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INTEGRABILITY: A DIFFICULT ANALYTICAL PROBLEM

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Abstract. Generically hamiltonian systems are nonintegrable. However there are few tools in order to prove that a given system is nonintegrable. For two degrees of freedom the usual methods rely upon the appearance of transversal homoclinic or heteroclinic orbits. The transversal character is shown through evaluation of integrals along orbits. Such computation requires the knowledgement of a one parameter family of periodic orbits and an explicit solution for the unperturbed (integrable) case. Due to the dependence of the form $\exp(-C/\varepsilon^k)$ of the angle measuring transversality with respect to the perturbation parameter, none of the approximations of perturbation theory is enough to establish nonintegrability.

§1. The meaning of integrability and nonintegrability. Let $H(q,p)$ be a hamiltonian of n degrees of freedom. For the sake of simplicity we take R^{2n} as phase space. A first integral F of the associated hamiltonian system is a smooth function such that the Poisson bracket (F,H) is identically zero. Let F_j, $j=1\div k$ smooth functions. We say that they are in involution if $(F_i,F_j)=0$ $\forall i,j$. From now on we take $F_1=H$. A hamiltonian system is said integrable if there are n functionally independent smooth global functions in involution. We refer to [1] and [3] for basic definitions and results. We recall that what Hamilton-Jacobi theory intends is to convert a given system in an integrable one.

Under the preceding conditions if on the level set $L=\{F_j=c_j,\ j=1\div n\}$ where $c_1, \dots, c_n$ are real values, the forms DF_j, $j=1\div n$ are independent and L is compact, then it is diffeomorphic to the n-dimensional torus T^n. (Taking out the compactness condition we get some $R^k \times T^{n-k}$; eventual dependence of DF_j, $j=1\div n$, along subsets can produce that L be a union of tori along such subsets). Then there exist action-angle variables (I,φ) in $D^n \times T^n$, a neighbourhood of T^n such that H can be expressed as H(I), the flow is given by $I(t) = I(0)$, $\varphi(t) = \varphi(0) + \omega t$ where $\omega = (D_I H)^T$ is the frequency vector. Therefore we get a linear flow on T^n, quasi periodic and dense on the torus if ω is incommensurable or nonresonant (i.e., $\omega_1, \ldots, \omega_n$ are Z - independent) or in a lower dimensional torus otherwise (in particular periodic if the Z - module generated by the ω's is 1-dimensional). The solution can be obtained through quadratures. The statements above constitute the Liouville-Arnold theorem.

Near the integrable systems the KAM theory [4] ensures the existence of slighty distorted invariant tori (the resonant ones). They do not fill completely the available phase space and if $n > 2$ some slow escape across the tori is possible: the so called Arnold diffusion.

In the situation opposite to the integrable systems we found the ergodic ones. A hamiltonian systems is called ergodiç if it is ergodic in (almost) all the levels of the energy. Letting aside (functions of) H the only first integrals are the constants. In the integrable case the flow is confined to a n-dimensional manifold almost everywhere (if $D_I H$ is nondegenerate) and dense there. In the ergodic case the flow is dense in their energy level. The real world is neither integrable non ergodic. A mathematical statement is this direction is due to Markus and Meyer.

Let $\mathcal{H}$ be the set of $\mathcal{C}^r$ hamiltonians with the $\mathcal{C}^r$ topology.It is a Baire space. A propiety $\mathcal{P}$ is called generic in $\mathcal{H}$ if $\mathcal{M} = \{H \in \mathcal{H} \mid H$ satisfies $\mathcal{P}\}$ is a set of the second category. Then the theorem [20] asserts: Hamiltonian systems are generically neither integrable non ergodic.

A standing problem is how many first integrals has, generically, a hamiltonian system. Some numerical experiments [12] and the destruction of symmetries by genericity seem to favourish the fact that only H remains as first integral.

