle group acting on* $\mathcal{H}^*$.

(iv) *For each group* $\Gamma_0(N)$, $\Gamma_0^+(N)$, $\Gamma_0^+(N)^*$, *a fundamental domain is illustrated in Figures 1, 8, 9 and 10; 1, 11, 12 and 13; 14, 15, 16 and 17.*

Γ	M_1	M_2	M_3	A_1	A_2	A_3	(e_1, e_2, e_3)
$\Gamma_0(1)$	T	$T^{-1}W_1$	W_1	v_1	v_2	v_3	$(\infty, 3, 2)$
$\Gamma_0(2)$	T	U^{-2}	$-U^2T^{-1}$	v_1	v_0	v_7	$(\infty, \infty, 2)$
$\Gamma_0(3)$	T	U^{-3}	$-U^3T^{-1}$	v_1	v_0	v_8	$(\infty, \infty, 3)$
$\Gamma_0(4)$	T	U^{-4}	$-U^4T^{-1}$	v_1	v_0	v_{17}	(∞, ∞, ∞)
$\Gamma_0^+(2)$	T	$T^{-1}W_2$	W_2	v_1	v_5	v_{13}	$(\infty, 4, 2)$
$\Gamma_0^+(3)$	T	$T^{-1}W_3$	W_3	v_1	v_6	v_{14}	$(\infty, 6, 2)$
$\Gamma_0^+(4)$	T	$T^{-1}W_4$	W_4	v_1	v_{16}	v_{15}	$(\infty, \infty, 2)$
$\Gamma_0^+(1)^*$	T^2	$T^{-1}W_1$	W_1T^{-1}	v_1	v_2	v_4	$(\infty, 3, 3)$
$\Gamma_0^+(2)^*$	T^2	$T^{-1}W_2$	W_2T^{-1}	v_1	v_5	v_7	$(\infty, 4, 4)$
$\Gamma_0^+(3)^*$	T^2	$T^{-1}W_3$	W_3T^{-1}	v_1	v_6	v_8	$(\infty, 6, 6)$
$\Gamma_0^+(4)^*$	T^2	$T^{-1}W_4$	W_4T^{-1}	v_1	v_{16}	v_{17}	(∞, ∞, ∞)

TABLE 4. Triangle presentations

group	identifications	mappings
$\Gamma_0(2)$	$[v_0, v_1] \sim [v_{18}, v_1]$	$z \mapsto T \cdot z$
	$[v_7, v_0] \sim [v_7, v_{18}]$	$z \mapsto U^2 T^{-1} \cdot z$
$\Gamma_0(3)$	$[v_0, v_1] \sim [v_{18}, v_1]$	$z \mapsto T \cdot z$
	$[v_8, v_0] \sim [v_8, v_{18}]$	$z \mapsto U^3 T^{-1} \cdot z$
$\Gamma_0(4)$	$[v_0, v_1] \sim [v_{18}, v_1]$	$z \mapsto T \cdot z$
	$[v_{17}, v_0] \sim [v_{17}, v_{18}]$	$z \mapsto U^4 T^{-1} \cdot z$
$\Gamma_0^+(1)$	$[v_2, v_1] \sim [v_4, v_1]$	$z \mapsto T \cdot z$
	$[v_3, v_2] \sim [v_3, v_4]$	$z \mapsto W_1 \cdot z$
$\Gamma_0^+(2)$	$[v_5, v_1] \sim [v_7, v_1]$	$z \mapsto T \cdot z$
	$[v_{13}, v_5] \sim [v_{13}, v_7]$	$z \mapsto W_2 \cdot z$
$\Gamma_0^+(3)$	$[v_6, v_1] \sim [v_8, v_1]$	$z \mapsto T \cdot z$
	$[v_{14}, v_6] \sim [v_{14}, v_8]$	$z \mapsto W_3 \cdot z$
$\Gamma_0^+(4)$	$[v_{16}, v_1] \sim [v_{17}, v_1]$	$z \mapsto T \cdot z$
	$[v_{15}, v_{16}] \sim [v_{15}, v_{17}]$	$z \mapsto W_4 \cdot z$
$\Gamma_0^+(1)^*$	$[v_2, v_1] \sim [v_{19}, v_1]$	$z \mapsto T^2 \cdot z$
	$[v_4, v_2] \sim [v_4, v_{19}]$	$z \mapsto TW_1 \cdot z$
$\Gamma_0^+(2)^*$	$[v_5, v_1] \sim [v_{20}, v_1]$	$z \mapsto T^2 \cdot z$

TABLE 5. Side identifications

	$[v_7, v_5] \sim [v_7, v_{20}]$	$z \mapsto TW_2 \cdot z$
$\Gamma_0^+(3)^*$	$[v_6, v_1] \sim [v_{21}, v_1]$	$z \mapsto T^2 \cdot z$
	$[v_8, v_6] \sim [v_8, v_{21}]$	$z \mapsto TW_3 \cdot z$
$\Gamma_0^+(4)^*$	$[v_{16}, v_1] \sim [v_{22}, v_1]$	$z \mapsto T^2 \cdot z$
	$[v_{17}, v_{16}] \sim [v_{17}, v_{22}]$	$z \mapsto TW_4 \cdot z$

Table 5. Side identifications (continued)

Proof: If in Figures 5, 6 and 7 we cut the hyperbolic triangles $[v_0, v_1, v_5]$, $[v_0, v_1, v_6]$, $[v_0, v_1, v_{16}]$, respectively, and paste them with transformation T, we obtain the fundamental domains for the groups $\Gamma_0(2)$, $\Gamma_0(3)$, $\Gamma_0(4)$ illustrated in Figures 8, 9 and 10. The identifications are described in Table 5.

Observe that each transformation W_N acts as an involution on the fundamental domains in Figures 5, 6 and 7. The action on their vertices is listed in Table 6. Moreover, the fixed point of W_N is $i/\sqrt{N}$, for $2 \leq N \leq 4$. It is now easy to conclude that Figures 11, 12 and 13 illustrate fundamental domains for $\Gamma_0^+(2)$, $\Gamma_0^+(3)$, and $\Gamma_0^+(4)$, respectively. The identifications are described in Table 5.

I	v_0	v_1	v_5	v_7	v_6	v_8	v_{16}	v_{17}
W_2	v_1	v_0	v_7	v_5	$*$	$*$	$*$	$*$
W_3	v_1	v_0	$*$	$*$	v_8	v_6	$*$	$*$
W_4	v_1	v_0	$*$	$*$	$*$	$*$	v_{17}	v_{16}

Table 6. Action of the involutions W_N

Our next task is to construct fundamental domains for the Fuchsian groups defining the curves $X_0^+(N)^*$, for $1 \leq N \leq 4$. For this purpose, we need to make explicit the epimorphism $p\colon F \to \widetilde{\Gamma}_0^+(N)$ appearing in Remark 1.9. By Lemma 3.1, each group $\widetilde{\Gamma}_0^+(N)$, for $1 \leq N \leq 4$, admits the following presentation:

$$\widetilde{\Gamma}_0^+(N) = \langle M_1, M_2, M_3;\ M_1M_2M_3 = -1_2,\ M_3^2 = -1_2,\ M_2^{e_2} = -1_2\rangle.$$

The last relation has only to be considered whenever it makes sense, namely for $N = 1, 2, 3$, because in these cases the condition $e_2 \neq \infty$ is satisfied. Now it is meaningful to consider the epimorphism

$$p\colon F \to \widetilde{\Gamma}_0^+(N) \to \Gamma_0^+(N),$$

which sends m_i to M_i (mod $\pm 1_2$). Clearly, $\ker p \subseteq \ker \chi$ since

$$\chi(m_1m_2m_3) = \chi(m_3^2) = \chi(m_2^{e_2}) = 1,$$

and we may consider the group $\ker \chi^*$, as was prescribed in Remark 1.9.

Lemma 3.7. *For $1 \leq N \leq 4$, we have*

(i) $\widetilde{\Gamma}_0^+(N)^* = \langle M_1^2, M_2, M_3M_1^{-1}\rangle \subseteq \widetilde{\Gamma}_0^+(N)$.

(ii) *A set of coset representatives of $\widetilde{\Gamma}_0^+(N)$ modulo $\widetilde{\Gamma}_0^+(N)^*$ is given by $\{1_2,\ M_1 = T\}$.*

Proof of Lemma 3.7: We only need to observe that $\ker \chi$ is the normal subgroup of F generated by $\{m_1^2, m_2, m_3m_1^{-1}\}$. Then their images under p in $\Gamma_0^+(N)$ will generate the group $\Gamma_0^+(N)^*$. □

Returning to the proof of Theorem 3.6, we present the groups $\Gamma_0^+(N)^*$ as triangle groups. The results are specified in Table 4. The second statement in Lemma 3.7 is all what we need to construct a fundamental domain for these groups. In fact, given a fundamental domain $\mathcal{D}$ for $\Gamma_0^+(N)$, $\mathcal{D} \cup T\mathcal{D}$ will be a fundamental domain for $\Gamma_0^+(N)^*$. The required identifications are described in Table 5. □

4. Equations for the coverings

The inclusions

$$\Gamma_0(N) \subseteq \Gamma_0^+(N), \quad \Gamma_0^+(N)^* \subseteq \Gamma_0^+(N), \quad \Gamma_0(N) \subseteq \Gamma_0(1), \quad \Gamma_0(4) \subseteq \Gamma_0(2)$$

give rise to coverings

$X_0(N) \to X_0^+(N)$	of degree 2,	for $2 \leq N \leq 4$;
$X_0^+(N)^* \to X_0^+(N)$	of degree 2,	for $1 \leq N \leq 4$;
$X_0(N) \to X_0(1)$	of degrees $3, 4, 6$,	for $N = 2, 3, 4$;
$X_0(4) \to X_0(2)$	of degree 2.	

The purpose of this section is to provide equations for all them. We shall begin by making explicit their ramification.

Γ	P_1	P_2	P_3	P_4	P_5	Q_0	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8
$\Gamma_0(1)$	v_1	v_3	v_4											
$\Gamma_0(2)$	v_1	v_7	v_0	v_3	v_4	v_0	v_1	v_{13}	v_7	v_3	v_4			
$\Gamma_0(3)$						v_0	v_1	v_{14}	v_8	v_3	v_7	v_4		
$\Gamma_0(4)$						v_0	v_1	v_{15}	v_{17}	v_3	v_7	v_4	v_8	v_{23}
$\Gamma_0^+(1)$	v_1	v_3	v_4											
$\Gamma_0^+(2)$	v_1	v_{13}	v_7											
$\Gamma_0^+(3)$	v_1	v_{14}	v_8											
$\Gamma_0^+(4)$	v_1	v_{15}	v_{17}											
$\Gamma_0^+(1)^*$						v_{19}	v_1	v_3	v_4					
$\Gamma_0^+(2)^*$						v_{20}	v_1	v_{13}	v_7					
$\Gamma_0^+(3)^*$						v_{21}	v_1	v_{14}	v_8					
$\Gamma_0^+(4)^*$						v_{22}	v_1	v_{15}	v_{17}					

Table 7. Ramification points

Proposition 4.1. *Consider the points P_i and Q_i defined in accordance with Table 7.*

(i) *The covering $X_0(N) \to X_0^+(N)$, $N = 2, 3, 4$, ramifies as $P_i = Q_i^2$, $i = 2, 3$, with $P_2, P_3 \in X_0^+(N)$ and $Q_2, Q_3 \in X_0(N)$. The point $P_1 \in X_0^+(N)$ splits as $P_1 = Q_0 Q_1$, with $Q_0, Q_1 \in X_0(N)$.*

(ii) *The covering $X_0^+(N)^* \to X_0^+(N)$, $1 \leq N \leq 4$, ramifies as $P_i = Q_i^2$, $i = 1, 2$, with $P_1, P_2 \in X_0^+(N)$ and $Q_1, Q_2 \in X_0(N)^*$. The point $P_3 \in X_0^+(N)$ splits as $P_3 = Q_0 Q_3$, with $Q_0, Q_3 \in X_0^+(N)^*$.*

(iii) *The covering $X_0(2) \to X_0(1)$ ramifies as $P_1 = Q_0^2 Q_1$, $P_2 = Q_4^2 Q_3$, $P_3 = Q_5^3$, with $P_i \in X_0(1)$, $Q_i \in X_0(2)$.*

(iv) *The covering $X_0(3) \to X_0(1)$ ramifies as $P_1 = Q_0^3 Q_1$, $P_2 = Q_4^2 Q_5^2$, $P_3 = Q_6^3 Q_3$, with $P_i \in X_0(1)$ and $Q_i \in X_0(3)$.*

(v) *The covering* $X_0(4) \to X_0(2)$ *ramifies as* $P_2 = Q_5^2$, $P_3 = Q_0^2$, *with* $P_2, P_3 \in X_0(2)$ *and* $Q_0, Q_5 \in X_0(4)$. *The points* $P_i \in X_0(2)$, $i = 1, 4, 5$, *split as* $P_1 = Q_1Q_3$, $P_4 = Q_4Q_8$, $P_5 = Q_6Q_7$, *with* $Q_i \in X_0(4)$.

(vi) *The covering* $X_0(4) \to X_0(1)$ *ramifies as* $P_1 = Q_0^4Q_1Q_3$, $P_2 = Q_4^2Q_5^2Q_8^2$, $P_3 = Q_6^3Q_7^3$, *with* $P_i \in X_0(1)$ *and* $Q_i \in X_0(4)$.

Definition 4.2. Let t_N, t_N^+, t_N^* be the uniformizing functions for the triangle groups $\Gamma_0(N)$, $\Gamma_0^+(N)$, $\Gamma_0^+(N)^*$ uniquely determined by their values at the three points chosen in accordance with Table 8. We shall call them triangle functions.

Theorem 4.3. *Each triangle function* t *defined in Definition 4.2 is a* Hauptmodul*:* $\mathbb{C}(X(\Gamma)) \simeq \mathbb{C}(t)$. *They fulfill the following algebraic relations:*

(i) $(t_2 - 4)^3 + 27t_1t_2^2 = 0$.

(ii) $(t_3 - 9)^3(t_3 - 1) + 64t_1t_3^3 = 0$.

(iii) $t_4^2 - 4t_2(t_4 - 1) = 0$.

(iv) $(t_4^2 - 16t_4 + 16)^3 + 108t_1t_4^4(t_4 - 1) = 0$.

(v) $t_1^+ = 1 - t_1$.

(vi) $t_N^2 + 2(1 - 2t_N^+)t_N + 1 = 0$, *for* $2 \leq N \leq 4$.

