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POWER-CANCELLATION OF CW-COMPLEXES WITH FEW CELLS

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Dedicated to Pere Menal, in memoriam

Abstract

In this paper, we use the fact that the rings of integer matrices have the power-substitution property in order to obtain a power-cancellation property for homotopy types of CW-complexes with one cell in dimensions 0 and $4n$ and a finite number of cells in dimension $2n$.

Introduction

Cancellation in homotopy is closely related to the genus. Recall that the *genus* of a homotopy type X is the set of all homotopy types Y such that the p-localizations of X and Y are homotopy equivalent for all primes $p : X_p \simeq Y_p$ [9]. If X and Y are in the same genus, we shall write $X \sim Y$. All the known examples of non-cancellation occur for homotopy types in the same genus. A relevant result in this line is the following.

Theorem 1 (Wilkerson [12]). *Let X, Y, W be nilpotent co-H-complexes with finitely generated homology. Then*

i) $X \vee Y \simeq X \vee W$ *implies* $Y \sim W$, *and*

ii) $\bigvee^k X \simeq \bigvee^k Y$ *implies* $X \sim Y$.

In [10] Mislin characterized the genus of nilpotent co-H-spaces with finitely generated homology in the following way

$$X \sim Y \quad \text{if and only if} \quad X \vee \bigvee S^{n_i} \simeq Y \vee \bigvee S^{m_j},$$

where the dimensions of the spheres are determined by the homotopy groups of X and Y. A similar result had been obtained by Molnar [11] for CW-complexes with two cells, $S^n \cup_f e^m$, with attaching map f of finite order in $\pi_{m-1}(S^n)$. See also [7].

In [1] Bokor develops matricial methods in order to deal with spaces $S^{2n} \cup_f e^{4n}$ such that the attaching map is of infinite order. This lead him to prove the following.

Theorem 2 (Bokor [1]). *Let*

$$X_i = S^{2n} \cup_{f_i} e^{4n}, \quad Y_i = S^{2n} \cup_{g_i} e^{4n}, \quad i = 1, ..., H.$$

Suppose that f_i, g_j represent elements of infinite order in $\pi_{4n-1}(\bigvee^k S^{2n})$ if and only if $i \leq h$ and $j \leq h'$. Then

$$\bigvee^H X_i \simeq \bigvee^H Y_j$$

if and only if

i) $h = h'$

ii) $\bigvee_{h+1}^{H} X_i \simeq \bigvee_{h+1}^{H} Y_i$

iii) *There exists a permutation τ of $\{1, \ldots, h\}$ such that $X_i \simeq Y_{\tau(i)}$ for all $i \leq h$.*

In [8] we tried to extend Bokor's result to complexes with one cell in dimensions 0 and $4n$ and k cells in dimension $2n$. For $n \geq 2$, the result turned out not to be true. However, if we restrict ourselves to the p-local case, the theorem holds. The proof uses a technical lemma that is also used in the proof of a cancellation property for the ring $\mathcal{M}_k(\mathbb{Z}_p)$ of matrices over the p-localization $\mathbb{Z}_p$ of the ring $\mathbb{Z}$ ([4, Th.2]). For our purpose, we state the following version of the main theorem in [8].

Theorem 3. *Let*

$$X_i = \bigvee^k S^{2n} \cup_{f_i} e^{4n}, \quad Y_i = \bigvee^k S^{2n} \cup_{g_i} e^{4n}, \quad i = 1, \ldots, H.$$

Suppose that f_i, g_j represent elements of infinite order in $\pi_{4n-1}(\bigvee^k S^{2n})$ if and only if $i \leq h$ and $j \leq h'$. Then

$$\bigvee^H X_i \sim \bigvee^H Y_i$$

if and only if

i) $h = h'$

ii) $\bigvee^{h} X_i \sim \bigvee^{h} Y_i$

iii) $\bigvee_{h+1}^{H} X_i \sim \bigvee_{h+1}^{H} Y_i$.