Let us look for the behaviour of the solutions in the nonintegrable case. We first consider the easiest nontrivial case: $n = 2$. The levels $H = e$ are 3-dimensional hypersurfaces H_e. We restrict our study to some fixed H_e. Let Σ be a 2-dimensional manifold in H_e transversal to the flow at a point P. Let us suppose that the orbit through P cuts again Σ transversally (this is the case if such orbit is periodic). Then we can define (locally) a map $T: \Sigma \circlearrowleft$ given by: find the next cut (Poincaré map). If $H = F_1, F_2$ are first integrals, the set $\Sigma \cap \{F_2 = c_2\}$ is a curve, except for degenerate cases. Therefore the iterates of P under T are on this curve. If H is perturbed and integrability is lost the points are scattered in a more or less narrow strip around the curve. It seems that they fill a region of positive measure according to some density (but with a host of holes smaller and smaller in it). If the system is ergodic they should be scattered through all Σ. In many examples the simulation shows the existence of different unrelated "stochastic" zones if $n = 2$. We suspect that they are related through very narrow channels for $n > 2$.

To see how the nonintegrability depends on global questions the following example is instructive. Let $T: \Sigma \circlearrowleft$ a Poincaré map. It is easy to show [4] that it is area preserving. We can learn about qualitative behaviour of flows if we study area preserving mappings (APM) from R^2 into itself. Suppose that T has a hyperbolic fixed point P: Spec $(DT(P)) = \{\lambda, 1/\lambda\}, |\lambda| > 1$. Then Hartman's theorem [14] assures that the behaviour near P is essentially the one given by the linear part. Even in this case the linearizing change of variables is analytic [30]. We have invariant stable and unstable manifolds ($W^s(P)$, $W^u(P)$) that can be globalized. In a similar way, some piece of analytic invariant curve γ near P can be extended by iteration of T and T^{-1}. However if G is a first integral near P (i.e. $G(T(P)) = G(P)$ in a neighbourhood U of P) this function can not be extended in general, for instance, if $W^s(P)$ comes near P again. This happens if $W^u(P) \cap W^s(P) \neq \emptyset$. A point belonging to both manifolds is called homoclinic. If P,Q are fixed hyperbolic and $S \in W^u(P) \cap W^s(Q)$ the point S is called heteroclinic and similar problems can occur.

The intersections of globalized invariant curves can produce cantorian sets. The lack of integrability due to the folding of W^u is related to the fact that W^u is a manifold but not a submanifold of R^2. Wintner [33] stated this fact saying that the integrals G (obtained locally and piecewise continuated) are nonisolating: they are not able to isolate the "inner" and "outer" parts of a set $G < g$ and the boundary of this set can have positive measure. Therefore, iterates of a point standing on $G = g$ "fill" a strip.

In a different approach, using perturbation theory, the problems of small divisors, overlapping of resonances, etc. are typical of nonintegrable systems. For two degrees of freedom the shift of Bernoulli can be included as a subsystem of the hamiltonian system [24]. Then, in particular, an infinity of periodic orbits (P.O.) of arbitrary high period exist as well as oscillatory and quasirandom motions.

§2. Some analytical results. We begin with a few historical comments. Newton formulated the n-body problem and found the 10 classical integrals. Some 200 years were elapsed in a fruitless search of additional integrals. Later in the XIX century, Bruns proved that no more independent first integrals can be found being algebraic functions of q and p, and Painlevé stated even the nonexistence of additional first integrals algebraic in p. Poincaré [26] in turn showed the lack of integrals analytical in q, p, μ for the restricted three body problem with mass parameter μ besides the Jacobi integral.

If we restrict ourselves to analytic hamiltonians near an equilibrium point then the existence of integrals is related to the problem of normal forms started by Birkhoff (see [23]). Let $H = H_2 + H_3 + H_4 + \dots$ an analytical function near an equilibrium point that we take as the origin. H_k is the homogeneous part of degree k. Suppose $H_2 = \frac{1}{2} \sum \alpha_j (q_j^2 + p_j^2)$ and define $J = \{k \in Z^n \mid (k,\alpha) = 0\}$ as the Z-module of the resonances (here (,) is the inner product).