(vii) $t_N^+ = (2t_N^* - 1)^2$, *for* $1 \leq N \leq 4$.

Proof: We keep the notations given in Proposition 4.1.

(i) Let $Q_4, Q_5 \in X_0(2)$, and set $a := t_2(Q_5)$, $b := t_2(Q_4)$. Then

$$\begin{aligned}\operatorname{div}\left(\left(1 - \frac{a}{t_2}\right)^2 (t_2 - a)\right) &= 3(Q_5) - 2(Q_0) - (Q_1)\\ &= (P_3) - (P_1) = \operatorname{div}(t_1),\\ \operatorname{div}\left(\left(1 - \frac{b}{t_2}\right)^2 (t_2 - 1)\right) &= 2(Q_4) + (Q_3) - 2(Q_0) - (Q_1)\\ &= (P_2) - (P_1) = \operatorname{div}(t_1 - 1).\end{aligned}$$

Therefore, there exist constants A, B such that

$$\begin{cases} t_1 = A\left(1 - \dfrac{a}{t_2}\right)^2 (t_2 - a) \\ t_1 - 1 = B\left(1 - \dfrac{b}{t_2}\right)^2 (t_2 - 1). \end{cases}$$

It follows that $a = 4$, $b = -8$, $A = B = -1/27$, and we obtain relation (i).

In the remaining cases the computations are very similar. □

t	Γ	A_1	A_2	A_3	$t(A_1)$	$t(A_2)$	$t(A_3)$
t_1	$\Gamma_0(1)$	v_1	v_2	v_3	∞	0	1
t_2	$\Gamma_0(2)$	v_1	v_0	v_7	∞	0	1
t_3	$\Gamma_0(3)$	v_1	v_0	v_8	∞	0	1
t_4	$\Gamma_0(4)$	v_1	v_0	v_{17}	∞	0	1
t_1^+	$\Gamma_0^+(1)$	v_1	v_3	v_4	∞	0	1
t_2^+	$\Gamma_0^+(2)$	v_1	v_{13}	v_7	∞	0	1
t_3^+	$\Gamma_0^+(3)$	v_1	v_{14}	v_8	∞	0	1
t_4^+	$\Gamma_0^+(4)$	v_1	v_{15}	v_{17}	∞	0	1
t_1^*	$\Gamma_0(1)^*$	v_1	v_4	v_{19}	∞	0	1
t_2^*	$\Gamma_0(2)^*$	v_1	v_7	v_{20}	∞	0	1
t_3^*	$\Gamma_0(3)^*$	v_1	v_8	v_{21}	∞	0	1
t_4^*	$\Gamma_0(4)^*$	v_1	v_{17}	v_{22}	∞	0	1

Table 8. Triangle functions

5. Uniformizing differential equations

The main tool in our approach to the differential treatment of the triangle automorphic functions will be Fuchs' theory on ordinary differential equations, together with a rational ordinary differential operator of order three, obtained via a suitable modification of an operator introduced by Schwarz [**24**] in 1873.

Definition 5.1. Let $f(z)$ be a non-constant smooth function and let $D(f,z)$ stand for the usual derivative.

(1) The *Schwarzian derivative* of f is defined as

$$Ds(f,z) = \frac{2D(f,z)D^3(f,z) - 3D^2(f,z)^2}{D(f,z)^2}.$$

(2) The *automorphic derivative* of f is defined as

$$Da(f,z) = \frac{Ds(f,z)}{D(f,z)^2}.$$

Although neither the Schwarzian derivative nor the automorphic derivative are derivations in the usual sense, they have some properties similar to those of the standard derivation.

Proposition 5.2. *Let $f(z)$, $g(z)$ be non-constant smooth functions whose composition $g \circ f$ is defined. Then the automorphic derivative satisfies the following chain rule:*

$$Da(g \circ f, z) = Da(g, f(z)) + \frac{Da(f,z)}{D(g, f(z))^2}.$$

A multivalued function defined on $\mathbf{P}^1(\mathbb{C})$ is said to be $\mathbf{PGL}(2,\mathbb{C})$-multivalued if any pair of its branches are always projectively related. Examples of $\mathbf{PGL}(2,\mathbb{C})$-multivalued functions occur by inversion of automorphic functions.

Proposition 5.3. *Suppose that $f(z) = w$ is a smooth function whose inverse function is $\mathbf{PGL}(2,\mathbb{C})$-multivalued. Then the Schwarzian derivative $Ds(f^{-1}, w)$ is univalued and*

$$Ds(f^{-1}, w) = -Da(f,z), \quad f^{-1}(w) = z.$$

As a first application of these properties, consider a homographic transformation

$$\gamma(z) = \frac{az+b}{cz+d}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{GL}(2,\mathbb{C}).$$

Then we have $Da(\gamma, z) = 0$ so that, for any function $f(z)$, we shall have

$$Da(f \circ \gamma, z) = Da(f, \gamma(z)).$$

In particular, for a group $\Gamma \subseteq \mathbf{PSL}(2,\mathbb{R})$ and for a Γ-automorphic function f, we obtain the Γ-invariance of $Da(f,z)$.

Proposition 5.4. *The automorphic derivative $Da(f,z)$ of a Γ-automorphic function, $f(z)$, is again a Γ-automorphic function. That is to say, the following equality holds:*

$$Da(f, \gamma(z)) = Da(f,z), \quad \textit{for any } \gamma \in \Gamma.$$

We see that automorphic derivatives fit automorphic functions just as Schwarzian derivatives fit their inverses. The following theorem goes back to Poincaré (cf. [**21**], [**16**]).

Theorem 5.5. *Let* Γ *be a Fuchsian group of the first kind. Let* $f(z) = w$ *be a non-constant* Γ*-automorphic function and let* $z = g(w)$ *be a branch of its inverse. Then the functions*

$$\eta_1(w) := \frac{g(w)}{D(g,w)^{1/2}}, \quad \eta_2(w) := \frac{1}{D(g,w)^{1/2}}$$

satisfy a linear differential equation

$$D^2(\eta, w) + A(w)\eta = 0,$$

where $A(w)$ *is an algebraic function of* w.

We see from Theorem 5.5 that the branches of the multivalued function f^{-1} can be obtained as quotients of two solutions of a specific linear second order differential equation. In its turn, the automorphic function f can be obtained as a solution of the non-linear third order differential equation

$$Da(w, z) + A(w) = 0.$$

Therefore, in order to calculate an automorphic function f it suffices to know its automorphic derivative $-A(w)$, although this may be a rather complicated task.

Let us restrict ourselves to the case of genus $g = 0$. Then the field of Γ-automorphic functions is generated over $\mathbb{C}$ by an automorphic function t. Since $Da(t, z)$ is also automorphic, there exists a rational function $R(t)$ such that

$$Da(t, z) + R(t) = 0.$$

Now, suppose that we are aware of a fundamental domain for the Γ-action on $\mathcal{H}$ given by a polygon whose sides are identified by pairs. We require the existence of a symmetry that cuts a half domain given by a polygon $\mathcal{P}$ containing exactly one representative of each vertex of the fundamental domain, and whose internal angles at its vertices are $\alpha_i\pi$. Suppose that the function t applies the boundary of $\mathcal{P}$ in $\mathbf{P}^1(\mathbb{R})$. Then

$$R(t) = \sum \frac{1-\alpha_i^2}{(t-a_i)^2} + \sum \frac{B_i}{t-a_i},$$

where B_i are constants and the summation extends over all the vertices of $\mathcal{P}$ where t takes finite values a_i.

At this point, two cases have to be considered. First, assume that the vertices of $\mathcal{P}$ have their images at finite distances from the z-origin. Then

(i) $\sum B_i = 0$,

(ii) $\sum a_i B_i + \sum(1-\alpha_i^2) = 0$,

(iii) $\sum a_i^2 B_i + \sum a_i(1-\alpha_i^2) = 0$.

Secondly, assume that one vertex of $\mathcal{P}$ with internal angle $\alpha\pi$ has its image at infinity. Then

(i) $\sum B_i = 0$,

(ii) $\sum a_i B_i + \sum(1-\alpha_i^2) - (1-\alpha^2) = 0$.

Remark. By Schwarz's symmetry principle (cf. [**9**]), we only need to define the function t in $\mathcal{P}$ so the angles have to be taken with respect to this half domain.

In general, the above relations between the constants B_i and the values a_i do not suffice to determine all the constants. But, in case of triangle groups, we can prescribe the values a_i of the function t at the vertices of a hyperbolic triangle and then the constants B_i will be fully determined. The differential equations involved in our problem are specified in the following theorem.

Theorem 5.6. *The triangle functions defined in Definition 4.2 satisfy the differential equations*

$$Da(t,z) + R(t) = 0,$$

where the rational functions $R(t)$ are listed in Table 9.

Proof: All the necessary data to compute the functions $R(t)$ are provided in the first columns of Table 9. We observe that the angles in the third column of Table 9 correspond to the triangles $[A_1, A_2, A_3]$ for the groups $\Gamma_0(N)$, and to the triangles $[A_1, A_3, A_2']$ for the groups $\Gamma_0^+(N)$ and $\Gamma_0^+(N)^*$ (see also Table 4). □

Γ	t	$(\alpha_1,\alpha_2,\alpha_3)$	(a_1,a_2,a_3)	$-Da(t,z)$
$\Gamma_0(1)$	t_1	$(0,\frac{1}{3},\frac{1}{2})$	$(\infty,0,1)$	$\dfrac{36t^2-41t+32}{36t^2(t-1)^2}$
$\Gamma_0(2)$	t_2	$(0,0,\frac{1}{2})$	$(\infty,0,1)$	$\dfrac{4t^2-5t+4}{4t^2(t-1)^2}$
$\Gamma_0(3)$	t_3	$(0,0,\frac{1}{3})$	$(\infty,0,1)$	$\dfrac{9t^2-10t+9}{9t^2(t-1)^2}$
$\Gamma_0(4)$	t_4	$(0,0,0)$	$(\infty,0,1)$	$\dfrac{t^2-t+1}{t^2(t-1)^2}$
$\Gamma_0^+(1)$	t_1^+	$(0,\frac{1}{2},\frac{1}{3})$	$(\infty,0,1)$	$\dfrac{36t^2-31t+27}{36t^2(t-1)^2}$
$\Gamma_0^+(2)$	t_2^+	$(0,\frac{1}{2},\frac{1}{4})$	$(\infty,0,1)$	$\dfrac{16t^2-13t+12}{16t^2(t-1)^2}$
$\Gamma_0^+(3)$	t_3^+	$(0,\frac{1}{2},\frac{1}{6})$	$(\infty,0,1)$	$\dfrac{36t^2-28t+27}{36t^2(t-1)^2}$
$\Gamma_0^+(4)$	t_4^+	$(0,\frac{1}{2},0)$	$(\infty,0,1)$	$\dfrac{4t^2-3t+3}{4t^2(t-1)^2}$
$\Gamma_0^+(1)^*$	t_1^*	$(0,\frac{1}{3},\frac{1}{3})$	$(\infty,0,1)$	$\dfrac{9t^2-9t+8}{9t^2(t-1)^2}$
$\Gamma_0^+(2)^*$	t_2^*	$(0,\frac{1}{4},\frac{1}{4})$	$(\infty,0,1)$	$\dfrac{16t^2-16t+15}{16t^2(t-1)^2}$
$\Gamma_0^+(3)^*$	t_3^*	$(0,\frac{1}{6},\frac{1}{6})$	$(\infty,0,1)$	$\dfrac{36t^2-36t+35}{36t^2(t-1)^2}$
$\Gamma_0^+(4)^*$	t_4^*	$(0,0,0)$	$(\infty,0,1)$	$\dfrac{t^2-t+1}{t^2(t-1)^2}$

Table 9. Automorphic derivatives of the triangle functions

6. Local charts at the elliptic points

Our goal is to obtain explicit expansions of the modular uniformizing functions around the elliptic points and around the cusps. The purpose of this section is a first choice of local uniformizing parameters adapted to our functions at the elliptic points of the fundamental domains considered in Section 3.

Suppose that v is an elliptic point in $\mathcal{H}$ of order e for the Γ-action. Let $\Gamma_v = \langle g \rangle$ be the isotropy group at v, generated by a transformation $g \in \mathbf{PSL}(2,\mathbb{R})$. Let $G \in \mathbf{SL}(2,\mathbb{R})$ be a matrix defining g. Since in all our cases $-1_2 \in \widetilde{\Gamma}$, we may take the matrix G of order $2e$. Since g is an elliptic transformation, the matrix G can be diagonalized. Let $H \in \mathbf{GL}(2,\mathbb{C})$ be such that $D := HGH^{-1} = \left[\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{smallmatrix}\right]$, where ζ is a $2e$-th primitive root of unity. We denote by h and d the homographies defined by H and D, respectively. Then

$$(*) \qquad h(g(z)) = d(h(z)) = \zeta^2 h(z).$$

By evaluating $(*)$ at the point $z = v$, we obtain $h(v) = \zeta^2 h(v)$. Since $e > 1$, we have $\zeta^2 \neq 1$ and deduce that $h(v) = 0$. Thus $h(z) = \dfrac{z-v}{cz+d}$. We extend g to a transformation of $\mathbf{P}^1(\mathbb{C})$. The equality $(*)$ evaluated at v and at its conjugate $\overline{v}$ yields

$$h(v) = h(g(v)) = \zeta^2 h(v), \quad h(\overline{v}) = h(g(\overline{v})) = \zeta^2 h(\overline{v}).$$

Since $h(v) = 0$ and h is a bijective mapping of $\mathbf{P}^1(\mathbb{C})$, we must have $h(\overline{v}) = \infty$. Hence,

$$h(z) = k\frac{z-v}{z-\overline{v}},$$

for some constant $k \in \mathbb{C}$ to be determined.

Now we can expand any Γ_v-automorphic function t around the point v as a power series T in the variable $h(z)$:

$$t(z) = T(h(z)) = \sum_{n=n_0}^{\infty} a_n h(z)^n.$$

We shall have

$$T(h(z)) = t(z) = t(g(z)) = T(h(g(z))) = T(\zeta^2 h(z)).$$

Thus $a_n = 0$ unless $n \equiv 0 \pmod{e}$.

Definition 6.1. A local parameter at an elliptic point $v \in \mathcal{H}$ for the Γ_v-action is any function

$$q(z) := \left(k\frac{z-v}{z-\overline{v}}\right)^e,$$

where e is the order of the group Γ_v and $k \in \mathbb{C}$ is any constant. The local parameter is said to be adapted to a function $t = \sum_{n=m}^{\infty} a_{ne}q^n$ when, moreover, $a_e = 1$ if $m \geq 0$; and $a_{-e} = 1$ otherwise.

