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ON THE RANGE SPACE OF YANO'S EXTRAPOLATION THEOREM AND NEW EXTRAPOLATION ESTIMATES AT INFINITY

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Abstract

Given a sublinear operator T satisfying that $\|Tf\|_{L^p(\nu)} \leq \frac{C}{p-1}\|f\|_{L^p(\mu)}$, for every $1 < p \leq p_0$, with C independent of f and p, it was proved in [**C**] that

$$\sup_{r>0} \frac{\int_{1/r}^\infty \lambda^\nu_{Tf}(y)\,dy}{1+\log^+ r} \lesssim \int_{\mathcal{M}} |f(x)|(1+\log^+|f(x)|)\,d\mu(x).$$

This estimate implies that $T\colon L\log L \to B$, where B is a rearrangement invariant space. The purpose of this note is to give several characterizations of the space B and study its associate space. This last information allows us to formulate an extrapolation result of Zygmund type for linear operators satisfying $\|Tf\|_{L^p(\nu)} \leq Cp\|f\|_{L^p(\mu)}$, for every $p \geq p_0$.

1. Introduction

In 1951, Yano (see [**Y**], [**Z**]) using the ideas of Titchmarsh in [**T**], proved that for every sublinear operator satisfying

$$\left(\int_{\mathcal{N}} |Tf(x)|^p\,d\nu(x)\right)^{1/p} \leq \frac{C}{p-1}\left(\int_{\mathcal{M}} |f(x)|^p\,d\mu(x)\right)^{1/p},$$

where $\mathcal{N}$ and $\mathcal{M}$ are two finite measure spaces, $T\colon L\log L(\mu) \longrightarrow L^1(\nu)$ is bounded. If the measures involved are not finite, then an easy modification of the above proofs, shows that $T\colon L\log L(\mu) \longrightarrow L^1_{\text{loc}}(\nu)$ and, in fact, $T\colon L\log L(\mu) \longrightarrow L^1(\nu)+L^\infty(\nu)$.

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Quite recently, it was proved, in [**C**], that under a weaker condition on the operator T, namely that

$$\left(\int_{\mathcal{N}} |T\chi_A(x)|^p \, d\nu(x)\right)^{1/p} \leq \frac{C}{p-1}\mu(A)^{1/p}, \tag{1}$$

for every measurable set $A \subset \mathcal{M}$ and every $1 < p \leq p_0$, with C independent of A and p, we have that, there exists a positive constant K, such that

$$\sup_{r>0} \frac{\int_{1/r}^{\infty} \lambda_{Tf}^{\nu}(y)\, dy}{1 + \log^+ r} \leq K \int_{\mathcal{M}} |f(x)|(1 + \log^+ |f(x)|)\, d\mu(x), \tag{2}$$

where λ_{Tf}^{ν} is the distribution function of Tf with respect to ν, and μ and ν are two σ-finite measures. This estimate allows us, as we shall see in this note, to improve Yano's theorem in the following sense: There exists a rearrangement invariant space $B(\nu) \subset L^1 + L^\infty$, $B(\nu) \neq L^1 + L^\infty$ and such that for every sublinear operator T satisfying (1), we have that

$$T\colon L\log L(\mu) \longrightarrow B(\nu).$$

Throughout this paper, a sublinear operator satisfying (1) shall be called Yano's operator. From (2), it is very easy to see that if we define

$$B(\nu) = \{f \text{ measurable}; \|f\|_{B(\nu)} < \infty\}, \tag{3}$$

where

$$\|f\|_{B(\nu)} = \inf\left\{\alpha > 0; \sup_{r>0} \frac{\int_r^{\infty} \lambda_f^{\nu}(\alpha y)\, dy}{1 + \log^+ \frac{1}{r}} \leq 1\right\},$$

then, every Yano's operator satisfies that

$$T\colon L\log L \longrightarrow B(\nu)$$

is bounded.

The purpose of this note is to study in detail the space $B(\nu)$, including the identification of its associate space.

