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THE FUNDAMENTAL THEOREM OF ALGEBRA BEFORE CARL FRIEDRICH GAUSS

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Abstract

This is a paper about the first attempts of the demonstration of the fundamental theorem of algebra.

Before, we analyze the tie between complex numbers and the number of roots of an equation of n-th degree.

In second paragraph we see the relation between the integration and fundamental theorem.

Finally, we observe the linear differential equation with constant coefficients and the Euler's position about the fundamental theorem and then we consider the d'Alembert's, Euler's and Laplace's demonstrations.

It is a synthesis paper dedicated to Pere Menal a collegue and a friend.

És quan dormo que hi veig clar

Josep Vicens FOIX

En la calle mayor de los que han muerto, el deber de vivir iré a gritar

Enrique BADOSA

To be or not to be.
That is the question.

William SHAKESPEARE

1. Introduction: The Complex Numbers

In the year 1545 Gerolamo Cardano wrote *Ars Magna*[1]. In this book Cardano offers us a process for solving cubic equations, learned from

[1]There are many interesting papers on complex numbers. See, for example, Jones, P. S. [**43**]; Molas, C.-Pérez, J. [**57**] and Remmert, R. [**67**]. Moreover, in this paper, our interest on complex numbers is limited only in their connexion with *algebra* and particulary with the *Fundamental Theorem of Algebra*.

Niccolò Tartaglia[2]. In his book it appears for the first time an special quadratic equation:

> *If some one says you, divide 10 into two parts, one of which multiplied into the other shall produce 30 or 40, it is evident this case or equation is impossible*[3].

Cardano says then

> *Putting aside the mental tortures involved, multiply* $5+\sqrt{-15}$ *by* $5-\sqrt{-15}$, *making* $25-(-15)$, *which is* $+15$. *Hence this product is* 40 ... *This is truly sophisticated* ...[4].

But, as Remmert remembers us, "it is not clear whether Cardano was led to complex numbers through cubic or quadratic equations"[5]. The sense of these words is the following: while quadratic equations

$$x^2 + b = ax, \text{ with } \Delta = \frac{1}{4}a^2 - b < 0,$$

have no real roots [and they are therefore *impossible equations*], cubic equations

$$x^3 = px + q, \text{ with } \Delta = \left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3 < 0,$$

have real roots which are given as sums of imaginary cubic roots[6]. This question was further developed by Rafael Bombelli in his *L'Algebra*, published in Bologna in 1572. Bombelli worked out the formal algebra of

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[2]Cardano's rule for cubic equation $x^3 = px + q$ is

$$x = \sqrt[3]{\frac{q}{2}+\sqrt{\Delta}} + \sqrt[3]{\frac{q}{2}-\sqrt{\Delta}}, \text{ where } \Delta = \left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3.$$

The history of the process for solving cubic equations is now perfectly known. See, for example, Burton, D. M. [**12**, 302–312]; Stillwell, J. [**76**, 59–62]; Vera, F. [**80**, 47–59] and van der Waerden, B. L. [**85**, 54–55].

See also Tartaglia, N. [**78**, 69 and 120].

[3]Cardano, G. [**16**, Ch. 37]. See also Struik, D. J. [**77**, 67].

The equation $x^2 - 10x = 40$ [or 30] has the solutions $5 \pm \sqrt{-15}$ [or $5 \pm \sqrt{-5}$] and both solutions are *formally corrects*, but in this time they have not any sense.

[4]Cardano, G. [**16**, Ch. 37]. See also Struik, D. J. [**77**, 69 and footnote 7].

The name *imaginary* is introduced by René Descartes, as we will see soon. But it is debt, perhaps, to following Cardano's words: "... you will nevertheless *imagine* $\sqrt{-15}$ to be the difference between ...", completing, in that case, the square.

It is interesting to observe that Cardano accompanied his result over this kind of quadratic equation with the comment: "the result in that case is as subtle as it is useless" [see Cardano, G. [**16**, Ch. 37, rule II] and also Struik, D. J. [**77**, 69]].

[5]Remmert, R. [**67**, 57].

[6]We can see van der Waerden, B. L. [**84**, 194]: It is not possible solve, by real radicals, an irreductible cubic equation over $\mathbb{Q}$ whose three roots are all real [*casus irreductibilis*, following Cardano].

complex numbers. He introduced (in actual notation) the complex unit[7] i and eight fundamental rules of computation[8]:

$$(+1) \cdot i = +i; (+1) \cdot (-i) = -i; (+i) \cdot (+i) = -1; (+i) \cdot (-i) = +1;$$
$$(-1) \cdot i = -i; (-1) \cdot (-i) = +i; (-i) \cdot (+i) = +1; (-i) \cdot (-i) = -1.$$

His principal aim consisted to reduce expressions as $\sqrt[3]{a+bi}$ to the form $c + di$[9], because then it should be possible to use formally the Cardano's expression by solving the *casus irreductibilis* $x^3 = 15x + 4$. Bombelli obtains, according the Cardano's expression,

$$x = \sqrt[3]{2 + 11\,i} + \sqrt[3]{2 - 11\,i}.$$

Hence $x = (2 + i) + (2 - i) = 4$.

François Viète wrote in 1591 a higher level paper, which relates algebra to trigonometry[10]. In this paper[11] Viète offers us his solution of the cubic equation by *circular functions*, which shows that solving the cubic is equivalent to trisecting an arbitrary angle[12]. He starts (in modern notation) from the identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

[or $z^3 - \frac{3}{4}z - \frac{1}{4}\cos 3\theta = 0$, where $z = \cos\theta$]. Suppose now that the cubic to be solved is given by

$$x^3 - px = q \quad [p, q > 0].$$

[7]It is perhaps interesting to remember that the symbol i for indicate *imaginary unit* is debt to Euler: "In the following I shall denote the expression $\sqrt{-1}$ by the letter i so that $ii = -1$" [Euler, L. [**25**, 130]]. See Kline, M. [**44**, 410]: "In his earlier work Euler used i (the first letter of *infinitus*) for an infinitely large quantity. After 1777 he used i for $\sqrt{-1}$".

[8]Really, Bombelly introduced *più di meno* [for $+i$] and *meno di meno* [for $-i$] and rules of calculation such as

meno di meno via meno di meno fa meno

which means $(-i) \cdot (-i) = -1$.

See Bombelli, R. [**9**, 169] or Bertolotti reprint, 133.

[9]Bombelli did not through too much on the nature of complex numbers, but he knows, for example, that

$$(2 \pm i)^3 = 2 \pm 11\,i,$$

so that

$$\sqrt[3]{2 \pm 11\,i} = 2 \pm i\,.$$

See Bombelli, R. [**9**, 110] or Bertolotti reprint, 140–141.

[10]This paper, "De aequatione recognitione et emendatione", written by Viète in 1591, was not published until 1615 by his Scottish friend Alexander Anderson.

[11]See Viète, F. [**81**, Ch. VI, Th. 3].

[12]See Hollingsdale, S. [**41**, 122–123].

If we introduce an arbitrary constant λ, setting $x = \lambda z$, then

$$z^3 - \frac{p}{\lambda^2} z - \frac{q}{\lambda^3} = 0.$$

We can now match coefficients in the two forms

$$\frac{p}{\lambda^2} = \frac{3}{4} \quad \text{and} \quad \frac{q}{\lambda^3} = \frac{1}{4}\cos 3\theta, \quad \text{so that} \quad \lambda = \sqrt{\frac{4p}{3}}.$$

With this value of λ, we can select a value of θ so that

$$\cos 3\theta = \frac{4q}{\lambda^3} = \frac{q/2}{\sqrt{(p/3)^3}}.$$

In the *casus irreductibilis*, we have

$$\Delta = \left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3 < 0 \quad \text{and then} \quad \left|\frac{q/2}{\sqrt{(p/3)^3}}\right| < 1$$

and thus the condition for three real roots ensures us that $|\cos 3\theta| < 1$, which is essential[13].

In 1637 René Descartes wrote *La Géométrie*[14]. This appendix was his only mathematical work; but a what work! It contains the birth of *analytic geometry*[15]. In Book III of his *La Géométrie* Descartes gives a brief summary of that was known about equations[16]. Between his

[13] Then he proves the equivalence: we have $\cos 3\theta = \mu$, where $\mu = \frac{q/2}{\sqrt{(p/3)^3}}$. Given μ, we can construct a triangle with angle $3\theta = \cos^{-1}\mu$. Trisection of this angle gives us the solution $z = \cos\theta$ of the equation. Conversely, the problem of trisecting an angle with cosine μ is equivalent to solve the cubic equation $4z^3 - 3z = \mu$.

[14] It is, as it is well known, the third appendix of his famous *Discours de la mèthode pour bien conduire sa raison et chercher la verité dans les sciences*. The other appendices are *La Dioptrique* and *Les Météors*. For a comment we can see Bos, H. J. M. [**10**], Milhaud, G. [**56**, 124–175], Pla, J. [**63**], or Scott, J. F. [**71**, 84–157].

[15] The *analytic geometry* was independently discovered by Pierre Fermat, a French amateur mathematician, in his "Ad locos planos et solidos isagoge" [**32**].

[16] "John Wallis in his *Algebra* [**86**] declared that there was little in Descartes which was no to be found in the *Artis Analyticæ Praxis* [**39**] of Harriot" [see Scott, J. F. [**71**, 138] and Wallis, J. [**87**, 126]]. But, says Scott [Scott, J. [**71**, 139]], "this statement is far from true".

algebraic assertions[17], we are interested in the following:

> *in every equation there are as many distint roots as is the number of dimensions of the unknown quantities*[18].

This is an important approach to *Fundamental Theorem of Algebra*, but it is not the first and perhaps never the more explicit.

The first writer to assert that "every such equation of the nth degree has n roots and no more" seems to have been Peter Roth[19]. The law was next set forth by a more prominent algebraist, Albert Girard, in 1629:

> *Every algebraic equation admits as many solutions as the denomination of the highest quantity indicates ...*[20]

Girard gives no proof or any indication of one. He merely explains his proposition by some examples, including that of the equation $x^4 - 4x + 3 = 0$ whose solutions are $1, 1, -1 + i\sqrt{2}, -1 - i\sqrt{2}$[21].

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[17]The other important assertions in Book III of *La Géométrie* are:

- A polynomial $P(x)$ which vanishes at c is always divisible by the factor $x - c$ and then

$$P(x) = (x - c) \cdot Q(x), \text{ where } \deg(Q(x)) = \deg(P(x)) - 1.$$

[This theorem was probably already known by Thomas Harriot, following Remmert, R. [**68**, 99 footnote 2].]

