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GRAU DE MATEMÀTIQUES

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ARAKELIAN'S THEOREM

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Contents

Abstract

The approximation by rational functions and polynomials is one of the topics that has been studied for a long time. The aim of this text is to study the uniform approximation by rational functions and polynomials based on three theorems: Runge, Mergelyan, and Arakelian. The first one concerns uniform approximation by rational functions on compact sets. Mergelyan's theorem is a generalization of Runge's theorem. Finally, Arakelian's theorem deals with uniform approximation by entire functions on possibly unbounded closed sets. We provide the proofs of these theorems and furthermore, we state connexions between them.

Resum

L'aproximació de funcions a partir de funcions racionals i polinomis és un dels temes més estudiats de la Teoria de l'Aproximació. L'objectiu d'aquest text és estudiar l'aproximació uniforme a partir de funcions racionals i polinomis. Per fer-ho desenvoluparem tres teoremes: els de Runge, Mergelyan i Arakelian. El primer es basa en l'aproximació uniforme a partir de funcions racionals en conjunts compactes, i el segon és una generalització de Runge. Finalment, el teorema d'Arakelian tracta l'aproximació uniforme per funcions enteres sobre conjunts tancats i potser no acotats. Veurem les demostracions dels tres teoremes i, a més a més, establirem conexions entre ells.

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Chapter 1

Introduction

The methods and results of complex approximation theory in the complex plane present a powerful instrument for investigating different problems in analytic functions. In particular, the approximation by rational functions and polynomials is one of the topics that has been studied for a long time.
It is often convenient in the study of holomorphic functions to compactify the complex plane by the adjunction of a new point, called ∞. Therefore we will work over the Riemann sphere, $\mathbb{S}^2$, which is the union of $\mathbb{R}^2$ and $\{\infty\}$.
Moreover, remember that a rational function f is, by definition, a quotient of two polynomials P and Q. We may assume that P and Q have no common factors. Then f has a pole at each zero of Q (the pole of f has the same order as the zero of Q). If we subtract the corresponding principal parts, we obtain a rational function whose only singularity is at ∞ and which is, therefore, a polynomial.
The aim of this text is to study the uniform approximation by rational functions and polynomials. We will analyze the classical approximation theorems of Runge, and Mergelyan, which is a generalization of Runge. Both of them require that the uniform approximation by entire functions have to be on a compact subset of the complex plane. The main focus of our work is Arakelian's theorem, which deals with uniform approximation by entire functions on possibly unbounded sets. It is interesting to point out that the proof of Arakelian's theorem uses the theorems of Runge and Mergelyan.
In 1885 Karl Weierstrass published one of the most important results in Approximation Theory, the well-known Weierstrass approximation theorem, which states that every continuous function defined on a compact set $[a, b] \subset \mathbb{R}$ can be uniformly approximated by polynomials. At the same time, Runge proved the first approximation theorem in the complex plane, and after a few years, Mergelyan generalized Runge's theorem. Finally, Arakelian published an approximation theorem that gives necessary and sufficient conditions over the sets to approximate

entire functions on closed subsets of the complex plane.
One wonders if every function can be uniformly approximated by polynomials in the complex plane, and the answer, in this case, is negative. Suppose that $\Omega \subseteq \mathbb{C}$ is an open domain, then an analytic function can not always be uniformly approximated by polynomials in a compact set of Ω. Polynomials are holomorphic, and hence any sequence of polynomials which converges uniformly on Ω converges to a holomorphic function on that set. However, this is not enough: on the interior of the domain the function is holomorphic, but it is not necessarily the uniform limit of any sequence of polynomials. For instance, take the annulus $K = \{z \in \mathbb{C} : 1/2 \leq z \leq 2\}$, let $f(z) = 1/z$ and let $\gamma(t) = e^{2\pi i t}$, $t \in [0, 1]$. Clearly, f is holomorphic in the interior of K, but it is not the uniform limit of any sequence of polynomials. This is because

$$\int_\gamma \frac{1}{z} dz = \int_0^1 \frac{1}{\gamma(t)} \gamma'(t) dt = \int_0^1 \frac{1}{e^{2\pi i t}} 2\pi i e^{2\pi i t} dt = 2\pi i.$$

However

$$\int_\gamma p(z) dz = 0 \quad \text{for all polynomial } p.$$

It was Runge, who introduced a condition to make the result valid in the complex plane: it is enough that the rational functions have their poles outside Ω.

Theorem 1.1. ***(Runge)***. *Suppose K is a compact set in the plane and $\{\alpha_j\}$ is a set which contains one point in each connected component of $\mathbb{S}^2 \backslash K$. If Ω is open, $K \subset \Omega$, and $f \in H(\Omega)$, for all $\epsilon > 0$, there exists a rational function R, all whose poles lie in $\{\alpha_j\}$, such that*

$$|f(z) - R(z)| < \epsilon$$

for every $z \in K$.

Note that $\mathbb{S}^2 \backslash K$ has at most countably many components. Note also that the preassigned point in the unbounded component of $\mathbb{S}^2 \backslash K$ may very well be ∞; in fact, this happens to be the most interesting choice.
We shall see two proofs of Runge's theorem. The first one requires some functional analysis, such as Banach spaces, a consequence of the Hahn-Banach theorem and dual spaces. The idea is to formulate Runge's theorem in terms of some functionals and use the Riesz representation theorem to prove the theorem. We can say that this is an abstract version because we make a double approximation and the proof is neither direct nor mechanical.

The second proof is based on the use of Banach algebras. Define $B(S)$ as the Banach algebra consisting of all rational functions whose poles lie on S.
In this case, the theorem of Runge states:

Theorem 1.2. *(**Runge**). Suppose that K is a compact subset of the complex plane, and that $S = \{\alpha_j\}$ is a set containing one point on each component of $\mathbb{S}^2 \backslash K$. Then $B(S)$ contains every function which is analytic on a neighborhood of K.*

Let us see some examples of Runge's theorem applications.

Example 1.3. *There is a sequence of polynomials p_n such that*

$$p_n(z) \longrightarrow 1 \quad \forall z \in \overline{\mathbb{D}}; \quad p_n(z) \longrightarrow 0 \quad \forall z \notin \overline{\mathbb{D}}$$

In order to see this, let

$$K_n = \overline{\mathbb{D}} \cup \{z; 1 + \frac{1}{n} \leq |z| \leq n, 0 \leq arg(z) \leq 2\pi - \frac{1}{n}\}$$

Since the complement of each K_n is connected, Runge's theorem applies to the function f that is 1 in a neighborhood of $\overline{\mathbb{D}}$ and 0 in a neighborhood of $K_n \backslash \overline{\mathbb{D}}$. Therefore, there are polynomials $p_n(z)$ such that

$$|p_n - 1| < 1 \quad on\ \overline{\mathbb{D}}$$

$$|p_n| < \frac{1}{n} \quad on\ K_n \backslash \overline{\mathbb{D}}$$

Example 1.4. *We construct now a sequence of polynomials P_n such that*

$$P_n(0) = 0, \quad P_n(z) \longrightarrow 1 \quad if\ z \neq 0$$

Writting $P_n(z) = zQ_n(z)$, it is enough to find polynomials $Q_n(z)$ such that

$$Q_n(z) \longrightarrow \frac{1}{z} \quad if\ z \neq 0$$

that is Q_n tends to $1/z$ pointwize in $\mathbb{C} \backslash \{0\}$.
In order to get Q_n, consider the compact sets

$$K_n = \left[\frac{1}{n}, n\right] \cup \{z : |z| \leq n, d(z, \mathbb{R}^+) \geq 1/n\}$$

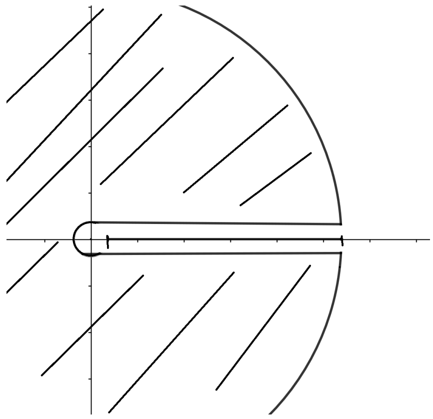


Figure 1.1: $K_2 = [1/2, 2] \cup \{z : |z| \leq 2, d(z, \mathbb{R}^+) \geq 1/2\}$, shaded region

The function $1/z$ *is holomorphic on a neighborhood of* K_n*, and* $\mathbb{C} \backslash K_n$ *is connected. Therefore, there is a polynimial* Q_n *such that*

$$|1/z - Q_n(z)| < 1/n \quad if\ z \in K_n\ .$$

Since $\bigcup_{n=1}^{\infty} K_n = \mathbb{C} \backslash \{0\}$ *and* $K_n \subset K_{n+1}$ *it turns out that*

$$Q_n(z) \longrightarrow \frac{1}{z} \quad if\ z \neq 0.$$

From now on we write $f \in \mathscr{C}(X) \cap H(\mathring{X})$ to denote that a function f is continuous on X and analytic in the interior of X.