This last information will allow us to formulate an extrapolation result of Zygmund type (see [**Z**, p. 119]) for linear operators satisfying

$$\|Tf\|_{L^p(\nu)} \leq Cp\|f\|_{L^p(\mu)},$$

for every p near ∞.

Some years ago, in the work of Jawerth and Milman (see [**JM1**], [**JM2**]), the extrapolation theory was extended to the setting of compatible couples of Banach spaces. More recently, in [**CM**], the authors have developed a new abstract extrapolation method, where the range space (of the previous method) has been improved.

Constants such as C will denote universal constants (independent of f and p and, whenever it makes sense, independent also of r) and may change from one occurrence to the next. As usual, the symbol $f \approx g$ will indicate the existence of an universal positive constant C so that $f/C \leq g \leq Cf$, while the symbol $f \lesssim g$ means that $f \leq Cg$. Throughout this paper $(\mathcal{N}, \nu)$ and $(\mathcal{M}, \mu)$ are two σ-finite measure spaces, we shall write $\|g\|_p$ to denote either $\|g\|_{L^p(\mu)}$ or $\|g\|_{L^p(\nu)}$, $\lambda_g^\nu(y) = \nu(\{x \in \mathcal{N};\, |g(x)| > y\})$ is the distribution function of g with respect to the measure ν, $g_\nu^*(t) = \inf\{s;\, \lambda_g^\nu(s) \leq t\}$ is the decreasing rearrangement (see [**BS**]), $f^{**}(t) = \dfrac{1}{t}\displaystyle\int_0^t f^*$ and we say that a function W satisfies the Δ_2 condition, if there exists a positive constant C so that $W(2t) \leq CW(t)$, for every t.

Finally, as usual, $L^0(\mathbb{R}^n)$ will denote the set of Lebesgue measurable functions on $\mathbb{R}^n$ and $x_+ := \max[x, 0]$.

2. On the range space B

Let $B = B(\nu)$ be the space defined in (3). Observe that

$$\int_r^\infty \lambda_f(y)\, dy = \int_{\mathcal{M}} P_r(|f(x)|)\, d\nu(x),$$

where $P_r(t) = (t - r)_+$. Therefore, the functional $\|\cdot\|_B$ is similar to a uniform (in r) Luxembourg norm. Since P_r is a convex function, the fact that it is a norm is an easy exercise. However, that B is a rearrangement invariant Banach function space is a consequence of the fact that B is a maximal Lorentz space (see Theorem 2.4 below).

Our first result proves that $B \subset L^1 + L^\infty$ and that $B \neq L^1 + L^\infty$.

Proposition 2.1. *For every $p > 1$, $B \subset L^1 + L^p$ with constant less than or equal to $Cp/(p-1)$.*

Proof: Let $f \in B$ such that $\|f\|_B = 1$. Then $\int_1^\infty \lambda_f(y)\, dy \leq C < \infty$ and hence, if we define $\overline{f} = f\chi_{\{|f|>1\}}$, we have that

$$\|\overline{f}\|_1 = \lambda_f(1) + \int_1^\infty \lambda_f(y)\, dy \leq C.$$

Now, if we set $\underline{f} = f - \overline{f}$ and take $p > 1$, then an integration by parts shows that

$$\begin{aligned}\|\underline{f}\|_p^p &= p\int_0^\infty y^{p-1}\lambda_{\underline{f}}(y)\,dy = p\int_0^1 y^{p-1}\lambda_f(y)\,dy\\ &= p(p-1)\int_0^1 y^{p-2}\left(\int_y^1 \lambda_f(s)\,ds\right)dy\\ &\lesssim p(p-1)\int_0^1 y^{p-2}\left(1+\log\frac{1}{y}\right)dy\\ &= p(p-1)\left(\frac{1}{p-1}+\frac{1}{(p-1)^2}\right) = \frac{p^2}{p-1},\end{aligned}$$

from which the result follows. □

Our next step is to give a different and useful characterization of the space B.