Theorem (Gustavson [13]): There is a formal change of variables $(q,p) \rightarrow (\xi,\eta)$ such that the new hamiltonian Γ is of the form (Gustavson Normal Form) $\Gamma = \sum c_{km} \varsigma^k \bar{\varsigma}^m$ where $\varsigma_r = \xi_r + i\eta_r$, $\varsigma^k = \varsigma_1^{k_1} \cdot \dots \cdot \varsigma_n^{k_n}$ and $k-m \in J$. (Equivalently $(H_2,\Gamma) = 0$).

If the dimension of J is r we get n-r formal first integrals. The question of obtaining more first integrals is sometimes refered as the search of the third integral [9] because for problems of galactic dynamics we already know the energy and momentum integrals.

If J=0 (α's Z-independent) then k=m and therefore $\Gamma = \Gamma(I)$ (Birkhoff N.F. [6]), where $I = (I_1, \dots, I_n)^T$, $I_r = \xi_r^2 + \eta_r^2$, and the system is formally integrable. What about convergence?

For $n = 2$ we have the following result (Siegel [29]): Let $\mathcal{H}$ be the set of analytic hamiltonians, $H \in \mathcal{H}$, $H = \sum_{k,m \in Z^2} c_{km} \xi^k \eta^m$. It is not a restriction (use scaling if necessary) to suppose $|c_{km}| < 1$. Define a very fine topology $\mathcal{T}$ in the following way: Given H and $\varepsilon = \{\varepsilon_{km}\}$ the ball of radius ε centered in H is the set $B_\varepsilon(H) = \{H^* \in \mathcal{H} \mid |c_{km} - c^*_{km}| < \varepsilon_{km} \ \forall k,m\}$.

Theorem: with the topology $\mathcal{T}$ the set of hamiltonian systems in $\mathcal{H}$ showing divergence when going to the B.N.F. is dense.

A coarser topology $\mathcal{T}'$ can be defined throug $B_{\varepsilon,N}(H) = \{H^* \in \mathcal{H} \mid |c_{km} - c^*_{km}| < \varepsilon_{km}$ for $|k| + |m| < N\}$. Then we can produce finite changes to B.N.F. without convergence problems. The set of integrable systems is dense in $\mathcal{H}$ with respect to $\mathcal{T}'$.

As examples of integrable systems we can display all the problems found in elementary textbooks in mechanics. Nonintegrability is displayed by systems with $n = 2$ possessing transversal homoclinic or heteroclinic orbits [2, 8, 15, 18, 19, 24, 32]. However for $n > 2$ there are examples with transversal homoclinic orbits that are integrable [10].

Nonintegrability is related to the divergence of the transformation to normal form. In fact, for $n = 2$, Rüssmann [28] proved that if $\alpha_2/\alpha_1 \notin Q$ and $G = G_2 + G_3 + \ldots$ is a first integral with $G_2 = \frac{1}{2} \sum \beta_j (q_j^2 + p_j^2)$, $\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} \neq 0$ then we have convergence when going to the B.N.F.

For the relation between divergence and destruction of invariant curves see [27].

§3. Detecting nonintegrability. Faced to a definite problem, how to decide about integrability? Here genericity is useless. A hopeless approach is trying to get enough first integrals. However this is not be recommended except if there is a strong evidence (numerically, see later) of such existence. That was the way Hénon followed to show that the Toda lattice with equal masses is integrable [16].

If we proceed numerically the Poincaré map is a useful device. If $n = 2$ and the iterates of a point are scattered along a line we have an evidence of nonintegrability. However, if the system is very near an integrable one it could be difficult to decide whether or not the points are on a curve. A much finer criterion is to look for transversal homoclinic points [18,19,24,31,32] or for a chain [4] of transversal heteroclinic points. We return later to that topic. If $n > 2$ to visualize the Poincaré map we need some "stroboscopic" device [22] or different cuts of Σ [11].