In this paper we use a property of the rings $\mathcal{M}_k(\mathbb{Z})$, the (left) power-substitution property, to prove the following theorems.

Theorem I. *Let*

$$X = \bigvee^{k} S^n \cup_f e^m\,, \quad Y = \bigvee^{k} S^n \cup_g e^m\,.$$

Assume that f, g are suspension elements of finite order in $\pi_{m-1}(\bigvee^{k} S^n)$. Then if $X \sim Y$, there is a positive integer t such that

$$\bigvee^{t} X \simeq \bigvee^{t} Y.$$

Observe that the converse holds by Theorem 1.

Theorem II. *Let*

$$X_i = \bigvee^{k} S^{2n} \cup_{f_i} e^{4n}\,, \quad Y_i = \bigvee^{k} S^{2n} \cup_{g_i} e^{4n}\,, \quad i = 1, \dots, H\,.$$

Assume that f_i, g_j represent elements of infinite order in $\pi_{4n-1}(\bigvee^k S^{2n})$ if and only if $i \leq h\,,\ j \leq h'$. Then if

$$\bigvee^{H} X_i \simeq \bigvee^{H} Y_i$$

we have $h = h'$ and there exist positive integers $s, t_1, \dots, t_h$ and a permutation τ of $\{1, \dots, h\}$ such that

$$\bigvee^{t_i} X_i \simeq \bigvee^{t_i} Y_{\tau(i)}$$

for all $i \leq h$, and

$$\bigvee^{s}\Big(\bigvee_{h+1}^{H} X_i\Big) \simeq \bigvee^{s}\Big(\bigvee_{h+1}^{H} Y_i\Big).$$

Power-substitution

A ring E satisfies the *"(left) power-substitution property"* if given $xa + b = 1$ in E there exist a positive integer t and a matrix $T \in \mathcal{M}_t(E)$ such that $I_t a + Tb$ is an unit in $\mathcal{M}_t(E)$. Here I_t is the identity $t \times t$ matrix. All subrings of the rationals $\mathbb{Q}$ and in particular $\mathbb{Z}$, as well as all the matrix rings $\mathcal{M}_n(\mathbb{Z})$, $n \geq 1$, satisfy power-substitution ([**5**, (2.9) and (3.4)]). So, given

$$XA + B = I_n$$

in $\mathcal{M}_n(\mathbb{Z})$, there exist a positive integer t and a matrix $T \in \mathcal{M}_{nt}(\mathbb{Z})$ such that

$$I_t \otimes A + T(I_t \otimes B)$$

is invertible. Here $\otimes$ denotes the Kronecker product

$$I_t \otimes A = \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A \end{pmatrix}$$

Power-substitution implies power-cancellation in the following sense. If A, B, C are right R-modules and $\mathrm{End}_R(A)$ has the power-substitution property, then $A \oplus B \cong A \oplus C$ implies $B^t \cong C^t$ for some positive integer t.

Proof of Theorem I

Since f and g are suspension elements, they represent elements in $\bigoplus^k \pi_{m-1}(S^n) \subset \pi_{m-1}(\bigvee^k S^n)$. So, f and g are determined by column matrices $x(f)$ and $x(g)$ with entries in $\pi_{m-1}(S^n)$. It was shown in [**7**] that the mapping cones of f and g, X and Y, are in the same genus if and only if there exists an integer matrix A such that

$$x(g) = Ax(f)$$

and $(\det A, l) = 1$, where l denotes the order of f.