The next theorem we study is Mergelyan's theorem, which is a generalization of Runge's theorem and gives the complete solution of the classical problem of approximation by polynomials.

Theorem 1.5. (Mergelyan) *Let K be a compact set in the plane such that the complement is connected, and suppose that* $f \in \mathscr{C}(K) \cap H(\mathring{K})$*. To each* ϵ *>0 there is a polynomial p such that*

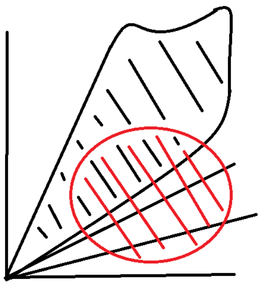
$$|f(z) - p(z)| < \epsilon$$

for all $z \in K$*.*

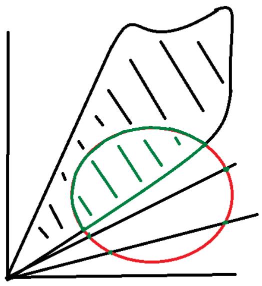
Notice that Runge's theorem applies only if f is analytic in a neighborhood of K, and therefore Mergelyan's theorem is considerably stronger because it only requires that f is analytic in the interior of K. In particular, if the interior of K is empty, any continuous function can be approximated uniformly by analytic polynomials, and if K is an interval this is the classical Weierstrass theorem.

It should also be noted that its proof is constructive and that there is no other way to prove it, for the moment. It is interesting to notice that the proof uses at some steps Runge's theorem.

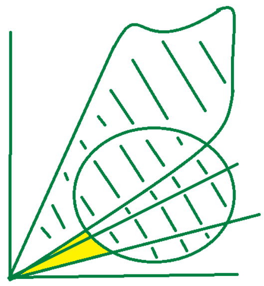
The last theorem we shall see is Arakelian's theorem, which deals with possibly unbounded closed sets. We shall introduce some new definitions:
If E is a closed subset of $\mathbb{C}$, we shall use the term "hole of E" to denote any bounded component of the complement of E.
To motivate the definition that follows, note that if E is a closed set without holes and D is a closed disc in $\mathbb{C}$, then the intersection $E \cap D$ obviously has no holes either, but the union $E \cup D$ may very well have some, even infinitely many.



(a) E is the set in black, and D is the red disk.



(b) $E \cup D$ is the set in green.



(c) $E \cup D$ is the set in green and the holes are in yellow.

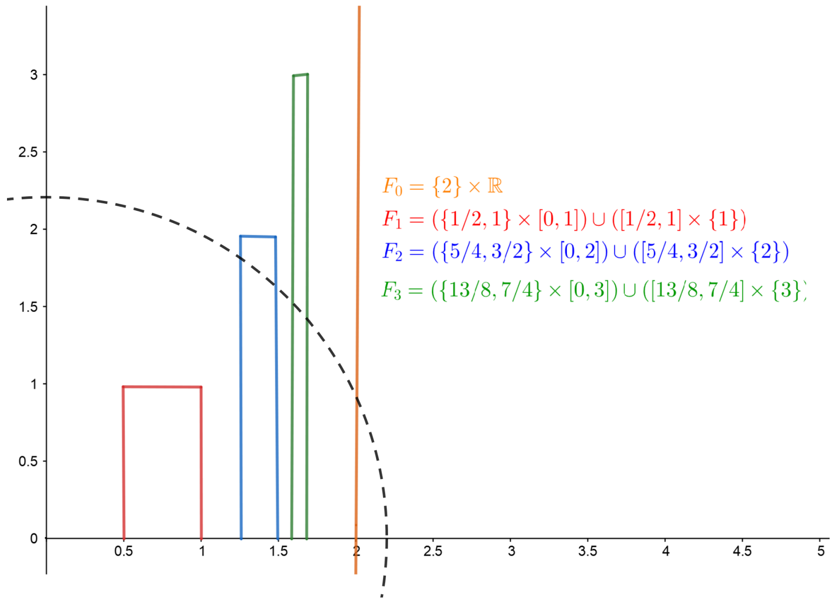
Definition 1.6. *A closed set $E \subset \mathbb{C}$, without holes, is an Arakelian set if, for every closed disc $D \subset \mathbb{C}$, the union of all holes of $E \cup D$ is a bounded set.*

Let's see an example of a set that is not Arakelian.

Example 1.7. *Let $\Omega = \mathbb{C}$. Now define $F_0 = \{2\} \times \mathbb{R}$ and*

$$F_n = \left(\{ \sum_{i=0}^{n-1} \frac{1}{2^i} - \frac{1}{2^n}, \sum_{i=0}^{n-1} \frac{1}{2^i} \} \times [0, n] \right) \cup \left([\sum_{i=0}^{n-1} \frac{1}{2^i} - \frac{1}{2^n}, \sum_{i=0}^{n-1} \frac{1}{2^i}] \times \{n\} \right)$$

It is easy to see that each F_n, $n = 0, 1, \cdots$ is an Arakelian set in $\mathbb{C}$. However, $F = \bigcup_{n=0}^{\infty} F_n$ is not Arakelian in $\mathbb{C}$, because despite the fact that F is closed and has no holes, the union of all holes in $\mathbb{C}$ of $\overline{D(0,r)} \cup F$, for $r \geq 2$, is unbounded.



(d) The high parts of the columns are the holes of $\overline{D(0,r)} \cup F$.

Notice that Arakelian sets are precisely closed sets without holes whose complement is "locally connected at infinity". However we chose the terminology of Arakelian set because it explicitly states the relevant property of E.

Theorem 1.8. (Arakelian) *If E is an Arakelian set, for all $f \in \mathscr{C}(E) \cap H(\mathring{E})$, and all $\epsilon > 0$, there is an entire function h such that*

$$|h(z) - f(z)| < \epsilon$$

for every $z \in E$.

In most applications the function that is to be approximated on E is actually holomorphic in a neighborhood of E. In that case the proof of Arakelian's theorem relies only on the classical approximation theorem of Runge. Using the terminology "hole" Runge's theorem states:
If K is a compact subset of $\mathbb{C}$, without holes, and f is holomorphic in a neighborhood of K, then f can be approximated, uniformly on K, by holomorphic polynomials.
Therefore, for functions that are holomorphic in a neighborhood of E, Arakelian's theorem turns out to be really elementary.
When E is compact, the complement of E is connected , and in this case Arakelian's theorem coincides with Mergelyan's theorem, which derives the same conclusion from a weaker assumption about f; f should be continuous on E and holomorphic in the interior of E.

So, Arakelian's theorem differs from Runge's and Mergelyan's theorem by the fact that it does not need compact sets, it applies to closed (maybe unbounded) sets. The idea of the proof is to exhaust E by increasing compact sets and approximate recursively f on each of them.

We state next a consequence of Arakelian's theorem, which might illustrate its relevance.

Corollary 1.9. *Let E be an Arakelian set with empty interior. Let $\omega : E \longrightarrow \mathbb{R}_+$ be a continuous function. Then for all $f \in \mathscr{C}(E)$ there exists an entire function h such that*

$$|h(z) - f(z)| < \omega(z) \quad z \in E.$$

For instance, let $E = \mathbb{R}$ and let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a continuous function. Take ω to be any positive continuous function such that

$$\lim_{|x| \to \infty} \omega(x) = 0$$

Then there exists $h \in H(\mathbb{C})$ such that

$$|f(x) - h(x)| < \omega(x) \quad \forall x \in \mathbb{R}.$$

Note that $f \in \mathscr{C}(\mathbb{R})$ might be quite irregular; for example f can fail to have a derivative at any point $x \in \mathbb{R}$, while $h(x)$ is analytic.

The text is divided into four parts.
The first chapter presents some preliminaries. We give some definitions and results that are needed in the following chapters. In particular, we include some basic properties of analytic functions, such as Green's Formula, the Maximum Principle, and consequences of Cauchy's Formula. We also state properties of analytic mappings such as the Riemann Mapping theorem and some functional analysis results, such as the Riesz Representation theorem and the Tietze's Extension theorem, which is essential in the proof of Mergelyan's theorem.
Chapter 2 is devoted to the study of Runge's theorem. We state the theorem in two different ways. One of them is classical and based on abstract results, such as the Hanh-Banach theorem and the Riesz Representation theorem, among others. The other one is done by elementary complex analysis.
Chapter 3 deals with Mergelyan's theorem. It should be noted that we shall use

Runge's theorem and some more specific arguments.
Finally, in Chapter 4 we present Arakelian's theorem, in which we make a recursive approximation, taking larger pieces of E. The way in which we construct the function that approximates f can be done in two different ways, by Runge's hypothesis or by Mergelyan's hypothesis. Both of them have the same structure, the difference between them is the hypothesis over the function that approximates recursively f. We also prove Corollary 1.9.