In order to obtain local parameters adapted to our functions, in Theorem 6.2 we review some relevant facts on the Schwarzian functions. For that purpose, let us consider the classical hypergeometric function defined by the series

$$F(a,b,c;w) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{w^n}{n!}, \quad (a)_n := a(a+1)\dots(a+n-1),$$

which converges for $|w| < 1$ (cf. [**19**]).

Theorem 6.2. *Assume that $c \neq 1$. The functions $F(a,b,c;w)$ and $w^{1-c}F(a-c+1,b-c+1,2-c;w)$ are two linearly independent solutions of the hypergeometric differential equation*

$$w(1-w)D^2(f,w) + (c-(1+a+b)w)D(f,w) - ab\,f = 0.$$

The function

$$z = s(a,b,c;w) := \frac{w^{1-c}F(a-c+1,b-c+1,2-c;w)}{F(a,b,c;w)}$$

maps the upper half w-plane $\overline{\mathcal{H}}$ onto a triangle in the z-plane. The vertices of this triangle can be expressed in terms of Euler's gamma function:

$$s(0) = 0,$$

$$s(\infty) = \exp\left(\pi i(1-c)\right)\frac{\Gamma(b)\Gamma(c-a)\Gamma(2-c)}{\Gamma(c)\Gamma(b-c+1)\Gamma(1-a)},$$

$$s(1) = \frac{\Gamma(2-c)\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(1-a)\Gamma(1-b)}.$$

The internal angles at these vertices are $\alpha\pi$, $\beta\pi$, $\gamma\pi$, where

$$\alpha = 1-c \neq 0, \quad \beta = b-a, \quad \gamma = c-a-b.$$

In the next theorem we compare the triangle $[s(0), s(\infty), s(1)]$ with those defining the triangle functions t. In this way, we shall obtain in a closed form the constants k defining local parameters adapted to our functions.

t	v	e_v	$t(v)$	a, c	ν_v	k_v
t_1	v_2	3	0	$\frac{1}{12}, \frac{2}{3}$	2^3	$\exp(\frac{\pi i}{3})\frac{1}{3}\frac{\Gamma(\frac{1}{12})\Gamma(\frac{7}{12})\Gamma(\frac{1}{3})}{\Gamma(\frac{11}{12})\Gamma(\frac{5}{12})\Gamma(\frac{2}{3})}$
t_1^+	v_3	2	0	$\frac{1}{12}, \frac{1}{2}$	$2 \cdot 3^2$	$\exp(\frac{\pi i}{2})\frac{1}{2}\frac{\Gamma(\frac{1}{12})\Gamma(\frac{5}{12})}{\Gamma(\frac{11}{12})\Gamma(\frac{7}{12})}$
t_2^+	v_{13}	2	0	$\frac{1}{8}, \frac{1}{2}$	2^3	$\exp(\frac{\pi i}{2})\frac{1}{2}\frac{\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{\Gamma(\frac{7}{8})\Gamma(\frac{5}{8})}$
t_3^+	v_{14}	2	0	$\frac{1}{6}, \frac{1}{2}$	3^2	$\exp(\frac{\pi i}{2})\frac{1}{2}\frac{\Gamma(\frac{1}{6})\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{6})\Gamma(\frac{2}{3})}$
t_4^+	v_{15}	2	0	$\frac{1}{4}, \frac{1}{2}$	2	$\exp(\frac{\pi i}{2})\frac{1}{2}\frac{\Gamma(\frac{1}{4})^2}{\Gamma(\frac{3}{4})^2}$
t_1^*	v_4	3	0	$\frac{1}{6}, \frac{2}{3}$	2	$\exp(\frac{\pi i}{3})\frac{1}{3}\frac{\Gamma(\frac{1}{6})\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{6})\Gamma(\frac{2}{3})}$
t_2^*	v_7	4	0	$\frac{1}{4}, \frac{3}{4}$	1	$\exp(\frac{\pi i}{4})\frac{1}{4}\frac{\Gamma(\frac{1}{4})^2}{\Gamma(\frac{3}{4})^2}$
t_3^*	v_8	6	0	$\frac{1}{3}, \frac{5}{6}$	1	$\exp(\frac{\pi i}{6})\frac{1}{6}\frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}$

TABLE 10. Local constants at the elliptic points v such that $t(v) = 0$

Theorem 6.3. *Let t be one of the triangle functions defined in Definition 4.2. Suppose that it is obtained from a hyperbolic triangle $[A, B, C]$ of internal angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$, and that it takes the values*

$$t(A) = 0, \quad t(B) = \infty, \quad t(C) = 1.$$

Suppose that $\alpha \neq 0$. Then the constant k_A, at an elliptic point A of order e_A, defining a local parameter adapted to t is listed in Table 10.

In the case that the function t takes the values

$$t(A) = 1, \quad t(B) = \infty, \quad t(C) = 0,$$

the corresponding constants k_A are listed in Table 11.

t	v	e_v	$t(v),$	a, c	ν_v	k_v
t_1	v_3	2	1	$\frac{1}{12}, \frac{1}{2}$	$2 \cdot 3^2$	$\frac{1}{2}\frac{\Gamma(\frac{1}{12})\Gamma(\frac{5}{12})}{\Gamma(\frac{11}{12})\Gamma(\frac{7}{12})}$
t_2	v_7	2	1	$\frac{1}{4}, \frac{1}{2}$	2	$\frac{1}{2}\frac{\Gamma(\frac{1}{4})^2}{\Gamma(\frac{3}{4})^2}$
t_3	v_8	3	1	$\frac{1}{3}, \frac{2}{3}$	2	$\frac{1}{3}\frac{\Gamma(\frac{1}{3})^3}{\Gamma(\frac{2}{3})^3}$
t_1^+	v_4	3	1	$\frac{1}{12}, \frac{2}{3}$	2^3	$\frac{1}{3}\frac{\Gamma(\frac{1}{12})\Gamma(\frac{7}{12})\Gamma(\frac{1}{3})}{\Gamma(\frac{11}{12})\Gamma(\frac{5}{12})\Gamma(\frac{2}{3})}$
t_2^+	v_7	4	1	$\frac{1}{8}, \frac{3}{4}$	1	$\frac{1}{4}\frac{\Gamma(\frac{1}{8})\Gamma(\frac{5}{8})\Gamma(\frac{1}{4})}{\Gamma(\frac{7}{8})\Gamma(\frac{3}{8})\Gamma(\frac{3}{4})}$
t_3^+	v_8	6	1	$\frac{1}{6}, \frac{5}{6}$	1	$\frac{1}{6}\frac{\Gamma(\frac{1}{6})^2\Gamma(\frac{2}{3})}{\Gamma(\frac{5}{6})^2\Gamma(\frac{1}{3})}$
t_1^*	v_{19}	3	1	$\frac{1}{6}, \frac{2}{3}$	2	$\frac{1}{3}\frac{\Gamma(\frac{1}{6})\Gamma(\frac{1}{3})}{\Gamma(\frac{5}{6})\Gamma(\frac{2}{3})}$
t_2^*	v_{20}	4	1	$\frac{1}{4}, \frac{3}{4}$	1	$\frac{1}{4}\frac{\Gamma(\frac{1}{4})^2}{\Gamma(\frac{3}{4})^2}$
t_3^*	v_{21}	6	1	$\frac{1}{3}, \frac{5}{6}$	1	$\frac{1}{6}\frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}$

Table 11. Local constants at the elliptic points v such that $t(v) = 1$

Proof: In all the cases to be considered, we have $B = v_1$ and $\beta = 0$. First we explain the results in Table 10. By formal integration of the differential equation of the third order in Theorem 5.6, and taking into account that $t(A) = 0$, it follows that there exists a normalized power series in two variables

$$r(X, Y) = \sum_{n=1}^{\infty} a_{ne} X^{en} Y^{en}, \quad a_e = 1,$$

and a constant $k \in \mathbb{C}$, such that

$$t(z) = r(k; h_1(z)) = \sum_{n=1}^{\infty} a_{ne} k^{en} h_1^{en}(z),$$

for any z in a neighbourhood of A. Here we take $h_1(z) := \dfrac{z-A}{z-\overline{A}}$.

Consider the Schwarzian function $s(a,b,c;w)$ determined by the angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$. Since r satisfies the conditions

$$r(k; h_1(A)) = 0, \quad r(k; h_1(B)) = \infty, \quad r(k; h_1(C)) = 1,$$

we can relate the inverse of the series defining $s(a,b,c;w)$ to the series defining $t(z)$. A direct computation of the first terms in both series suffices to establish the following lemma.

Lemma 6.4. *Let $u(a,b,c;z)$ be the inverse series of $s(a,b,c;w)$. Then*

$$r(1; h_1(z)) = u(a,b,c; h_1(z))$$

for $a = \frac{1}{2}(1-\alpha-\beta-\gamma)$, $b = \frac{1}{2}(1-\alpha+\beta-\gamma)$, $c = 1-\alpha$, and any $z \in \mathbb{C}$ in the convergence domain.

To continue the calculation of k, we may use either the condition $t(C) = 1$ or $t(B) = \infty$. In the first case, we obtain

$$1 = t(C) = r(k; h_1(C)) = r(1; k\ h_1(C)),$$

and

$$k\ h_1(C) = s(a,b,c;1) = \frac{\Gamma(2-c)\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(1-a)\Gamma(1-b)}.$$

We can conclude that

$$k = \frac{C-\overline{A}}{C-A} \frac{\Gamma(2-c)\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(1-a)\Gamma(1-b)}.$$

In the second case, we obtain

$$k = \exp\left(\pi i(1-c)\right) \frac{\Gamma(b)\Gamma(c-a)\Gamma(2-c)}{\Gamma(c)\Gamma(b-c+1)\Gamma(1-a)}.$$

Both values of k are equal and we take $k_A = k$.

At this point, it would be natural to consider the adapted local parameter

$$q_A(z) = \left(k_A \frac{z-A}{z-\overline{A}}\right)^{e_A}$$

as an uniformizing variable in the neighbourhood of the point A. By doing this, we would obtain a series development

$$t(z) = \sum_{n=1}^{\infty} b_n q^n, \quad b_n := a_{ne}, \quad b_1 = 1.$$

In order to get series with integral coefficients, the parameters q_A will be modified in Section 7.

When $t(A) = 1$, we obtain Table 11 by proceeding along the same lines. □

7. Expansions at the elliptic points

Each of the uniformizing functions considered in the preceding sections will be developed at the neighbourhood of each of the elliptic vertices of the defining triangle.

Case $t(v) = 0$. We begin by studying those functions t which take the value zero at some elliptic point v. Let $q(z)$ be the local parameter adapted to a function t chosen in accordance with Table 10. First we consider developments of the shape

$$t(z) = \sum_{n=1}^{\infty} b'_n \frac{q(z)^n}{(en)!}, \quad b'_1 = e!$$

Next we renormalize the function q. We replace q by $\nu^{-1}q$, where the values of ν are listed in Table 10. Thus,

$$t(z) = \sum_{n=1}^{\infty} b''_n \frac{q(z)^n}{(en)!}, \quad b''_1 = \nu e!$$

Finally we define the factor $\mathfrak{n}_0 = \nu e!$ and normalize the generating function t by $j(v, q_v; z) := \mathfrak{n}_0^{-1} t(z)$ so that

$$j(v, q_v; z) = \sum_{n=1}^{\infty} c_n \frac{q_v(z)^n}{(en)!}, \quad c_1 = 1, \quad q_v(z) = \frac{1}{\nu_v}\left(k_v \frac{z - v}{z - \overline{v}}\right)^{e_v},$$

where the values of e_v, k_v are listed in Table 10.

We note that each generating function $j(v, q_v; z)$ is a representative of the homothety class of the corresponding function t in Table 8. These representatives depend on the point v. The relations between them will be compiled in Table 16.

Coefficients c_n $(1 \leq n \leq 10)$ of $j_1(v_2, q_{v_2}; z)$:

$$1 = 1$$
$$-345 = -3 \cdot 5 \cdot 23$$
$$240003 = 3^4 \cdot 2963$$
$$-286541145 = -3^4 \cdot 5 \cdot 11 \cdot 64319$$
$$531355048470 = 2 \cdot 3^5 \cdot 5 \cdot 218664629$$
$$-1431567508360320 = -2^7 \cdot 3^8 \cdot 5 \cdot 17 \cdot 20054549$$
$$5337775894717036800 = 2^8 \cdot 3^8 \cdot 5^2 \cdot 10733 \cdot 11843749$$
$$-26546056702161728244480 = -2^8 \cdot 3^9 \cdot 5 \cdot 11 \cdot 23 \cdot 4164647368009$$
$$171034212264597883762560000 = 2^{10} \cdot 3^{14} \cdot 5^4 \cdot 71 \cdot 787 \cdot 999936383$$
$$-1394346733163593859989651968000 = -2^{12} \cdot 3^{13} \cdot 5^3 \cdot 23 \cdot 29 \cdot 2560935717202529$$

•

Coefficients c_n $(1 \leq n \leq 10)$ of $j_1^+(v_3, q_{v_3}; z)$:

$$1 = 1$$
$$-92 = -2^2 \cdot 23$$
$$12454 = 2 \cdot 13 \cdot 479$$
$$-2230368 = -2^5 \cdot 3 \cdot 7 \cdot 3319$$
$$512222616 = 2^3 \cdot 3^5 \cdot 263489$$
$$-144878909472 = -2^5 \cdot 3^5 \cdot 11 \cdot 1693777$$
$$49442305079664 = 2^4 \cdot 3^6 \cdot 7 \cdot 605554393$$
$$-19925125158693888 = -2^{10} \cdot 3^8 \cdot 7 \cdot 29 \cdot 41 \cdot 349 \cdot 1021$$
$$9349543945456131456 = 2^7 \cdot 3^9 \cdot 13 \cdot 285460362413$$
$$-5039552099183446743552 = -2^9 \cdot 3^{11} \cdot 19 \cdot 174329 \cdot 16775093$$

•

Coefficients c_n $(1 \leq n \leq 10)$ of $j_2^+(v_{13}, q_{v_{13}}; z)$:

$$1 = 1$$
$$-44 = -2^2 \cdot 11$$
$$3070 = 2 \cdot 5 \cdot 307$$
$$-298592 = -2^5 \cdot 7 \cdot 31 \cdot 43$$
$$38370520 = 2^3 \cdot 5 \cdot 959263$$
$$-6253696160 = -2^5 \cdot 5 \cdot 17 \cdot 509 \cdot 4517$$
$$1253004761008 = 2^4 \cdot 7 \cdot 11^2 \cdot 13 \cdot 7112233$$
$$-301902712294400 = -2^{10} \cdot 5^2 \cdot 7 \cdot 1684724957$$
$$85866490414622080 = 2^7 \cdot 5 \cdot 23 \cdot 5833321359689$$
$$-28407641837085831680 = -2^9 \cdot 5 \cdot 1709 \cdot 8681 \cdot 747968657$$