Let B be an integer matrix such that $AB = \det A \; I_k$, and take r, s such that $r \det A + sl = 1$. Since $\mathbb{Z}$ has the power-substitution property, there exist an integer c and a matrix C such that

$$I_c \det A + C\,l \quad \text{is invertible.}$$

Then

$$I_k \otimes (I_c \det A) + (I_k \otimes C\,l) = (I_c \otimes B)(I_c \otimes A) + (I_k \otimes C\,l) = V$$

is invertible and, by the power-substitution property of $\mathcal{M}_{ck}(\mathbb{Z})$, we obtain an integer d and a matrix D such that

$$I_d \otimes (I_c \otimes A) + D(I_d \otimes V^{-1}(I_k \otimes C\,l)) = U$$

has an inverse. So applying power-substitution once more, we obtain an invertible matrix of the form

$$Q = I_e \otimes (I_{cd} \otimes A) + E(I_e \otimes U^{-1}D(I_d \otimes V^{-1}(I_k \otimes C\,l))) = I_t \otimes A + T\,l$$

where $t = cde$ and $T = E(I_e \otimes U^{-1}D(I_d \otimes V^{-1}(I_k \otimes C)))$.

Now

$$Qx(f \vee \ldots \vee f) = Q \begin{pmatrix} x(f) & 0 & \cdots & 0 \\ 0 & x(f) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x(f) \end{pmatrix} =$$

$$= \begin{pmatrix} Ax(f) & 0 & \cdots & 0 \\ 0 & Ax(f) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & Ax(f) \end{pmatrix} = x(g \vee \cdots \vee g)\,.$$

Hence

$$\bigvee^{t} X \simeq \bigvee^{t} Y. \quad \blacksquare$$

Proof of Theorem II

First of all, recall that the homotopy type of each $X_i = \bigvee^{k} S^{2n} \cup_{f_i} e^{4n}$ is determined by the $k \times k$ integer matrix $H(f_i)$ of the associated Hilton-Hopf quadratic form and one one-column matrix $x(\sum f_i)$, the entries of which are suspension elements in $\pi_{4n}(S^{2n+1})$ [1]. Moreover, $\bigvee^H X_i \simeq \bigvee^H Y_i$ if and only if there exists a homotopy commutative diagram

$$\begin{array}{ccc} \bigvee^{H} S^{4n-1} & \xrightarrow{f_1 \vee \cdots \vee f_H} & \bigvee^{H}(\bigvee^{k} S^{2n}) \\ \vartheta \big\downarrow & & \big\downarrow \varphi \\ \bigvee^{H} S^{4n-1} & \xrightarrow{g_1 \vee \cdots \vee g_H} & \bigvee^{H}(\bigvee^{k} S^{2n}) \end{array}$$

with ϑ and φ homotopy equivalences. Let $\theta_{i,j}$ be the degree of

$$S^{4n-1} \xrightarrow{in_j} \bigvee^H S^{4n-1} \xrightarrow{\vartheta} \bigvee^H S^{4n-1} \xrightarrow{pr_i} S^{4n-1}.$$

The matrix $(\theta_{ij}) = \theta$ characterizes the homotopy class of ϑ.

Let now ϕ_{ij} be the matrix of

$$\bigvee^k S^{2n} \xrightarrow{in_j} \bigvee^H (\bigvee^k S^{2n}) \xrightarrow{\varphi} \bigvee^H (\bigvee^k S^{2n}) \xrightarrow{pr_i} \bigvee^k S^{2n}.$$

The homotopy commutativity of the previous diagram is equivalent to the following condition [1], [8].

$$(*) \qquad \begin{array}{ll} \phi_{ji} H(f_i) \phi_{ji}^t = \theta_{ji} H(g_j) & \text{for all} \quad i, j \\ \phi_{ji} H(f_i) \phi_{li}^t = 0 & \text{if} \quad l \neq j \\ \phi_{ji} x(\sum f_i) = \theta_{ji} x(\sum g_j) & \text{for all} \quad i, j \end{array}$$

where ϕ_{ji}^t is the transpose of ϕ_{ji}.

We know that f_i is of finite order if and only if $H(f_i) = 0$. Hence, $\theta_{ji} = 0$ for $j \leq h'$, $i > h$. Thus, $\det \theta = \pm 1$ implies $h \geq h'$ and by symmetry $h = h'$.