A dimension-independent method consists in the computation of the Lyapunov numbers. Let ϕ_t be a (hamiltonian) flow and $D\phi_t$ the differential with respect to initial conditions ($D\phi_t$ is the solution of the first order variational equations; in coordinates $D\phi_t = A(t) \in \mathcal{L}(R^{2n}, R^{2n})$, $A(0) = I$). The maximal Lyapunov number $\ell_1(P)$ is the maximum rate of growth of the length of a tangent vector at P under $D\phi_t$, i.e. $\ell_1(P) = \lim \ln \|A(t)\|_2/t$ where we recall that $\|A(t)\|_2 = (\rho(A(t)\,A^T(t))^{1/2}$ (ρ = spectral radius). The remaining Lyapunov numbers ℓ_j, $j = 2 \div 2n$ are defined in the following way: let $\hat{\ell}_j$ be the maximum rate of growth of the j-dimensional measure of a j-dimensional subspace of the tangent space at P under the action of $D\phi_t$. Then $\ell_j = \hat{\ell}_j / \hat{\ell}_{j-1}$. See [5] for the effective computation of all the Lyapunov numbers (taking care of scaling, orthogonalization, etc.). he important fact for detecting nonintegrability relies in the

Theorem: Integrability $\Rightarrow$ all the Lyapunov numbers are zero.

Proof: (See also [7]). We restrict ourselves to the case where the Liouville-Arnold theorem applies. From $\dot{\varphi} = H_I$, $\dot{I} = 0$, we get the variational equations $\dot{A} = \begin{pmatrix} M & N \\ P & Q \end{pmatrix}^{\cdot} = \begin{pmatrix} 0 & H_{II} \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} M & N \\ P & Q \end{pmatrix}$, $M(0) = Q(0) = I$, $N(0) = P(0) = 0$. The solutions are $P = 0$, $Q = I$, $M = I$, $N = H_{II}t$, and $AA^T = \begin{pmatrix} I + N^2 & N \\ N & I \end{pmatrix} = O(t^2)$ from where the result follows because $\lim \ln \|A(t)\|_2^j / t = 0$.

We see that integrable systems have a "parabolic" character in the same sense that a fixed parabolic point of a diffeomorphism.

Following a result of Pesin [25] the entropy of the flow is given by $h = \int_{\text{phase space}} \sum_{\ell_i > 0} \ell_i(P)$. However a direct computation of h can be harder than that of the Lyapunov numbers.

For $n = 2$ nonintegrability follows if the Poincaré map has the smale horseshoe embedded as a subsystem [24]. At some level of energy h for a hamiltonian it is possible to show the existence of transversal homoclinic and chains of heteroclinic orbits and, therefore, of such embedding by simple topological considerations. See f.i. [8] for the Hénon-Heiles (HH) problem and for the potential $\frac{1}{2}(q_1^2 + q_2^2) - \frac{1}{2} q_1^2 \times q_2^2$. However for those and other systems nonintegrability is detected numerically for smaller values of h, far away of the value for which the zero velocity curve becomes open.

The HH problem is obtained through perturbation of a harmonic oscillator. With a suitable scaling we have $H = H_2 + \varepsilon H_3$ on the level $H = 1$, where $H_2 = \frac{1}{2}(q_1^2+q_2^2+p_1^2+p_2^2)$ is the harmonic oscillator and H_3 a homogeneous third order term (or, in general, an analytic function beginning with terms of third order at least). We realize that is a resonant hamiltonian, J being here $\{(k,-k) \mid k \in Z\}$. For $\varepsilon > 0$ all the orbits are periodic. In fact H=1 is S^3 and it is easily obtained than the space of orbits is S^2. For a n-dimensional harmonic oscillator we have S^{2n-1} and $P^{n-1}(\mathbb{C})$, respectively. Perturbation methods or the Gustavson N.F. allow to establish the existence of a finite number (except for degenerate cases) of families of simple P.O. [17, 32] . We have a map near the identity in S^2 that can be seen as the approximate time one flow of an integrable hamiltonian system. The rest points are associated to families of P.O. of the original hamiltonian. For the HH problem there are 8 such families. The stability status of some of the orbits can be a delicate question because they appear as parabolic up to high order term, i.e., the eigenvalues of the associated Poincaré map being of the form $1 + O(\varepsilon^4)$, we need several terms to detect the hyperbolicity. The effect of this "slow" hyperbolicity is seen through the following result.