- *Descartes' rule of signs*: we can determinate from this also the number of true and false roots that any equation can have, as follows: *Every equation can have as many true roots as it contains changes of signs, from + to − or from − to +; and as many false roots as the number of times two + signs or two − signs are found in succession.* [This law was apparently known by Cardano [Cantor, M. [[**15**, II, 539], but a satisfactory statement is possibly due to Harriot [Harriot, [**39**, 18, 268]]. See also Smith, E. D. [**73**, II, 471].]
  [On limitations or mistakes in Descartes' rule see, for example, Scott, J. F. [**71**, 140].]
  This rule was formulated in a more precise manner by Isaac Newton in his *Arithmetica Universalis*, composed between 1673 and 1683, perhaps for Newton's lectures at Cambridge, but first published in 1707. Newton's rule counts moreover complex roots.
  This Newton's work contains also the formulas, usually known as *Newton's identities*, for sums of the power of the roots of polynomial equations.

[18]Descartes, R. [**19**, 372]. English translation in Smith, D.E.-Latham, M. [**75**, 159].

[19]Peter Roth, who name also appears as Rothe, was a Nürnberg Rechenmeister, died at Nürnberg in 1617. He wrote, in 1600, his *Arithmetica philosophica*, where we can find the quoted statement.

[20]See Girard, A. [**38**] in Viète and alii [**83**, 139] and in Struik, D. J. [**77**, 85]. See also Tropfke, J. [**79**, III(2), 95] for further details.

[21]There are opposed opinions about the real content in these formulations. Whilst for Smith [Smith, D. E. [**73**, II, 471]] "this law was more clearly expressed by Descartes

Later another mathematician, named Rahn [or Rohnius], also gave a clear statement of the law in his *Teutschen Algebra* [**66**].

The question about these formulations of the Theorem is the following: *these algebraists accepted real and complex numbers and only them as solutions of equations?* The answer is not easy nor clear. Girard accepts the "*impossible solutions*" with these words

> *Someone could also ask what these impossible solutions are. I would answer that they are good for three things: for the certaintly of the general rule, for being sure that there are no other solutions, and for its utility*[22].

Descartes, by his side, realized the fact that an equation of the nth degree has exactly n roots[23]. But, for Descartes, the *imaginary roots* do never correspond any real quantity[24].

[**19**], who not only stated the law but distinguished between real and imaginary roots and between positive and negative real roots in making the total number", for Remmert [Remmert, R. [**68**, 100]], contrarily, "Descartes takes a rather vague position on the thesis put forward by Girard".

[22]Girard, A. [**38**] in Viète and alii [**83**, 141]. In other side [Viète and alii [**83**, 142]] he says : "Thus we can give three names to the other solutions, seeing that there are some which are greater than nothing, other less than nothing, and other enveloped, as those which have $\sqrt{-}$, like $\sqrt{-3}$ or other similar numbers."

Remmert, R. [**68**, 99], goes further. He says: "He thus leaves open the possibility of solutions which are not complex". Remmert thinks that, in his ambiguity, Girard leaves an open door to the solutions more complicated than the complex. The problem consists to know the exact sense of the Girard's words "*impossible solutions*" because, for him, "there are no other solutions". [About this question see also Gilain, C. [**37**, 93–95].]

[23]This assert is debt to the Descartes' text [see Descartes, R. [**19**, 380]. English translation in Smith, D.E.-Latham, M. [**75**, 175]]:

> *Neither the true nor false roots are real; sometimes they are imaginary; that is, while we can always conceive of as many roots for each equation as I have already assigned, yet there is not always a definite quantity corresponding to each root so conceived of. Thus, while we may conceive of the equation*
>
> $$x^3 - 6x^2 + 13x - 10 = 0$$
>
> *as having three roots, yet there is only one real root, 2, while the other two, however we may increase, diminish, or multiply them in accordance with the rules just laid down, remains always imaginary.*

In this text there is a rather interesting classification signifying that we may have positive and negative roots that are imaginary.

It seems that for Descartes the roots are always real or imaginary and no other kind of root is possible. [About with this oppinion, see Gilain, C. [**37**, 95–97].]

The use of word *imaginary* in his actual sense begin here [see Smith, D.E.-Latham, M. [**75**, 175, footnote 207]].

[24]Descartes confess that one is quite unable to visualize imaginary quantities [see

This impossibility or difficulty for visualizing imaginary quantities was perhaps the reason which carried the English mathematician John Wallis to give a *geometrical interpretation* in his *Treatise of Algebra* of 1685[25].

He says: "The Geometrical Effection, therefore answering to this Equation

$$a \cdot a \mp b \cdot a + c = 0$$

may be this"[26].

Smith, D.E.-Latham, M. [**75**, 187]]. As says Remmert, R. [**67**, 58], "Newton regarded complex quantities as indication of the insolubility problem". In the Newton's own words: "But it is just that the Roots of Equation should be impossible, lest they should exhibit the cases of Problems that are impossible as if they were possible" [Newton, I. [**59**] 2nd ed., 193].

[25]This representation is quoted in Smith, D. E. [**74**, 46–54]. [See also Stillwell, J. C. [**76**, 191–192]]. In a letter to Collins, May 6, 1673, Wallis suggests a construction a little different from any of the constructions found in his *Algebra* [see Cajori, F. [**13**]]. We shall see this alternative construction here:

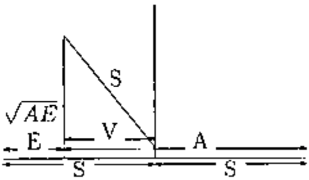

Figure 1a

"This imaginable root in a quadratic equation I have had thoughts long since of designing geometrically, and have had several projects the that purpose. One of them was this:
Supposing a quadratic equation

$$A^2 - 2SA + AE = 0.$$

If $S\left[=\frac{A+E}{2}\right]$ be bigger than $\sqrt{AE}$ [that is $S^2 > AE$], the roots are $S \pm \sqrt{S^2 - AE} = \begin{cases} A \\ E \end{cases}$, putting ..., $S = \frac{1}{2}Z$ and ... $V = \frac{1}{2}X$, where $V\left[=\sqrt{S^2 - AE}\right]$ added to and takes from S, yields $S + V = A$, $S - V = E$, that is, [the roots are] $S \pm \sqrt{V^2}$ [see figure 1a].
But if AE be bigger than S^2, the roots are $S \pm \sqrt{S^2 - AE} = \left[S \pm \sqrt{V^2}\right]$, where

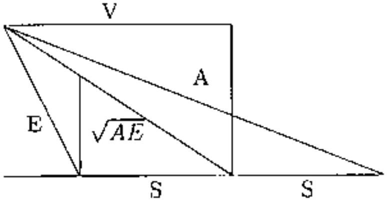

Figure 1b

$\sqrt{AE}$, which was the sine, now become the secant, and V, that was the cosine, is now the tangent [see figure 1b]. For $S^2 - AE = V^2$, the difference of the plane S^2 and AE, the greater is to be expressed by the hypotenuse, and the lesser by the perpendicular."

[26]Wallis, J. [**87**] in Smith, D. E. [**74**, 52]. Wallis calls the independent term ae. It is the product of two roots a and e of the equation $a \cdot a \mp b \cdot a + ae = 0$.

Before this, Wallis offers us the following calculation for solving

$$a \cdot a - 2a\sqrt{175} + 256 = 0.$$

On $AC\alpha = b$ bisected in C, erect a Perpendicular $CP = \sqrt{c}$. And taking $PB = \frac{1}{2}b$ make a Rectangular Triangle [figure 3a].

If $PC = \sqrt{c} < \frac{1}{2}b = PB$, then the solutions are *Real* and are precisely AB and $B\alpha$.

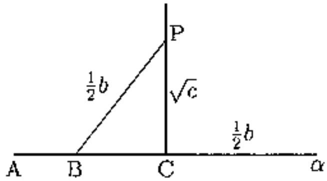

Figure 3a

[In this case $AB = \frac{1}{2}b - \sqrt{\frac{1}{4}b^2 - c}$ and $\alpha B = \frac{1}{2}b + \sqrt{\frac{1}{4}b^2 - c}$ [see Smith, D. E. [**74**, 53]].]

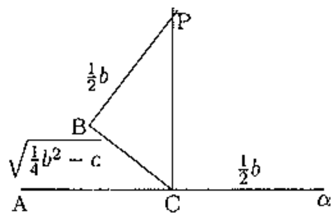

Figure 3b

But if $PC > PB$, "the above construction fails" and "the Right Angle will be at B". Then the solutions are *Imaginary* and are AB and αB [see figure 3b]. [Now $AB = \frac{1}{2}b - i\sqrt{c - \frac{1}{4}b^2}, \alpha B = \frac{1}{2}b + i\sqrt{c - \frac{1}{4}b^2}$. Wallis uses the later BC to obtain the *imaginary part* of the solution.]

This geometrically representation was not accepted by the mathematicians[27] and would be still necessary to wait a hundred years to obtaining

The solutions are $\alpha = \sqrt{175} + \sqrt{-81}$ and $c = \sqrt{175} - \sqrt{-81}$. The geometrical representation is [following Walllis, J. [**87**] in Smith, D. E. [**74**, 50–51]]:

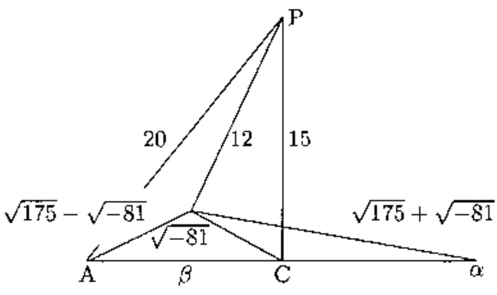

Figure 2

[27]In Stillwell, J. [**76**, 192], we can see the Wallis' figures and the modern

the correct and acceptable representation[28]. We shall not comment this work.

2. The technique of integration and complex quantities

The eighteenth century use of the integral concept was limited. Newton represented the transcendental functions as series and integrated these functions term by term[29]. Gottfried Wilhelm Leibniz and Johann Bernoulli treated the integral as the inverse of the differential[30].

In this context the decomposition of rational fractions [or functions] into partial [or simple] fractions made possible a decisive step in integral calculus[31].

The problem was calculate the integral

$$\int \frac{P}{Q} dx,$$

where P and Q are polynomials and $\deg(P) < \deg Q$ and, for getting it, Gottfried Wilhelm Leibniz and Johann Bernoulli, together other mathematicians of his time, saw the necessity to express every real polynomial as product of real factors of first and second degree[32]. This fact shows us that they had very much confidence in the *Fundamental Theorem of Algebra*[33].

representation.