•

Coefficients c_n $(1 \leq n \leq 10)$ of $j_3^+(v_{14}, q_{v_{14}}; z)$:

$$1 = 1$$
$$-52 = -2^2 \cdot 13$$
$$4480 = 2^7 \cdot 5 \cdot 7$$
$$-554304 = -2^6 \cdot 3 \cdot 2887$$
$$92257920 = 2^7 \cdot 3^3 \cdot 5 \cdot 19 \cdot 281$$
$$-19756154880 = -2^{16} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 29$$
$$5262440804352 = 2^{12} \cdot 3^4 \cdot 15861427$$
$$-1700564767948800 = -2^{14} \cdot 3^6 \cdot 5^2 \cdot 13 \cdot 307 \cdot 1427$$
$$653791044336353280 = 2^{24} \cdot 3^7 \cdot 5 \cdot 7 \cdot 17 \cdot 29947$$
$$-294289935697699799040 = -2^{17} \cdot 3^8 \cdot 5 \cdot 68442433349$$

•

Coefficients c_n $(1 \leq n \leq 10)$ of $j_4^+(v_{15}, q_{v_{15}}; z)$:

$$1 = 1$$
$$-12 = -2^2 \cdot 3$$
$$246 = 2 \cdot 3 \cdot 41$$
$$-7392 = -2^5 \cdot 3 \cdot 7 \cdot 11$$
$$302616 = 2^3 \cdot 3^4 \cdot 467$$
$$-16090272 = -2^5 \cdot 3^3 \cdot 11 \cdot 1693$$
$$1072529136 = 2^4 \cdot 3^2 \cdot 7 \cdot 1064017$$
$$-87266737152 = -2^1 0 \cdot 3^3 \cdot 7 \cdot 211 \cdot 2137$$
$$8490208669056 = 2^7 \cdot 3^4 \cdot 818885867$$
$$-971360853484032 = -2^9 \cdot 3^5 \cdot 11 \cdot 19 \cdot 1667 \cdot 22409$$

•

Coefficients c_n $(1 \leq n \leq 10)$ of $j_1^*(v_4, q_{v_4}; z)$:

$$1 = 1$$
$$-105 = -3 \cdot 5 \cdot 7$$
$$28323 = 3^3 \cdot 1049$$
$$-14750505 = -3^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 43$$
$$13052864070 = 2 \cdot 3^5 \cdot 5 \cdot 37 \cdot 145177$$
$$-17861621435280 = -2^4 \cdot 3^7 \cdot 5 \cdot 7 \cdot 17 \cdot 857897$$
$$35571783638602800 = 2^4 \cdot 3^8 \cdot 5^2 \cdot 13554253787$$
$$-98319566382392844720 = -2^4 \cdot 3^9 \cdot 5 \cdot 7 \cdot 11^2 \cdot 23 \cdot 29 \cdot 110522177$$
$$364025405491786199160000 = 2^6 \cdot 3^{12} \cdot 5^4 \cdot 17124450573619$$
$$-1753919978389416255755232000 = -2^8 \cdot 3^{13} \cdot 5^3 \cdot 7 \cdot 29 \cdot 64403 \cdot 2629549493$$

•

Coefficients c_n $(1 \leq n \leq 10)$ of $j_2^*(v_7, q_{v_7}; z)$:

$$
\begin{aligned}
1 &= 1 \\
-784 &= 2^4 \cdot 7^2 \\
2825856 &= 2^7 \cdot 3^2 \cdot 11 \cdot 223 \\
-28700872704 &= -2^{12} \cdot 3^2 \cdot 7^2 \cdot 15889 \\
651710959681536 &= 2^{15} \cdot 3^4 \cdot 19 \cdot 129223093 \\
-28556273929878503424 &= -2^{19} \cdot 3^4 \cdot 7^2 \cdot 11 \cdot 23 \cdot 31 \cdot 239 \cdot 7321 \\
2181991075583891305660416 &= 2^{22} \cdot 3^8 \cdot 7 \cdot 11327263088227 \\
-270314448732146703022575058944 &= -2^{28} \cdot 3^8 \cdot 7^2 \cdot 31 \cdot 1523 \cdot 5701 \cdot 11637257 \\
5132915362169491891909587977386496 &= 2^{31} \cdot 3^8 \cdot 7^2 \cdot 11 \cdot 107 \cdot 63167204434636859 \\
-1429600012074175595380791212254058407526 4 &= -2^{35} \cdot 3^8 \cdot 7^3 \cdot 19 \cdot 691 \cdot 14082147496175175569
\end{aligned}
$$

•

Coefficients c_n $(1 \leq n \leq 10)$ of $j_3^*(v_8, q_{v_8}; z)$:

$$
\begin{aligned}
1 \\
-323136 \\
1410262327296 \\
-3491898820964410982 4 \\
3242281034771640857552486400 \\
-8773249886209669675649594906640384 00 \\
58303131696560343863577707909261710684323 8400 \\
-84158287978943479920962523392339331281417526218588160 0 \\
2404459716373062611539035974816629561651569452652135123517440000 \\
-126414468440322296379225509113752480971576509566340091838088301091225600 00
\end{aligned}
$$

and their factorizations:

$$
\begin{aligned}
&1 \\
&-2^6 \cdot 3^3 \cdot 11 \cdot 17 \\
&2^{13} \cdot 3^6 \cdot 17 \cdot 29 \cdot 479 \\
&-2^{18} \cdot 3^8 \cdot 11 \cdot 23 \cdot 29 \cdot 31 \cdot 3881 \\
&2^{23} \cdot 3^{12} \cdot 5^2 \cdot 17 \cdot 29 \cdot 59009070139 \\
&-2^{31} \cdot 3^{15} \cdot 5^2 \cdot 11 \cdot 17 \cdot 53 \cdot 279137 \cdot 411658343 \\
&2^{37} \cdot 3^{17} \cdot 5^2 \cdot 41 \cdot 61 \cdot 257 \cdot 84523 \cdot 24185718087541 \\
&-2^{42} \cdot 3^{20} \cdot 5^2 \cdot 11 \cdot 17 \cdot 23 \cdot 29 \cdot 47 \cdot 211 \cdot 134857 \cdot 33656888643251 \\
&2^{47} \cdot 3^{24} \cdot 5^4 \cdot 17 \cdot 29 \cdot 31 \cdot 53 \cdot 2221 \cdot 538002801750197242414215 07 \\
&-2^{53} \cdot 3^{26} \cdot 5^4 \cdot 11 \cdot 29 \cdot 59 \cdot 5297 \cdot 142061 \cdot 62377338077857838099246320871
\end{aligned}
$$

Case $t(v) = 1$. Next we study those uniformizing functions t which take the value 1 at some elliptic point. Let $q(z)$ be the local parameter adapted to a function t chosen in accordance with Table 11. First we consider developments of the shape

$$t(z) = \sum_{n=0}^{\infty} b'_n \frac{q(z)^n}{(en)!}, \quad b'_1 = e!$$

Next we renormalize the function q. We replace q by $\nu^{-1}q$, where the values of ν are listed in Table 11. Thus,

$$t(z) = \sum_{n=0}^{\infty} b''_n \frac{q(z)^n}{(en)!}, \quad b''_1 = \nu e!$$

Finally we define the factor $\mathfrak{n}_1 = \nu e!$ and normalize the generating function t by $j(v, q_v; z) := \mathfrak{n}_1^{-1} t(z)$ so that

$$j(v, q_v; z) = \sum_{n=0}^{\infty} c_n \frac{q_v(z)^n}{(en)!}, \quad c_1 = 1, \quad q_v(z) = \frac{1}{\nu_v}\left(k_v \frac{z-v}{z-\overline{v}}\right)^{e_v},$$

where the values of e_v, k_v are listed in Table 11.

Coefficients c_n $(0 \le n \le 10)$ of $j_1(v_3, q_{v_3}; z)$:

$$\begin{aligned}
1/36 &= 2^{-2} \cdot 3^{-2}\\
1 &= 1\\
92 &= 2^2 \cdot 23\\
12454 &= 2 \cdot 13 \cdot 479\\
2230368 &= 2^5 \cdot 3 \cdot 7 \cdot 3319\\
512222616 &= 2^3 \cdot 3^5 \cdot 263489\\
144878909472 &= 2^5 \cdot 3^5 \cdot 11 \cdot 1693777\\
49442305079664 &= 2^4 \cdot 3^6 \cdot 7 \cdot 605554393\\
19925125158693888 &= 2^{10} \cdot 3^8 \cdot 7 \cdot 29 \cdot 41 \cdot 349 \cdot 1021\\
9349543945456131456 &= 2^7 \cdot 3^9 \cdot 13 \cdot 285460362413\\
5039552099183446743552 &= 2^9 \cdot 3^{11} \cdot 19 \cdot 174329 \cdot 16775093
\end{aligned}$$

•

Coefficients c_n $(0 \leq n \leq 10)$ of $j_2(v_7, q_{v_7}; z)$:

$$
\begin{aligned}
1/4 &= 2^{-2} \\
1 &= 1 \\
12 &= 2^2 \cdot 3 \\
246 &= 2 \cdot 3 \cdot 41 \\
7392 &= 2^5 \cdot 3 \cdot 7 \cdot 11 \\
302616 &= 2^3 \cdot 3^4 \cdot 467 \\
16090272 &= 2^5 \cdot 3^3 \cdot 11 \cdot 1693 \\
1072529136 &= 2^4 \cdot 3^2 \cdot 7 \cdot 1064017 \\
87266737152 &= 2^{10} \cdot 3^3 \cdot 7 \cdot 211 \cdot 2137 \\
8490208669056 &= 2^7 \cdot 3^4 \cdot 818885867 \\
971360853484032 &= 2^9 \cdot 3^5 \cdot 11 \cdot 19 \cdot 1667 \cdot 22409
\end{aligned}
$$

•

Coefficients c_n $(0 \leq n \leq 10)$ of $j_3(v_8, q_{v_8}; z)$:

$$
\begin{aligned}
1/12 &= 2^{-2} \cdot 3^{-1} \\
1 &= 1 \\
120 &= 2^3 \cdot 3 \cdot 5 \\
41472 &= 2^9 \cdot 3^4 \\
29652480 &= 2^9 \cdot 3^4 \cdot 5 \cdot 11 \cdot 13 \\
37408158720 &= 2^{10} \cdot 3^5 \cdot 5 \cdot 107 \cdot 281 \\
75362891857920 &= 2^{17} \cdot 3^8 \cdot 5 \cdot 17 \cdot 1031 \\
226060382778163200 &= 2^{19} \cdot 3^8 \cdot 5^2 \cdot 1399 \cdot 1879 \\
959160755899827486720 &= 2^{21} \cdot 3^9 \cdot 5 \cdot 11 \cdot 13 \cdot 23 \cdot 1412981 \\
5535414863241908060160000 &= 2^{22} \cdot 3^{14} \cdot 5^4 \cdot 373 \cdot 1183597 \\
42120765101624397070860288000 &= 2^{25} \cdot 3^{13} \cdot 5^3 \cdot 29 \cdot 217201108541
\end{aligned}
$$

•

Coefficients c_n $(0 \leq n \leq 10)$ of $j_1^+(v_4, q_{v_4}; z)$:

$$\begin{aligned}
1/48 &= 2^{-4} \cdot 3^{-1} \\
1 &= 1 \\
345 &= 3 \cdot 5 \cdot 23 \\
240003 &= 3^4 \cdot 2963 \\
286541145 &= 3^4 \cdot 5 \cdot 11 \cdot 64319 \\
531355048470 &= 2 \cdot 3^5 \cdot 5 \cdot 218664629 \\
1431567508360320 &= 2^7 \cdot 3^8 \cdot 5 \cdot 17 \cdot 20054549 \\
5337775894717036800 &= 2^8 \cdot 3^8 \cdot 5^2 \cdot 10733 \cdot 11843749 \\
26546056702161728244480 &= 2^8 \cdot 3^9 \cdot 5 \cdot 11 \cdot 23 \cdot 4164647368009 \\
171034212264597883762560000 &= 2^{10} \cdot 3^{14} \cdot 5^4 \cdot 71 \cdot 787 \cdot 999936383 \\
1394346733163593859989651968000 &= 2^{12} \cdot 3^{13} \cdot 5^3 \cdot 23 \cdot 29 \cdot 2560935717202529
\end{aligned}$$

•

Coefficients c_n $(0 \leq n \leq 10)$ of $j_2^+(v_7, q_{v_7}; z)$:

$$\begin{aligned}
1/24 &= 2^{-3} \cdot 3^{-1} \\
1 &= 1 \\
616 &= 2^3 \cdot 7 \cdot 11 \\
1340856 &= 2^3 \cdot 3^2 \cdot 11 \cdot 1693 \\
7272228096 &= 2^8 \cdot 3^2 \cdot 7 \cdot 211 \cdot 2137 \\
80946737790336 &= 2^7 \cdot 3^4 \cdot 11 \cdot 19 \cdot 1667 \cdot 22409 \\
1634351239998360576 &= 2^{10} \cdot 3^4 \cdot 7 \cdot 11 \cdot 23 \cdot 11126071849 \\
54908784316988465826816 &= 2^{10} \cdot 3^8 \cdot 7 \cdot 15493 \cdot 75359545969 \\
287981640619871309895735 7056 &= 2^{17} \cdot 3^8 \cdot 7 \cdot 11^2 \cdot 31 \cdot 503 \cdot 653 \cdot 388292111 \\
224454096766537769412039538999296 &= 2^{15} \cdot 3^8 \cdot 7^2 \cdot 11 \cdot 19369517332698577 45393 \\
2499792131027152649338931516557728 1536 &= 2^{18} \cdot 3^8 \cdot 7 \cdot 19 \cdot 149 \cdot 433 \cdot 58031 \cdot 29188251300945769
\end{aligned}$$