Theorem: Let β be the angle at one homoclinic point between $W^u(P)$ and $W^s(P)$ where P is a hyperbolic point of a planar diffeomorphism T depending on a parameter ε. If the eigenvalues of DT(P) are $\lambda = 1 \pm O(\varepsilon^k)$ then $\beta \simeq A\,\varepsilon^r \cdot \exp(-B/\varepsilon^k)$ for $\varepsilon \downarrow 0$, where A,B,r are constants. Equivalently we can put $\beta(\varepsilon) \simeq A\,\varepsilon^r \exp(-\frac{B}{\ln\lambda(\varepsilon)})$. A similar behaviour is found for suitable heteroclinic points.

As a consequence of the theorem $\beta(\varepsilon) \ll \varepsilon^m$ for all natural m when $\varepsilon \downarrow 0$ and using a theory of perturbations with respect to ε we can not find β analytically.

Examples of analytical computation of transversality of homoclinic (heteroclinic) points are found in [2] , related to the problem of diffusion, [24] (for the Sitnikov problem) and [18,19] for the restricted problems of three bodies, planar and collinear, and general collinear problem. In the first and third references the computation is obtained through the use of a second perturbation parameter.

Proof of the theorem: we suppose that one of the branches $W_P^{u,1}$ of the unstable manifold of P coincides with one of the branches $W_P^{s,2}$ of the stable manifold, or that we have coincidence $W_P^{u,1} \equiv W_Q^{s,2}$ for the heteroclinic case. As we obtain the coincidence taking only a finite number of terms of the B.N.F. or of the G.N.F. we must compute the variation of the manifold due to the suppressed terms. We get an expresion of the form $I \doteq \varepsilon^s \int_R \cos t \cdot f(t)$ where f(t) is of the type $\exp(-|\ln\lambda \cdot t|)$ for $t \to \pm\infty$, and $\ln\lambda \simeq \varepsilon^k$. Scaling $\tau = \varepsilon^k t$ we get $I = \varepsilon^{s-k} \int_R \cos(\tau\,\varepsilon^{-k})\, f(\tau)\, d\tau = \mathrm{Re} \int_R \varepsilon^{s-k} \exp(i\tau\varepsilon^{-k}) f(\tau)\, d\tau = \varepsilon^{s-k}\, \mathrm{Re}\,(2\pi i \sum \mathrm{Res})$ where the summation is extended to the residues of f in the upper semiplane. Let us suppose that the poles are $a_j + ib_j$, $b_j > 0$ with residue $c_j + id_j$. Then $I = \varepsilon^{s-k} \sum_j \mathrm{Re}\, 2\pi i (c_j + id_j) \exp((ia_j - b_j)\,\varepsilon^{-k})$ and the dominant term is of the form stated in the theorem.

With this theorem in mind we return to the HH problem. In the numerical survey where the problem is introduced [15] it is reported that for ε small ($\varepsilon^2 = h$) it seems to be a foliation by inv riant curves. For $h \simeq 0.11$ some curves dissappear and at $h = 0.16$ no invariant curve remains (except, perhaps, at a very small scale). A numerical computation of the related angle β produces the values of $\alpha = \text{tg}\ \beta/2$ as a function of ε given in table 1. The related results for the $\frac{1}{2}(q_1^2 + q_2^2) - \frac{1}{2}\varepsilon q_1^2 q_2^2$ potential on the level $H = 1$ ($\varepsilon = h$) for which the lack of integrability was nondetected in [15] are given in table 2.