[28] The satisfactory *geometrically representation of complex quantities* was carried by the Norwegian mathematician Caspar Wessel in 1797 and independently by the Swiss Jean Robert Argand in 1806. This last work, despite its considerable merit, remained unnoticed until a French translation appercad in 1897.

[29] See Pla, J. [**64**, 9–20].

[30] See Kline, M. [**44**, 406]: "If $dy = f'(x) \cdot dx$, then $y = f(x)$. That is, a Newtonian antiderivative was chosen as the integral, but differentials were used in place of Newton's derivatives".

[31] The existence of an integral was never questioned.

[32] The *Arithmetica Universalis* of Isaac Newton contains, as we have said before, the substance of Newton's lectures from 1673 to 1683 at Cambridge. In it are found many important results in equations theory, such as the fact that the imaginary roots of a real polynomial "must occur in conjugate pairs". This fact is a very important result and it was naturally accepted by the mathematicians of the end of seventeenth century. But, following Leibniz, this fact presents difficulties, as we shall see next.

[33] See Leibniz, G. W. [**49**], [**51**] and Bernoulli, Jh. [**6**].

The chance did that in 1702, July 10, Johann Bernoulli, thinking to enunciate him a new result, wrote to Leibniz that had found the integral of differential quantities $\frac{p}{q}dx$, where p and q are polynomials. But Leibniz responded: "No only I have already the solution of this problem, but moreover I have it from the first years in which I practiced the higher geometry. In this result I have seen an essential component of

The exact Bernoulli's text is:

> *Let the differential be $pdx : q$ which p and q express rational quantities composed arbitrarily of a single variable x and constants; one seeks the integral or the algebraic sum or the means of reducing it to the quadrature of the hyperbola or the circle, the one or the other always being possible*[34].

And next he says that $\frac{dx}{x\pm a}$ is the differential of logarithm of $x \pm a$. Therefore

$$\int \frac{a\,dx}{x+f} + \int \frac{b\,dx}{x+g} + \int \frac{c\,dx}{x+h} + \cdots = \\ = \log\left\{(x+f)^a \cdot (x+g)^b \cdot (x+h)^c \cdot \cdots\right\}.$$

But the remarkable question is that "complex numbers made their entry to the theory of circular functions". The process is the following: he observes that "one transforms the differential $\frac{adz}{b^2-z^2}$ into a logarithmic differential $\frac{adt}{2bt}$ and reciprocally"[35] and, as a corollary, "one transforms the differential $\frac{adz}{b^2+z^2}$ in the same way into $-\frac{adt}{2bit}$, an imaginary logarithmic differential and reciprocally"[36]. But then he observes that $\frac{adz}{b^2+z^2}$ can

my science of the infinite, and hence of the integral analysis ..." [Leibniz, G. W. [**49**, 703]].

"When, in 1746, Jean le Rond d'Alembert drew people's attention to the need to prove that theorem, he was to cite Bernoullis' paper as a particulary important use of it" [see Fauvel, J.-Gray, J. [**31**, 435]].

[34] Bernoulli, Jh. [**6**] in *Opera Omnia*, I, 393 or Fauvel, J.-Gray, J. [**31**, 439].

[35] He uses the change of variable $z = b\frac{t-1}{t+1}$ and observes that $\frac{adz}{b^2-z^2}$ goes over into $\frac{adt}{2bt}$ [see Fauvel, J.-Gray, J. [**31**, 438]]. How does Johann Bernoulli obtain this result? It is clear that Johann Bernoulli knows the integral of rational functions as $\frac{a}{b^2-z^2}$, because he knows the decomposition of the rational functions into simple functions:

$$\frac{a}{b^2-z^2} = \frac{a}{2b}\cdot\frac{1}{b-z} + \frac{a}{2b}\cdot\frac{1}{b+z}.$$

And then he applies his technique and obtains

$$\int \frac{a\,dz}{b^2-z^2} = \frac{a}{2b}\cdot\log\frac{b+z}{b-z} = \frac{a}{2b}\cdot\int\frac{dt}{t}, \text{ where } t = \frac{b+z}{b-z},$$

and then $z = b\cdot\frac{t-1}{t+1}$.

[36] Similarily, the differential $\frac{adz}{b^2+z^2}$ goes over, by the substitution $z = bi\frac{t-1}{t+1}$, into $-\frac{adt}{2bit}$ [see Fauvel, J.-Gray, J. [**31**, 438]].

Remember that, in 1699, Jakob Bernoulli had evaluated [see Bernoulli, Jk. [1699], 868–870] the integral of $\frac{a^2dx}{a^2-x^2}$, using the change of variable $x = \frac{a}{2}\cdot\frac{b^2-t^2}{b^2+t^2}$; this converts the integrand $\frac{a^2dx}{a^2-x^2}$ into $\frac{dt}{2at}$. [See Kline, M. [**44**, 407].]

be transformed also [using now $z = \sqrt{\frac{1}{t} - b^2}$] into the differential of "a sector or circular arc $-\frac{adt}{2\sqrt{t-b^2t^2}}$ and reciprocally". Finally he observes that the integral of

$$\frac{adz}{b^2+z^2}$$

> *depends on the quadrature of the circle, and moreover*
>
> $$\frac{adz}{b^2+z^2} = \frac{1}{2b}\cdot\frac{adz}{b+iz} + \frac{1}{2b}\cdot\frac{adz}{b-iz}$$
>
> *which are two differentials of imaginary logarithms: one sees that imaginary logarithms can be taken for real circular sectors because the compensation which imaginary quantities makes on being added together of destroying themselves in such a way that their sums is always real*[37].

We have observed there the introduction of *imaginary logarithmic differential* into the integration of rational functions[38].

[37] Bernoulli, Jh. [6] in Fauvel, J.-Gray, J. [**31**, 439]. In fact Bernoulli obtains

$$\tan^{-1} z = \frac{1}{2i}\cdot\log\frac{i-z}{i+z}.$$

In this sense it is interesting to note that, several years later, in 1712, Johann Bernoulli carried out the integration to obtain an algebraic relation between $\tan n\theta$ and $\tan\theta$. His argument is as follows. Given

$$y = \tan n\theta, \quad x = \tan\theta,$$

we have

$$n\,\theta = \tan^{-1} y = n\cdot\tan^{-1} x;$$

hence, taking differentials,

$$n\,d\theta = \frac{dy}{1+y^2} = n\cdot\frac{dx}{1+x^2}$$

and then

$$\left[\frac{1}{y+i} - \frac{1}{y-i}\right]\cdot dy = n\left[\frac{1}{x+i} - \frac{1}{x-i}\right]\cdot dx.$$

Integration gives

$$\log\frac{y+i}{y-i} = \log\left[\frac{x+i}{x-i}\right]^n$$

and whence

$$(x-i)^n\cdot(y+i) = (x+i)^n\cdot(y-i).$$

[38] We do not explain the history of *imaginary logarithms*. But there are many papers on complex logarithms as, for example, Cajori, F. [**14**], Kline, M. [**44**, 407–408]; Naux, I. [**58**] and Stillwell, J. [**76**, 220–222].

But this situation is not easier than it seems. In his presentation about the integral of rational functions, Leibniz shows us a difficulty, a limitation or merely a question. It is always possible decompose a real polynomial into a product of real lineal factors or real quadratic factors?[39] or, every polynomial has always a real and complex root and, with every complex root, has also the conjugate complex root? Although always Leibniz is clear and rotund when he says

> *As soon as I had found my Arithmetic Quadrature, reducing the quadrature of circle into a rational quadrature and observing that the sum*
>
> $$\int \frac{dx}{1+x^2}$$
>
> *depends of the quadrature of the circle, I immediately observed that a time reduced to the summation of a rational expression, all quadrature can be converted in many kinds of summation of the more simple. And I will show, by a decomposition proceeding of a new genus because it must be in this manner. This proceeding consists to convert a product of factors into a sum; this is, to transform a fraction with a denominator of higher degree, equal to product of roots, into a sum of fractions with simple denominators,*[40]

when he must integrate $\int \frac{dx}{x^4+a^4}$ he finds a problem. It is possible obtain $\frac{1}{x^4+a^4}$ to multiply $\frac{1}{x^2+ia^2}$ by $\frac{1}{x^2-ia^2}$, but they are not real. And it is not possible to obtain a real decomposition, because

[39]This assert is absolutely clear in Newton, I. [**59**], as we have seen in the footnote 32.

[40]Leibniz, G. W. [**50**, 351–352].

In this work Leibniz obtains naturally the integration of rational functions, as for example

$$\int \frac{dx}{x^2-1} = \frac{1}{2}\cdot\int \frac{dx}{x-1} - \frac{1}{2}\cdot\int \frac{dx}{x+1},$$

although "$\int \frac{dy}{y}$ is the quadrature of the hyperbola".

Next year Leibniz studies the case in which the roots are not simple and therefore the sum is transformated into the sum of fractions with multiple denominators [see Leibniz, G. W. [**51**]].

$$x^4 + a^4 = \left[x + a\sqrt{i}\right] \cdot \left[x - a\sqrt{i}\right] \cdot \left[x + a\sqrt{-i}\right] \cdot \left[x - a\sqrt{-i}\right]^{41}$$

and therefore it is not possible to reduce $\int \frac{dx}{x^4+a^4}$ to the quadrature of the circle nor to the quadrature of the hyperbola. It would be necessary to introduce the quadrature of $\int \frac{dx}{x^4+a^4}$ as a new function[42].

There is neither hesitation about the importance which Leibniz granted the complex numbers and his contributions, "when they were almost forgotten", were remarkable[43]. Between these it is interesting to observe that he obtained *an imaginary decomposition of a positive real number* which surprised his contemporaries and enriched the theory of imaginaries:

$$\sqrt{6} = \sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}}^{44}.$$

[41] Leibniz does not observe that

$$x^4 + a^4 = [x^2 + a\sqrt{2}x + a^2] \cdot [x^2 - a\sqrt{2}x + a^2].$$

The possible mistake is debt to have begun by the *complex conjugate decomposition*

$$x^4 + a^4 = [x^2 + ia^2] \cdot [x^2 - ia^2].$$

[42] We have already introduced the quadrature of the hyperbola $\int \frac{dx}{x+a}$ and the quadrature of circle $\int \frac{dx}{x^2+a^2}$. Then, says Leibniz, "I wait that we will be able to follow this progression and we will found the problems related with $\int \frac{dx}{x^4+a^4}, \int \frac{dx}{x^8+a^8}, \ldots$" [see Leibniz, G. W. [**50**, 360]].