Lemma 2.2. *For every $s > 0$,*

(a)
$$\int_{f^{**}(s)}^\infty \lambda_f(y)\,dy \le \int_0^s f^*(t)\,dt,$$

(b)
$$\int_0^s f^*(t)\,dt \le 2\int_{\frac{1}{2}f^{**}(s)}^\infty \lambda_f(y)\,dy.$$

Proof: (a) Using that $\lambda_f = \lambda_{f^*}$ and Fubini's theorem, we obtain that

$$\begin{aligned}\int_{f^{**}(s)}^\infty \lambda_f(y)\,dy &= \int_0^\infty \Big(f^*(t) - f^{**}(s)\Big)_+\,dt = \int_0^s \Big(f^*(t) - f^{**}(s)\Big)_+\,dt\\ &\le \int_0^s f^*(t)\,dt.\end{aligned}$$

(b) By the distribution formula proved in [**CS1**], we have that

$$\begin{aligned}\int_0^s f^*(t)\,dt &= \int_0^\infty \min\Big(\lambda_f(y), s\Big)\,dy \le \int_0^{\frac{1}{2}f^{**}(s)} s\,dy + \int_{\frac{1}{2}f^{**}(s)}^\infty \lambda_f(y)\,dy\\ &= \frac{1}{2}\int_0^s f^*(t)\,dt + \int_{\frac{1}{2}f^{**}(s)}^\infty \lambda_f(y)\,dy,\end{aligned}$$

from which the result follows. □

Lemma 2.3.

$$\sup_{s>0} \frac{\int_0^s f^*(t)\,dt}{1+\log^+ \frac{s}{\int_0^s f^*}} \approx \sup_{r>0} \frac{\int_r^\infty \lambda_f(y)\,dy}{1+\log^+ \frac{1}{r}}.$$

Proof: Given $s>0$, we have, by Lemma 2.2(b), that

$$\frac{\int_0^s f^*(t)\,dt}{1+\log^+ \frac{1}{f^{**}(s)}} \leq 2\frac{\int_{\frac{1}{2}f^{**}(s)}^\infty \lambda_f(y)\,dy}{1+\log^+ \frac{1}{f^{**}(s)}} \leq 2\sup_{r>0} \frac{1+\log^+ \frac{1}{r}}{1+\log^+ \frac{1}{2r}} \frac{\int_r^\infty \lambda_f(y)\,dy}{1+\log^+ \frac{1}{r}}$$
$$\lesssim \sup_{r>0} \frac{\int_r^\infty \lambda_f(y)\,dy}{1+\log^+ \frac{1}{r}},$$

and therefore the inequality $\lesssim$ follows.

Conversely, if $\sup_{s>0} \frac{\int_0^s f^*(t)\,dt}{1+\log^+ \frac{s}{\int_0^s f^*}} < \infty$, then necessarily $f^{**}(+\infty) = 0$, and hence, if $r < \|f\|_\infty = \sup_s f^{**}(s)$, we have that $0 = f^{**}(+\infty) = \inf_s f^{**}(s) < r < \sup_s f^{**}(s)$ and by continuity, there exists s so that $r = f^{**}(s)$. Then using Lemma 2.2(a),

$$\frac{\int_r^\infty \lambda_f(y)\,dy}{1+\log^+ \frac{1}{r}} = \frac{\int_{f^{**}(s)}^\infty \lambda_f(y)\,dy}{1+\log^+ \frac{1}{f^{**}(s)}} \leq \frac{\int_0^s f^*(t)\,dt}{1+\log^+ \frac{1}{f^{**}(s)}}.$$

If $r \geq \|f\|_\infty$, then $\int_r^\infty \lambda_f(y)\,dy = 0$ and the result follows immediately. □

Given a concave function $\varphi(t)$, we recall that the maximal Lorentz space is defined (see [**BS**, p. 69]) by

$$\|f\|_{M(\varphi)} = \sup_{t>0} \Big(\varphi(t) f^{**}(t)\Big),$$

and, for a positive locally integrable weight v, the Lorentz space $\Lambda^1(v)$ is defined by

$$\|f\|_{\Lambda^1(v)} = \int_0^\infty f^*(t) v(t)\,dt.$$

Theorem 2.4. *The space B coincides with the maximal Lorentz space $M(\varphi)$ with equivalent norms, where $\varphi(t) = t/(1+\log^+ t)$.*