We see that in fact we have nonintegrability for all h but it is hard (or even impossible) to detect it for small energy due to the extremely small angle β. One can ask for the importance of very small angles. For instance, for HH and $h = 0.01$ a rough extrapolation gives $\beta = O(10^{-250})$, and this is nonsense for the physical and numerical points of view. We can proceed in the converse way. The width of the "random" zone is of the order of β. We define a γ-approximate first integral (for a diffeomorphism $T: M \circlearrowleft$) as a function F such that $|F(T^k(Q)) - F(Q)| < \gamma,\ \forall k \in Z,\ \forall Q \in M$. Then we say that T is γ-integrable, i.e., we neglect zones of width $O(\gamma)$. We can ask for the maximum value of h in the HH problem for which the system is γ-integrable. We set $\gamma = A\,\varepsilon^r \exp(-B/\varepsilon^4)$ with values of A,B,r obtained analytically (as in [18]) or through a rough numerical estimate. For instance, using the second approach values $A = 1.74E4$, $B = 0.0554$, $r = 8$ are obtained. Then $\gamma = 10^{-20}$ gives $\varepsilon \simeq 0.19$ $\Rightarrow h = 0.036$, i.e., if zones of width $O(10^{-20})$ are taken as curves, we can say that HH is integrable up to $h = 0.036$.

Another analytical method to make apparent the nonintegrability is to show the existence of solenoids. Let $S^1 \xleftarrow{h_o} S^1 \xleftarrow{h_1} S^1 \xleftarrow{h_2} \dots$ a sequence of maps $h_j: S^1 \longrightarrow S^1$, $h_j(z) = z^{a_j}$ where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. The solenoid Σ_a is the projective limit, i.e. $\{(z_o, z_1, z_2, \dots) \in S^1 x S^1 x S^1 x \dots \mid z_j = h_j(z_{j+1}),\ j \geq 0\}$. Two solenoids are homeomorphic ($\Sigma a \equiv \Sigma b$) if for every power of a prime p^r and every k there is a m such that $p^r | a_o a_1 \dots a_k \Rightarrow p^r | b_o b_1 \dots b_m$ and conversely. Σa is a compact abelian topological group, connected, one-dimensional and without torsion and the flow in Σa is Bohr almost periodic. If a system has embedded solenoids it is nonintegrable. A genericity result is

Theorem [21]: There exist a generic set $\mathcal{R}$ (with the $\mathcal{C}^r$ topology, $r \geq 4$) in $\mathcal{H}$ such that for every $H \in \mathcal{R}$ and every solenoid Σa there is a minimal set, under the hamiltonian flow, homeomorphic to Σa.

The intuitive idea associated to the described solenoids is the existence of islands inside the islands. Near an elliptic fixed point we have a stable island (we think in the case $n=2$ for simplicity). Inside the invariant curves given by Moser twist theorem there are elliptic periodic points where the whole structure is repeated. However it can be very difficult to check the existence of such chains of islands, smaller and smaller, for a concrete system.

ε	α	ε	α
.40	.5805	.30	.1221E-2
.39	.4211	.29	.3591E-3
.38	.2829	.28	.8488E-4
.37	.1800	.27	.1545E-4
.36	.1099	.26	.2056E-5
.35	.6412E-1	.25	.187E-6
.34	.3538E-1	.24	.107E-7
.33	.1811E-1	.23	.35E-9
.32	.8414E-2	.22	.8E-11
.31	.3457E-2		

Table 1

ε	α	ε	α
.6	.1541E-1	-.6	.3615E-4
.575	.1030E-1	-.575	.2107E-4
.55	.6489E-2	-.55	.1164E-4
.525	.3822E-2	-.525	.6052E-5
.5	.2085E-2	-.5	.2933E-5
.475	.1042E-2	-.475	.1309E-5
.45	.4707E-3	-.45	.5312E-6
.425	.1890E-3	-.425	.1924E-6
.4	.6606E-4	-.4	.609E-7
.375	.1956E-4	-.375	.163E-7
.35	.473E-5	-.35	.359E-8
.325	.895E-5	-.325	.59E-9
.3	.124E-6	-.3	.3E-10

Table 2

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