[43] Moreover, for Leibniz, complex numbers are the natural consequence of have accepted real numbers: "From the irrationals are born the impossible or imaginary quantities whose nature is very strange but whose usefulness is not to be despised" [see Leibniz, G. W. [**50**, 51]].

[44] See a letter from Leibniz to Huygens, writen in 1674 or 1675 [Gerhardt, C. I. [**36**, 563] and see also Hofmann, J.E. [1972], 147 and McClenon, R. B. [**55**]]: "I once came upon two equations of this kind $x^2 + y^2 = b, x \cdot y = c$". He obtains then

$$y = \sqrt{\frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2}} \quad \text{and} \quad x^2 - \frac{b}{2} + \sqrt{\frac{b^2}{4} - c^2} = 0 \quad \text{or} \quad x = \sqrt{\frac{b}{2} - \sqrt{\frac{b^2}{4} - c^2}}.$$

Then

$$d = x + y = \sqrt{\frac{b}{2} + \sqrt{\frac{b}{4} - c^2}} + \sqrt{\frac{b^2}{2} - \sqrt{\frac{b^2}{4} - c^2}} \qquad \text{or} \quad d^2 = b + 2c.$$

Moreover, as says Boyer, "Leibniz did not write the square roots of complex numbers in standard complex form, nor was he able to prove his conjecture that

$$f(x+iy)+f(x-iy) \quad \text{is real},$$

if $f(z)$ is a real polynomial."[45]

Finally in an unpublished Leibniz's paper[46] appears the so-called *de Moivre's formula*. He does not explain how he found it, but it is comprehensible to us as

$$2y = \sqrt[n]{x+\sqrt{x^2-1}} + \sqrt[n]{x-\sqrt{x^2-1}},$$

where $x = \cos\theta, y = \cos\frac{\theta}{n}$[47]. But these important mathematical contributions did not enough to clarify the nature and reality of the complex numbers.

Finally

$$(*) \qquad \sqrt{b+2c} = \sqrt{\frac{b}{2}+\sqrt{\frac{b}{4}-c^2}} + \sqrt{\frac{b^2}{2}-\sqrt{\frac{b^2}{4}-c^2}}$$

If we put $b=2$ and $c=2$, there results $\sqrt{6} = \sqrt{1+\sqrt{-3}} + \sqrt{1-\sqrt{-3}}$.

But that what results really surprising is the use of Cardano's rule by obtaining this kind of results. Taking Albert Girard's equation

$$x^3-13x-12=0,$$

whose true root is 4 and the sum of roots is zero, Leibniz obtains

$$4 = 2+\sqrt{-\frac{1}{3}}+2-\sqrt{-\frac{1}{3}} = \sqrt[3]{6+\sqrt{\frac{-1225}{27}}} + \sqrt[3]{6-\sqrt{\frac{-1225}{27}}}.$$

By using the equation

$$x^3-48x-72=0,$$

he shows finally that

$$-6 = \sqrt[3]{36+\sqrt{-2800}} + \sqrt[3]{36-\sqrt{-2800}}$$

Hofmann says us "the identity $(*)$ is implicit in Euclide's book X, 47–54 [if $4c^2 < b^2$], but "nobody noticed it at the time".

[45]Boyer, C. B. [**11**, 444]. This conjecture is done by Leibniz in Gerhardt, C. I. [**36**, 550].

[46]Leibniz, G. W. [**49**].

[47]See Hofmann, J.E. [1972], 145–146; Schneider, I. [**72**, 224–229] and Stillwell, J.

Leibniz adventures his mistic nature, saying: "The nature, mother of the eternal diversities, or the divine spirit, are zaelous of her variety by accepting one and only one pattern for all things. By these reasons she has invented this elegant and admirable proceeding. This wonder of Analysis, prodigy of the universe of ideas, a kind of hermaphrodite between existence and non-existence, which we have named *imaginary roots*"[48].

This mysterious character stood during several centuries, may be until the Euler's time with the contributions of the own Euler and d'Alembert.

Kline is absolutely clear in this sense:

> *Complex numbers were more of a bane to the eighteenth-century mathematicians. These numbers were practically ignored from their introduction by Cardan until about 1700. Then complex numbers were used to integrate by the methode of partial fractions, which was followed by the lengthy controversy about complex numbers and the logarithms of negative and complex numbers. Despite his correct resolution of the problem of the logarithms of complex numbers, neither Euler nor the other mathematicians were clear about those numbers.*
>
> *Euler tried to understand what complex numbers really are, and in his "Vollständige Anleitung zur Algebra", which first appeared in Russian in 1768–69 and in Germany in 1770 and is the best algebra text of the eighteenth century, says,*
>
>> Because all conceivable numbers are either greater than zero or less than 0 or equal to 0, then it is clear that the square roots of negative numbers cannot be included among the possible numbers [real numbers]. Consequently we must say that these are impossible numbers. And this circumstance leads us to the concept of such numbers, which by their nature are impossible, and ordinarily are called imaginary or fancied numbers, because they exist only in the imagination
>
> *Euler made mistakes with complex numbers. In this Algebra he writes* $\sqrt{-1}\cdot\sqrt{-4} = \sqrt{4} = 2$, *because* $\sqrt{a}\cdot\sqrt{b} = \sqrt{a\cdot b}$. *He also gives* $i^i = 0.2078795763$, *but misses other values of this quantity*[49].

[**76**, 56–57]. Moreover Leibniz is conscious of this result and "when it appeared in De Moivres's paper in the *Philosophical Transactions*, **20**, n^o 240 of May 1698 (published in 1699), Leibniz —quite modestly— put in his rightful claim of authority in *Acta Eroditorum* (May 1700): 199–208 [Gerhardt, C. I. [**35**, V, 346–347]]" [see Hofmann, J.E. [1972], 146, footnote 17].

[48] Leibniz, G. W. [**49**, 357].

[49] Kline, M. [**44**, 594].

3. The three first attempts to prove the Fundamental Theorem of Algebra

One possible enunciate of the *Fundamental Theorem of Algebra*[50] is:

Every polynomial $P(x)$ with real coefficients has a complex root.

Before 1799, year in what Karl Friedrich Gauss gave his first rigorous proof of *Fundamental Theorem of Algebra*[51], three important mathematicians had already made three attempts to prove the Theorem. The first is debt to a French mathematician and philosopher, Jean le Rond d'Alembert, and was published in 1748, but elaborated in 1746. Three years later, in 1749, Leonhard Euler gave an algebraic demonstration, very different of the d'Alembert's demonstration. This demonstration was completed by Joseph Louis Lagrange in 1772[52]. Several years later another French mathematician, Pierre Simon Laplace, tried to prove the Theorem. It was the year 1795[53].

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[50] There are excellent papers about the *Fundamental Theorem of Algebra*. See, for example, Bashmakova, I. [**4**], Dieudonné, J. et alii [**20**, 68–71], Gilain, C. [**37**], Houzel, C. [**42**], Petrova, S. S. [**61**], Remmert, R. [**67**] and van der Waerden, B.L. [1980], 94–102.

The Givains's text offers us a distintion between the *Fundamental Theorem of Algebra* —sometimes known as the d'*Alembert's Theorem*— and the *Theorem of linear factorization* —sometimes known as the *Kronecker Theorem*— very clever for understand posterior developments and clarify the different kinds of demonstrations [see Gilain, C. [**37**, 92]].

But I think that, historically, this distintion is not clear. The former mathematicians to Gauss was not conscious of that fact.

[51] Gauss considered the Theorem so important that he gave *four proofs*; the principles on which the first is based was discovered by Gauss in October 1797, but the proof was not published until 1799. In this proof, similar to d'Alembert's attempt of proof, he does not introduce *complex numbers*. He proves the Theorem in the form:

> *Every polynomial $P(x)$ with real coefficients can be factored into linear or quadratic factors.*

The second and third proofs of Theorem were published in 1816. The second proof is purely algebraic, following perhaps the Euler's intention. The forth proof is based in the same principle of the first and was published in 1849. In this proof Gauss uses already complex numbers more freely because, he says, "they are now common knowledge". In the third proof he used, in fact, that what we today know as the *Cauchy integral theorem*.

A half century dedicated by Gauss to prove the Theorem.

Following these different demonstrations we can find precisely the differences noted by Gilain.

[52] The Euler and Lagrange attempts were published, respectively, in 1751 and 1774.

[53] Pierre Simon Laplace made an attempt to prove the Theorem, quite different from the Euler-Lagrange attempt but also algebraic, in his *Leçons de mathématiques donnés a l'Ecole Normal*, published in 1812.

Really therefore was Euler the first of these three mathematicians which asserted the true of the Theorem. So in a letter to Nikolaus Bernoulli, Euler ennuniates the *factorization theorem* for real polynomials, closing the question posed by Leibniz[54].

[54] We have already seen that "does not seem to have ocurred to Leibniz that $\sqrt{i}$ could be of the form $a + b\,i$, because if he had seen that

$$\sqrt{i} = \frac{1}{2}\sqrt{2} \cdot [1 + i] \quad and \quad \sqrt{-i} = \frac{1}{2}\sqrt{2} \cdot [1 - i],$$

he would have noticed that the product of the factors

$$[X + a\sqrt{i}] \cdot [X + a\sqrt{-i}] \text{ and } [X - a\sqrt{i}] \cdot [X - a\sqrt{-i}]$$

are both reals and then he would have obtained

$$X^4 + a^4 = [X^2 + a\sqrt{2}X + a^2] \cdot [X^2 - a\sqrt{2}X + a^2].$$

So he wuold have avoid his mistake. It is remarkable that he should not have been led to this factorization by the simple advice for writing $X^4 + a^4 = [X^2 + a^2]^2 - 2a^2X^2$" [see Remmert, R. [**67**, 100]].