Now write the matrices of ϑ and φ in the form

$$\theta = \begin{pmatrix} \Theta_1 & 0 \\ \Theta_3 & \Theta_4 \end{pmatrix} \quad , \quad \phi = \begin{pmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{pmatrix}.$$

They are invertibles. Thus

$$C_1 \Phi_1 + C_2 \Phi_3 = I \qquad C_3 \Phi_2 + C_4 \Phi_4 = I$$

where $\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = \phi^{-1}$. By the power-substitution property of $\mathcal{M}_{kh}(\mathbb{Z})$ and $\mathcal{M}_{k(H-h)}(\mathbb{Z})$, we can find positive integers r, s and matrices R, S such that

$$\begin{aligned} A &= I_r \otimes \Phi_1 + R(I_r \otimes C_2 \Phi_3) \\ B &= I_s \otimes \Phi_4 + S(I_s \otimes C_3 \Phi_2) \end{aligned}$$

are invertibles. Consider the diagrams

$$\begin{array}{ccc}
\bigvee^{r}(\bigvee^{h} S^{4n-1}) & \xrightarrow{\bigvee^{r}(\bigvee^{h} f_i)} & \bigvee^{r}(\bigvee^{h}(\bigvee^{k} S^{2n})) \\
\zeta\downarrow & & \downarrow\alpha \\
\bigvee^{r}(\bigvee^{h} S^{4n-1}) & \xrightarrow{\bigvee^{r}(\bigvee^{h} g_i)} & \bigvee^{r}(\bigvee^{h}(\bigvee^{k} S^{2n}))
\end{array}$$

and

$$\begin{array}{ccc}
\bigvee^{s}(\bigvee_{h+1}^{H} S^{4n-1}) & \xrightarrow{\bigvee^{s}(\bigvee_{h+1}^{H} f_i)} & \bigvee^{s}(\bigvee_{h+1}^{H}(\bigvee^{k} S^{2n})) \\
\xi\downarrow & & \downarrow\beta \\
\bigvee^{s}(\bigvee_{h+1}^{H} S^{4n-1}) & \xrightarrow{\bigvee^{s}(\bigvee_{h+1}^{H} g_i)} & \bigvee^{s}(\bigvee_{h+1}^{H}(\bigvee^{k} S^{2n}))
\end{array}$$

where ζ, ξ, α and β are homotopy equivalences with matrices $I_r \otimes \Theta_1$, $I_s \otimes \Theta_4$, A and B respectively. By checking the matricial conditions mentioned above $(*)$, we can see that these diagrams are homotopy commutative. (See [8] for the details in a quite similar case). So, these diagrams induce homotopy equivalences

$$\bigvee^{r}(\bigvee^{h} X_i) \simeq \bigvee^{r}(\bigvee^{h} Y_i) \quad \text{and} \quad \bigvee^{s}(\bigvee_{h+1}^{H} X_i) \simeq \bigvee^{s}(\bigvee_{h+1}^{H} Y_i)$$

We may now assume that all the given spaces X_i and Y_i, $i = 1, \ldots, H$, have attaching maps of infinite order. Let r be the maximum rank of the matrices $H(f_i)$ and $H(g_i)$ and assume that the rank of $H(g_1)$ is r. Since $\det\theta = \pm 1$, $\theta_{1l} \neq 0$, for some l and, hence, $\phi_{1l} H(f_l) \phi_{1l}^t = \theta_{1l} H(g_1)$ has rank r. Thus $rk\, H(f_l) = r$. Now, from the rest of the matricial conditions $(*)$, one gets $\theta_{jl} = 0$ if $j \neq 1$, and $\theta_{1l} = \pm 1$. For simplicity we suppose $l = 1$.