•

Coefficients c_n $(0 \leq n \leq 10)$ of $j_3^+(v_8, q_{v_8}; z)$:

$$1/720$$
$$1$$
$$247104$$
$$628024098816$$
$$7993006299165229056$$
$$35100637584686997559050240 0$$
$$42289377174585337023336621382041600$$
$$1195681539914982080774724983663993542410240 0$$
$$708171014923171486526862508390426456789348924784640 0$$
$$805859259188827031603148718810926174833537137496675057664000 0$$
$$164596430586129186089051644113836378310785317244031369204413654630400 00$$

and their factorizations:

$$2^{-4} \cdot 3^{-2} \cdot 5^{-1}$$
$$1$$
$$2^{6} \cdot 3^{3} \cdot 11 \cdot 13$$
$$2^{14} \cdot 3^{7} \cdot 17 \cdot 1031$$
$$2^{18} \cdot 3^{8} \cdot 11 \cdot 13 \cdot 23 \cdot 1412981$$
$$2^{22} \cdot 3^{12} \cdot 5^{2} \cdot 29 \cdot 217201108541$$
$$2^{32} \cdot 3^{16} \cdot 5^{2} \cdot 11 \cdot 13 \cdot 17 \cdot 774857 \cdot 4857187$$
$$2^{38} \cdot 3^{17} \cdot 5^{2} \cdot 41 \cdot 33589 \cdot 1161781 \cdot 8421106297$$
$$2^{42} \cdot 3^{20} \cdot 5^{2} \cdot 11 \cdot 13 \cdot 23 \cdot 47 \cdot 5171 \cdot 23108776762609624673$$
$$2^{46} \cdot 3^{2}6 \cdot 5^{4} \cdot 17 \cdot 53 \cdot 10264159 \cdot 779468551294552406313 1$$
$$2^{52} \cdot 3^{26} \cdot 5^{4} \cdot 11 \cdot 13 \cdot 29 \cdot 59 \cdot 365809361 \cdot 1025890823597 \cdot 25054514492881$$

•

Coefficients c_n $(0 \le n \le 10)$ of $j_1^*(v_{19}, q_{v_{19}}; z)$:

$$
\begin{aligned}
1/12 &= 2^{-2} \cdot 3^{-1} \\
1 &= 1 \\
105 &= 3 \cdot 5 \cdot 7 \\
28323 &= 3^3 \cdot 1049 \\
14750505 &= 3^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 43 \\
13052864070 &= 2 \cdot 3^5 \cdot 5 \cdot 37 \cdot 145177 \\
17861621435280 &= 2^4 \cdot 3^7 \cdot 5 \cdot 7 \cdot 17 \cdot 857897 \\
35571783638602800 &= 2^4 \cdot 3^8 \cdot 5^2 \cdot 13554253787 \\
98319566382392844720 &= 2^4 \cdot 3^9 \cdot 5 \cdot 7 \cdot 11^2 \cdot 23 \cdot 29 \cdot 110522177 \\
364025405491786199160000 &= 2^6 \cdot 3^{12} \cdot 5^4 \cdot 17124450573619 \\
1753919978389416255755232000 &= 2^8 \cdot 3^{13} \cdot 5^3 \cdot 7 \cdot 29 \cdot 64403 \cdot 2629549493
\end{aligned}
$$

•

Coefficients c_n $(0 \le n \le 10)$ of $j_2^*(v_{20}, q_{v_{20}}; z)$:

$$
\begin{aligned}
1/24 &= 2^{-3} \cdot 3^{-1} \\
1 &= 1 \\
784 &= 2^4 \cdot 7^2 \\
2825856 &= 2^7 \cdot 3^2 \cdot 11 \cdot 223 \\
28700872704 &= 2^{12} \cdot 3^2 \cdot 7^2 \cdot 15889 \\
651710959681536 &= 2^{15} \cdot 3^4 \cdot 19 \cdot 129223093 \\
2855627392987850342 4 &= 2^{19} \cdot 3^4 \cdot 7^2 \cdot 11 \cdot 23 \cdot 31 \cdot 239 \cdot 7321 \\
2181991075583891305660416 &= 2^{22} \cdot 3^8 \cdot 7 \cdot 11327263088227 \\
270314448732146703022575058944 &= 2^{28} \cdot 3^8 \cdot 7^2 \cdot 31 \cdot 1523 \cdot 5701 \cdot 11637257 \\
5132915362169491891909587977738649 6 &= 2^{31} \cdot 3^8 \cdot 7^2 \cdot 11 \cdot 107 \cdot 63167204434636859 \\
1429600012074175595380791212254058407526 4 &= 2^{35} \cdot 3^8 \cdot 7^3 \cdot 19 \cdot 691 \cdot 14082147496175175569
\end{aligned}
$$

•

Coefficients c_n $(0 \leq n \leq 10)$ of $j_3^*(v_{21}, q_{21}; z)$:

$$1/720$$
$$1$$
$$323136$$
$$1410262327296$$
$$3491898820964410 9824$$
$$32422810347716408575524864 00$$
$$877324988620966967564959490664038400$$
$$583031316965603438635777079092617106843238400$$
$$84158287978943479920962523392339331281417526218588 1600$$
$$24044597163730626115390359748166295616515694526521351235174400 00$$
$$12641446844032229637922550911375248097157650956634009183808830109122560000$$

and their factorizations:

$$2^{-4} \cdot 3^{-2} \cdot 5^{-1}$$
$$1$$
$$2^{6} \cdot 3^{3} \cdot 11 \cdot 17$$
$$2^{13} \cdot 3^{6} \cdot 17 \cdot 29 \cdot 479$$
$$2^{18} \cdot 3^{8} \cdot 11 \cdot 23 \cdot 29 \cdot 31 \cdot 3881$$
$$2^{23} \cdot 3^{12} \cdot 5^{2} \cdot 17 \cdot 29 \cdot 59009070139$$
$$2^{31} \cdot 3^{15} \cdot 5^{2} \cdot 11 \cdot 17 \cdot 53 \cdot 279137 \cdot 411658343$$
$$2^{37} \cdot 3^{17} \cdot 5^{2} \cdot 41 \cdot 61 \cdot 257 \cdot 84523 \cdot 24185718087541$$
$$2^{42} \cdot 3^{20} \cdot 5^{2} \cdot 11 \cdot 17 \cdot 23 \cdot 29 \cdot 47 \cdot 211 \cdot 134857 \cdot 33656888643251$$
$$2^{47} \cdot 3^{24} \cdot 5^{4} \cdot 17 \cdot 29 \cdot 31 \cdot 53 \cdot 2221 \cdot 538002801750197242414215 07$$
$$2^{53} \cdot 3^{26} \cdot 5^{4} \cdot 11 \cdot 29 \cdot 59 \cdot 5297 \cdot 142061 \cdot 62377338077857838099246320871$$

8. Local charts at the cusps

The general form of a local parameter $q(z)$ at a cusp v is

$$q(z) = \exp\left(\pi i \left(\frac{az+b}{cz+d}\right)\right), \quad q(v) = 0, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{GL}(2, \mathbb{C}).$$

But its invariance under the corresponding isotropy groups imposes several constraints that we will discuss now. We shall denote by G a matrix such that $\widetilde{\Gamma}_v = \langle \pm G \rangle$.

Γ	v	Γ_v	$h_{0,v}(z)$	$\log q_v(z)$
$\Gamma_0(N)$	v_1	T	$2z$	$\pi i(k+2z)$
$\Gamma_0^+(N)$	v_1	T	$2z$	$\pi i(k+2z)$
$\Gamma_0^+(N)^*$	v_1	T^2	z	$\pi i(k+z)$
$\Gamma_0(N)$	v_0	U^{-N}	$-\dfrac{2}{Nz}$	$\pi i(k-\frac{2}{Nz})$
$\Gamma_0(4)$	v_{17}	$-U^4T^{-1}$	$-\dfrac{1}{2z-1}$	$\pi i\left(k-\frac{1}{2\left(z-\frac{1}{2}\right)}\right)$
$\Gamma_0^+(4)$	v_{17}	W_4T^{-1}	$-\dfrac{2}{2z-1}$	$\pi i\left(k-\frac{1}{z-\frac{1}{2}}\right)$
$\Gamma_0^+(4)^*$	v_{17}	W_4T^{-1}	$-\dfrac{2}{2z-1}$	$\pi i\left(k-\frac{1}{z-\frac{1}{2}}\right)$
$\Gamma_0^+(4)^*$	v_{22}	TW_4T^{-2}	$-\dfrac{2}{2z-3}$	$\pi i\left(k-\frac{1}{z-\frac{3}{2}}\right)$

Table 12. Local parameters at the cusps

Case 1. $v = v_1 = i\infty$. Let us write $G = T^m = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$. Since $q(v_1) = 0$, we have $c = 0$, and we can take $q(z) = \exp(\pi i(az+b))$. Moreover, the equality

$$\exp(\pi i(az+b)) = q(z) = q(T^m \cdot z) = \exp(\pi i(az+am+b))$$

tells us that $\exp(\pi iam) = 1$ and, therefore, $am \in 2\mathbb{Z}$. If $a = 2n/m$ with $n \in \mathbb{Z}$, $n \geq 2$, then $q\left(z + \dfrac{m}{n}\right) = q(z)$ and $\begin{bmatrix} 1 & m/n \\ 0 & 1 \end{bmatrix}$ would be in $\widetilde{\Gamma}_v$. But this is not possible, and we must have $a = \pm 2/m$. Since we want $q(z)$ to be directly conformal, we take $q(z) = \exp\left(\pi i\left(\frac{2z}{m}+b\right)\right)$. We shall rewrite this expression as $q(z) = \exp(\pi i(k + h_0(z)))$, where $h_0(z) := \frac{2z}{m}$ and $k \in \mathbb{C}$. As in the case of the elliptic points, it is worth noting that an indeterminacy remains, namely, the constant k.

Case 2. $v \in \mathbb{R}$. Let us write $G = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Since $\dfrac{Av+B}{Cv+D} = v$, we have $Cv^2 + (D-A)v - B = 0$, and v equals the double root of the polynomial $Cz^2 + (D-A)z - B$. Thus

$$Cz^2 + (D-A)z - B = C(z-v)^2, \quad B - vD = (Cv - A)v.$$

Since $q(v) = 0$, we have $cv + d = 0$, and we can take

$$q(z) := \exp\left(\pi i \left(a + \frac{z}{z-v}\right)\right).$$

Now

$$q(G \cdot z) = q(z) \Leftrightarrow \frac{b}{G \cdot z - v} - \frac{b}{z-v} \in 2\mathbb{Z}.$$

The term in the right hand side equals $b\dfrac{C}{A - vC}$. The invariance under G implies that $b \in \dfrac{2(A-vC)}{C}\mathbb{Z}$. As in Case 1, we deduce that $b = \pm\dfrac{2(A-vC)}{C}$. We define $q(z) := \exp\left(\pi i \left(a \pm \frac{2(A-vC)}{C(z-v)}\right)\right)$. In each case, the sign will be chosen so that $q(z)$ becomes a directly conformal mapping. We shall rewrite the final expression as

$$q(z) = \exp\left(\pi i (k + h_0(z))\right),$$

where the functions h_0 are listed in Table 12. Here also an indeterminacy remains, namely, the constant k.

Summarizing the results obtained we state the following definition.

Definition 8.1. A local parameter at a cusp v for the Γ_v-action is any function

$$q(z) = \exp\left(\pi i (k + h_0(z))\right)$$

defined in accordance with Table 12. The entries in the third column of that table denote a matrix whose class generates the isotropy group Γ_v at the point v located in the second column. Here, $k \in \mathbb{C}$ is a constant. The local parameter $q(z)$ is said to be adapted to a function $t = \sum_{n=m}^{\infty} b_n q^n$ when, moreover, $b_1 = 1$ if $m \geq 0$ and $b_{-1} = 1$ otherwise.

In order to obtain local parameters at the cusps adapted to our functions, we review in Theorem 8.2 a result due to Carathéodory [**6**]. An account of it can be found in [**27**].

Theorem 8.2. *Let $A, B, C \in \overline{\mathcal{H}}$ define a triangle $[A, B, C]$ with internal angles 0, $\frac{\pi}{r}$, $\frac{\pi}{s}$, where $r, s \in \mathbb{N} \cup \{\infty\}$. Suppose that $A = i\infty$, $\mathrm{Re}(B) = 0$, $\mathrm{Re}(C) = 1$. Let $u\colon \mathcal{H} \to \mathbf{P}^1(\mathbb{C})$ be the meromorphic function that maps the interior of the triangle biholomorphically on $\mathcal{H}$ and satisfies*

$$u(A) = 0, \quad u(B) = 1, \quad u(C) = \infty.$$

Then, in a neighbourhood of A, we have $u(z) = \sum_{n=1}^{\infty} a_n \exp(\pi i n z)$, where

$$\log(a_1) = \log(b) + \log(d) - \frac{1}{2}\sum_{n=1}^{b-1} \cos\left(\frac{2an\pi}{b}\right) \log\left(2 - 2\cos\left(\frac{2n\pi}{b}\right)\right) - \frac{1}{2}\sum_{n=1}^{d-1} \cos\left(\frac{2cn\pi}{d}\right) \log\left(2 - 2\cos\left(\frac{2n\pi}{d}\right)\right).$$

Here

$$\frac{a}{b} := \frac{1}{2}\left(1 + \frac{1}{r} - \frac{1}{s}\right), \quad \frac{c}{d} := \frac{1}{2}\left(1 + \frac{1}{r} + \frac{1}{s}\right)$$

be reduced fractions.

Theorem 8.3. *Let t be one of the triangle functions defined in Definition 4.2. Suppose that it is obtained from a hyperbolic triangle $[A_1,A_2,A_3]$ of internal angles $\alpha_1\pi$, $\alpha_2\pi$, $\alpha_3\pi$, with $\alpha_1 = 0$, and that it takes the values*

$$t(A_1) = 0, \quad t(A_2) = 1, \quad t(A_3) = \infty;$$

respectively $t(A_1) = 1$, $t(A_2) = \infty$, $t(A_3) = 0$; respectively $t(A_1) = \infty$, $t(A_2) = 0$, $t(A_3) = 1$. Then the constant k_{A_1} defining a local parameter adapted to t is listed in Table 13; respectively in Table 14; respectively in Table 15.

Proof: In this proof we keep the notation given in Theorem 8.2.