Chapter 2

Preliminaries

2.1 Notation

We denote that f is holomorphic in a neighborhood of E by $f \in H(\widehat{E})$. The notation $\mathscr{C}_0(X)$ always denotes the class of all continuous f in X which vanish at infinity and $\mathscr{C}_c(X)$, the collection of all continuous complex functions on X whose support is compact. Notice that $\mathscr{C}_c(X) \subset \mathscr{C}_0(X)$, and that the two classes coincide if X is compact. In this case, we write $\mathscr{C}(X)$ for either of them. Futhermore the space of all continuously differentiable functions in the plane with compact support is denoted by $\mathscr{C}_c^1(\mathbb{C})$.
$\mathscr{C}^k(\overline{\Omega})$ is the subspace of functions in $\mathscr{C}^k(\Omega)$ whose derivates up to the k-th order have continuous extentions to $\overline{\Omega}$.
We use $\mathbb{D}$ to denote the unit disk, i.e, $\mathbb{D} = D(0,1)$. The Lebesgue measure in $\mathbb{R}^2$ is denoted by $d\lambda$. In chapter 3, we use the notation $B(S)$ to denotes the closed subalgebra of $\mathscr{C}(K)$ that contains every rational function with poles in S.

2.2 Properties of Analytic Functions

We indentify $\mathbb{C}$ with $\mathbb{R}^2$ through the indentity $z = x + iy$, $(x,y) \in \mathbb{R}^2$. Observe that the equality

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

can be written in terms of dz and $d\bar{z}$ as

$$df = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)d\bar{z}$$

This motivates the definition of the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{2.1}$$

so that

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$$

We state Green's formula in these terms.

Green's Formula. *Let Ω be an open set in $\mathbb{C}$ and let ω be a bounded open set such that $f, g \in \mathscr{C}^1(\overline{\omega})$. Then*

$$\int_{\partial\omega} f dz + g d\bar{z} = 2i \int_{\omega} \left(\frac{\partial f}{\partial \bar{z}} - \frac{\partial g}{\partial z}\right) d\lambda(z).$$

We shall also use the following well-known results.

Theorem 2.1. *Suppose μ is a complex (finite) measure on a measurable space, φ is a complex measurable function on X and Ω is an open set in the plane which does not intersect $\varphi(X)$. Then the function*

$$f(z) = \int_X \frac{d\mu(\zeta)}{\varphi(\zeta) - z}, \quad z \in \Omega$$

is represented by power series in Ω.

Theorem 2.2. *If K is a compact subset of a plane open set $\Omega(\neq \emptyset)$, then there is a cycle Γ in $\Omega \backslash K$ such that the Cauchy formula*

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

holds for every $f \in H(\Omega)$ and for every $z \in K$.

Analytic Continuation Principle. *Let $\Omega \subseteq \mathbb{C}$ be a connected open set in $\mathbb{C}$ and let $f \in H(\Omega)$. If the set*

$$Z(f) = \{z \in \Omega\,; f(z) = 0\}$$

has a limit point in Ω, then $f \equiv 0$ in Ω.

Cauchy-Pompeiu formula. *Let ω be a bounded open set in $\mathbb{C}$. Suppose $f \in \mathscr{C}^1(\overline{\omega})$. Then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial\omega} \frac{f(\zeta)d\zeta}{\zeta - z} - \frac{1}{\pi} \int_{\omega} \frac{\partial f}{\partial \overline{\zeta}}(\zeta) \frac{d\lambda(\zeta)}{\zeta - z} \quad z \in \mathbb{C}\,. \tag{2.2}$$

In particular, if $f \in H(\omega) \cap \mathscr{C}^1(\overline{\omega})$, then

$$f(z) = \frac{1}{2\pi i} \int_{\partial\omega} \frac{f(\zeta)d\zeta}{\zeta - z}\,.$$

Next we show that the Cauchy-Pompeiu formula provides a solution in to the equation $\overline{\partial} u = \phi$ when $\phi \in \mathscr{C}^1_c(\mathbb{C})$.

Theorem 2.3. *Given $\phi \in \mathscr{C}^1_c(\mathbb{C})$ define*

$$u(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\phi(\zeta)}{\zeta - z} d\lambda(\zeta) \quad z \in \mathbb{C}. \tag{2.3}$$

Then $u \in \mathscr{C}^1(\mathbb{C})$ and

$$\frac{\partial u}{\partial \overline{z}} = \phi. \tag{2.4}$$

Proof. By translation, we have $u(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\phi(\zeta+z)}{\zeta} d\lambda(\zeta)$. Since $1/\zeta$ is integrable on any compact set, u is continuous. Now let $h \in \mathbb{R}, h \neq 0$. Then

$$\frac{u(z+h) - u(z)}{h} = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{\zeta} \frac{\phi(\zeta + z + h) - \phi(\zeta - z)}{h} d\lambda(\zeta)\,.$$

Notice that for fixed z, ζ, and writting $\zeta = \xi + i\eta$, $\xi, \eta \in \mathbb{R}$,

$$\frac{\phi(\zeta + z + h) - \phi(\zeta - z)}{h} \longrightarrow \left(\frac{\partial \phi}{\partial \xi}\right)(\zeta + z) \quad \text{as } h \to 0.$$

Moreover, since ϕ is continuously differentiable and has compact support, this convergence is uniform in ζ for z in any compact subset of $\mathbb{C}$. Since $|\zeta|^{-1}$ is integrable on any compact set, we conclude that

$$\begin{aligned} \frac{\partial u}{\partial x}(z) &= \lim_{h \to 0} \frac{1}{h}(u(z+h) - u(z)) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{\zeta} \frac{\partial \phi}{\partial \xi}(\zeta + z) d\lambda(\zeta) \\ &= -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \phi}{\partial \xi}(\zeta) \frac{1}{\zeta - z} d\lambda(\zeta) \end{aligned} \tag{2.5}$$

and that this limit is uniform for z in any compact set in $\mathbb{C}$. In particular, $\frac{\partial u}{\partial x}$ is continuous. Similarly,

$$\frac{\partial u}{\partial y}(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{1}{\zeta}\frac{\partial \phi}{\partial \eta}(\zeta + z)d\lambda(\zeta) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \phi}{\partial \eta}(\zeta)\frac{1}{\zeta - z}d\lambda(\zeta), \tag{2.6}$$

and is continuous. It follows that $u \in \mathscr{C}^1(\mathbb{C})$. Finally by (2.1) and (2.2), we get that

$$\frac{\partial u}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \phi}{\partial \overline{\zeta}}(\zeta)\frac{1}{\zeta - z}d\lambda(\zeta) = \phi(\zeta)\ ,$$

as we claimed. □

2.3 Conformal Mappings

We give a very brief section with a couple of results on conformal mapping that will be used later on. In the proof of Mergelyan's theorem we shall use them. The next theorem is probably the best known conformal mapping result. It allows to reduce certain problems in simply connected domains into the corresponding problems in $\mathbb{D}$.

The Riemann Mapping Theorem. *If $\Omega \subset \mathbb{C}$ is open, simply connected and $\mathbb{C}\backslash\Omega$ is nonempty, then Ω is conformally equivalent to $\mathbb{D}$. Moreover, the conformal biholomorphism $g : \mathbb{D} \longrightarrow \Omega$ is unique if we fix $g(0)$ and take $g\prime(0) > 0$.*

The following result deals with a normalized class of holomorphic mappings.

Lemma 2.4. *Let $\mathcal{S}$ be the class of injective functions in $\mathbb{D}$ with $f(0) = 0$ and $f\prime(0) = 1$. Suppose that $f(z) = z + a_2z^2 + \cdots \in \mathcal{S}$. Then*

(a) There is a function $g \in \mathcal{S}$ such that $g^2(z) = f(z^2)$.

(b) $|a_2| \leq 2$.

Next result, a corollary of the previuous one, concerns injective functions taking 0 to ∞.

Corollary 2.5. *If $h(z) = \frac{1}{z} + c_0 + c_1z + \cdots$ is injective in $\mathbb{D}$ and avoids the values ω_1, ω_2, then $|\omega_1 - \omega_2| \leq 4$.*

Proof. By assumption, $\frac{1}{h(z)-w_j} = z + (w_j - c_0)z^2 + \cdots \in \mathcal{S}$, so $|w_j - c_0| \leq 2$ by Lemma 2.4 (b). This implies that $|w_1 - w_2| \leq 4$. □

2.4 Some Functional Analysis

We start with a well-known consequence of the Hahn-Banach theorem.

