Energy Transduction in Periodically Driven Non-Hermitian Systems

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We show a new mechanism to extract energy from nonequilibrium fluctuations typical of periodically driven non-Hermitian systems. The transduction of energy between the driving force and the system is revealed by an *anomalous* behavior of the susceptibility, leading to a diminution of the dissipated power and consequently to an improvement of the transport properties. The general framework is illustrated by the analysis of some relevant cases.

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In the past years a growing interest in mechanisms for energy transduction by rectification of unbiased thermal fluctuations has arisen, partly motivated by problems from cell biology. Several phenomenological models, categorized as thermal ratchets or Brownian motors, have been proposed. These engines operate at molecular level [1], although their potential implementation in a larger scale would be of evident interest. In this context, a number of methods for particle separation have been recently proposed based on several variants of the ratchet concept [2]. Motivated by this interest, we investigate how to take advantage of nonequilibrium fluctuations to optimize the energy consumption. We propose a new mechanism for energy transduction in a class of nonequilibrium systems when they are acted upon by a weak periodic force. The coupling between the external driving and the out-of-equilibrium fluctuations leads to a minimization of the dissipated power. The minimum occurs when the frequency of the external force matches a characteristic frequency of the system, thus manifesting a resonant behavior.

Let us consider the class of differential equations

$$\partial_t \Psi(\vec{x}, t) = [\mathcal{L}_0 + \lambda(t)\mathcal{L}_1]\Psi(\vec{x}, t), \tag{1}$$

describing the dynamics of a probability density or of a field $\Psi(\vec{x}, t)$, where $\vec{x}$ represents a coordinate. The dynamics is governed by the non-Hermitian operator $\mathcal{L}_0$ and is influenced by the action of a periodic force which introduces the perturbation $\lambda(t)\mathcal{L}_1$, with $\lambda(t) = \lambda_0 e^{i\omega t}$.

We will analyze the response of the system to the external perturbation by using linear response theory (LRT). Accordingly, Eq. (1) can be solved formally yielding

$$\begin{aligned}\Psi(\vec{x}, t) &= \Psi_0(\vec{x}) \\ &\quad + \int_{t_0}^{t} d\tau\, \lambda(\tau) e^{(t-\tau)\mathcal{L}_0}[\mathcal{L}_1\Psi_0(\vec{x})] \\ &\equiv \Psi_0(\vec{x}) + \Delta\Psi(\vec{x}, t),\end{aligned} \tag{2}$$

where $\Psi_0(\vec{x})$ is the initial condition, which corresponds to the stationary state of Eq. (1) in the absence of the external force [3]. Expansion of the term $\mathcal{L}_1\Psi_0(\vec{x})$ in a series of the eigenfunctions of the operator $\mathcal{L}_0$, ϕ_n with eigenvalue a_n; $n = 0, 1, \ldots,$

$$\mathcal{L}_1\Psi_0(\vec{x}) = \sum_{n=0}^{\infty}\{c_n\phi_n(\vec{x}) + c_n^*\phi_n^*(\vec{x})\}, \tag{3}$$

where c_n are the corresponding coefficients in this expansion, then leads to

$$\Delta\vec{x}(t) = \int \vec{x}\Delta\Psi(\vec{x}, t)\, d\vec{x} = \int_{t_0}^{t} d\tau\, \vec{\chi}(t - \tau)\lambda(\tau), \tag{4}$$

which defines the susceptibility $\vec{\chi}(t)$.

We will assume the existence of a dominant time scale governing the relaxation process, which corresponds to the $n = 1$ mode in the expansion (3). Since the remaining modes decay faster we can truncate the series retaining only the first term. Thus, considering only contributions of the first mode, $\vec{\chi}(t) = \vec{A}e^{a_1 t} + \text{c.c.}$, with $\vec{A}$ defined as

$$\vec{A} = c_1 \int \vec{x}\phi_1(\vec{x})\, d\vec{x}, \tag{5}$$

the explicit expression of $\vec{\chi}(\omega)$ follows from the Fourier transform of $\vec{\chi}(t)$,

$$\vec{\chi}(\omega) = \frac{\vec{A}}{I_1}\frac{1}{\beta - i(\alpha + 1)} + \frac{\vec{A}^*}{I_1}\frac{1}{\beta - i(\alpha - 1)}, \tag{6}$$

where $a_1 \equiv R_1 + iI_1$, $\beta \equiv R_1/I_1$, and $\alpha \equiv \omega/I_1$, with R_1 and I_1 being the real and imaginary parts of a_1, respectively. In Fig. 1, we have plotted the modulus of the susceptibility as a function of α for different values of β. During the relaxation process of nonequilibrium fluctuations, the susceptibility undergoes a resonant behavior when the frequency of the force matches the imaginary part of the first eigenvalue of the nonperturbed operator $\mathcal{L}_0$. This behavior reveals the resonant coupling between the periodic force and the nonequilibrium source, responsible for the non-Hermitian nature of $\mathcal{L}_0$.