Case $t(A_1) = 0$. By formal integration of the differential equation in Theorem 5.6, and taking into account that $t(A_1) = 0$, it follows that there exists a normalized power series and a constant $k \in \mathbb{C}$, such that

$$t(z) = \sum_{n=1}^{\infty} b_n q^n(z), \quad b_1 = 1, \quad q(z) = \exp\left(\pi i(k + h_0(z))\right),$$

for any z in a neighbourhood of A_1. Here we take $h_0(z)$ in accordance with Table 12. To calculate the value of k, we consider an homographic transformation h mapping the triangle $[A_1, A_2, A_3]$ to a triangle $[A, B, C]$ fulfilling the conditions in Theorem 8.2. An easy computation shows that $h(z) = 1 + h_0(z)$. Since

$$t(A_1) = u(A) = 0, \quad t(A_2) = u(B) = 1, \quad t(A_3) = u(C) = \infty,$$

we shall have $u(h(z)) = t(z)$, and the value of the constant k will be obtained from the equality

$$u(h(z)) = \sum_{n=1}^{\infty} a_n \exp(\pi i n h(z)) = \sum_{n=1}^{\infty} b_n q(z)^n, \quad q(z) = \exp(\pi i (k + h_0(z))).$$

It turns out that $k = (\log(a_1) + \pi i)/(\pi i)$. The corresponding values of k are listed in Table 13.

Case $t(A_1) = 1$. Now it follows that there exists a normalized power series and a constant $k \in \mathbb{C}$, such that

$$t(z) = \sum_{n=0}^{\infty} b_n q^n(z), \quad b_1 = 1, \quad q(z) = \exp(\pi i (k + h_0(z))),$$

for any z in a neighbourhood of A_1. To calculate the value of k, we consider the homographic transformation $h_0(z)$, which maps the triangle $[A_1, A_2, A_3]$ to a triangle $[A, B, C]$ fulfilling the conditions in Theorem 8.2. Since

$$\left(1 - \frac{1}{t}\right)(A_1) = 0, \quad \left(1 - \frac{1}{t}\right)(A_2) = 1, \quad \left(1 - \frac{1}{t}\right)(A_3) = \infty,$$

we shall have $u(h_0(z)) = (1 - 1/t)(z)$ and the value of the constant k will be obtained from the equality

$$\sum_{n=1}^{\infty} a_n \exp(\pi i n h(z)) = \left(1 - \frac{1}{t}\right)(z) = \sum_{n=1}^{\infty} b'_n q^n(z).$$

But $b'_1 = b_1$. It turns out that $k = \log(a_1)/(\pi i)$. The corresponding values of k are listed in Table 14.

Case $t(A_1) = \infty$. In this case, there exists a normalized power series and a constant $k \in \mathbb{C}$, such that

$$t(z) = \frac{1}{q} + \sum_{n=0}^{\infty} b_n q^n(z), \quad q(z) = \exp\left(\pi i(k + h_0(z))\right),$$

for any z in a neighbourhood of A_1. To calculate the value of k, we define the homographic transformations $h(z) = 1 + h_0(z)$ for $t = t_1$; $h(z) = h_0(z)$ for $t = t_N$, $2 \leq N \leq 4$, and for $t = t_N^+$, $1 \leq N \leq 4$; $h(z) = -\frac{1}{2} + h_0(z)$ for $t = t_N^*$, $1 \leq N \leq 4$. They map the respective triangles $[A_1, A_2, A_3]$ to triangles $[A, B, C]$ fulfilling the conditions in Theorem 8.2. Now we use the functions $1/(1-t)$, instead of t or $1 - 1/t$ in the preceding cases, to obtain the values of k. They are listed in Table 15.

t	v	$[A_1, A_2, A_3]$	$\frac{a}{b}, \frac{c}{d}$	ν_v	$\pi i\, k_v$
t_2	v_0	$[v_0, v_7, v_1]$	$\frac{3}{4}, \frac{3}{4}$	2^6	$6\log(2) + \pi i$
t_3	v_0	$[v_0, v_8, v_1]$	$\frac{2}{3}, \frac{2}{3}$	3^3	$3\log(3) + \pi i$
t_4	v_0	$[v_0, v_{17}, v_1]$	$\frac{1}{2}, \frac{1}{2}$	2^4	$4\log(2) + \pi i$
t_4^*	v_{17}	$[v_{17}, v_{22}, v_1]$	$\frac{1}{2}, \frac{1}{2}$	2^4	$4\log(2) + \pi i$

Table 13. Local constants at the cusps v such that $t(v) = 0$

t	v	$[A_1, A_2, A_3]$	$\frac{a}{b}, \frac{c}{d}$	ν_v	$\pi i\, k_v$
t_4	v_{17}	$[v_{17}, v_1, v_0]$	$\frac{1}{2}, \frac{1}{2}$	2^4	$4\log(2)$
t_4^+	v_{17}	$[v_{17}, v_1, v_{15}]$	$\frac{1}{4}, \frac{3}{4}$	2^6	$6\log(2)$
t_4^*	v_{22}	$[v_{22}, v_1, v_{17}]$	$\frac{1}{2}, \frac{1}{2}$	2^4	$4\log(2)$

Table 14. Local constants at the cusps v such that $t(v) = 1$

t	v	$[A_1, A_2, A_3]$	$\frac{a}{b}, \frac{c}{d}$	ν_v^{-1}	$\pi i\, k_v$
t_1	v_1	$[v_1, v_2, v_3]$	$\frac{5}{12}, \frac{11}{12}$	$2^6 \cdot 3^3$	$3\log(12)$
t_2	v_1	$[v_1, v_0, v_7]$	$\frac{1}{4}, \frac{3}{4}$	2^6	$6\log(2) + \pi i$
t_3	v_1	$[v_1, v_0, v_8]$	$\frac{1}{3}, \frac{2}{3}$	3^3	$3\log(3) + \pi i$
t_4	v_1	$[v_1, v_0, v_{17}]$	$\frac{1}{2}, \frac{1}{2}$	2^4	$4\log(2) + \pi i$
t_1^+	v_1	$[v_1, v_3, v_4]$	$\frac{7}{12}, \frac{11}{12}$	$2^6 \cdot 3^3$	$3\log(12) + \pi i$
t_2^+	v_1	$[v_1, v_{13}, v_7]$	$\frac{5}{8}, \frac{7}{8}$	2^8	$8\log(2) + \pi i$
t_3^+	v_1	$[v_1, v_{14}, v_8]$	$\frac{2}{3}, \frac{5}{6}$	$2^2 \cdot 3^3$	$\log(108) + \pi i$
t_4^+	v_1	$[v_1, v_{15}, v_{17}]$	$\frac{3}{4}, \frac{3}{4}$	2^6	$6\log(2) + \pi i$
t_1^*	v_1	$[v_1, v_4, v_{19}]$	$\frac{1}{2}, \frac{5}{6}$	$2^4 \cdot 3^2$	$\log(48\sqrt{3}) + \dfrac{\pi i}{2}$
t_2^*	v_1	$[v_1, v_7, v_{20}]$	$\frac{1}{2}, \frac{3}{4}$	2^5	$5\log(2) + \dfrac{\pi i}{2}$
t_3^*	v_1	$[v_1, v_8, v_{21}]$	$\frac{1}{2}, \frac{2}{3}$	$2^2 \cdot 3^2$	$\log(12\sqrt{3}) + \dfrac{\pi i}{2}$
t_4^*	v_1	$[v_1, v_{17}, v_{22}]$	$\frac{1}{2}, \frac{1}{2}$	2^4	$4\log(2) + \dfrac{\pi i}{2}$

TABLE 15. Local constants at the cusps v such that $t(v) = \infty$

9. Expansions at the cusps

Each of the uniformizing functions considered in the preceding sections will be developed in the neighbourhood of each of the cusps of their defining triangle.

Case $t(v) = 0$. We begin by studying those uniformizing functions t which take the value 0 at some cusp. Let q be the adapted local parameter chosen in accordance with Table 13. First we consider developments of the shape

$$t(z) = \sum_{n=1}^{\infty} b_n q(z)^n, \quad b_1 = 1.$$

Next we renormalize the function q. We replace q by $\nu^{-1}q$, where the values of ν are listed in Table 13. Thus,

$$t(z) = \sum_{n=1}^{\infty} b'_n q(z)^n, \quad b'_1 = \nu.$$

Finally we define the factor $\mathfrak{n}_0 = \nu$ and normalize the generating function t by $j(v, q_v; z) := \mathfrak{n}_0^{-1} t(z)$ so that

$$j(v, q_v; z) = \sum_{n=1}^{\infty} c_n q_v(z)^n, \quad c_1 = 1, \quad q_v(z) = \frac{1}{\nu_v} \exp\left(\pi i (k_v + h_{0,v}(z))\right).$$

Coefficients c_n $(1 \leq n \leq 10)$ of $j_2(v_0, q_{v_0}; z)$:

$$1 = 1$$
$$-24 = -2^3 \cdot 3$$
$$300 = 2^2 \cdot 3 \cdot 5^2$$
$$-2624 = -2^6 \cdot 41$$
$$18126 = 2 \cdot 3^2 \cdot 19 \cdot 53$$
$$-105504 = -2^5 \cdot 3 \cdot 7 \cdot 157$$
$$538296 = 2^3 \cdot 3 \cdot 11 \cdot 2039$$
$$-2471424 = -2^9 \cdot 3 \cdot 1609$$
$$10400997 = 3 \cdot 659 \cdot 5261$$
$$-40674128 = -2^4 \cdot 11 \cdot 59 \cdot 3917$$

•

Coefficients c_n $(1 \leq n \leq 10)$ of $j_3(v_0, q_{v_0}; z)$:

$$1 = 1$$
$$-12 = -2^2 \cdot 3$$
$$90 = 2 \cdot 3^2 \cdot 5$$
$$-508 = -2^2 \cdot 127$$
$$2391 = 3 \cdot 797$$
$$-9828 = -2^2 \cdot 3^3 \cdot 7 \cdot 13$$
$$36428 = 2^2 \cdot 7 \cdot 1301$$
$$-124188 = -2^2 \cdot 3 \cdot 79 \cdot 131$$
$$395199 = 3^4 \cdot 7 \cdot 17 \cdot 41$$
$$-1186344 = -2^3 \cdot 3^2 \cdot 16477$$

•

Coefficients c_n $(1 \leq n \leq 10)$ of $j_4(v_0, q_{v_0}; z)$:

$$1 = 1$$
$$-8 = -2^3$$
$$44 = 2^2 \cdot 11$$
$$-192 = -2^6 \cdot 3$$
$$718 = 2 \cdot 359$$
$$-2400 = -2^5 \cdot 3 \cdot 5^2$$
$$7352 = 2^3 \cdot 919$$
$$-20992 = -2^9 \cdot 41$$
$$56549 = 193 \cdot 293$$
$$-145008 = -2^4 \cdot 3^2 \cdot 19 \cdot 53$$

•

Coefficients c_n $(1 \leq n \leq 10)$ of $j_4^*(v_{17}, q_{v_{17}}; z)$. These coefficients coincide with those of $j_4(v_0, q_{v_0}; z)$, already computed.

Case $t(v) = 1$. Next we study those uniformizing functions t which take the value 1 at some cusp. Let q be the adapted local parameter chosen in accordance with Table 14. First we consider developments of the shape

$$t(z) = \sum_{n=0}^{\infty} b_n q(z)^n, \quad b_1 = 1.$$

Next we renormalize the function q. We replace q by $\nu^{-1}q$, where the values of ν are listed in Table 14. Thus,

$$t(z) = \sum_{n=0}^{\infty} b'_n q(z)^n, \quad b'_1 = \nu.$$

Finally we define $\mathfrak{n}_1 = \nu$ and normalize the generating function t by $j(v, q_v; z) := \mathfrak{n}_1^{-1} t(z)$ so that

$$j(v, q_v; z) = \sum_{n=0}^{\infty} c_n q_v(z)^n, \quad c_1 = 1, \quad q_v(z) = \frac{1}{\nu_v} \exp\left(\pi i (k_v + h_{0,v}(z))\right).$$

Coefficients c_n $(0 \leq n \leq 10)$ of $j_4(v_{17}, q_{v_{17}}; z)$:

$$1/16 = 2^{-4}$$
$$1 = 1$$
$$8 = 2^3$$
$$44 = 2^2 \cdot 11$$
$$192 = 2^6 \cdot 3$$
$$718 = 2 \cdot 359$$
$$2400 = 2^5 \cdot 3 \cdot 5^2$$
$$7352 = 2^3 \cdot 919$$
$$20992 = 2^9 \cdot 41$$
$$56549 = 193 \cdot 293$$
$$145008 = 2^4 \cdot 3^2 \cdot 19 \cdot 53$$

•

Coefficients c_n $(0 \leq n \leq 10)$ of $j_4^+(v_{17}, q_{v_{17}}; z)$:

$$1/64 = 2^{-6}$$
$$1 = 1$$
$$24 = 2^3 \cdot 3$$
$$300 = 2^2 \cdot 3 \cdot 5^2$$
$$2624 = 2^6 \cdot 41$$
$$18126 = 2 \cdot 3^2 \cdot 19 \cdot 53$$
$$105504 = 2^5 \cdot 3 \cdot 7 \cdot 157$$
$$538296 = 2^3 \cdot 3 \cdot 11 \cdot 2039$$
$$2471424 = 2^9 \cdot 3 \cdot 1609$$
$$10400997 = 3 \cdot 659 \cdot 5261$$
$$40674128 = 2^4 \cdot 11 \cdot 59 \cdot 3917$$

•

Coefficients c_n $(0 \leq n \leq 10)$ of $j_4^*(v_{22}, q_{v_{22}}; z)$. These coefficients coincide with those of $j_4(v_{17}, q_{v_{17}}; z)$, already computed.