Theorem 2.6. *Let K be a compact set, $M \subseteq \mathscr{C}(K)$ a subspace and let $f_0 \in \mathscr{C}(K)$. Then $f_0 \in \overline{M}$ if and only if there is no bounded linear functional $T : \mathscr{C}(K) \longrightarrow \mathbb{C}$ such that $T(f) = 0$ for all $f \in M$ and $T(f_0) \neq 0$.*

Definition 2.7. *A Banach algebra, A, is an algebra over $\mathbb{C}$ with a norm such that*

(i) $\|xy\| \leq \|x\|\|y\| \quad x, y \in A,$

(ii) A is a complete metric space to this norm.

Theorem 2.8. *Let X be a locally compact space. Then the dual space of $\mathscr{C}_b(X) = \{f \mid f : X \longrightarrow \mathbb{C}$ is continuous and bounded$\}$ is the space of Radon measures with bounded variation.*

Another well-known result that we shall use is the following.

Riesz Representation theorem. *If X is a locally compact Hausdorff space, then every bounded linear functional Φ on $\mathscr{C}_0(X)$ is represented by an unique regular complex Borel measure μ, in the sense that*

$$\Phi_f = \int_X f d\mu$$

for every $f \in \mathscr{C}_0(X)$. Moreover, the norm of Φ is the total variation of μ:

$$\|\Phi\| = |\mu|(X)$$

Tietze's Extension Theorem. *Suppose K is a compact subset of a locally compact Hausdorff space X, and $f \in \mathscr{C}(K)$. Then there exists an $F \in \mathscr{C}_c(X)$ such that $F(x) = f(x)$ for all $x \in K$.*

Proof. First of all, we assume that f is real, $-1 \leq f \leq 1$. Let W be an open set with compact closure so that $K \subset W$. Put

$$K^+ = \{x \in K : f(x) \geq 1/3\}, \quad K^- = \{x \in K : f(x) \leq -1/3\}.$$

Then K^+ and K^- are disjoint compact subsets of W. As a consequence of Urysohn's lemma there is a function $f_1 \in \mathscr{C}_c(X)$ such that the support of f_1 lies in W and such that

$$f_1(x) = 1/3 \text{ on } K^+, \quad f_1(x) = -1/3 \text{ on } K^-.$$

So $-1/3 \le f_1(x) \le 1/3$ for all $x \in X$. Thus

$$|f(x) - f_1(x)| \le 2/3 \text{ on } K, \quad |f_1(x)| \le 1/3 \text{ on } X.$$

Repeat this construction with $f - f_1$ in place of f: there exists $f_2 \in \mathscr{C}_c(X)$, with support in W, so that

$$|f(x) - f_1(x) - f_2(x)| \le \left(\frac{2}{3}\right)^2 \text{ on } K, \quad |f_2(x)| \le \frac{1}{3} \times \frac{2}{3} \text{ on } X.$$

In this way we obtain functions $f_n \in \mathscr{C}_c(X)$, with supports in W, such that

$$|f(x) - f_1(x) - \cdots - f_n(x)| \le \left(\frac{2}{3}\right)^n \text{ on } K, \quad |f_n(x)| \le \frac{1}{3} \times \left(\frac{2}{3}\right)^{n-1} \text{ on } X. \tag{2.7}$$

Now put $F = f_1 + f_2 + \cdots$. By (2.7), the series converges to f on K, and it converges uniformly on X. Hence F is continuous. Also, the support of F lies in $\overline{W}$. $\square$

Chapter 3

Runge's Theorem

Carl Runge (30 August 1856 - 3 January 1927) was a German mathematician, physicist, and spectroscopist. He spent the first few years of his life in Havana, although the family moved to Germany. At the age of 19, Runge enrolled at the University of Munich to study literature. However, after 6 weeks of course, he changed studies to pursue a career in physics and mathematics. Carl spent much of his professional career in Germany. He decided to travel to Berlin to attend various lectures on mathematics. After hearing several of Weierstrass' lectures, he decided to focus on pure mathematics.
In 1880, Runge received his doctorate from the University of Munich and took his secondary school teachers examination. Carl, who often regarded to himself as a Weierstrass disciple, worked feverishly on obtaining a general procedure for the numerical solution of algebraic equations in which the roots were expressed as infinite series of rational functions of the coefficients. With this success, he continued to work on a variety of problems in algebra and function theory. Soon after, he obtained a chair position at Hannover, where he remained for 18 years. A year after arriving to Hannover, Runge underwent a thorough reorientation in his research habits and interest in mathematics. He moved away from pure mathematics to study the wavelengths of spectral lines of elements, immersing himself in problems of spectroscopy and astrophysics. However, Runge did not receive the academic appointments he deserved until he was in the twilight of his career. In 1904, with the influence of Planck and Felix Klein, Runge was appointed to Gottingen as the chair of mathematics, where he remained until he retired in 1925. Some of his researches are known nowadays as the Runge's phenomenon and Runge's approximation theorem. Futhermore he was co-developer of the Runge-Kutta method, in the field of numerical analysis.

In this chapter we study the possibility of approximating analytic functions

with polynomials and, more generally, by rational functions. Notice that a polynomial is a rational function with a pole at ∞. It will be proved that polynomials and rational functions approximate all holomorphic functions.
We show two different proofs: the first one is a slightly different version in which Banach spaces and complex measures are essential and the second one gives an elementary proof by functional analysis.

3.1 Proof by complex measures

This proof of Runge's theorem (Theorem 1.1) is abstract and requires at least the Riesz Representation theorem, a consequence of the Hahn-Banach theorem, Fubini's theorem and the Analytic Continuation Principle. The strategy of the proof is making a double approximation.

Notice that in Runge's theorem we take $X = K$, which is a compact set, so $\mathscr{C}_0(X) = \mathscr{C}_c(X) = \mathscr{C}(X)$. Therefore, in this case, the Riesz representation theorem characterizes the dual of $\mathscr{C}(K)$. So, every bounded linear functional ϕ on $\mathscr{C}(K)$ is represented by μ:

$$\phi(f) = \int_K f d\mu \quad \forall f \in \mathscr{C}(K).$$

Proof. We consider the Banach space $\mathscr{C}(K)$ with the supremum norm, $\|.\|_\infty$. Let M be the subspace of $\mathscr{C}(K)$ which consists of the restriccions to K of those rational functions which have all their poles in $\{\alpha_j\}$.
We will be done once we prove that f is in the closure of M, since

$$f \in \overline{M} \Leftrightarrow \exists R_n \in M \text{ such that } \lim_n \|f - R_n\|_\infty = \lim_n \sup_{z\in K}\{f(z) - R(z)\} = 0$$
$$\Leftrightarrow \forall \epsilon > 0,\ \exists R_n \in M \text{ such that } |f(z) - R(z)| < \epsilon \quad \forall z \in K.$$

By Theorem 2.6, proving that $f \in \overline{M}$ is equivalent to saying that every bounded linear functional on $\mathscr{C}(K)$ which vanishes on M also vanishes at f. By the Riesz representation theorem it is enough to prove the following:
If $T : \mathscr{C}(K) \longrightarrow \mathbb{C}$ is a bounded linear functional then there exists a complex Borel measure μ on K such that

$$T(R) = \int_K R d\mu = 0 \quad \forall R \in M \tag{3.1}$$

implies

$$T(f) = \int_K f d\mu = 0 \quad \forall f \in \mathscr{C}(K). \tag{3.2}$$

So let us assume that μ satisfies (3.1). Define, for $z \in \mathbb{S}^2 \backslash K$

$$h(z) = \int_K \frac{d\mu(\zeta)}{\zeta - z}. \tag{3.3}$$

By Theorem 2.1 applied to the case $X = K$, $\varphi(\zeta) = \zeta$, we deduce that h is represented by power series in $\mathbb{S}^2 \backslash K$. In particular $h \in H(\mathbb{S}^2 \backslash K)$.