The appearance of this resonance leads to a diminution of the dissipation in the relaxation process of the fluctuations of $\vec{x}$. To illustrate our assertion, let us suppose that Eq. (1) represents the non-Hermitian Fokker-Planck equation

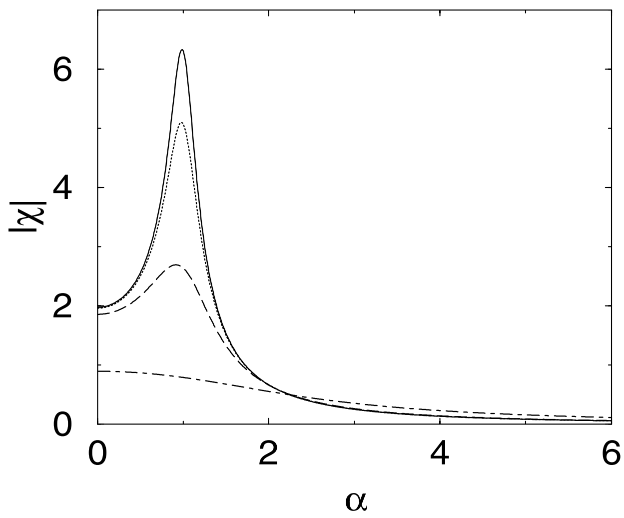

FIG. 1. Nondimensional modulus of the susceptibility as a function of the parameter α. The solid line corresponds to $\beta = 0.1$, dotted line to $\beta = 0.5$, and dashed line to $\beta = 1$. The resonance fades away practically for $\beta \approx 10$ (dot-dashed line).

$$\partial_t \Psi(\vec{x}, t) = -\nabla_{\vec{x}} \cdot [\vec{v}(\vec{x})\Psi - D\nabla_{\vec{x}}\Psi] - \lambda(t)\nabla_{\vec{x}} \cdot [b\Psi\nabla_{\vec{x}}U(\vec{x})] \equiv \mathcal{L}_0\Psi + \lambda(t)\mathcal{L}_1\Psi\,, \quad (7)$$

where $\vec{v}(\vec{x})$ is a nonpotential drift, b is a mobility, $D = K_BTb$ is the corresponding diffusion coefficient, and $U(\vec{x})$ is the potential related to the external force. Among the physical realizations of the model described by Eq. (7), we can quote the case of a Brownian particle advected by a constant drift $\vec{v}$ acted upon by a force $\vec{F}(\vec{x}, t)$, or a field-responsive particle in a vortex flow under the influence of an oscillating magnetic field [4].

We are interested in analyzing the energy dissipated by the system in the dynamic process governed by Eq. (7). The dissipated power is

$$P = -\int d\vec{x}\, \vec{J} \cdot \nabla_{\vec{x}}\mu\,, \quad (8)$$

where $\vec{J} = -D\nabla_{\vec{x}}\Psi - b\lambda(t)\Psi\nabla_{\vec{x}}U(\vec{x})$ is the diffusion current and $\mu = K_BT\ln\Psi(\vec{x}, t) + \lambda(t)U(\vec{x})$ is the corresponding chemical potential, including the power supplied by the external force. The externally supplied power is given by

$$P_F = -\lambda(t)\int d\vec{x}\, \vec{J} \cdot \nabla_{\vec{x}}U = \int d\vec{x}\, \vec{J} \cdot \vec{F}(\vec{x}, t)\,. \quad (9)$$

