Soliton Collision in General Relativity

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(Recieved 1 April 1983)

An exact solution of the Einstein equations in vacuum representing two pairs of gravitational solitons propagating on an expanding universe is given and studied. It is suggested that the solitons evolve from quasiparticles to pure gravitational waves. Two of the four solitons collide and the focusing produced on null rays is studied. Although the space-time following the collision is highly distorted, null rays do not focus to a singularity.

PACS numbers: 04.20.Jb, 11.10.Lm, 98.80.Dr

Solitons are found in general relativity as in other fields of nonlinear physics. We discuss a family of solutions of the Einstein equations in vacuum that can be interpreted as describing the propagation of four gravitational solitons and the collision of two of them on an expanding cosmological background. The family is the "four-soliton solution" generated via the inverse-scattering technique of Belinskii and Zakharov[1] taking as "seed" metric the homogeneous Kasner cosmological solution. The solution obtained has the Kasner singularity at $t=0$ and is completely regular afterwards, it is inhomogeneous through the z axis, and it contains four solitons propagating on the Kasner background.

We shall be interested in two main problems: first, the characterization of the solitons through the Riemann tensor and their scalar invariants; second, the collision of solitons which will be studied by means of the expansion and shear of null congruences propagating on the metric field, and which provides an example of the interaction of nonlinear waves in general relativity.

The metric considered admits two commuting orthogonal spacelike Killing vectors; it is essentially an Einstein-Rosen metric but differs in its global structure and interpretation. It involves only three "soliton" parameters and a "seed" parameter (δ). It is the diagonal, vacuum, four-soliton solution

$$ds^2=f(t,z)(dz^2-dt^2)+g_{11}(t,z)dx^2+g_{22}(t,z)dy^2\,,\quad g_{11}=t^{1+\delta}\sigma_1\sigma_2,\quad g_{22}=t^2/g_{11}\,,$$
$$f=t^{(\delta^2-1)/2-8}\frac{\sigma_1{}^{\delta}\sigma_2{}^{\delta}}{H_1H_2(1-\sigma_1)^2(1-\sigma_2)^2}\left\{\left[(\sigma_1+\sigma_2)t^2-\frac{8z_1z_2\sigma_1\sigma_2}{(1+\sigma_1)(1+\sigma_2)}\right]^2-\frac{64w_1{}^2w_2{}^2\sigma_1{}^2\sigma_2{}^2}{(1-\sigma_1)^2(1-\sigma_2)^2}\right\}^2, \tag{1}$$

with

$$H_i=(1-\sigma_i)^2+\frac{16w_i{}^2\sigma_i{}^2}{(1-\sigma_i)^2t^2}\,,\quad z_i=z_i{}^0-z\ (i=1,2).$$

The functions $\sigma_i(t,z)$ are obtained from the complex equation ("pole trajectory" equation[1])

$$\mu_i{}^2-2(z_i-iw_i)\mu_i+t^2=0\,,\quad \sigma_i=\mu_i\bar{\mu}_i/t^2,$$

which has one root in the range $(0,1)$ and the other, its inverse, in $(1,\infty)$. We take the first root for σ_1 and the second for σ_2. Analytical expressions for $\sigma_i(t,z)$ can be given explicitly[2] but they are rather lengthy; we note that $\sigma_1\sigma_2\sim 1$ in the limits $t\to\infty$ ($|z|\ll t$) and $z\to\infty$ ($t\ll|z|$).

The "soliton" parameters $z_i{}^0$ and w_i are arbitrary real constants which indicate the "origins" and "widths" of the two pairs of solitons: At $t=0$ one pair of solitons is localized at $z_1{}^0$ and the other pair at $z_2{}^0$ (only $z_2{}^0-z_1{}^0$ is relevant) and for small values of $w_i{}^2\ll 1$ the solitons are very localized.

The parameter $\delta\geqslant 0$ comes from the Kasner seed metric:

$$ds^2=t^{(\delta^2-1)/2}(dz^2-dt^2)+t^{1+\delta}dx^2+t^{1-\delta}dy^2. \tag{2}$$

The value $\delta=0$ corresponds to the axisymmetric Kasner metric. For $\delta>1$ the z axis is expanding and for $\delta<1$ it is contracting. Flat space is given by $\delta=1$; we shall not consider that possibility in what follows.