Let us write the matrices of ϑ and φ in the form

$$\theta = \begin{pmatrix} \theta_{11} & \Theta_3 \\ 0 & \Theta_4 \end{pmatrix} \quad , \quad \phi = \begin{pmatrix} \phi_{11} & \Phi_2 \\ \Phi_3 & \Phi_4 \end{pmatrix}$$

and let $\phi^{-1} = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$. Thus

$$C_1\phi_{11} + C_2\Phi_3 = I \quad \text{and} \quad C_3\Phi_2 + C_4\Phi_4 = I\,.$$

As before, we can find invertible matrices of the form

$$A = I_r \otimes \phi_{11} + R(I_r \otimes C_2\Phi_3) \ \text{ and } \ B = I_s \otimes \Phi_4 + S(I_s \otimes C_3\Phi_2)\,.$$

Consider homotopy commutative diagrams

$$\begin{array}{ccc}
\bigvee\limits^{r} S^{4n-1} & \xrightarrow{\ \bigvee\limits^{r} f_1\ } & \bigvee\limits^{r}(\bigvee\limits^{k} S^{2n}) \\
{\scriptstyle\zeta}\big\downarrow & & \big\downarrow{\scriptstyle\alpha} \\
\bigvee\limits^{r} S^{4n-1} & \xrightarrow{\ \bigvee\limits^{r} g_1\ } & \bigvee\limits^{r}(\bigvee\limits^{k} S^{2n})
\end{array}$$

$$\begin{array}{ccc}
\bigvee\limits^{s}(\bigvee\limits_{2}^{H} S^{4n-1}) & \xrightarrow{\ \bigvee\limits^{s}(\bigvee\limits_{2}^{H} f_i)\ } & \bigvee\limits^{s}(\bigvee\limits_{2}^{H}(\bigvee\limits^{k} S^{2n})) \\
{\scriptstyle\xi}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\
\bigvee\limits^{s}(\bigvee\limits_{2}^{H} S^{4n-1}) & \xrightarrow{\ \bigvee\limits^{s}(\bigvee\limits_{2}^{H} g_i)\ } & \bigvee\limits^{s}(\bigvee\limits_{2}^{H}(\bigvee\limits^{k} S^{2n}))
\end{array}$$

where ζ, ξ, α and β are homotopy equivalences with matrices $\theta_{11}I_r$, $I_s \otimes \Theta_4$, A and B respectively. These diagrams induce homotopy equivalences

$$\bigvee^{r} X_1 \simeq \bigvee^{r} Y_1 \quad \text{and} \quad \bigvee^{s}(\bigvee_{2}^{H} X_i) \simeq \bigvee^{s}(\bigvee_{2}^{H} Y_i)\,.$$

This, by induction, completes the proof of Theorem II. ■

Remark. The following question naturally arises in the study of cancellation: is $X \sim Y$ equivalent to $\bigvee^{t} X \simeq \bigvee^{t} Y$? (See [**6**] for an algebraic result of this kind). Theorem I and Theorem 1 give a positive answer for some co-H-spaces with few cells. For mapping cones X and Y of elements of infinite order in $\pi_{4n-1}(\bigvee^k S^{2n})$, it follows from [**8**] Theorem 1 that $\bigvee^t X \simeq \bigvee^t Y$ implies $X \sim Y$. In this case, however, the converse is not always true. If, for instance, f and g are such that

$$H(f) = \begin{pmatrix} 2p & 0 \\ 0 & 2q \end{pmatrix}, \quad H(g) = \begin{pmatrix} 2 & 0 \\ 0 & 2pq \end{pmatrix}, \quad \sum f = 0 \text{ and } \sum g = 0,$$

where $p \neq q$ are prime integers, then $X \sim Y$ and $X \not\simeq Y$ (see [**2**]). But, since $H(f)$ and $H(g)$ are of maximum rank, it follows from [8] Theorem 2 that $\bigvee^t X \not\simeq \bigvee^t Y$ for all integers t.

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