By expanding $\vec{F}(\vec{x}, t)$ in a series of the eigenfunctions of $\mathcal{L}_0$ and substituting this equation into Eq. (9), we achieve

$$P_F = \vec{F}_0(t) \cdot \left\{\frac{d\langle\vec{x}\rangle}{dt} - \langle\vec{v}(\vec{x})\rangle\right\} + \sum_{n\neq 0} 2\Omega\, \mathrm{Re}\{\vec{F}_n(t) \cdot \vec{J}_n^*\}\,, \quad (10)$$

where $\vec{F}_n(t)$ are the coefficients of the expansion of $\vec{F}(\vec{x}, t)$, and Ω is the volume of the system. To obtain Eq. (10) we have used the result

$$\int d\vec{x}\, \vec{J} = \frac{d\langle\vec{x}\rangle}{dt} - \langle\vec{v}(\vec{x})\rangle\,. \quad (11)$$

This expression follows from the Fokker-Planck equation (7) through the definition of $\vec{J}$.

The quantity of interest is the time-averaged dissipated power,

$$\overline{P}_F(\omega) = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\, P_F\,. \quad (12)$$

For the particular case of the Brownian particle, P_F reads

$$P_F = \vec{F}_0(t) \cdot \left(\frac{d\langle\vec{x}\rangle}{dt} - \vec{v}\right) + \sum_{n\neq 0} 2\Omega\, \mathrm{Re}\{\vec{F}_n(t) \cdot \vec{J}_n^*\}\,. \quad (13)$$

In Fig. 2a we have plotted the corresponding $\overline{P}_F(\omega)$. To this end we have assumed that the motion of the particle takes place in two dimensions, with periodic boundary conditions; thus, the set of eigenfunctions $\phi_{\vec{k}}(\vec{x})$ are the Fourier modes with eigenvalues $a_{\vec{k}} = -Dk^2 - i\vec{v} \cdot \vec{k}$. We have considered a force of the form $\vec{F}(\vec{x}, t) = \lambda(t)[1 + \cos(\vec{k} \cdot \vec{x})]\hat{v}$, where $\hat{v}$ is the unit vector pointing along the direction of the drift. The figure shows that $\overline{P}_F(\omega)$ achieves its minimum value at the resonant

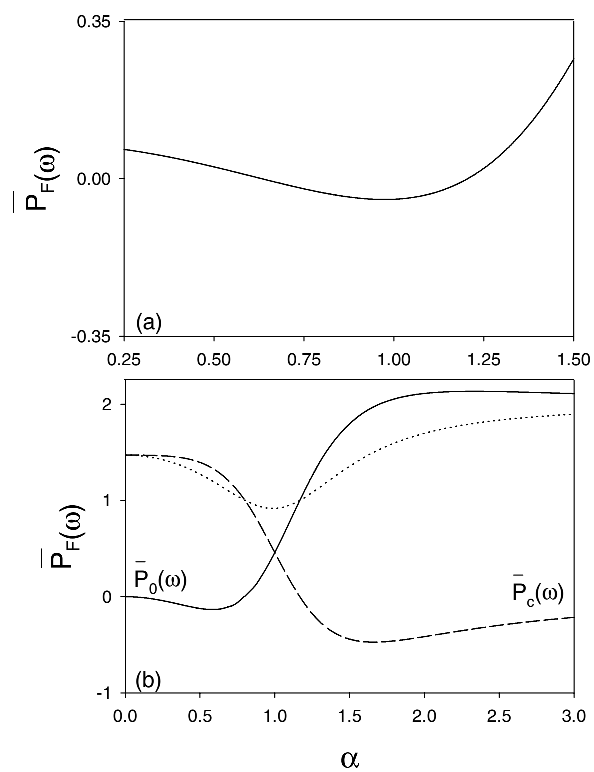

FIG. 2. (a) Nondimensional dissipated power for a particle under a constant drift as a function of the nondimensional parameter α. (b) Same for a field-responsive fluid. The dotted line corresponds to the total dissipation $\overline{P}_F(\omega)$.

frequency. The negative character of this quantity indicates that the system is acting as a generator.

In Fig. 2b we have represented that quantity for the more complex case of the mesoscopic dynamics of a field-responsive Brownian particle in a vortex flow with vorticity $\vec{\omega}_0$, under an oscillating magnetic field, $\vec{H}(t)$ [4]. In this case, the power supplied by the external force is

$$P_F = \vec{H}(t) \cdot \left\{ \frac{d\langle \vec{M} \rangle}{dt} - \vec{\omega}_0 \times \langle \vec{M} \rangle \right\}, \qquad (14)$$

where $\vec{M}$ is the magnetization. This figure shows that in this case the transduction of energy occurs in two different regimes. In the low frequency regime, $\overline{P}_0$ (the power dissipated in the Debye relaxation [5], achieves negative values while $\overline{P}_c$, the energy dissipated due to the coupling between the drift and the external force) takes negative values for high frequencies. In both cases the averaged dissipated power $\overline{P}_F(\omega)$ exhibits a minimum value at the resonant frequency, showing the resonant character of the mechanism for energy transduction.