See also Kline, M. [**44**, 597–598]: "... Leibniz did not believe that every polynomial with real coefficients could be decomposed into linear and quadratic factors. Euler took the correct position. In a letter to Nikolaus Bernoulli of October 1, 1742, Euler affirmed without proof that a polynomial of *arbitrary degree* with real coefficients could be so expressed [see Euler, L. [1862], I, 525]. Nikolaus did not believe the assertion to be correct and gave the example of

$$x^4 - 4x^3 + 2x^2 + 4x + 4$$

with the imaginary roots $1 + \sqrt{2 + \sqrt{-3}}$, $1 - \sqrt{2 + \sqrt{-3}}$, $1 + \sqrt{2 - \sqrt{-3}}$, $1 - \sqrt{2 - \sqrt{-3}}$, which he said contradicts Euler's assertion [see Euler, L. [**27**, II, 695]]". On December 15, 1742, Euler into a letter to Goldbach [see Euler, L. [**27**, I, 170–171]], after assert that he doubted once when he saw this example, did it doubt once seen the example, "pointed out the *complex roots occur in conjugate pairs*, so the produt of $x - [a + b\,i]$ and $x - [a - b\,i]$, wherein $a + b\,i$ and $a - b\,i$ are a conjugate pair, gives a quadratic expression with real coefficients. Euler then showed that his was true for Bernoulli's example. But Goldbach, too, rejected the idea that every polynomial with real coefficients can be factored into real factors and gave the example $x^4 + 72x - 20$ [see the letter from Goldbach to Euler of february 5, 1743 in Euler, L. [**27**, I, 193]]. Euler then showed Goldbach that the later had made a mistake and that he [Euler] had proved this theorem for polynomials up to the sixth degree. However, Goldbach was not convinced, because Euler did not succeded in giving a general proof of this assertion".

The reader interested to follow the succession of these letters can see, for example, Gilain, C. [**37**, 106–108].

Next year, in a very important paper[55], Euler thinks about the homogeneous nth-order differential equation with constant coefficients

$$[1] \qquad 0 = Ay + B\,\frac{dy}{dx} + C\,\frac{d^2y}{d^2x} + D\,\frac{d^3y}{d^3x} + \cdots + L\,\frac{d^ny}{d^nx},$$

where $A, B, C, D, \ldots, L$ are constants. He points out that the general solution of [1] must contain n arbitrary constants and the solution will be a sum of n particular solutions y_j, every one multiplied by an arbitrary constant. So the general solution of y has the form

$$[2] \qquad y = C_1\,y_1 + C_2\,y_2 + \cdots + C_n\,y_n.$$

Then he makes in [1] the substitution

$$y = e^{\left[\int r\,dx\right]}, \quad \text{with } r \text{ constant},$$

and obtains the polynomial equation in r,

$$[3] \qquad A + B\,r + C\,r^2 + \cdots + L\,r^n = 0.$$

In fact, the general solution depends of the factorization of the polynomial [3] and of the nature of its roots —reals or complex; simple or multiple—, and indirectly his result depends essentially of the Fundamental Theorem[56].

[55] Euler, L. **[22]**.

[56] Each root r_j of the polinomial equation [3] furnishes a partial solution into the sum [2] in accordance with the nature of each root $r_j, j = 1, \ldots, n$:

- if r_j is a real simple root of [3], then it furnishes into the sum [2] the sumand

$$z_j = D_j\,e^{r_j\,x};$$

- if r_j is a multiple real root of multiplicity k, the k equal roots r_j furnish into the sum [2] the sumand

$$z_{j,k} = e^{r_j\,x}\,\left[D_0 + D_1\,x + \cdots + D_{k-1}\,x^{k-1}\right]$$

- if $r_j = \alpha_j + i\,\beta_j$ is a simple complex root of [3], then it and its conjugate $\overline{r_j} = \alpha_j - i\,\beta_j$ furnish into the sum [2] the sumand

$$z_j^* = e^{\alpha_j\,x}\,[D_1^*\,\cos\beta_j\,x + D_2^*\,\sin\beta_j\,x]$$

and finally,

- si $r_j = \alpha_j + i\,\beta_j$ is a multiple complex root of multiplicity k, then the k equal roots $r_j = \alpha_j + i\,\beta_j$ and their k conjugate roots furnish into the sum [2] the

But, as we have already said, the first attempt of demonstration of the *Fundamental Theorem of Algebra* is debt to d'Alembert[57].

3.1. The d'Alembert's attempt.

Really d'Alembert proves the existence of the root of $P(x)$ in two steps[58]:

1. There is the *minimum* x_0 of the module $|P(x)|$;
2. The *d'Alembert's lemma*: if $P(x_0) \neq 0$, then any neighborhood of

sumand

$$z^*_{j,k} = \sum_{\ell=0}^{k-1} e^{\alpha_j x} x^\ell \left[D_1^{*\,\ell} \cos\beta_j x + D_2^{*\,\ell} \sin\beta_j x\right]$$

Somewhat later [Euler, L. [**24**]] he treated the nonhomogeneous nth-order linear differential equation

$$X(x) = Ay + B\,\frac{dy}{dx} + C\,\frac{d^2y}{d^2x}.$$

[57]D'Alembert remembers the Johann Bernoulli's text and then he says: "Nobody, what I know, have went more far [in the question of the decomposition of polynomials], if we exclude mister Euler, which in the tome VII of *Miscellanea Berolinensia* declares that he has demostrated the proposition in the general case. But I seem me that Euler never has published yet on this theorem [d'Alembert, J. le Rond [**2**, 183]].

[58]See d'Alembert, J. le Rond [**2**] and Petrova, S. S. [**62**]. In the d'Alembert's words:

> *In order to reduce in general a differential rational function to the quadrature of the hyperbola or to that of the circle, it is necessary, according to the method of M. Bernoulli* [*Mem. Acad. Paris*, 1702], *to show that every rational polynomial, without a divisor composed of a variable x and of constants, can always be divided, when it is of even degree, into trinomial factors $xx + fx + g, xx + hx + i$, etc., of which all coeficients $f, g, h, i, \dots$ are real. It is clear that this difficulty affects only the polynomial that cannot be divided by any binomial $x + a, x + b$, etc., because we can always by divison reduce to zero all the real binomials, if two are any, and it can easily be seen that the products of there binomials will give real factors $xx + fx + g$ [see Struik, D. J. [**77**, 89, footnote 1]]*.

x_0 contains a point x_1 such that $|P(x_1)| < |P(x_0)|$[59].

Then, if 1 and 2 are true and x_0 is the point in which $|P(x)|$ atteints the minimum, then $|P(x_0)| = 0$. This is the sketch of the d'Alembert's proof[60].

The second step is, for d'Alembert, the more important[61] and the proof offered by d'Alembert depends essentially on the Newton's method

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[59]D'Alembert accepts without demonstration the step 1 and the Newton's method. A simple elementary proof of d'*Alembert lemma* was given by Argand in 1806. This mathematician was one of the co-discoverers of the geometric representation of complex numbers. He represents the complex numbers as a vectors into the plan. Then

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is a vector OA_{n+1}. The demonstration consists to see that it is possible to choose x such that the point A_{n+1} coincides with O. By seeing this, he explains

$$P(x_0 + \Delta x) = P(x_0) + A \Delta x + \text{ terms in } (\Delta x)^2, (\Delta x)^3, \cdots = P(x_0) + A \Delta x + \epsilon$$

where A is constant and $|\epsilon|$ is small compared to $|\Delta x|$ when $|\Delta x|$ is small. Then, choosing the adequate direction of vector Δx, it is possible obtain that $A \Delta x$ was opposite in direction to $P(x_0)$. Then

$$|P(x_0 + \Delta x)| < |P(x_0)|.$$

[See Dörrie, H. [**21**, 108–112], or Stillwell, J. [**76**, 197–200].]

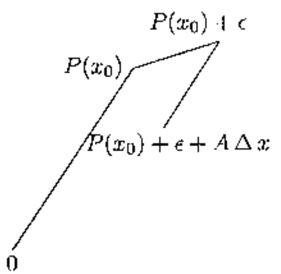

Figure 4

[60]By seeing a complete proof of this kind, see, for example, Aleksandrov et alii [**1**]; Dörrie, H. [**21**, 108–112], or Rey Pastor, J. et alii [**69**, 239–241].

[61]The first step was naturally accepted in the eighteenth century. The rigorous demonstration can be seen into Cauchy, A. [1821], Ch. X: "For every polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{C}[x],$$

there is a $x_0 \in \mathbb{C}$ such that $|P(x_0)| = \inf \left| P \left\langle \mathbb{C} \right\rangle \right|$.

of polygon[62]. Applying this, d'Alembert obtains

$$(*) \qquad x - x_0 = \sum_{k \geq 0} c_k \cdot [y - y_0]^{q_k}.$$

The equation $(*)$ shows that, if y is a real point very close to y_0, it is the image of any x which appears into the form $p + q\sqrt{-1}$[63]. Then the demonstration of the Theorem is founded if we can prove that $y_0 = 0$ is the image of any x [which will be naturally real or imaginary].

D'Alembert examines the set of real images y and takes the minimum y_0 which associate x is of complex form. But following the development $(*)$, all real number y very close to y_0 must be also an image of the complex numbers x. Then, if $y_0 \neq 0$, there is an image closer to zero than y_0. Contradiction. This contradiction establishes the Theorem.

It is interesting to note two important facts which were observed by d'Alembert into his work. The first are corollaries I and II and proposition III[64] and says: "if a complex number $a + b\sqrt{-1}$ is a root of the polynomial $P(x)$, then $a - b\sqrt{-1}$ is another rooot of $P(x)$ and then $P(x)$ can always be decomposed into quadratic factors of the kind $xx+mx+n$".

The second fact, contained in the demonstration but not mencioned explicitly[65], is: "if $P(x)$ is a real polynomial and we substitute x by a complex number $z = z_1 + i\,z_2$, where z_1, z_2 are real numbers, then we obtain $P(z) = Q_1(z_1) - i\,Q_2(z_2)$, where $Q_1(x)$ and $Q_2(y)$ are real polynomials. Then $P(z) = 0$ iff $Q_1(z) = 0$ and $Q_2(z) = 0$"[66].

3.2. The Euler-Lagrange's attempt.

The idea of Euler's demonstration[67] was to decompose every monic polynomial with real coefficients $P(x)$ of degree $2^n \geq 4$ into a product

[62] See Newton, I. [**59**] and Stillwell, J. [**76**, 125–126]. The sense of this theorem is the following: "To every pair (x_0, y_0) of complex numbers with $y_0 - P(x_0) = 0$, there correspond an increasing series $\{q_k\}$ of rational numbers such that

$$x - x_0 = \sum_{k \geq 0} c_k \cdot [y - y_0]^{q_k}$$

in a *neighborhood of* y_0". This theorem was proved rigorously by Pusieux in 1850.

It is possible to avoid this theorem like we can see, for example, in Dörrie, H. [**21**, 108–112].

[63] see d'Alembert, J. le Rond [**2**, 189].

[64] See d'Alembert, J. le Rond [**2**, 190–191].

[65] See d'Alembert, J. le Rond [**2**, 186–187].