Proof: Let $\alpha > 0$ satisfying

$$\sup_{r>0} \frac{\int_r^\infty \lambda_{f/\alpha}(y)\,dy}{1+\log^+ \frac{1}{r}} \leq 1.$$

Then, by Lemma 2.3, there exists a positive constant C so that

$$\sup_{s>0} \frac{\int_0^s \frac{f^*(t)}{\alpha}\,dt}{1+\log^+ \frac{s}{\int_0^s (f^*/\alpha)}} \leq C,$$

and thus, if $\Phi(t) = t/\big(1+\log^+(1/t)\big)$, we obtain that

$$\sup_{s>0} s\Phi\left(\frac{f^{**}(s)}{\alpha}\right) \leq C.$$

Consequently, for every $s>0$, $f^{**}(s) \leq \alpha\Phi^{-1}(C/s)$ and, hence

$$\alpha \geq \sup_{s>0} \frac{f^{**}(s)}{\Phi^{-1}(C/s)}.$$

From this, the fact that $\Phi^{-1}(t) \approx t(1+\log^+(1/t))$ and that this function satisfies the Δ_2 condition, we conclude that

$$\|f\|_B \geq \sup_{s>0} \frac{s}{(1+\log^+ s)} f^{**}(s).$$

The converse follows similarly. □

Remark 2.5. If M is the Hardy-Littlewood maximal operator,

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

where the supremum is taken over all cubes containing x, it is known (see for example [**CS2**]) that, for every $0<\alpha<1$, every function f and $y>0$,

$$\begin{aligned} \frac{1}{y}\int_{\{x:|f(x)|>y\}} |f(x)|\,dx &\leq 2\lambda_{Mf}(y) \\ &\leq \frac{2}{(1-\alpha)y}\int_{\{x:|f(x)|>\alpha y\}} |f(x)|\,dx. \end{aligned} \tag{4}$$

Therefore, taking $\alpha = 1/2$,

$$\begin{aligned}\int_r^\infty \lambda_{Mf}(y)\,dy &\lesssim \int_r^\infty \frac{1}{y}\left(\int_{\{x\colon\, |f(x)|>y/2\}} |f(x)|\,dx\right) dy\\ &= \int_{\{|f(x)|>\frac{r}{2}\}} |f(x)|\left(\int_r^{2|f(x)|} \frac{1}{y}\,dy\right) dx\\ &= \int_{\{|f(x)|>\frac{r}{2}\}} |f(x)|\log\left(\frac{2|f(x)|}{r}\right) dx\\ &= \int_{\mathbb{R}^n} |f(x)|\log^+\left(2\frac{|f(x)|}{r}\right) dx.\end{aligned}$$

Similarly, if we now use the first inequality in (4), we obtain

$$A := \sup_{r>0} \frac{\int |f(x)|\log^+\left(\frac{|f(x)|}{r}\right) dx}{1+\log^+\frac{1}{r}} \approx \sup_{r>0} \frac{\int_r^\infty \lambda_{Mf}(y)\,dy}{1+\log^+\frac{1}{r}}.$$

Taking $r = 1$, we obtain that $A \geq \int |f(x)|\log^+(2|f(x)|)\,dx$, and if, $|f(x)| \leq 1$, we have, by dominated convergence theorem that

$$A \geq \lim_{r\to\infty} \frac{\int |f(x)|\log^+\big(2r|f(x)|\big)\,dx}{1+\log^+(1/r)} \geq \int_{|f(x)|\leq 1} |f(x)|\,dx,$$

and thus,

$$\int_{\mathbb{R}^n} |f(x)|\Big(1+\log^+ 2|f(x)|\Big)\,dx \lesssim A.$$

Since, obviously A satisfies the converse inequality we conclude that

$$\sup_{r>0} \frac{\int_r^\infty \lambda_{Mf}(y)\,dy}{1+\log^+\frac{1}{r}} \approx \int_{\mathbb{R}^n} |f(x)|\Big(1+\log^+|f(x)|\Big)\,dx,$$

and, therefore, the range space B is optimal for the Hardy-Littlewood maximal operator in the following sense:

Proposition 2.6. *If there exists a Banach space $E \subset L^0(\mathbb{R}^n)$, such that for every Yano's operator T on $L^0(\mathbb{R}^n)$, we have that $T\colon L\log L(\mathbb{R}^n) \to E$ is bounded, then*

$$\|Mf\|_E \lesssim \|Mf\|_B.$$

In particular, if E is a rearrangement invariant space, $\|f^{**}\|_E \lesssim \|f^{**}\|_B$.

Observe that if we were able to prove that $\|f^*\|_E \lesssim \|f^*\|_B$, we would have obtained the optimality of the range space B, in Yano's theorem, in the setting of rearrangement invariant spaces.

3. Associate space of B and extrapolation results at infinity

Given a Banach space X, the associate space X^* is defined as the set of measurable functions g so that

$$\|g\|_{X^*} = \sup_f \frac{\int_{\mathcal{N}} f(x)g(x)\,d\nu(x)}{\|f\|_X} < \infty.$$

If X is a Banach function space, then by Lorentz-Luxembourg theorem (see [**BS**, p. 10]), $X = X^{**}$; that is, the associate of X^* is X.

Also, if X is a rearrangement invariant space and the measure ν is resonant, we have that

$$\|g\|_{X^*} = \frac{\int_0^\infty f^*(t)g^*(t)\,dt}{\|f\|_X}.$$

In this section, we shall assume that the measure is resonant. In [**Z**, p. 119], it was proved that if T is a linear operator so that

$$\|Tf\|_{L^p(\nu)} \le Cp\|f\|_{L^p(\mu)} \tag{5}$$

for p big enough, $\mu(\mathcal{M}) < \infty$ and $\nu(\mathcal{N}) < \infty$, then

$$T\colon L^\infty(\mu) \to L(\exp, \nu),$$

where

$$L(\exp, \nu) = \left\{ f;\, \exists\, \lambda > 0,\, \int_{\mathcal{N}} e^{\lambda|f(x)|}\,d\nu(x) < \infty \right\}.$$

Now, it T satisfies (5) (and we shall say then that T is a Zygmund's operator), then the adjoint operator T^* satisfies that

$$\|T^* f\|_{L^{p'}(\mu)} \le \frac{C}{p'-1}\|f\|_{L^{p'}(\nu)}$$

for $1 < p' \le p_0$ and hence, T^* is a Yano's operator. Therefore,

$$T^*\colon L\log L \longrightarrow M(\varphi),$$

and we can deduce the following result.

Theorem 3.1. *If T is a Zygmund operator then*

$$T\colon (M(\varphi))^* \longrightarrow (L\log L)^*.$$

Now, the purpose of this section is to identify the two spaces appearing in Theorem 3.1 and conclude some endpoint estimate at $p = \infty$ for such operators. We emphasize that our measures are σ-finite and resonant but not necessarily finite.

Proposition 3.2. *If* $\varphi(t) = t/(1+\log^+ t)$*, then*

$$(M(\varphi))^* = \Lambda^1\big(\min(t^{-1},1)\big) \cap L^\infty.$$

Proof: We have to compute

$$\begin{aligned}\|g\|_{(M(\varphi))^*} &= \sup_f \frac{\int_0^\infty f^*(t)g^*(t)\,dt}{\sup_{t>0}\frac{t}{(1+\log^+ t)}f^{**}(t)}\\ &= \sup_{\int_0^t f^* \le 1+\log^+ t} \int_0^\infty f^*(t)g^*(t)\,dt.\end{aligned}$$