The analysis of the energy dissipation allows us to study the transport properties. The presence of external driving forces the system to move with a net velocity, $\vec{v}_m$, different from the drift $\vec{v}(\vec{x})$, thus leading to the appearance of a drag force, $\vec{F}_d = -\vec{\vec{\kappa}} \cdot \vec{v}_m$ [6], with $\vec{\vec{\kappa}}$ a friction tensor accounting for dissipation in the system. The resulting dissipated power is $\overline{P}_F = \vec{v}_m \cdot \vec{\vec{\kappa}} \cdot \vec{v}_m$. In the case of the Brownian particle advected by a constant drift, the friction is given by $\kappa_B = v_m^{-2}\overline{P}_F$, whose behavior is essentially shown in Fig. 2a.

For the field-responsive particle the dissipated power given through Eq. (14) consists of two independent contributions corresponding to longitudinal ($\overline{P}_0$) and transversal ($\overline{P}_c$) effects, with respect to the direction of the magnetic field. Consequently, associated with the last one, which corresponds to the viscous dissipation occurring when the magnetic field acts on the fluid, we can define a friction coefficient as

$$\kappa_F = -\frac{1}{(v_m^T)^2}\, \overline{\vec{\omega}_0 \cdot (\langle \vec{M} \rangle \times \vec{H})}, \qquad (15)$$

where $v_m^T = \omega_0 a$ is the transversal component of $\vec{v}_m$, with a the radius of the particle and ω_0 the modulus of the vorticity. It is interesting to analyze the behavior of this quantity in terms of the parameter β, which is given in this case by D_r/ω_0, with D_r being the rotational diffusion coefficient. As can be seen in Fig. 3, as β grows the friction coefficient κ_F becomes positive in the entire frequency range. In fact the frequency for which $\kappa_F = 0$ goes to infinity in the limit of a Hermitian dynamics. Figure 1 shows something genuinely analogous of non-Hermitian systems, that is, the resonance disappears when β grows.

The anomalous behavior exhibited by the friction coefficient is a direct consequence of the form of the susceptibility. In equilibrium, the fluctuation-dissipation theorem, implying that $\omega \, \mathrm{Im}\chi_x(\omega) \geq 0$, manifests such that during the relaxation of the fluctuations around an equilibrium state the system always dissipates energy. Nonetheless, in the nonequilibrium case shown in Fig. 4 the imaginary part of the susceptibility achieves negative values for positive frequencies, thus violating the aforementioned inequality. In this figure, obtained for the particular case of the Brownian particle in a constant drift, the analytical results are compared with numerical results from the corresponding Langevin equation by means of a second order Runge-Kutta method [7]. For the field-responsive particle, we obtain the same behavior, which essentially corresponds to $\overline{P}_0$, plotted in Fig. 2b. This fact indicates that the system is generating energy instead of dissipating power, which manifests at a macroscopic level through the diminution of the friction coefficient. An anomalous behavior of the response was first discussed in the context of current generation by noise-induced symmetry-breaking in coupled Brownian motors [8]. These papers were focused

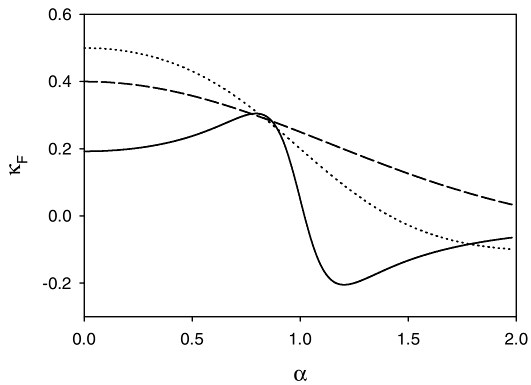

FIG. 3. Nondimensional friction coefficient as a function of the parameter α for different values of β. The solid line corresponds to $\beta = 0.1$, dotted line to $\beta = 0.5$, and dashed line to $\beta = 1$.