Case $t(v) = \infty$. Finally we study the uniformizing functions t at the cusp $v_0 = i\infty$, where all of them have a pole. Let q be the adapted local parameter chosen in accordance with Table 15. First we consider developments of the shape

$$t(z) = \frac{1}{q(z)} + \sum_{n=0}^{\infty} b_n q(z)^n.$$

Next we renormalize the function q. We replace q by $\nu^{-1}q$, where the values of ν^{-1} are listed in Table 15. Thus,

$$t(z) = \frac{\nu}{q(z)} + \sum_{n=0}^{\infty} b'_n q(z)^n.$$

Finally we define the factor $\mathfrak{n}_\infty = \nu$ and normalize the generating function t by $j(v,q;z) := \mathfrak{n}_\infty^{-1} t(z)$ so that

$$j(v,q_v;z) = \frac{1}{q_v(z)} + \sum_{n=0}^{\infty} c_n q_v(z)^n, \quad q_v(z) = \frac{1}{\nu_v} \exp\left(\pi i(k_v + h_{0,v}(z))\right).$$

Coefficients c_n $(-1 \le n \le 10)$ of $j_1(v_1, q_{v_1}; z)$:

$$\begin{aligned}
1 &= 1 \\
744 &= 2^3 \cdot 3 \cdot 31 \\
196884 &= 2^2 \cdot 3^3 \cdot 1823 \\
21493760 &= 2^{11} \cdot 5 \cdot 2099 \\
864299970 &= 2 \cdot 3^5 \cdot 5 \cdot 355679 \\
20245856256 &= 2^{14} \cdot 3^3 \cdot 45767 \\
333202640600 &= 2^3 \cdot 5^2 \cdot 2143 \cdot 777421 \\
4252023300096 &= 2^{13} \cdot 3^6 \cdot 11 \cdot 13^2 \cdot 383 \\
44656994071935 &= 3^3 \cdot 5 \cdot 7 \cdot 271 \cdot 174376673 \\
401490886656000 &= 2^{17} \cdot 3 \cdot 5^3 \cdot 199 \cdot 41047 \\
3176440229784420 &= 2^2 \cdot 3^7 \cdot 5 \cdot 4723 \cdot 15376021 \\
22567393309593600 &= 2^{12} \cdot 3^5 \cdot 5^2 \cdot 13^2 \cdot 5366467
\end{aligned}$$

•

Coefficients c_n $(-1 \leq n \leq 10)$ of $j_2(v_1, q_{v_1}; z)$:

$$
\begin{aligned}
1 &= 1 \\
24 &= 2^3 \cdot 3 \\
276 &= 2^2 \cdot 3 \cdot 23 \\
2048 &= 2^{11} \\
11202 &= 2 \cdot 3 \cdot 1867 \\
49152 &= 2^{14} \cdot 3 \\
184024 &= 2^3 \cdot 23003 \\
614400 &= 2^{13} \cdot 3 \cdot 5^2 \\
1881471 &= 3 \cdot 337 \cdot 1861 \\
5373952 &= 2^{17} \cdot 41 \\
14478180 &= 2^2 \cdot 3 \cdot 5 \cdot 241303 \\
37122048 &= 2^{12} \cdot 3^2 \cdot 19 \cdot 53
\end{aligned}
$$

•

Coefficients c_n $(-1 \leq n \leq 10)$ of $j_3(v_1, q_{v_1}; z)$:

$$
\begin{aligned}
1 &= 1 \\
12 &= 2^2 \cdot 3 \\
54 &= 2 \cdot 3^3 \\
76 &= 2^2 \cdot 19 \\
-243 &= -3^5 \\
-1188 &= -2^2 \cdot 3^3 \cdot 11 \\
-1384 &= -2^3 \cdot 173 \\
2916 &= 2^2 \cdot 3^6 \\
11934 &= 2 \cdot 3^3 \cdot 13 \cdot 17 \\
11580 &= 2^2 \cdot 3 \cdot 5 \cdot 193 \\
-21870 &= -2 \cdot 3^7 \cdot 5 \\
-79704 &= -2^3 \cdot 3^5 \cdot 41
\end{aligned}
$$

•

Coefficients c_n $(-1 \leq n \leq 10)$ of $j_4(v_1, q_{v_1}; z)$:

$$1 = 1$$
$$8 = 2^3$$
$$20 = 2^2 \cdot 5$$
$$0 = 0$$
$$-62 = -2 \cdot 31$$
$$0 = 0$$
$$216 = 2^3 \cdot 3^3$$
$$0 = 0$$
$$-641 = -641$$
$$0 = 0$$
$$1636 = 2^2 \cdot 409$$
$$0 = 0$$

•

Coefficients c_n $(-1 \leq n \leq 10)$ of $j_1^+(v_1, q_{v_1}; z)$:

$$1 = 1$$
$$984 = 2^3 \cdot 3 \cdot 41$$
$$196884 = 2^2 \cdot 3^3 \cdot 1823$$
$$-21493760 = -2^{11} \cdot 5 \cdot 2099$$
$$864299970 = 2 \cdot 3^5 \cdot 5 \cdot 355679$$
$$-20245856256 = -2^{14} \cdot 3^3 \cdot 45767$$
$$333202640600 = 2^3 \cdot 5^2 \cdot 2143 \cdot 777421$$
$$-4252023300096 = -2^{13} \cdot 3^6 \cdot 11 \cdot 13^2 \cdot 383$$
$$44656994071935 = 3^3 \cdot 5 \cdot 7 \cdot 271 \cdot 174376673$$
$$-401490886656000 = -2^{17} \cdot 3 \cdot 5^3 \cdot 199 \cdot 41047$$
$$3176440229784420 = 2^2 \cdot 3^7 \cdot 5 \cdot 4723 \cdot 15376021$$
$$-22567393309593600 = -2^{12} \cdot 3^5 \cdot 5^2 \cdot 13^2 \cdot 5366467$$

•

Coefficients c_n $(-1 \leq n \leq 10)$ of $j_2^+(v_1, q_{v_1}; z)$:

$$
\begin{aligned}
1 &= 1\\
152 &= 2^3 \cdot 19\\
4372 &= 2^2 \cdot 1093\\
-96256 &= -2^{11} \cdot 47\\
1240002 &= 2 \cdot 3^3 \cdot 22963\\
-10698752 &= -2^{14} \cdot 653\\
74428120 &= 2^3 \cdot 5 \cdot 13 \cdot 41 \cdot 3491\\
-431529984 &= -2^{13} \cdot 33 \cdot 1951\\
2206741887 &= 3^4 \cdot 7 \cdot 1801 \cdot 2161\\
-10117578752 &= -2^{17} \cdot 77191\\
42616961892 &= 2^2 \cdot 3^5 \cdot 59 \cdot 743129\\
-166564106240 &= -2^{12} \cdot 5 \cdot 7 \cdot 1063 \cdot 1093
\end{aligned}
$$

•

Coefficients c_n $(-1 \leq n \leq 10)$ of $j_3^+(v_1, q_{v_1}; z)$:

$$
\begin{aligned}
1 &= 1\\
66 &= 2 \cdot 3 \cdot 11\\
783 &= 3^3 \cdot 29\\
-8672 &= -2^5 \cdot 271\\
65367 &= 3^5 \cdot 269\\
-371520 &= -2^6 \cdot 3^3 \cdot 5 \cdot 43\\
1741655 &= 5 \cdot 163 \cdot 2137\\
-7161696 &= -2^5 \cdot 3^6 \cdot 307\\
26567946 &= 2 \cdot 3^3 \cdot 53 \cdot 9283\\
-90521472 &= -2^7 \cdot 3 \cdot 19^2 \cdot 653\\
288078201 &= 3^7 \cdot 157 \cdot 839\\
-864924480 &= -2^6 \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 227
\end{aligned}
$$

•

Coefficients c_n $(-1 \leq n \leq 10)$ of $j_4^+(v_1, q_{v_1}; z)$:

$$
\begin{aligned}
1 &= 1 \\
40 &= 2^3 \cdot 5 \\
276 &= 2^2 \cdot 3 \cdot 23 \\
-2048 &= -2^{11} \\
11202 &= 2 \cdot 3 \cdot 1867 \\
-49152 &= -2^{14} \cdot 3 \\
184024 &= 2^3 \cdot 23003 \\
-614400 &= -2^{13} \cdot 3 \cdot 5^2 \\
1881471 &= 3 \cdot 337 \cdot 1861 \\
-5373952 &= -2^{17} \cdot 41 \\
14478180 &= 2^2 \cdot 3 \cdot 5 \cdot 241303 \\
-37122048 &= -2^{12} \cdot 3^2 \cdot 19 \cdot 53
\end{aligned}
$$

•

Coefficients c_n $(-1 \leq n \leq 10)$ of $j_1^*(v_1, q_{v_1}; z)$:

$$
\begin{aligned}
1 &= 1 \\
72 &= 2^3 \cdot 3^2 \\
1476 &= 2^2 \cdot 3^2 \cdot 41 \\
0 &= 0 \\
-203310 &= -2 \cdot 3^4 \cdot 5 \cdot 251 \\
0 &= 0 \\
9919800 &= 2^3 \cdot 3^3 \cdot 5^2 \cdot 11 \cdot 167 \\
0 &= 0 \\
-304954065 &= -3^5 \cdot 5 \cdot 13 \cdot 43 \cdot 449 \\
0 &= 0 \\
7035202836 &= 2^2 \cdot 3^8 \cdot 268069 \\
0 &= 0
\end{aligned}
$$

•

Coefficients c_n $(-1 \leq n \leq 10)$ of $j_2^*(v_1, q_{v_1}; z)$:

$$\begin{aligned} 1 &= 1 \\ 16 &= 2^4 \\ 76 &= 2^2 \cdot 19 \\ 0 &= 0 \\ -702 &= -2 \cdot 3^3 \cdot 13 \\ 0 &= 0 \\ 5224 &= 2^3 \cdot 653 \\ 0 &= 0 \\ -23425 &= -5^2 \cdot 937 \\ 0 &= 0 \\ 98172 &= 2^2 \cdot 3^5 \cdot 101 \\ 0 &= 0 \end{aligned}$$

•

Coefficients c_n $(-1 \leq n \leq 10)$ of $j_3^*(v_1, q_{v_1}; z)$:

$$\begin{aligned} 1 &= 1 \\ 18 &= 2 \cdot 3^2 \\ 99 &= 3^2 \cdot 11 \\ 0 &= 0 \\ -1377 &= -3^4 \cdot 17 \\ 0 &= 0 \\ 19251 &= 3^3 \cdot 23 \cdot 31 \\ 0 &= 0 \\ -206550 &= -2 \cdot 3^5 \cdot 5^2 \cdot 17 \\ 0 &= 0 \\ 1817397 &= 3^8 \cdot 277 \\ 0 &= 0 \end{aligned}$$

•

Coefficients c_n $(-1 \leq n \leq 10)$ of $j_4^*(v_1, q_{v_1}; z)$. These coefficients coincide with those of $j_4(v_1, q_{v_1}; z)$, already computed.

Table 16 lists the local triangle functions together with their normalizing factors. We conclude by observing that our local function $j_1(v_1, q_{v_1})$ recovers the classical elliptic function:

$$j_1(i\infty, \exp(2\pi i z); z) = j(z),$$

and that $\mathfrak{n}_\infty^{-1} = 1728$.

Γ	t	$\mathfrak{n}_\infty^{-1}$	$t(v)=\infty$	$\mathfrak{n}_0$	$t(v)=0$	$\mathfrak{n}_1$	$t(v)=1$
$\Gamma_0(1)$	t_1	1728	$j_1(v_1,q_{v_1})$	48	$j_1(v_2,q_{v_2})$	36	$j_1(v_3,q_{v_3})$
$\Gamma_0(2)$	t_2	64	$j_2(v_1,q_{v_1})$	64	$j_2(v_0,q_{v_0})$	4	$j_2(v_7,q_{v_7})$
$\Gamma_0(3)$	t_3	27	$j_3(v_1,q_{v_1})$	27	$j_3(v_0,q_{v_0})$	12	$j_3(v_8,q_{v_8})$
$\Gamma_0(4)$	t_4	16	$j_4(v_1,q_{v_1})$	16	$j_4(v_0,q_{v_0})$	16	$j_4(v_{17},q_{v_{17}})$
$\Gamma_0^+(1)$	t_1^+	1728	$j_1^+(v_1,q_{v_1})$	36	$j_1^+(v_3,q_{v_3})$	48	$j_1^+(v_4,q_{v_4})$
$\Gamma_0^+(2)$	t_2^+	256	$j_2^+(v_1,q_{v_1})$	16	$j_2^+(v_{13},q_{v_{13}})$	24	$j_2^+(v_7,q_{v_7})$
$\Gamma_0^+(3)$	t_3^+	108	$j_3^+(v_1,q_{v_1})$	18	$j_3^+(v_{14},q_{v_{14}})$	720	$j_3^+(v_8,q_{v_8})$
$\Gamma_0^+(4)$	t_4^+	64	$j_4^+(v_1,q_{v_1})$	4	$j_4^+(v_{15},q_{v_{15}})$	64	$j_4^+(v_{17},q_{v_{17}})$
$\Gamma_0(1)^*$	t_1^*	144	$j_1^*(v_1,q_{v_1})$	12	$j_1^*(v_4,q_{v_4})$	12	$j_1^*(v_{19},q_{v_{19}})$
$\Gamma_0(2)^*$	t_2^*	32	$j_2^*(v_1,q_{v_1})$	24	$j_2^*(v_7,q_{v_7})$	24	$j_2^*(v_{20},q_{v_{20}})$
$\Gamma_0(3)^*$	t_3^*	36	$j_3^*(v_1,q_{v_1})$	720	$j_3^*(v_8,q_{v_8})$	720	$j_3^*(v_{21},q_{v_{21}})$
$\Gamma_0(4)^*$	t_4^*	16	$j_4^*(v_1,q_{v_1})$	16	$j_4^*(v_{17},q_{v_{17}})$	16	$j_4^*(v_{22},q_{v_{22}})$