Let V_j be the component of $\mathbb{S}^2 \backslash K$ which contains α_j, and suppose $r > 0$ is taken so that $D(\alpha_j; r) \subset V_j$.
If $\alpha_j \neq \infty$ and z is fixed in $D(\alpha_j; r)$, then

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{\zeta - \alpha_j - (z - \alpha_j)} = \frac{1}{(\zeta - \alpha_j)} \left(\frac{1}{1 - \frac{z - \alpha_j}{\zeta - \alpha_j}} \right) = \frac{1}{\zeta - \alpha_j} \sum_{n=0}^{\infty} \left(\frac{z - \alpha_j}{\zeta - \alpha_j} \right)^n \\ &= \sum_{n=0}^{\infty} \frac{(z - \alpha_j)^n}{(\zeta - \alpha_j)^{n+1}} = \lim_{N \to \infty} \sum_{n=0}^{N} \frac{(z - \alpha_j)^n}{(\zeta - \alpha_j)^{n+1}} \end{aligned} \tag{3.4}$$

converges uniformly for $\zeta \in K$. Each of the functions on the right hand side of (3.4) is rational with poles only on $\{\alpha_j\}$, so by (3.1)

$$h(z) = \int_K \frac{d\mu(\zeta)}{\zeta - z} = \int_K \sum_{n=0}^{\infty} \frac{(z - \alpha_j)^n}{(\zeta - \alpha_j)^{n+1}} d\mu(\zeta) = \sum_{n=0}^{\infty} \underbrace{\int_K \frac{(z - \alpha_j)^n}{(\zeta - \alpha_j)^{n+1}} d\mu(\zeta)}_{0} = 0$$

for $z \in \bigcup_j D(\alpha_j, r)$. By the analytic continuation principle, $h(z) = 0$ for all $z \in V_j$.
For the case $\alpha_j = \infty$, (3.4) is replaced by

$$\frac{1}{\zeta - z} = -\frac{1}{z} \left(\frac{1}{1 - \frac{\zeta}{z}} \right) = -\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{\zeta}{z} \right)^n = -\sum_{n=0}^{\infty} \frac{\zeta^n}{z^{n+1}} = -\lim_{N \to \infty} \sum_{n=0}^{N} z^{-n-1} \zeta^n \tag{3.5}$$

for $\zeta \in K, |z| > r$.
This implies that $h(z) = 0$ in $D(\infty; r)$, hence in V_j.
We have thus proved from (3.1) that

$$h(z) = 0, \quad z \in \mathbb{S}^2 \backslash K. \tag{3.6}$$

Now choose a cycle Γ in $\Omega \backslash K$, as in Theorem 2.2, and integrate this Cauchy integral representation of f with respect to μ. An application of Fubini's theorem combined with (3.6), gives

$$\begin{aligned} T(f) = \int_k f(\zeta) d\mu(\zeta) &= \int_K d\mu(\zeta) \left(\frac{1}{2\pi i} \int_\Gamma \frac{f(\omega)}{\omega - \zeta} d\omega \right) = \frac{1}{2\pi i} \int_\Gamma f(\omega) d\omega \underbrace{\int_K \frac{d\mu(\zeta)}{\omega - \zeta}}_{-h(\omega)} \\ &= -\frac{1}{2\pi i} \int_\Gamma f(\omega) h(\omega) d\omega = 0. \end{aligned}$$

The last equality depends on the fact that $\Gamma^* \subset \Omega \backslash K$, where $h(z) = 0$. Thus (3.2) holds, and the proof is complete. $\square$

3.2 Proof by functional analysis

The proof of Runge's theorem (Theorem 1.2) we give is elementary and it is motivated by the fact that $B(S)$ is a Banach algebra, not just a Banach space. Remember that $B(S)$ is the closed subalgebra of $\mathscr{C}(K)$ that contains every rational function with poles in S.
We base this proof on three lemmas. This next lemma provides the first step in obtaining approximation by rational functions.

Lemma 3.1. *Let γ be a rectifiable curve and let K be a compact set such that $K \cap \gamma = \varnothing$. If f is a continuous function on γ and $\epsilon > 0$ then there is a rational function $R(z)$ having all its poles on γ and such that*

$$\left| \int_\gamma \frac{f(w)}{w - z} dw - R(z) \right| < \epsilon$$

for all $z \in K$.
Note: This is equal to seeing that every function anaytic on a neighborhood of K is a uniform limit on K of rational functions, all whose pols lie in $\mathbb{S}^2 \backslash K$.

Proof. Since $K \cap \gamma = \varnothing$ there is a number r such that $0 < r < d(K, \gamma)$. If γ is defined on $[0, 1]$, then for $0 \le s, t \le 1, |\gamma(s) - z| > r$, and for any $z \in K$

$$\begin{aligned} \left| \frac{f(\gamma(t))}{\gamma(t) - z} - \frac{f(\gamma(s))}{\gamma(s) - z} \right| \le & \frac{1}{r^2} |f(\gamma(t))| |\gamma(s) - \gamma(t)| + \frac{1}{r^2} |\gamma(t)| |f(\gamma(s)) - f(\gamma(t))| \\ & + \frac{|z|}{r^2} |f(\gamma(s)) - f(\gamma(t))| \end{aligned}$$

There is a constant $c > 0$ such that $|z| \le c$ for all $z \in K$, $|\gamma(t)| \le c$ and $|f(\gamma(t))| \le c$ for all $t \in [0, 1]$. Then, for all $t, s \in [0, 1]$ and for all $z \in K$,

$$\left| \frac{f(\gamma(t))}{\gamma(t) - z} - \frac{f(\gamma(s))}{\gamma(s) - z} \right| \le \frac{c}{r^2} |\gamma(s) - \gamma(t)| + \frac{2c}{r^2} |f(\gamma(s)) - f(\gamma(t))|.$$

Since both γ and $f \circ \gamma$ are uniformly continuous on $[0, 1]$, there is a partition
$\{0 = t_0 < t_1 < ... < t_n = 1\}$ such that for $t_{j-1} \le t \le t_j, 1 \le j \le n, z \in K$

$$\left| \frac{f(\gamma(t))}{\gamma(t) - z} - \frac{f(\gamma(t_j))}{\gamma(t_j) - z} \right| \le \frac{\epsilon}{length(\gamma)} \ . \tag{3.7}$$

Define $R(z)$ to be the rational function

$$R(z) = \sum_{j=1}^{n} \frac{f(\gamma(t_{j-1}))(\gamma(t_j) - \gamma(t_{j-1}))}{\gamma(t_{j-1}) - z}$$

The poles of $R(z)$ are $\gamma(0), \gamma(t_1), ..., \gamma(t_{n-1})$. Using (3.7) we get, for all $z \in K$

$$\begin{aligned}\left| \int_\gamma \frac{f(w)}{w-z} dw - R(z) \right| &= \left| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left(\frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(t_{j-1}))}{\gamma(t_{j-1})-z} \right) d\gamma(t) \right| \\ &\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left| \frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(t_{j-1}))}{\gamma(t_{j-1})-z} \right| |\gamma'(t)| dt \\ &\leq \frac{\epsilon}{length(\gamma)} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} |\gamma'(t)| dt = \epsilon \,.\end{aligned}$$

□

We shall need the following elementary result.

Lemma 3.2. *If ρ and U are open subsets of $\mathbb{C}$, $\rho \subseteq U$ and $\partial\rho \cap U = \varnothing$ then ρ contains every component of U which it meets.*

Proof. Let $s \in H \cap \rho$ and let G be the component of ρ such that $s \in G$. Then $H \cup G$ is connected and $H \cup G \subseteq U$. Since H is a component of U, $G \subset H$. Since H is connected, it must either equal G or contain a boundary point of G. But $\partial G \subset \partial\rho$, then if $x \in \partial G$ also $x \in \partial\rho$ and $x \notin U$, because by hypothesis $\partial\rho \cap U = \varnothing$. So, $\partial G \cap H = \varnothing$ and $H = G$. □

Lemma 3.3. *If a does not belong to K then $(z-a)^{-1}$ belongs to $B(S)$.*

Proof. We just consider the case in which $\infty \notin S$. In the other case we can make a perturbation of ∞, called α_0, so that $\alpha_0 \notin S$. This way we reduce the proof to one case.
Let $U = \mathbb{C} \backslash K$ and $V = \{a \in \mathbb{C} : (z-a)^{-1} \in B(S)\}$, so $S \subseteq V \subseteq U$.
We shall see next that V is open. More precisaly, we shall see that

$$If \quad a \in V \quad and \quad |b-a| < d(a, K) \; then \; b \in V. \tag{3.8}$$

The condition on b gives the existence of $r \in (0,1)$ such that $|b-a| < r|z-a|$ for all $z \in K$. Hence $\frac{|b-a|}{|z-a|} < r < 1$, for all $z \in K$ and

$$\frac{1}{z-b} = \frac{1}{z-a} \frac{1}{(1 - \frac{b-a}{z-a})} = \frac{1}{z-a} \sum_{n=0}^{\infty} \left(\frac{b-a}{z-a} \right)^n \tag{3.9}$$

converges uniformly on K by the Weierstrass M-test.

If $Q_n(z) = \sum_{k=0}^{\infty}(\frac{b-a}{z-a})^k$ we see that $(z-a)^{-1}Q_n(z) \in B(S)$, since $a \in V$ and $B(S)$ is an algebra.
Since $B(S)$ is closed and the series above converges uniformly

$$\frac{1}{z-b} = \frac{1}{z-a}Q_n(z) \in B(S),$$

and therefore $b \in V$, as claimed.