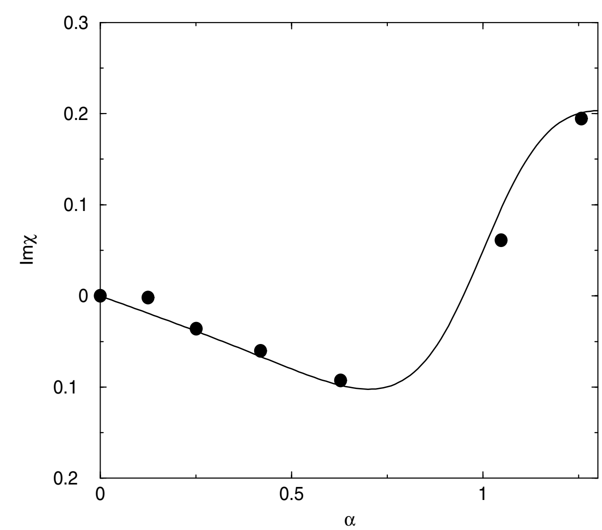

FIG. 4. Nondimensional imaginary part of the susceptibility for a Brownian particle advected by a constant drift as a function of the parameter α. The solid line corresponds to the analytical computation, whereas the dots have been obtained from numerical simulations.

on specific models, which questioned under which circumstances this behavior arises. In this Letter, we have found that the conditions for these phenomena occur in a quite general class of systems.

Notice that our results differ from the ones obtained when LRT is applied to nonequilibrium Hermitian systems. For example, in Ref. [3] fluctuation-dissipation–type relationships are derived, by assuming that the relaxation occurs as in equilibrium. In the present context, this assumption does not hold, since we are dealing with non-Hermitian systems whose eigenvalue spectrum is complex. Consequently, perturbations do not relax exponentially as they do in equilibrium.

The theoretical framework discussed in this paper can be applied to a number of problems formulated in terms of non-Hermitian dynamics. Among them, one could mention the transport of classical particles advected by a quenched [9] velocity field. This process, governed by a Fokker-Planck equation with random drift, models the diffusion in porous media. It has been recently shown [9] that, when the velocity field displays correlations in both longitudinal and transversal directions, the eigenvalues occupy a finite area in the complex plane. Consequently, this system evolves according to a non-Hermitian dynamics. If the particles can respond to an external field, the same phenomenology described in this paper holds; i.e., as a consequence of the diminution of the dissipation, transport through the porous medium becomes enhanced. The phenomenon we study may also arise in population biology problems, in particular in the generalization of the Malthus-Verhulst growth model proposed by Nelson and Shnerb [10]. The linearization of this model around its steady state yields a non-Hermitian evolution equation. When periodic driving is introduced as a time dependence of the resources of the medium, the minimum achieved by the dissipated energy is now related to a resonant optimization of these resources.

Additionally, our approach might unify the explanation of other resonant transport phenomena previously reported. The Senftleben-Beenakker effect observed in gases of polyatomic molecules shares the phenomenology inherent to our model. This effect occurs when the gas is under the action of both a constant magnetic field and an oscillating field parallel to the first one [11]. Larmor precession causes the non-Hermiticity of the Boltzmann equation describing the dynamics of this system. The resonant frequency is related to Larmor's frequency. Under these conditions, the viscosity of the gas manifests a non-monotonous behavior similar to the one depicted in Fig. 3. Another example exhibiting analogous characteristics is the negative viscosity effect observed in field-responsive fluids [4], under a nonpotential flow and submitted to an ac field. This effect consists of a diminution of the viscosity due to the presence of the periodic field. The field-responsive phase acts as a transmitter of energy between the external force and the system, improving the transport.

In summary, we have proposed a mechanism for energy transduction in nonequilibrium systems, based on the possibility of extracting energy from the relaxation process of out-of-equilibrium fluctuations. Under the action of an oscillating force, systems which evolve according to non-Hermitian dynamics act as transducers. Consequently, the energy dissipated in the system diminishes achieving its minimum value when the frequency of external driving matches the resonant frequency. This diminution of the dissipated energy has a strong influence on the macroscopic properties of the system, leading to an enhancement of the transport or more generally to an optimization of the consumption of energy. Because of the intrinsic nonequilibrium nature of the fluctuations, energy transduction does not require any further ingredient, as occurs in ratchetlike engines, in which the presence of a parity-symmetry-breaking potential is an unavoidable condition for transferring energy.

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