Now, the last supremum was identified in [**CPSS**], where it was proved that

$$\sup_{\int_0^t f^* \le 1+\log^+ t} \int_0^\infty f^*(t)g^*(t)\,dt \approx \sup_{t>0} g^{**}(t)(1+\log^+ t) + \int_1^\infty \frac{1}{t}g^*(t)\,dt,$$

and since, for every $t>0$,

$$\begin{aligned}g^{**}(t)(1+\log^+ t) &\lesssim \|g\|_\infty + g^{**}(t)\left(\int_1^t \frac{ds}{s}\right)_+ \le \|g\|_\infty + \left(\int_1^t g^{**}(s)\frac{ds}{s}\right)_+\\ &\le \|g\|_\infty + \int_1^\infty g^{**}(s)\frac{ds}{s} \approx \|g\|_\infty + \int_1^\infty g^*(s)\frac{ds}{s},\end{aligned}$$

we obtain the result. □

Proposition 3.3. *We have that*

$$(L\log L)^* = M(\Psi),$$

with $\Psi(t) = 1/\big(1+\log^+(1/t)\big)$.

Proof: In [**BS**, p. 243] it is proved that if $\mu(\mathcal{M}) = 1$, then $L\log L(\mu) = \Lambda^1(\log^+(1/t))$ with equivalent norms. A slight modification of this result (see also [**OP**]), shows that, for a general measure space, $L\log L(\mu) = \Lambda^1(1+\log^+(1/t))$. Then, using Theorem 2.12 in [**CS1**],

$$\begin{aligned}\|g\|_{(L\log L)^*} &= \sup_f \frac{\int_0^\infty f^*(s)g^*(s)\,ds}{\int_0^\infty f^*(s)(1+\log^+(1/s))\,ds}\\ &= \sup_{r>0} \frac{\int_0^r g^*(s)\,ds}{\int_0^r (1+\log^+(1/s))\,ds} \approx \sup_{r>0} \frac{\int_0^r g^*(s)\,ds}{r(1+\log^+(1/r))}\\ &= \sup_{r>0} \frac{g^{**}(r)}{(1+\log^+(1/r))}.\end{aligned}$$

□

Therefore, we deduce, from Theorem 3.1, the following result:

Corollary 3.4. *If T is a Zygmund operator, then*

$$\sup_{t>0} \frac{(Tf)^{**}(t)}{(1+\log^+(1/t))} \lesssim \int_0^\infty f^*(t) \min\left(\frac{1}{t}, 1\right) dt + \|f\|_\infty,$$

equivalently

$$\sup_{t>0} \frac{(Tf)^{*}(t)}{(1+\log^+(1/t))} \lesssim \int_0^\infty f^*(t) \min\left(\frac{1}{t}, 1\right) dt + \|f\|_\infty,$$

or

$$\sup_{t>0} \frac{(Tf)^{**}(t)}{(1+\log^+(1/t))} \lesssim \int_1^\infty f^{**}(t) \frac{dt}{t} + \|f\|_\infty.$$

Proof: The proof of the first part is an immediate consequence of Theorem 3.1 and Propositions 3.2 and 3.3. The second inequality follows also easily, since

$$\sup_{t>0} \frac{(Tf)^{**}(t)}{(1+\log^+(1/t))} \approx \sup_{t>0} \frac{(Tf)^{*}(t)}{(1+\log^+(1/t))},$$

and the last one can be deduced using that

$$\int_1^\infty f^{**}(t) \frac{dt}{t} = \int_0^\infty f^*(t) \min\left(\frac{1}{t}, 1\right) dt. \qquad \square$$

Remark 3.5. i) Observe that if $\mu(\mathcal{M}) = \nu(\mathcal{N}) = 1$, then, the above inequalities say (see [**BS**, p. 246]) that $T\colon L^\infty \to L(\exp)$ as proved in [**Z**, p. 119].

ii) Finally, let us just comment that obvious changes show that similar results can be obtained for sublinear operators satisfying

$$\|Tf\|_p \leq C(p-1)^{-\alpha} \|f\|_p,$$

for $\alpha > 0$ and p near 1, and, for linear operators such that

$$\|Tf\|_p \leq C p^\alpha \|f\|_p,$$

where again $\alpha > 0$ and p near ∞.

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