To finish the proof we use Lemma 3.2.
If $b \in \partial V$, let $\{a_n\}$ be a sequence in V with $b = \lim_n a_n$. Since $b \notin V$ it follows from (3.8) that $|b - a_n| \geq d(a_n, K)$. Then $d(b,K) \leq d(b, a_n) + d(a_n, K) \leq 2|b - a_n| \to 0$ as $n \to \infty$ or $b \in K$. Thus $\partial V \cap U = \emptyset$.
If H is a component of $U = \mathbb{C}\backslash K$ we deduce that $H \cap S \neq \emptyset$, so $H \cap V \neq \emptyset$, and $H \subset V$. But H was arbitrary so $U \subset V$ or $V = U$.
Then $V = \{a \in \mathbb{C} : (z-a)^{-1} \in B(S)\} = \mathbb{C}\backslash K$. So if $a \notin K$ then $(z-a)^{-1} \in B(S)$.
□

We are finally ready to prove Runge's theorem in this language.

Proof. If $f \in H(G)$, where G is an open set and $K \subset G$, then for each $\epsilon > 0$ Lemma 3.1 provides a $R(z)$ rational function with poles in $\mathbb{C}\backslash K$ such that

$$|f(z) - R(z)| < \epsilon, \quad \forall z \in K.$$

But Lemma 3.3 and the fact that $B(S)$ is an algebra gives that $f \in B(S)$. □

Chapter 4

Mergelyan's Theorem

Sergey Mergelyan (19 May 1928-20 August 2008) was an Armenian scientist. Before the Second World War, Mergelyan lived in Russia and Ukraine, however in 1941 his family moved to Yeveran, where he studied. At the age of 16, he graduated from high school and immediately entered the Physics and Mathematics Faculty of the Yerevan State University (YSU). After YSU, Mergelyan entered the postgraduate study at Steklov Institute of Mathematics to Mstislav Vsevolodovich Keldysh. He wrote his tesis for the degree on Physical and Mathematics Sciences for a year and a half, and after the defense took place he became the youngest doctor of physical and mathematical sciences in the USSR at the age of 20.
In 1964, Mergelyan was appointed head of the Department of Complex Analysis at the Mathematical Institute. In the same year, he became a professor of the Mechanics and Mathematics Faculty of the Moscow State University, however, after four years, he left the post of professor of the faculty.
Mergelyan's main works include results functions of complex variables, theory of approximation, and potential and harmonic functions. In particular, at the age of 23, he formulated and proved the famous result from complex analysis called Mergelyan's theorem , which is the generalization of the Weierstrass approximation theorem and Runge's theorem, that we shall see in this chapter. He also solved another famous problem, the Sergei Natanovich Bernstein Approximation Problem. Therefore, Mergelyan is the author of major contributions in Approximation Theory.

To prove Mergelyan's theorem we will use basicaly two results: an approximation of the identity to regularize the original function and a result which gives precise approximations of the Cauchy's Kernel (theorem 4.2).

4.1 Proof of Mergelyan's theorem

First of all, by an application of Tietze's extension theorem, we can extend f to a continuous function with compact support in $\mathbb{C}$. Let $\omega(\delta)$ be the modulus of continuity of f,

$$\omega(\delta) = \sup\{|f(z) - f(w)|; |z - w| < \delta\}.$$

Since f is uniformly continuous , $\lim_{\delta\to 0} \omega(\delta) = 0$. Hence, it is enough to find, for each δ, a polynomial p such that

$$|f(z) - p(z)| \leq C\omega(\delta), \quad z \in K, \tag{4.1}$$

with $C > 0$ independent of δ.

We shall construct an approximation of the identity, which will be necessary in the proof.

Theorem 4.1. *There exists $\phi \in \mathscr{C}^1(\mathbb{C})$ such that:*

(i) supp $\phi \subseteq \mathbb{D}$,

(ii) $\int \phi d\lambda = 1$,

(iii) $\int_{\mathbb{C}} \overline{\partial}\phi = 0$,

(iv) The functions $\phi_\delta(z) := \frac{1}{\delta^2}\phi\left(\frac{z}{\delta}\right)$, for $\delta > 0$, are approximations of the identity, in the sense that for all $f \in \mathscr{C}(\mathbb{C})$,

$$|f(z) - (f * \phi_\delta)(z)| \leq \omega(\delta).$$

Proof. . Define $\phi(z) = a(|z|^2)$ where $a(r) = \frac{3}{\pi}(1-r)^2$ for $0 \leq r \leq 1$.
Clearly *supp* $\phi \subseteq \mathbb{D}$. Let's check condition (ii) :

$$\begin{aligned}\int \phi(z)d\lambda(z) &= \int_{|z|\leq 1} a(|z|^2)d\lambda(z) = \int_0^{2\pi}\int_0^1 \frac{3}{\pi}(1-r^2)^2 r dr d\theta \\ &= 2\pi\int_0^1 \frac{3}{\pi}(1-r^2)^2 r dr = 3\int_0^1 (1-r^2)2r dr = 3\int_0^1 (1-t)^2 dt = 1.\end{aligned}$$

Let us prove condition (iii). By definition $\phi(z) = a(|z|^2) = a(z\overline{z})$, then $\frac{\partial\phi}{\partial\overline{z}} = a'(|z|^2)z$. Therefore,

$$\begin{aligned}\int_{D(0,r)} \frac{\partial\phi}{\partial\overline{z}}(\omega)d\lambda(\omega) &= \int_0^{2\pi}\int_0^r a'(\rho^2)\rho^2 e^{i\theta} d\rho d\theta \\ &= \underbrace{\left(\int_0^{2\pi} e^{i\theta}d\theta\right)}_{0}\left(\int_0^r a'(\rho^2)\rho^2 d\rho\right) = 0.\end{aligned}$$

Finally, we shall prove condition (iv). First of all we define a smooth function Φ as the convolution of f and ϕ_δ :

$$\Phi(z) = (f * \phi_\delta)(z) = \int \phi_\delta(z-\omega) f(\omega) d\lambda(\omega) = \int \phi_\delta(\omega) f(z-\omega) d\lambda(\omega).$$

Notice that

$$\int \phi_\delta(z) d\lambda(z) = \frac{1}{\delta^2} \int \phi\left(\frac{z}{\delta}\right) d\lambda(z) = \frac{1}{\delta^2} \int \phi(\omega) \delta^2 d\lambda(\omega) = 1 \tag{4.2}$$

where $\omega = \frac{z}{\delta}$.
Our goal is prove that

$$|f(z) - \Phi(z)| \leq \omega(\delta). \tag{4.3}$$

Since

$$f(z) - \Phi(z) = \int (f(z) - f(z-\omega)) \phi_\delta(\omega) d\lambda(\omega), \quad (for\ all\ z)$$

by (4.2) and since $|z-(z-\omega)| = |\omega| < \delta$ implies $|f(z) - f(z-\omega)| < \omega(\delta)$, it follows that

$$|f(z) - \Phi(z)| = \int |f(z) - f(z-\omega)| \phi_\delta(\omega) d\lambda(\omega) \leq \omega(\delta) \int \phi_\delta(\omega) d\lambda(\omega) = \omega(\delta)\,.$$

Therefore, we have (4.3) and condition (iv). So we have proved the last condition. □

Let ϕ be a function as in Theorem 4.1 and define $\Phi = (f * \phi_\delta)$.
Now, we shall see that

$$\left| \frac{\partial \Phi}{\partial \bar{z}} \right| \leq \frac{C\omega(\delta)}{\delta} \tag{4.4}$$

By Green's formula and the Cauchy's Integral theorem we get that

$$\begin{aligned} \int_{D(0,1/2)} \frac{\partial \phi_\delta}{\partial \overline{\omega}}(\omega) f(z) d\lambda(\omega) &= f(z) \int_{D(0,1/2)} \frac{\partial \phi_\delta}{\partial \overline{\omega}}(\omega) d\lambda(\omega) \\ &= f(z) \frac{1}{2i} \int_{\partial D(0,1/2)} \phi_\delta(\omega) d\lambda(\omega) \\ &= f(z) \frac{1}{2i} \cdot 0 = 0. \end{aligned} \tag{4.5}$$

Therefore,

$$\frac{\partial \Phi}{\partial \bar{z}}(z) = \int \frac{\partial \phi_\delta}{\partial \overline{\omega}}(\omega) f(z-\omega) d\lambda(\omega) = \int \frac{\partial \phi_\delta}{\partial \overline{\omega}}(\omega)(f(z-\omega) - f(z)) d\lambda(\omega)$$

So, we have that

$$\left| \frac{\partial \Phi}{\partial \bar{z}}(z) \right| \leq \int \left| \frac{\partial \phi_\delta}{\partial \overline{\omega}}(\omega) \right| |(f(z-\omega) - f(z))| d\lambda(\omega) \leq \omega(\delta) \int \left| \frac{\partial \phi_\delta}{\partial \overline{\omega}}(\omega) \right| d\lambda(\omega)\,.$$