[66] This fact is essential in the first Gauss' demonstration [see, for example, Hollingsdale, S. [**41**, 319–322]].

[67] See Euler, L. [**23**] and Lagrange, J.-L. [**45**].

$P_1(x) \cdot P_2(x)$ of two monic polynomials with real coefficients of degree $m = 2^{n-1}$.

Thus, if $P(x)$ is a polynomial of the form

$$P(x) = x^{2m} + B\,x^{2m-2} + C\,x^{2m-3} + \dots,$$

the polynomials $P_1(x), P_2(x)$ now take the form

$$x^m + u\,x^{m-1} + \alpha\,x^{m-2} + \beta\,x^{m-3} + \cdots$$
$$x^m - u\,x^{m-1} + \lambda\,x^{m-2} + \mu\,x^{m-3} + \cdots$$

Then Euler asserts that $\alpha, \beta, \dots, \lambda, \mu, \dots$ are *real functions* in B, $C, \dots, u$, and that, by elimination of $\alpha, \beta, \dots, \lambda, \mu, \dots$, is obtained a monic real polynomial in u of degree $\binom{2m}{m}$ *whose constant term is negative*. Now this polynomial in u has a zero u by the *intermediate value theorem* as Euler clearly knew[68]. Now we can follow quickly the Euler's steps[69]:

1. If the equation has a root of the form $x + y\sqrt{-1}$, then there is also another of the form $x - y\sqrt{-1}$[70];
2. Every equation of odd degree has a least one root;
3. Every equation of even degree with negative absolute term has at least one positive and one negative root[71].

But it is the forth theorem which gives us the key of his ideas:

Every equation of the forth degree, as

$$x^4 + A\,x^3 + B\,x^2 + C\,x + D = 0$$

can always be decomposed into two real factors in the second degree.

First, setting $x = y - \frac{1}{4}A$, he obtains that every equation of the forth degree can be of the form $x^4 + M\,x^2 + N\,x + P = 0$. If we decompose this equation in two equations of the second degree, we have

$$[x^2 + u\,x + \alpha] \cdot [x^2 - u\,x + \beta] = 0.$$

[68]This fact is also employed by Laplace in his demonstration as we will see next.
[69]See Struik, D.J. [**77**, 99–102].
[70]Then the polynomial has a factor of the form $xx + px + q$.

Euler gives an example of how to decompose an equation of the forth degree into two quadratic factors.

So Euler gives answer to the former problem posed by Nikolaus Bernoulli and Goldbach [see footnote 54].

[71]We have there a partial proof of the *Bolzano-Cauchy theorem* on *Intermediate Value*.

If we compare this product with the proposed equation, we shall find

$$M = \alpha + \beta - u^2, \quad N = [\beta - \alpha]\, u, \quad P = \alpha\,\beta$$

from which we derive

$$u^6 + 2M\, u^4 + [M^2 - 4P]\, u^2 - N^2 = 0,$$

"from which the value of u must be found. And since the absolute term $-N \cdot N$ is essentially negative, we have hope that this equation has at least two real values"[72].

Among the corollaries to Theorem 4 there is the statement that the resolution into real factors is now also proved for the fifth degree, and Scholium II points out that, if the roots of the given fourth-degree equation are x_1, x_2, x_3, x_4, then the sixth-degree equation in u, u being the sum of two roots of the given equation, will have the six roots $x_1+x_2, x_1+x_3, x_1+x_4, x_2+x_3, x_2+x_4, x_3+x_4$. Since $x_1+x_2+x_3+x_4 = 0$, we can write for u the values $u_1, u_2, u_3, -u_1, -u_2, -u_3$, and the equation in u becomes

$$[u^2 - u_1^2] \cdot [u^2 - u_2^2] \cdot [u^2 - u_3^2] = 0^{73}.$$

---

[72]When we take one of them as u, then the values of α and β will also be real, seeing that

$$2\beta = uu + M + \frac{N}{u}, \quad 2\alpha = uu + M - \frac{N}{u}.$$

[73]We can observe that the fourth roots x_1, x_2, x_3, x_4 of the equation

$$[1] \qquad x^4 + M\, x^2 + N\, x + P = 0$$

satisfies

$$[2] \qquad x_1 + x_2 + x_3 + x_4 = 0.$$

Then u can have $\binom{4}{2} = 6$ different values. Then u satisfies an equation of the sixth degree which coefficients are reals

$$[3] \qquad F_6(u) = 0.$$

We have $u_1 = x_1 + x_2, u_2 = x_1 + x_3, u_3 = x_1 + x_4, u_4 = x_2 + x_3, u_5 = x_2 + x_4, u_6 = x_3 + x_4$ and then

$$u_1 = -u_6, u_2 = -u_5, u_3 = -u_4$$

and then the equation [3] has the form

$$F_6(u) = [u^2 - u_1^2] \cdot [u^2 - u_2^2] \cdot [u^2 - u_3^2].$$

Next to, into the theorem 5, he establishes

> *Every equation of degree 8 can always be resolved into two real factors of the forth degree*[74].

The problem consists to see that not only u, but also the other coefficients $\alpha, \beta, \gamma, \delta, \epsilon, \psi$ are reals, a reasoning which Lagrange and, more later, Gauss objected.

Lagrange takes this equation but he observes that when u takes the value 0 into the rational expressions of the other coefficients of $P_1(x)$ and $P_2(x)$ as fonction of u, it is possible obtain undefined coefficinets of the form $\frac{0}{0}$. For avoid this, he takes as unknown [when $a_n = 1$], $v = 2n + a_{n-1}$ and then observes that the "imaginary roots" of the

---

His constant term is $-u_1^2 u_2^2 u_3^2$. The product $u_1^2 u_2^2 u_3^2$ is real? There is. Euler does not explain this with detail. He says only that this product is real because the *fundamental theorem of the theory of symmetric functions*.

We can reasoning this: Despite this product was not a symmetric fonction of the symbols x_1, x_2, x_3, x_4, it is unvariable when we do all possible permutations of the roots of the equation [1], under the condition [2], between the roots of the equation [1]. Really this product can be obtained of the following:

$$u_1^2 u_2^2 u_3^2 = \frac{1}{4} \left\{ \begin{matrix} (x_1 + x_2) \cdot (x_1 + x_3) \cdot (x_1 + x_4) + (x_1 + x_2) \cdot (x_2 + x_3) \cdot (x_2 + x_4) + \\ + (x_4 + x_1) \cdot (x_4 + x_2) \cdot (x_4 + x_3) + (x_3 + x_4) \cdot (x_3 + x_2) \cdot (x_3 + x_1) \end{matrix} \right\}.$$

Remember that the *fundamental theorem of the theory of symmetric functions* says:

> *Every rational fonction of roots of an algebraic equation*
>
> $$\varphi(x_1, x_2, \ldots, x_n)$$
>
> *which takes k different values when it makes all possible permutations of roots, satisfies an algebraic equation of degree k whose coefficients are rational fonctions of the coefficients of the given equation.*
>
> *Then, if $k = 1$, the fonction $\varphi(x)$ satisfies a rational expression of the coefficients of the given equation.*

Euler uses largely this fundamental theorem, but he only develop, with a sufficient rigour, for the general case of the second degree equations, but the theorem in his general form was proved firstly by Lagrange in his transcendental paper *Reflexions sur la resolution algebrique des equations* [1771]. So it will be necessary hope the Lagrange's apports by obtaining the general result.

[74]First the term x^7 is eliminated, so that the two supposed factors can be written $x^4 - u\,x^3 + \alpha\,x^2 + \beta\,x + \gamma$ and $x^4 + u\,x^3 + \delta\,x^2 + \epsilon\,x + \psi$. Since u expresses the sum of four roots of the eight-degree equation, it can have $\frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} = 70$ values, and it will satisfy an equation of the form

$$0 = [u^2 - p^2] \cdot [u^2 - q^2] \cdot [u^2 - r^2] \cdot [u^2 - s^2] \cdots$$

with 35 factors. The absolute term is negative, and the reasoning continues as before.

equation in the unknown v are the expressions

$$v_\sigma = \sum_{k=1}^{r} z_{\sigma(k)} - \sum_{k=1}^{r} z_{\sigma(k+r)}$$

where σ runs over the set S_n of all permutations of set $\{1, 2, \ldots, n\}$. It is easy see that the product of v_σ is always ≤ 0. Next he avoids the case in which the product is zero, substituing v_σ for a useful combination of the coefficients of P_1 with real coefficients and then using his results contained in a paper of 1770–1771[75] on permutations of an equation, finishes rightly the demonstration[76].

3.3. The Laplace's attempt.

In the year 1795, Pierre Simon Laplace made an attempt to prove the Fundamental Theorem[77]. This attempt was completely algebraic, but quite different from the Euler-Lagrange attempt. This mathematician and politician assumes, as his predecessors, that the roots of polynomials "exist"[78].

Laplace says[79]

> *Of this it results a demonstration very simple of this general theorem which we have ennounced before and which says that every equation of even degree can be solved into real factors of second degree.*

His prove is the following: Let be $x_1, x_2, \ldots, x_n$, where $n = 2^k q, k \leq 1, q \in 2\mathbb{N} + 1$, the roots of the polynomial

$$P(x) = x^n - b_1 x^{n-1} + b_2 x^{n-2} + \cdots + (-1)^n b_n \in \mathbb{R}[x],\ n \leq 1.$$

[75]Lagrange, J.-L. [**45**], [1773]. These papers are the most important works on algebraic equations in the eighteenth century. See Dieudonné, J. et alii [**20**, I, 70].

[76]In 1815 Gauss objectes "... this question has been treated as the only problem was determinate the form of roots and its existence is accepted without demonstration. But this manner of raisoning is completely illusory and it constitutes a veritable *petitio principis*" [*Opera Omnia*, III, 105–106]. He gives us a demonstration —his second demonstration— following the Euler's ideas, but he avoids to apply the *imaginary roots* because nothing "guarantees it existence". [See Dieudonné, J. et alii [**20**, I, 71]; Fauvel, J.-Gray, J. [**31**, 490–491]; Smith, D. E. [**74**, 292–306] or Remmert, R. [**68**, 104–106].

[77]See "Leçons de mathématiques donnés a l'Ecole Normale", *Oeuvres completes*, **14**, 10–111, especially 63–65. For an actual proof and comentaries, see Remmert, R. [1190b], 120–122.

[78]This existence is naturally a platonic existence.

[79]Laplace, P.-S. [**47**, 63].