In order to understand the behavior of the metric it is convenient to define the following asymptotic space-time regions: the "far region," $t\ll|z|\to\infty$; the "light-cone region," $|z|\simeq t\to\infty$, and the "interaction region," $|z|\ll t\to\infty$. In all those asymptotic regions the metric tends towards the Kasner background. However, the longitudinal expansion, which is given by the coefficient f, is the Kasner background value f^k in the far region only. In the light-cone and interaction regions the longitudinal expansion is f^k

multiplied by a factor depending on the soliton parameters. For instance, for the solutions for which $w_i^2 \sim w^2 \ll 1$, in the interaction region the asymptotic values are $f \sim f^k/w^4$, while in the light-cone region they are $f \sim f^k/w^4$ along the light cones of the inner solitons (those which collided) and $f \sim f^k/w^2$ along the light cones of the outer solitons.

The soliton structure and the intrinsic properties of the metric (1) can be seen from the Riemann tensor. This tensor can be easily calculated by iterative use of the formulas for the derivatives of σ_i, which can be deduced from the pole trajectory equations.[1] For a metric of type (1) the Riemann tensor has only three independent components: One can write it in terms of an "electric," E, and a "magnetic," B, 3×3 matrices. The only nonzero components are $E_{11}=e_1$, $E_{22}=e_2$, $E_{33}=e_3=-(e_1+e_2)$, and $B_{12}=B_{21}=b$ (for the Kasner metric the Riemann tensor depends only on t, and $b=0$). The soliton structure appears in the Riemann components: For small values of w^2 the four solitons are narrow pulses and the limit $w^2 \to 0$ produces δ functions.

Frame-independent soliton properties are seen from the two nonzero curvature scalar invariants: $I_1=(e_1^2+e_2^2+e_3^2-2b^2)/2$ and $I_2=(e_1^3+e_2^3+e_3^3+3b^2e_3)/6$. The evolution in time of the ratio I_1/I_1^k (I_1^k is the Kasner value) is shown in Fig. 1 for a model in which $\delta=0$. Before the collision four solitons with small tails are seen. After the collision the outer solitons are clear but the intensities of the inner solitons are very small because of the $1/w^4$ factor of the longitudinal expansion. The same factor makes the value of I_1 in the interaction region very small too.

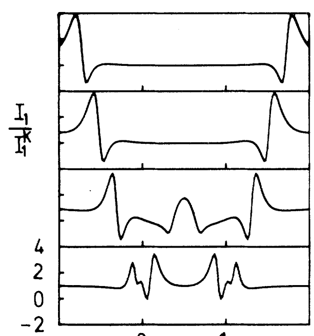

FIG. 1. Time evolution of the ratio of the first curvature scalar to the Kasner value, I_1/I_1^k for a model in which the background in the axisymmetric Kasner metric, $\delta=0$. The curves are represented against the propagation axis (z axis). The widths and origins of the solitons are $w_1=w_2=0.05$ and $z_1^0=0$, $z_2^0=1$. The different curves from bottom to top of figure represent the time sequence $t=0.1$, 0.3 (before collision), 0.5 (collision time: t_c), and 0.7 (after collision).

The behavior of the Riemann tensor and its scalar invariants indicates that metric (1) represents intrinsic inhomogeneities of the gravitational field propagating on a Kasner space-time.

From the algebraic classification of the metric in the light-cone region one can see that the solitons, which are localized there, seem to evolve towards pure gravitational waves. In fact, in that region the leading terms of the Riemann tensor are $e_1 \sim -e_2 \sim b$ which is typical of pure gravitational waves: type-N metrics in the Petrov classification [in general, however, metrics (1) are globally of Petrov type I].

Further features of metric (1) can be seen by projecting the Riemann tensor over a null tetrad $\vec{n}$, $\vec{l}$, $\vec{m}$, and $\vec{m}^*$: $\vec{n}=2^{-1/2}f^{-1}(\partial_t+\partial_z)$, $\vec{l}=2^{-1/2}(\partial_t-\partial_z)$, and $\vec{m}=2^{-1/2}(g_{11}^{-1/2}\partial_x+ig_{22}^{-1/2}\partial_y)$ where $\vec{n}$ is a geodesic null vector field that lies tangent to the rightward-traveling solitons. With that tetrad the nonnull Riemann components are[3] $\Psi_0=[(e_2-e_1)/2+b]f$, $\Psi_2=-e_3/2$, $\Psi_4=[(e_2-e_1)/2-b]f$. Ψ_0 and Ψ_4 contain the radiative part of the field whereas Ψ_2 contains its Coulomb part. Comparing with the homogeneous background we see that solitons induce inhomogeneities in those fields; in particular, Ψ_4 will induce time-varying tidal forces to test particles on the plane orthogonal to the propagation of the rightward solitons.