Notice that $supp\ \phi_\delta \subseteq D(0, \delta/2)$. Also $\phi_\delta(z) = \delta^{-2}\phi(\frac{z}{\delta})$ and therefore

$$\frac{\partial \phi_\delta}{\partial \bar{z}}(z) = \frac{1}{\delta^2}\frac{\partial \phi}{\partial \bar{z}}\left(\frac{z}{\delta}\right)\frac{1}{\delta}.$$

Since $\phi \in \mathscr{C}_c(\mathbb{C})$ there exists $C(\phi) > 0$ such that $\left|\frac{\partial \phi}{\partial \bar{z}}\right| \leq C(\phi)$, and therefore we have $\left|\frac{\partial \phi_\delta}{\partial \bar{z}}\right| \leq \frac{1}{\delta^2}C(\phi)\frac{1}{\delta}$. As a result we get

$$\int_{D(0,\frac{\delta}{2})}\left|\frac{\partial \phi_\delta}{\partial \bar{z}}(\omega)\right| d\lambda(\omega) \leq \frac{1}{\delta^2}C(\phi)\frac{1}{\delta}\lambda(D(0,\frac{\delta}{2})) \leq \frac{C}{\delta}. \tag{4.6}$$

Hence, by (4.6) we set the estimate

$$\left|\frac{\partial \Phi}{\partial \bar{z}}(z)\right| \leq \omega(\delta)\int\left|\frac{\partial \phi_\delta}{\partial \bar{\omega}}(\omega)\right| d\lambda(\omega) \leq \frac{C\omega(\delta)}{\delta},$$

as we claimed.
Thus, we have approximated f so far by the function Φ, which at least is analytic at points in K that have distance to ∂K bigger than δ.
Let H denote the support of $\frac{\partial \Phi}{\partial \bar{\omega}}$.
The following theorem, which is a technical result, is crucial to prove Mergelyan's theorem. We will prove it at the end of this section.

Theorem 4.2. *Let K be a compact set in $\mathbb{C}$. There is an open neighborhood Ω of K and a continuous function $r(\zeta, z)$ defined for $\zeta \in H$ and $z \in \Omega$ such that $r(\zeta, z)$ is holomorphic for $z \in \Omega$, and there exists $C > 0$ independent of $\delta > 0$ such that*

$$(i) \quad \left|r(\zeta, z) - \frac{1}{\zeta - z}\right| \leq \frac{C\delta^2}{|\zeta - z|^3},$$
$$(ii) \quad |r(\zeta, z)| \leq \frac{C}{\delta}.$$

Taking this theorem for granted, it is now easy to conclude the proof of the Mergelyan's theorem.
As a consequence of Cauchy-Pompeiu formula, we have that

$$\Phi(z) = -\frac{1}{\pi}\int_H \frac{(\partial\Phi/\partial\bar{\zeta})(\zeta)d\lambda(\zeta)}{\zeta - z}.$$

Now the function

$$F(z) = -\frac{1}{\pi}\int_H r(\zeta, z)\frac{\partial \Phi}{\partial \bar{\zeta}}(\zeta)d\lambda(\zeta)$$

is analytic in $K \subset \Omega$, and by (4.4) we have that

$$
\begin{aligned}
|F(z) - \Phi(z)| &\leq \frac{1}{\pi} \int_H \left| \frac{\partial \Phi}{\partial \overline{\zeta}}(\zeta) \right| \left| r(\zeta, z) - \frac{1}{\zeta - z} \right| d\lambda(\zeta) \\
&\leq \frac{C\omega(\delta)}{\delta} \int_H \left| r(\zeta, z) - \frac{1}{\zeta - z} \right| d\lambda(\zeta) \,.
\end{aligned}
$$

We split the estimate of this last integral in two parts:

$$
\begin{aligned}
I &= \int_{H \cap \{\zeta : |\zeta - z| \leq \delta\}} \left| r(\zeta, z) - \frac{1}{\zeta - z} \right| d\lambda(\zeta) \\
II &= \int_{H \cap \{\zeta : |\zeta - z| > \delta\}} \left| r(\zeta, z) - \frac{1}{\zeta - z} \right| d\lambda(\zeta) \,.
\end{aligned}
$$

From now on the constant $C > 0$ may be different on each place.
In (I), since $|\zeta - z| \leq \delta$ and using Theorem 4.2 (ii), we estimate the integrand by $C/\delta + |\zeta - z|^{-1}$:

$$
\left| r(\zeta, z) - \frac{1}{\zeta - z} \right| \leq |r(\zeta, z)| + \frac{1}{|\zeta - z|} \leq \frac{C}{\delta} + \frac{1}{|\zeta - z|} \,.
$$

Taking polar coordinates

$$
\begin{aligned}
(I) &\leq \int_{H \cap \{\zeta : |\zeta - z| \leq \delta\}} \frac{C}{\delta} d\lambda(\zeta) + \int_{H \cap \{\zeta : |\zeta - z| \leq \delta\}} \frac{1}{|\zeta - z|} d\lambda(\zeta) \\
&\leq \frac{C}{\delta} \int_{\{\zeta : |\zeta - z| \leq \delta\}} d\lambda(\zeta) + \int_{\{\zeta : |\zeta - z| \leq \delta\}} \frac{d\lambda(\zeta)}{|\zeta - z|} \\
&= \frac{C}{\delta} \pi \delta^2 + \int_{D(0,\delta)} \frac{d\lambda(u)}{|u|} \\
&= C\delta + \int_0^\delta \int_0^{2\pi} \frac{r dr d\theta}{r} \\
&= C\delta + 2\pi\delta = C\delta \,.
\end{aligned}
$$

In (II), since $|\zeta - z| > \delta$ and using Theorem 4.2 (i), we estimate the integrand by $\frac{C\delta^2}{|\zeta - z|^3}$.
So, taking polar coordinates again we get

$$
\begin{aligned}
(II) &\leq \int_{H \cap \{\zeta : |\zeta - z| > \delta\}} \frac{C\delta^2}{|\zeta - z|^3} d\lambda(\zeta) \\
&\leq C\delta^2 \int_{|\zeta - z| > \delta} \frac{d\lambda(\zeta)}{|\zeta - z|^3} = C\delta^2 \int_{|u| > \delta} \frac{d\lambda(u)}{|u|^3} \\
&= C\delta^2 \int_\delta^\infty \int_0^{2\pi} \frac{r dr d\theta}{r^3} = 2\pi C\delta^2 \int_\delta^\infty \frac{dr}{r^2} \\
&= 2\pi C\delta^2 \frac{1}{\delta} = C\delta \,.
\end{aligned}
$$

Finally

$$|F(z) - \Phi(z)| \leq \frac{C\omega(\delta)}{\delta} C\delta = C\omega(\delta). \tag{4.7}$$

Now, by condition (iv) of Theorem 4.1 and (4.7) we have that for $z \in K$

$$|f(z) - F(z)| \leq |f(z) - \Phi(z)| + |\Phi(z) - F(z)| \leq \omega(\delta) + C\omega(\delta) \leq C\omega(\delta).$$

Since F is analytic in a neighborhood of K, we can apply Runge's theorem and obtain a polynomial p such that (4.1) holds, i.e $|F(z) - p(z)| < \omega(\delta)$ for $z \in K$. Finally

$$|f(z) - p(z)| \leq |f(z) - F(z)| + |F(z) - p(z)| \leq C\omega(\delta)$$

as claimed.

4.2 Proof of Theorem 4.2

We start with a local approximation of the Cauchy Kernel.

Lemma 4.3. *Let D be a disk with radius δ and $E \subseteq \mathbb{C}$ a connected compact subset with $diam(E) > \delta$ such that $\mathbb{S}^2 \backslash E$ is also connected. Then there is a smooth function $r(\zeta, z)$ defined for $z \in \mathbb{S}^2 \backslash E$ and $\zeta \in D$ that is analytic in z and for some $C > 0$ independent of δ:*

$$(i) \quad \left| r(\zeta, z) - \frac{1}{\zeta - z} \right| \leq \frac{C\delta^2}{|\zeta - z|^3} \tag{4.8}$$

and

$$(ii) \quad |r(\zeta, z)| \leq \frac{C}{\delta} \tag{4.9}$$

Proof. We may assume that $\delta = 1$ and that $\mathbb{D}$ is the unit disk. We need to find a bounded analytic function $g(z)$ in $\mathbb{S}^2 \backslash E$ such that

$$\lim_{|z| \to \infty} z g(z) = 1.$$

This condition ensures that $g(\infty) = 0$ and $g\prime(\infty) = 1$: defining $G(w) = g(1/w)$ and $G(0) = g(\infty)$ we have

$$G\prime(0) = g\prime(\infty) = \lim_{\omega \to 0} \frac{G(\omega) - G(0)}{\omega} = \lim_{z \to 0} \frac{g(z) - g(\infty)}{\frac{1}{z}} = \lim_{z \to \infty} z(g(z) - g(\infty))\,.$$