The equation $Q_t(x)$ which roots are $x_i + x_j + t\,(x_i\,x_j)$, where $t \in \mathbb{R}$ arbitrary and $i < j$, has a degree of the form $2^{k-1}\,q'$, where $q' \in 2\mathbb{N}+1$[80]. Then Laplace proceeds by induction on k:

- if $k = 1$, the new polynomial $Q_t(x)$ will have an odd degree and then it will be a least a real root $x_i + x_j + t\,(x_i\,x_j)$[81].

  It is clear that there is infinitely many real values t such that, for a same x_i and x_j,

  $$x_i + x_j + t\,(x_i\,x_j) \in \mathbb{R}.$$

  Then there are $t_1 \neq t_2, t_1, t_2 \in \mathbb{R}$, such that $x_i + x_j + t_1\,(x_i\,x_j), x_i + x_j + t_2\,(x_i\,x_j) \in \mathbb{R}$. Then the quantities

  $$[t_1 - t_2]\,(x_i\,x_j), \quad x_i\,x_j \quad \textit{and} \quad x_i + x_j$$

  are all real. So the factor $x^2 - [x_i + x_j]\,x + x_i\,x_j$ will be a real factor of second degree of $P(x)$;
- if $k > 1$, then $P(x)$ will have a real factor of second degree if *every equation of degree* $2^{k-1}\,q'$ *has a factor of second degree*, because infinitely many

  $$x_i + x_j + t\,(x_i\,x_j), i < j, t \in \mathbb{R}$$

  will be complex numbers [that is: they are of the form $\alpha + i\,\beta, \alpha, \beta \in \mathbb{R}$] and then, following the precedent reasoning, there are two roots x_i, x_j of $P(x)$ such that $x_i + x_j, x_i \cdot x_j \in \mathbb{C}$. Therefore the factor

  $$x^2 - [x_i + x_j]\,x + x_i\,x_j \in \mathbb{C}[x]$$

  and it divides exactly $P(x)$. Then

  $$x^2 - [\,\overline{x_i + x_j}\,]\,x + \overline{x_i\,x_j} \in \mathbb{C}[x]$$

  divides also $P(x)$. Thus the following real polynomial of forth degree

  $$\begin{aligned} &[x^2 - [x_i + x_j]\,x + x_i\,x_j]\,[x^2 - [\,\overline{x_i + x_j}\,]\,x + \overline{x_i\,x_j}] = \\ &= [x^2 - \mathrm{Re}(x_i + x_j)\,x + \mathrm{Re}(x_i\,x_j)]^2 + [\mathrm{Im}(x_i\,x_j) - \mathrm{Im}(x_i + x_j)]^2. \end{aligned}$$

  This quantity, "as we have seen"[82], can be solved in two real factors of second degree[83].

[80] Its degree is exactely $2^k\,q\,[2^k\,q - 1]/2 = 2^{k-1}\,q'$, where $q' \in 2\mathbb{N} + 1$.

[81] Laplace applies the following corollary of the *Intermediate value Theorem*: "Every polynomial of odd degree has at least one real root".

[82] See Laplace, P.-S. [**47**, 60–63].

[83] Laplace considers the case in which the two factors

$$[x^2 - [x_i + x_j]\,x + x_i\,x_j], [x^2 - [\,\overline{x_i + x_j}\,]\,x + \overline{x_i\,x_j}]$$

Then the problem is finished because $P(x)$ has a real factor of second degree *iff* every real equation of degree $2^{k-1}\,q', q' \in 2\mathbb{N}+1$ has a simmilar factor, and then [for the same reason] *iff* every equation of $2^{k-2}\,q'', q'' \in 2\mathbb{N}+1$ has a simmilar factor and following we establish the proof[84].

References

1. ALEKSANDROV, A. D., KOLMOGOROV, A. N., LAURENTIEV, M. A. ET ALII., "*Mathematics: Its content, methods, and meaning,*" M.I.T. Press, Cambridge, Massachussets, 1963.
2. ALEMBERT, JEAN LE ROND D', Recherches sur le calcul intégral, *Actes de l'Academie des Scienes de Paris* (1746), 182–224.
3. ARGAND, JEAN ROBERT, "*Essais sur une manière de représenter les quantités imaginaires dans les construccions géométriques,*" Paris, 1806. Reprint by Albert Blanchard, Paris, 1971.
4. BASHMAKOVA, ISABELLA G., Le théorème fondamental de l'algèbre et la construction des corps algébriques, *Archives Internationelles d'Histoire des Scienes* **13** (1960), 211–222.
5. BERNOULLI, JAKOB, "*Opera,*" 2 vols., ed. by G. Cramer, Geneva, 1744. Reprint by Birkhaüser, 1968.

have a common factor. This factor must be necessary a factor of the difference of two polynomials and then it must be

$$\operatorname{Im}(x_i + x_j)\,x + \operatorname{Im}(x_i\,x_j).$$

If we divide $P(x)$ by this polynomial of first degree, we will have a polynomial with odd degree and then it will have a real root r. The product

$$[\operatorname{Im}(x_i + x_j)\,x + \operatorname{Im}(x_i\,x_j)] \cdot [x - r]$$

constitutes the factor of second degree found.

[84]This proof has a mistake, like we can see in Remmert, R. [**68**, 122]. It is necessary to see that the polynomial

$$Q_t(x) = \prod_{1 \le i < j \le n} [x - (x_i + x_j) + t\,(x_i\,x_j)] \in \mathbb{R}[x]$$

[that is: all coefficients are reals].

This fact is an easy consequence of the *main theorem on symmetric functions* which was proved by Newton in 1673. This theorem says that the coefficients of $Q_t(x)$ are real because "they are real polynomials in the elementary symmetric functions of $x_1, x_2, \ldots, x_n$": that is, in the real numbers $b_1, \ldots, b_n$.

6. BERNOULLI, JOHANN, Solution d'un problème concernant le calcul integral, avec quelques abregés par rapport à ce calcul, *Mémoires de l'Academie Royales des Sciences*, Paris (1702), *Acta Eroditorum*, January 1703, in *"Opera Omnia"* **II**, pp. 393 400. Partial English translation in Fauvel, J.-Gray, J., 1987, pp. 436–439.

7. BERNOULLI, JOHANN, Angulorum arcuumque sectio indefinita, *Acta Eroditorum* **31** (1712), 274–277, in *"Opera Omnia"* **I**, pp. 511–514.

8. BERNOULLI, JOHANN, "*Opera Omnia*," 4 vols., ed. by G. Cramer, Lausanne and Geneva, 1742. Reprint by Olms Verlag, Hildesheim, New York, 1968.

9. BOMBELLI, RAFAEL, "*L'Algebra*," Bologna, 1572. Reprint by Bertolloti, Feltrinelli Editore, Milano, 1966.

10. BOS, H. J. M., On the representation of curves in Descartes' *Géométrie*, *Archive for History of Exact Sciences* **24** (1981), 295–338.

11. BOYER, CARL B., "*A History of Mathematics*," John Wiley & Sons, New York, 1968.

12. BURTON, DAVID M., "*The History of Mathematics. An Introduction*," Allyn and Bacon, Newton, Massachusetts, 1985.

13. CAJORI, FLORIAN, Historical notes on the graphic representation of imaginaries before the time of Wessel, *American Mathematical Montly* **19** (1912), 167–171.

14. CAJORI, FLORIAN, History of the exponential and logarithmic concepts, *American Mathematical Montly* **20** (1913), 5–14; 35–47; 75–84; 107–117; 148–151; 173–182; 205–210.

15. CANTOR, MORITZ, "*Vorlesung über Geschichte der Mathematik*," 4 vols; vol. I (3rd ed., 1907), vol. II (2nd ed., 1900), vol. III (2nd ed., 1901), vol. IV (1908), Teubner, Leipzig, 1900.

16. CARDANO, GEROLAMO, "*Ars Magna*," 1545. English translation in Witmer, R. T., 1968.

17. CAUCHY, AUGUST-LOUIS, Sur les racines imaginaires des équations, in "*Oeuvres*," I, Ser. 2, 1820, pp. 258–263.

18. CAUCHY, AUGUST-LOUIS, "*Oeuvres de A. Cauchy*," Ser. 1, 13 vols, Gauthier-Villar, Paris, pp. 1882–1901. Ser. 2, 14 vols, Gauthier-Villars, Paris, 1908-1938.

19. DESCARTES, RENÉ, "*La Géométrie*," Leyden, 1637. English translation in Smith, D.E.-Latham, M., 1925.

20. DIEUDONNÉ, JEAN AND ALII, "*Abrégé d'histoire des mathématiques, 1700-1900*," 2 vols, Hermann, Paris, 1978. Reprint shortened in one volum, Hermann, Paris, 1989.

21. DÖRRIE, HEINRICH, "*Triumph der Mathematik*," Physica Verlag, Würrzburg, 1958. English translation by David Antin: *"100 Great Problem in Elementary Mathematics"*, Dover, New York, 1965.

22. EULER, LEONHARD, De intregratione æquationum differentialium altiorum gradnum, *Miscellania berolinensia* **7** (1743), 193–242, in *"Opera Omnia"* **XXII**, ser. 1, pp. 108–149.

23. EULER, LEONHARD, Recherches sur les racines imaginaires des équations, *Histoire de l'Académie Royale des Sciences et Belles Lettres,* Anné MDCCXLIX, Berlin 1751, 222–228, in *"Opera Omnia"* **VI**, ser. 1, 1749, pp. 78–147.

24. EULER, LEONHARD, Methodus æquationes differentialibus, quæcertis tantum casibus integrationem admittunt, *Novi Commentarii Academia Scientifiarum Petrp.* **3** (1750), 3–35, pub. 1753 in *"Opera Omnia"* **XXII**, ser. 1, pp. 181–213.

25. EULER, LEONHARD, De formulis differentialibus angularibus, published in the posthumous vol. IV in Euler, L., 1794, IV.

26. EULER, LEONHARD, Institutiones calculi integralis, in "*Opera Omnia,*" XIX, ser. 1, 1794, pp. 129–140.

27. EULER, LEONHARD, "*Correspondance mathématique et physique de quelques célèbres géomètres du XVIII^e siècl,*" 2 vols., edited by P. H. Fuss, Petersbourg, 1843.

28. EULER, LEONHARD, "*Leonhardi Euleri Opera postuma mathematica et physica,*" 2 vols., edited by P. H. and N. Fuss, Petersbourg, 1962.

29. EULER, LEONHARD, "*Opera Omnia,*" ed. by Soc. Scient. Natur. Helveticæ, Teubner, Leipzig and Berlin, 1911–1985, Füssli, Zürich, four series.