In Fig. 2, the evolution of the field Ψ_4 is shown. It indicates the gravitational strength of the rightward-traveling solitons. The soliton structure as a nondispersive wave is clear: The solitons are localized in their light-cone regions; the amplitude of Ψ_4 decreases there as $t^{-1/2}$ but this is a consequence of the background expansion and

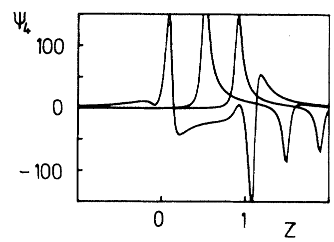

FIG. 2. Time evolution of the Riemann tensor component ψ_4 giving the gravitational strength of the rightward-traveling solitons. Same model and parameters as in Fig. 1. The Kasner background value is $\psi_4^k=0$. The three curves from left to right of figure represent the time sequence $t=0.1$, $t_c=0.5$, and $t=0.9$.

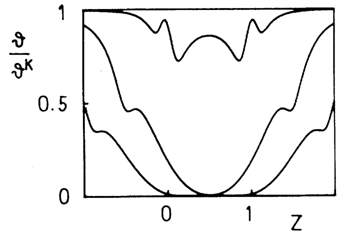

FIG. 3. Time evolution of the expansion ratio θ/θ^k produced by the solitons on the null geodesics generated by the null vector $\vec{n}$. Same model as in Fig. 1 but with soliton widths $w_1 = w_2 = 0.1$. The three curves from top to bottom of figure represent the time sequence $t = 0.1$, $t_c = 0.5$, and $t = 0.9$.

does not imply dispersion. Initially the solitons show tails and a complicated structure; as time increases the tails disappear. Ψ_0 has a similar behavior with the leftward traveling solitons, and Ψ_2, that produces tidal forces on test particles aligned in the direction of propagation of the waves, becomes much less localized than Ψ_0 and Ψ_4 as time increases. Consequently the tranverse part of the field becomes more important than the longitudinal one: The radiation component dominates over the Coulomb component.

The above results suggest to us the interpretation of the gravitational solitons as quasiparticles which evolve towards pure gravitational waves. Thus the soliton collision can not be interpreted as the collision of pure gravitational waves, but as the collision of gravitational fields with radiative and nonradiative (Coulomblike) components.

Finally we shall investigate the focusing effect on null congruences resulting from the collision of solitons. It is well known that the collision of plane gravitational waves produces curvature singularities in the interaction region,[4] but those singularities can be avoided if the collision takes place in an expanding cosmology.[5]

Given the geodesic null vector $\vec{n}$, the metric (1) is uniquely characterized[3] by the expansion, $\theta = 1/2\sqrt{2}ft$, and shear, σ, produced on the null congruence defined by $\vec{n}$. The Kasner solution (2) produces a homogeneous expansion, θ^k, and shear on such a null congruence as a consequence of its overall expansion on the x-y plane. The solitons will produce inhomogeneities on θ and σ and as a consequence of their energy they will focus the null rays. In fact, in Fig. 3, the ratio of the expansions θ/θ^k of the null rays is shown. Initially the focusing produced by the four solitons is clear. By the Raychaudhuri equation this can be interpreted as due to the gravitational "energy" of the solitons which can be measured by the θ^2 and σ^2 they produce. After the collision θ is getting small. This can be interpreted as focusing produced by the gravitational energy of the collision. This is quite different from what one would expect from the simple superposition of linear waves; it is a consequence of the nonlinear interaction of the solitons. The expansion, however, is still positive, $\theta > 0$, and there will be no convergence of null rays to a singularity. The overall expansion of the background prevents the formation of singularities in the future.

We are grateful to Bernard Schultz for helpful discussions. One of us (J.I.) would like to thank the Patronato de la Universidad del País Vasco for financial support. One of us (E.V.) would like to acknowledge the hospitality of the University College, Cardiff, Wales.

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