TABLE 16. Local triangle functions $j(v,q_v;z)=\mathfrak{n}^{-1}t(z)$

List of figures

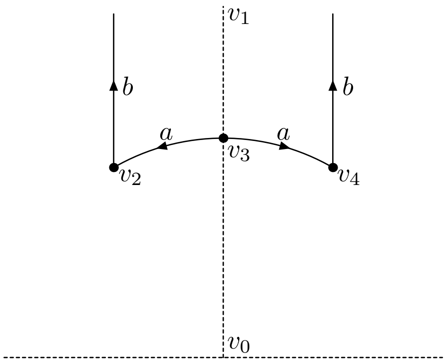

Figure 1. Fundamental domain for $\Gamma_0(1)$

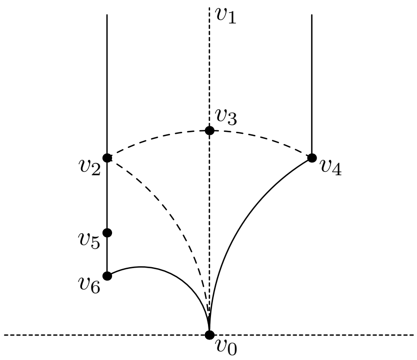

Figure 2. Fundamental domain for $\Gamma_0(2)$

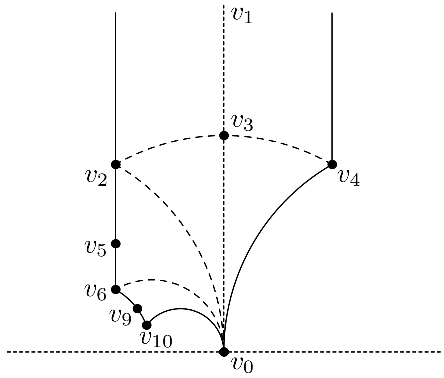

Figure 3. Fundamental domain for $\Gamma_0(3)$

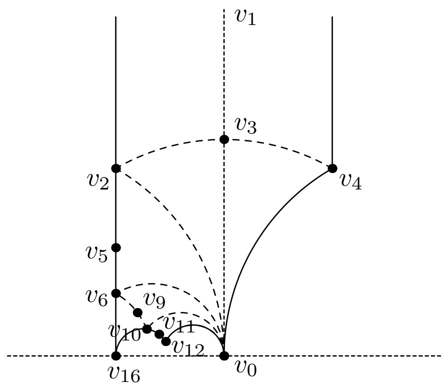

Figure 4. Fundamental domain for $\Gamma_0(4)$

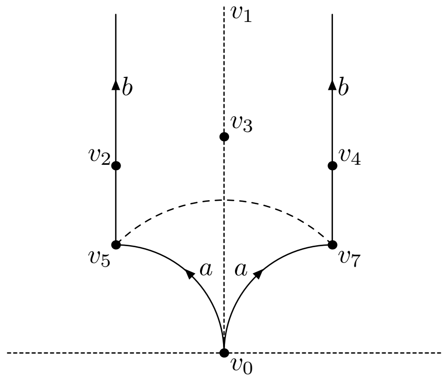

FIGURE 5. Fundamental domain for $\Gamma_0(2)$

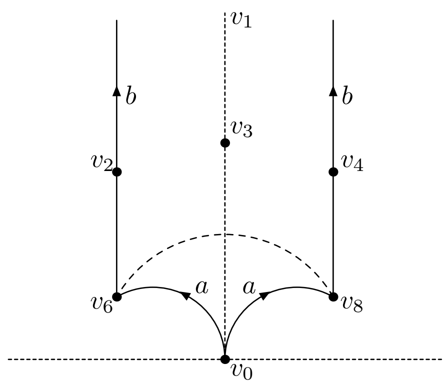

FIGURE 6. Fundamental domain for $\Gamma_0(3)$

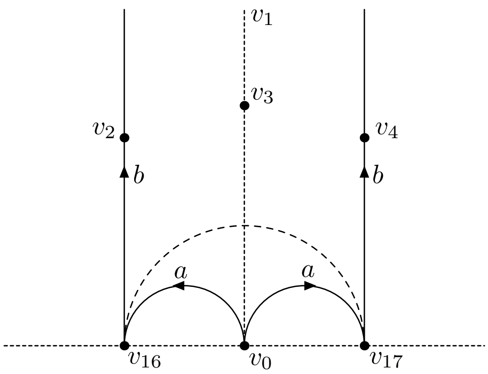

FIGURE 7. Fundamental domain for $\Gamma_0(4)$

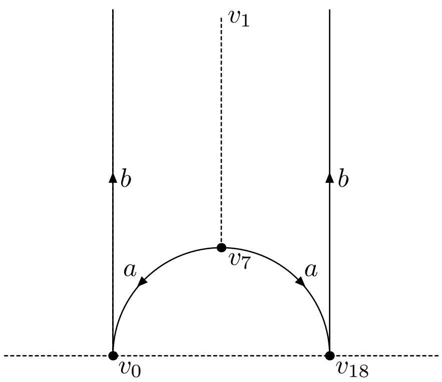

FIGURE 8. Fundamental domain for $\Gamma_0(2)$

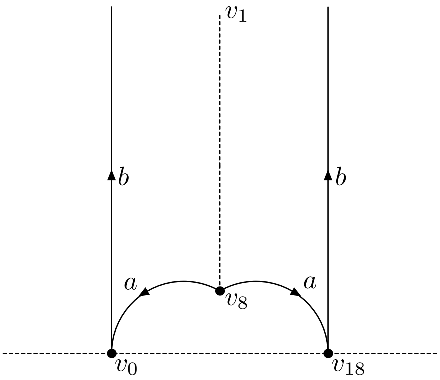

Figure 9. Fundamental domain for $\Gamma_0(3)$

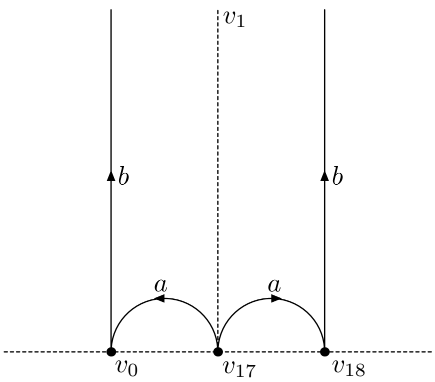

Figure 10. Fundamental domain for $\Gamma_0(4)$

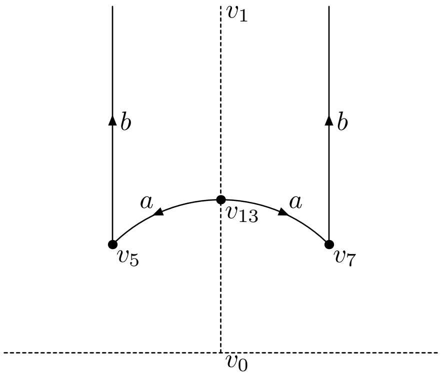

Figure 11. Fundamental domain for $\Gamma_0^+(2)$

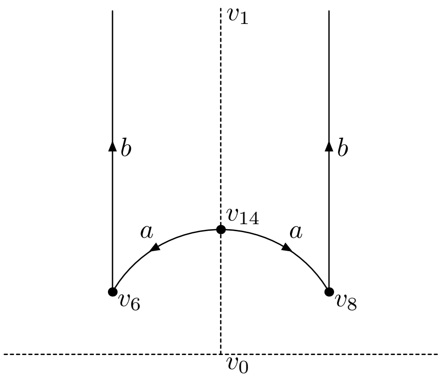

Figure 12. Fundamental domain for $\Gamma_0^+(3)$

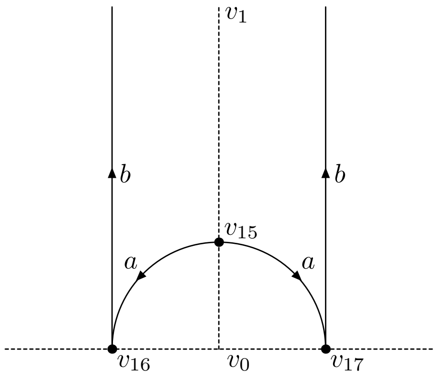

Figure 13. Fundamental domain for $\Gamma_0^+(4)$

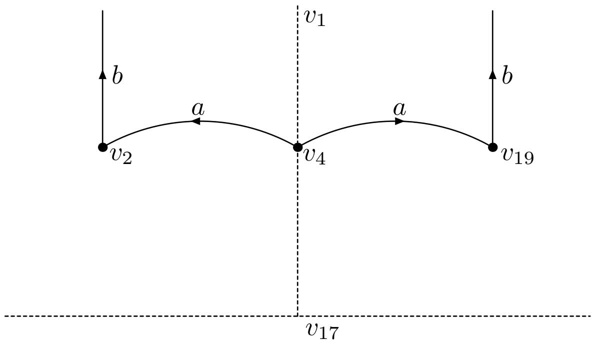

Figure 14. Fundamental domain for $\Gamma_0^+(1)^*$

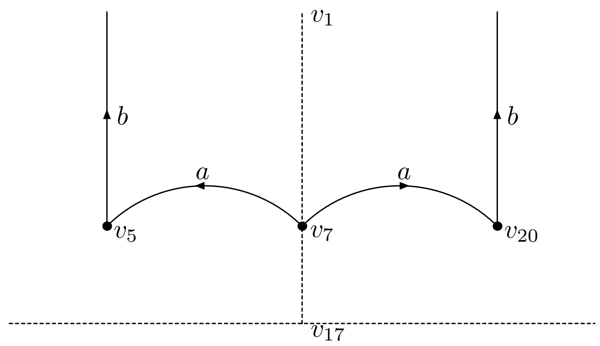

FIGURE 15. Fundamental domain for $\Gamma_0^+(2)^*$

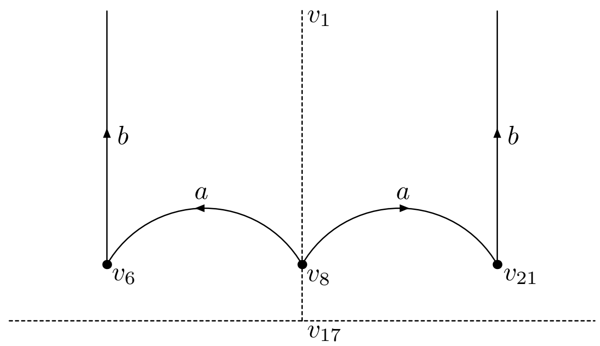

FIGURE 16. Fundamental domain for $\Gamma_0^+(3)^*$

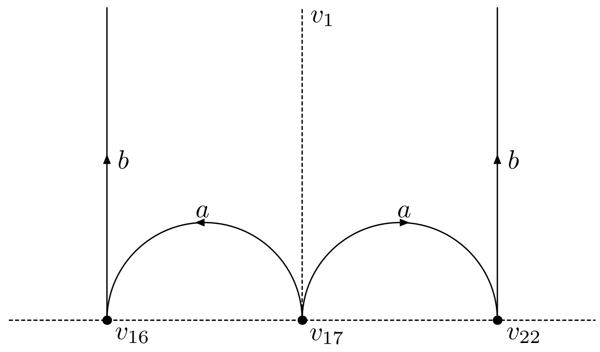

FIGURE 17. Fundamental domain for $\Gamma_0^+(4)^*$

References

[1] M. ALSINA AND P. BAYER, *"Quaternion orders, quadratic forms, and Shimura curves"*, CRM Monograph Series **22**, American Mathematical Society, Providence, RI, 2004.

[2] P. BAYER, Uniformization of certain Shimura curves, in: *Differential Galois theory* (Będlewo, 2001), Banach Center Publ. **58**, Polish Acad. Sci., Warsaw, 2002, pp. 13–26.

[3] P. BAYER AND J. GUÀRDIA, Uniformization of Fermat curves, *Ramanujan J.* **12** (2006), 207–223.

[4] P. BAYER AND A. TRAVESA, (EDS.), *Corbes modulars: Taules*, Notes del Seminari de Teoria de Nombres **1**, Barcelona (1992).

[5] P. BAYER AND A. TRAVESA, Uniformizing functions for certain Shimura curves, in the case $D = 6$, *Acta Arith.* **126(4)** (2007), 315–339.

[6] C. CARATHÉODORY, *"Funktionentheorie"*, Bd. II., 2te revidierte Aufl. Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, Mathematische Ree **9**, Birkhäuser Verlag, Basel-Stuttgart, 1961.

[7] D. A. COX, The arithmetic-geometric mean of Gauss, *Enseign. Math. (2)* **30(3–4)** (1984), 275–330.

[8] R. DEDEKIND, Schreiben an Herrn *Borchardt* über die Theorie der elliptischen Modul-Functionen, *J. reine angew. Math.* **83** (1877), 265–292.

[9] A. R. FORSYTH, *"Theory of functions of a complex variable"*, Cambridge University Press, Cambridge, 1893.

[10] A. R. FORSYTH, *"Theory of Differential Equations"*, vol. III, Cambridge University Press, Cambridge, 1902.

[11] L. FUCHS, Zur Theorie der linearen Differentialgleichungen mit veränderlichen Coefficienten, *J. reine angew. Math.* **66** (1866), 121–160.

[12] J. HARNAD AND J. MCKAY, Modular solutions to equations of generalized Halphen type, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **456(1994)** (2000), 261–294.

[13] Y. IHARA, Schwarzian equations, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **21** (1974), 97–118.

[14] F. KLEIN, Ueber die Transformation der elliptischen Functionen und die Auflösung der Gleichungen fünften Grades, *Math. Ann.* **14** (1879), 111–145.

[15] F. KLEIN, *"Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert"*, Two volumes reprinted as one, With a foreword by R. Courant and O. Neugebauer, Springer-Verlag, Berlin-New York, 1979.

[16] J. LEHNER, *"Discontinuous groups and automorphic functions"*, Mathematical Surveys **VIII**, American Mathematical Society, Providence, R.I., 1964.

[17] W. MAGNUS, *"Noneuclidean tesselations and their groups"*, Pure and Applied Mathematics **61**, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974.

[18] J. MCKAY AND A. SEBBAR, Fuchsian groups, automorphic functions and Schwarzians, *Math. Ann.* **318(2)** (2000), 255–275.

[19] N. NEHARI, *"Conformal mapping"*, McGraw-Hill Book Co., Inc., New York, Toronto, London, 1952.

[20] H. PETERSSON, Über die eindeutige Bestimmung und die Erweiterungsfähigkeit von gewissen Grenzkreisgruppen, *Abh. Math. Sem. Univ. Hamburg* **12** (1938), 189–199.

[21] H. POINCARÉ, *"Œuvres"*, Tome II, Gauthier-Villars, Paris, 1916.

[22] J. RALEIGH, On the Fourier coefficients of triangle functions, *Acta Arith.* **8** (1962/1963), 107–111.

[23] G. SANSONE AND J. GERRETSEN, *"Lectures on the theory of functions of a complex variable. II: Geometric theory"*, Wolters-Noordhoff Publishing, Groningen, 1969.

[24] H. A. SCHWARZ, Ueber diejenigen Fälle, in welchen die *Gauss*ische hypergeometrische Reihe eine *algebraische* Function ihres vierten Elementes darstellt, *J. reine angew. Math.* **75** (1873), 292–335.

[25] G. Shimura, *"Introduction to the arithmetic theory of automorphic functions"*, Kanô Memorial Lectures 1, Publications of the Mathematical Society of Japan **11**, Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, N.J., 1971.

[26] K. Takeuchi, Arithmetic triangle groups, *J. Math. Soc. Japan* **29(1)** (1977), 91–106.

[27] J. Wolfart, Eine arithmetische Eigenschaft automorpher Formen zu gewissen nicht-arithmetischen Gruppen, *Math. Ann.* **262(1)** (1983), 1–21.

[28] J. Wolfart, Taylorentwicklungen automorpher Formen und ein Transzendenzproblem aus der Uniformisierungstheorie, *Abh. Math. Sem. Univ. Hamburg* **54** (1984), 25–33.

[29] M. Yoshida, *"Fuchsian differential equations"*, With special emphasis on the Gauss-Schwarz theory, Aspects of Mathematics **E11**, Friedr. Vieweg & Sohn, Braunschweig, 1987.

Facultat de Matemàtiques
Universitat de Barcelona
Gran Via de les Corts Catalanes 585
E-08007 Barcelona
Spain
E-mail address: bayer@ub.edu
E-mail address: travesa@ub.edu