30. EWING, JOHN H. (EDITOR), "*Numbers,*" Springer-Verlag, New York, 1990. It is an English translation by H. S. L. Orde of Second edition of the German original *Zahlen*, Springer-Verlag, 1988.

31. FAUVEL, JOHN-GRAY, JEREMY (EDITORS), "*The History of Mathematics —A reader—,*" MacMillan, New York, 1987.

32. FERMAT, PIERRE, Ad locos planos et solidos isagoge, in "*Oeuvres,*" 1629, pp. 92–103.

33. FERMAT, PIERRE, "*Oeuvres,*" 4 vols., ed. P. Tannery, Gauthiers-Villars, Paris, 1896.

34. GAUSS, CARL FRIEDRICH, "*Werke,*" 12 vols., ed. by Königliche Gesellschaft für Wissenschaften, Göttingen, Leipzig and Berlin, 1863-1933.

35. GERHARD, C. I. (EDITOR), "*Leibniziens mathematische Scriften,*" 7 vols., Halle, 1850-1863. Reprint by Olms Vrelag, Hidelsheim, 1962.

36. GERHARD, C. I. (EDITOR), "*Briefwechsel von G.W. Leibniz mit Mathematikern,*" Berlin, 1899.

37. GILAIN, CHRISTIAN, Sur l'histoire du théorème fondamental de l'algèbgre: théorie des équations et calcul integral, *Archive for History of Exact Sciences* **42** (1991), 91–136.

38. GIRARD, ALBERT, "*L'Invention nouvelle de l'Algèbre,*" Amsterdam, 1629. Partial English translation by Ellen Black in Viète at alii, 1986, pp. 105–198.

39. HARRIOT, THOMAS, "*Artis Analyticæ,*" Praxis, London, 1631.

40. HOFMANN, JOSEPH E., "*Leibniz in Paris, 1672–1676,*" Cambridge University Press, Massachusetts, 1972.

41. HOLLINGSDALE, STUART, "*Makers of Mathematics,*" Penguin Books, London, 1989.

42. HOUZEL, CHRISTIAN, D'Alembert et le thèoréme fondamental de l'algèbre, in "*Jean d'Alembert, savant et philosophe: portrait à plusieurs voix,*" Editions des archives contemporaines, Paris, 1989, pp. 351–360.

43. JONES, PHILLIP S., Complex Numbers: an example of recurring themes in the development of mathematics, I and II, *The Mathematical Teacher* **47** (1954), 106–114, 257–262.

44. KLINE, MORRIS, "*Mathematical thought from ancient to modern times,*" Oxford University Press, New York, 1972. There is an edition in three volums (paperback), Oxford University Press, New York, 1990.

45. LAGRANGE, JOSEPH LOUIS, Sur la forme des racines imaginaires des équations, *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles Lettres,* Anné MDCCLXXII, Berlin 1774, 222–228, in *"Oeuvres complètes"*, III, 1772, pp. 477–516.

46. LAGRANGE, JOSEPH LOUIS, "*Oeuvres complètes,*" 14 vols, ed. by J. Serret, Gauthier-Villars, Paris, 1867–1892.

47. LAPLACE, PIERRE SIMON, Leçons de mathématiques donnés à l'Ecole normale en 1795, in "*Oeuvres complètes,*" **XIV**, 1795, pp. 10–177.

48. LAPLACE, PIERRE SIMON, "*Oeuvres complètes,*" 14 vols, Gauthier-Villars, Paris, 1878-1912.

49. LEIBNIZ, GOTTFRIED WILHELM, "*De Bisectione Laterum,*" 1675. See Schneider, I., 1968, 224.

50. LEIBNIZ, GOTTFRIED WILHELM, Specimen novum analyseos pro scientia infinity circa summas et quadraturas, *Acta Eroditorum,* May 1700 in Gerhardt, C.I., 1859, 350–361. French translation by Marc

Parmentier in Leibniz, G.W., 1989, 387–401.

51. LEIBNIZ, GOTTFRIED WILHELM, Extrait d'une lettre de Mr. Leibniz a Mr. Varignon, tiré du *Journal of Savant* de l'anné 1702, in Leibniz, G.W., 1983.

52. LEIBNIZ, GOTTFRIED WILHEM, Continuatio Analyseos Quadratorum Rationalium, *Acta Froditorum*, January 1703, in Gerhardt, C.I., 1850, V, 361-377. French translation by Marc Parmentier in Leibniz, G.W., 1989, 402–422.

53. LEIBNIZ, GOTTFRIED WILHELM, "*Oeuvre concernant le calcul infinitésimal*," 1983. Translated to French by Jean Peyroux, Blanchard, Paris.

54. LEIBNIZ, GOTTFRIED WILHELM, La naissance du calcul différentiel, 26 articles des *Acta Froditorum*, Introduction, translation and notes by Marc Parmentier, Vrin, Paris (1989).

55. MCCLENON, R. B., A contribution of Leibniz to the history of complex numbers, *American Mathematical Montly* **30** (1923), 369–374.

56. MILHAUD, GASCON, "*Descartes savant*," Alcan, Paris, 1921.

57. MOLAS, CORI AND PÉREZ, JULI, El naixement dels nombres complexos, *Butlletí de la Societat Catalana de Matemàtiques* **5** (1990), 37-66.

58. NAUX, I., "*L'histoire des logarithms*," Blanchard, Paris, 1966.

59. NEWTON, ISAAC, De methodis serierum et fluxionum, in "*Newton*," I, 1967-1981, III, pp. 32–353.

60. NEWTON, ISAAC, "*Arithmetica Universalis*," 1707, Second edition (1728), Cambridge, in "*Newton*", I., 1967, V, pp. 54–532.

61. NEWTON, ISAAC, "*Mathematical Papers of Isaac Newton*," 8 vols., edited by Whiteside, D.T. Cambridge University Press, Cambridge, 1967-1981.

62. PETROVA, S. S., Sur l'Histoire des démonstrations analytiques du théorème fondamental de l'algèbre, *Historia Mathematica* **1** (1974), 255–261.

63. PLA, JOSEP, *La Géométrie* de Descartes com un exemple de *La Méthode*, (catalan), *Actas del III Congreso de los Lenguajes Naturales y los Lenguajes Formales* **III** (1987), 821–863.

64. PLA, JOSEP, Les sèries en Newton, *Butlletí de la Societat Catalana de Matemàtiques* **4** (1989), 9–21.

65. PUISEUX, V.-A., Recherches sur les fonctions algébriques, *Journal of Mathematics* **15** (1850), 365–480.

66. RAHN, JOHANN HEINRICH. H., "*Teutschen Algebra*," Zürich, 1659.

67. REMMERT, REINHOLD, "*Complex Numbers,*" in Ewing, J. H., 1990, pp. 55–96.

68. REMMERT, REINHOLD, "*The Fundamental Theorem of Algebra,*" in Ewing, J. H., 1990, pp. 97–122.

69. REY PASTOR, JULIO-PI CALLEJA, PEDRO-TREJO AND CÉSAR A., "*Análisis Matemático,*" Editorial Kapelusz S.R.I., Buenos Aires, Argentina, 1951.

70. ROTH, PETER, "*Arithmetica philosophica,*" Nürnberg, 1600.

71. SCOTT, JOHN F., "*The scientific work of René Descartes,*" Taylor and Francis, London, 1952.

72. SCHNEIDER, I., Der Mathematiker Abraham de Moivre (1667-1754), *Archive for History of Exact Sciences* **5** (1968), 177–317.

73. SMITH, DAVID EUGENE, "*History of Mathematics,*" 2 vols (first volum (1923), second volum (1925)), Constable and Company, Ltd. London, 1925. Reprint by Dover, New York, 1957.

74. SMITH, DAVID EUGENE (EDITOR), "*A Source Book in Mathematics,*" 2 vols., McGraw-Hill, New York, 1929. Reprint by Dover. New York, 1959.

75. SMITH, DAVID EUGENE-LATHAM, MARCIA, "*The Geometry of René Descartes,*" Dover, New York, 1954.

76. STILLWELL, JOHN C., "*Mathematics and its History,*" Springer-Verlag, New York, 1989.

77. STRUIK, DIRK J., "*Source Book in Mathematics, 1200-1800,*" Harvard University Press, Cambridge, Massachusetts, 1969.

78. TARTAGLIA, NICCOLÒ, "*Quesiti et Inventione Diverse,*" Facsimile of 1559 edition, ed. A. Masotti, Ateneo di Brescia, Brescia, 1546. Partial English translation in Fauvel, J.-Grey, J., 1987, pp. 254–256.

79. TROPFKE, JOHANNES, "*Geschichte der Elementar-Mathematik in systematischer Darstellung,*" 7 vols., third edition of vol. III, 1937.

80. VERA, FRANCISCO, "*20 Matemáticos Célebres,*" Los libros del Mirasol, Argentina, 1961.

81. VIÈTE, FRANÇOIS, De æquatione recognitione et emendatione, in "*Opera Mathematica,*" 1646, pp. 82–162. There is also an English translation in Witmer, R.T., 1983, pp. 159–310.

82. VIÈTE, FRANÇOIS, "*Opera Mathematica,*" published by Franciscus van Schooten with commentaries, Leiden, 1646. Reprint by Georg Olms Verlag, Hildesheim, New York, 1970.

83. VIÈTE, F. ET ALII, "*Three Treatises by Viète, Girard and de-Beaune,*" Golden Hint Press, Annapolis, Maryland, 1986.

84. WAERDEN, BARTEL LEENERT VAN DER, "*Modern Algebra*," Frederick Ungar, New York, 1949.

85. WAERDEN, BARTEL LEENERT VAN DER, "*A History of Algebra*," Springer-Verlag, New York, 1985.

86. WALLIS, JOHN, "*Algebra*," London, 1673. In *"Opera Omnia"*, II, pp. 1–482.

87. WALLIS, JOHN, "*Treatise of Algebra*," 1685. English translation by Wallis of the *"Algebra"*, London.

88. WALLIS, JOHN, "*Opera Omnia*," 3 vols, Georg Olms Verlag, Hidelsheim, New York, 1972.

89. WESSEL, CASPAR, Om diretioneus analytiske Betegning et Forsög anwendt foremelig til plane of sphaeriske Polygoners Oplössing, *Danske Selsk. Skr. N. Sarml.* **5** (1797). Partial English translation in Smith, D. E., 1929, I, 55–66.

90. WITMER, R. T., "*The great art*," Cambridge, Massachusetts, 1968.

91. WITMER, R. T., "*The analytic art*," Kate State University Press, Kent, Ohio, 1983.

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