Then we have that

$$\begin{aligned} g : \mathbb{S}^2 \backslash E &\longrightarrow \mathbb{S}^2 \\ \infty &\longrightarrow 0 \\ g\prime(\infty) &= 1\,. \end{aligned}$$

It follows from the Riemann mapping theorem that such function exists and that if the norm $\sup_{z\in\mathbb{S}^2\setminus E}|g| = \|g\|_{\mathbb{S}^2\setminus E}$ is minimal then g is in fact a bijection onto some disk $D(0,t)$.
Now, define

$$\begin{aligned} h :& \mathbb{S}^2\setminus tE \longrightarrow \mathbb{D} \\ & z \longrightarrow \frac{1}{t}g\left(\frac{z}{t}\right) . \end{aligned}$$

An application of Cororllary 2.5 to h^{-1} shows that ω_1, ω_2 are avoided if and only if $\omega_1, \omega_2 \in tE$. So, if $\omega_1, \omega_2 \in tE$ then we get $|\omega_1 - \omega_2| \leq 4$. Hence, we have that

$$diam(tE) = \max\{|\omega_1 - \omega_2|; \omega_1, \omega_2 \in tE\} \leq 4$$

By hypotesis and the assumption that $\delta = 1$ we also know that

$$diam(tE) = t \cdot diam(E) > t\delta = t\,.$$

So, $t \leq 4$ and since $\|g\|_{\mathbb{S}^2\setminus E} \leq t$, then $|g(z)| \leq t < 4$.
For fixed $\zeta \in \mathbb{D}$ and $|z-\zeta| > 2$ we have

$$g(z) = \frac{1}{z-\zeta} + \frac{a_2(\zeta)}{(z-\zeta)^2} + \mathcal{O}\left(\frac{1}{|z-\zeta|^3}\right) .$$

Define

$$\begin{aligned} r(\zeta, z) &= g(z) - a_2(\zeta)g^2(z) \\ &= \frac{1}{z-\zeta} + \frac{a_2(\zeta)}{(z-\zeta)^2} + \mathcal{O}(|z-\zeta|^{-3}) - a_2(\zeta)\left(\frac{1}{(z-\zeta)^2} + \mathcal{O}(|z-\zeta|^{-3})\right) \\ &= \frac{1}{z-\zeta} + \mathcal{O}(|z-\zeta|^{-3})\,. \end{aligned}$$

Then

$$\left| r(\zeta, z) - \frac{1}{z-\zeta} \right| = \mathcal{O}(|z-\zeta|^{-3}) \; as \; z \longrightarrow \infty.$$

In order to prove (4.9) notice that from the definition of $r(\zeta, z)$ and the fact that $|g| \leq 4$ we have

$$|r(\zeta, z)| \leq |g(z)| + |a_2(\zeta)||g^2(z)| \leq 4 + 16|a_2(\zeta)|$$

We need to estimate

$$a_2(\zeta) = \frac{1}{2\pi i}\int_{|z|=R}(z-\zeta)g(z)dz = b - \zeta.$$

Notice first that

$$\frac{1}{2\pi i}\int_{|z|=R}\zeta g(z)dz = -\zeta\frac{1}{2\pi i}\int_{|\omega|=\frac{1}{R}} g\left(\frac{1}{\omega}\right)\frac{d\omega}{\omega^2} = \zeta\,\underbrace{g\prime(\infty)}_{1} = \zeta\,.$$

Now define $b = \frac{1}{2\pi i}\int_{|z|=R} zg(z)dz$. Our goal is to see that $|b| \leq 4$, so that therefore

$$|a_2(\zeta)| = |b-\zeta| \leq 5\,.$$

To see this, change the path of integration to the unit circle; we obtain

$$|b| = \frac{1}{2\pi}\left|\int_{|z|=R} zg(z)dz\right| \leq 4\frac{1}{2\pi}2\pi \leq 4.$$

Then, we have that

$$|r(\zeta,z)| \leq |g(z)| + |a_2(\zeta)||g^2(z)| \leq 4 + 16\cdot 5 = 84\,.$$

In particular, we get $C > 0$ such that the inequality (i) holds.
Finally, the function $G : \mathbb{S}^2\backslash E \longrightarrow \mathbb{S}^2$ defined by

$$G(z) = (z-\zeta)^3\left(r(\zeta,z) - \frac{1}{z-\zeta}\right)$$

is analytic (it is bounded when $z \to \infty$ and hence it has removable singularity at ∞), and it is bounded by some constant C in $\partial(\mathbb{S}^2\backslash E)$:

$$|G(z)| = \left|(z-\zeta)^3\left(r(\zeta,z) - \frac{1}{z-\zeta}\right)\right| \leq |z-\zeta|^3|r(\zeta,z)| + |z-\zeta|^2 < 2^3\cdot 84 + 2^2.$$

Therefore by the maximum principle $G(z)$ is bounded by the same constant in $\mathbb{S}^2\backslash E$ and (4.8) holds.

□

Now we can prove Theorem 4.2.

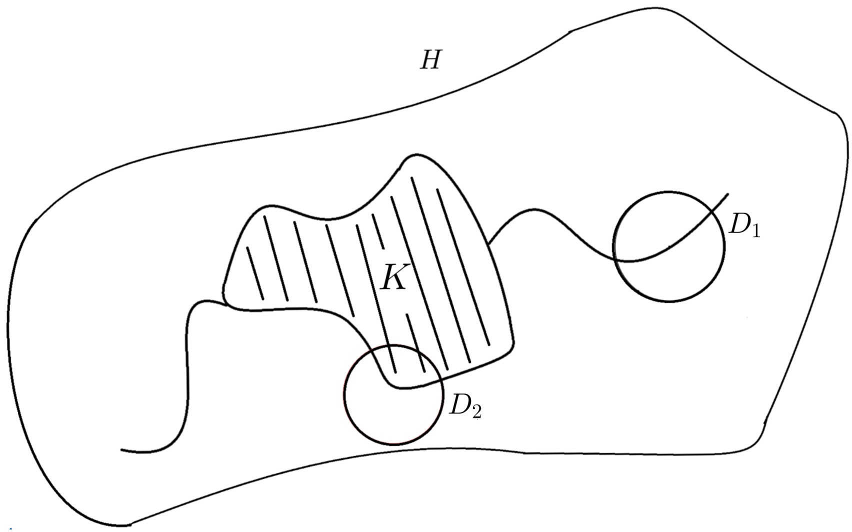


Figure 4.1: D_1 and D_2 are disks of radius 2δ and centers outside of K.

Proof. Cover H by a finit number of disks D_j with radius 2δ and centers outside K. Moreover, by Lemma 4.3, the complement of K is connected in each D_j, so we can find a set E_j such that $diam(E_j) \geq 2\delta$ and $E_j \cap K = \emptyset$ (there must be a curve from the center to the boundary that does not intersect K).

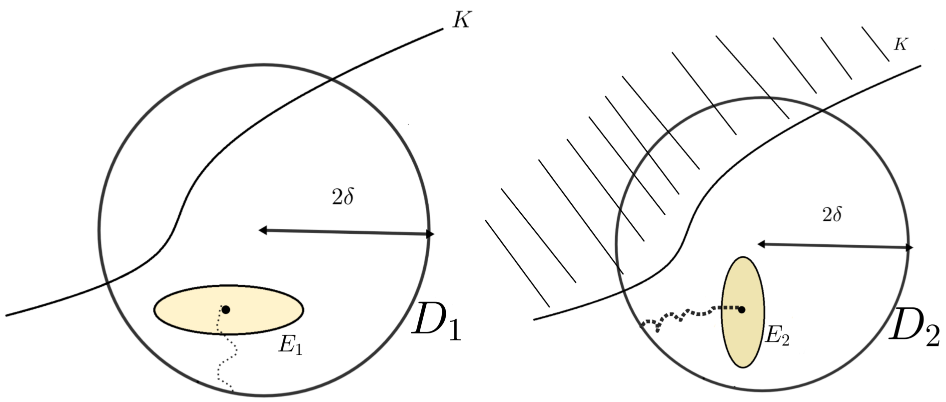


Figure 4.2: we can see that there is a curve from the center of E_j to the boundary of D_j that does not intersect K, in two situations.

For each D_j, let $r_j(\zeta, z)$ be the function given by Lemma 4.3.
Let $\Omega = \bigcap_j (\mathbb{P} \backslash E_j)$. Then clearly $K \subset \Omega$. Let ϕ_j be a partition of unity subordinate to the open cover D_j of the compact set H, i.e, ϕ_j is a collection of functions such that

- $0 \leq \phi_j \leq 1$ for $j = 1, \cdots, n$.
- The support of ϕ_j lies in D_j.
- $\sum_{j=1}^{n} \phi_j(\zeta) = 1$, for $\zeta \in H$.

Then the function

$$r(\zeta, z) = \sum_{j=1}^{n} \phi_j(\zeta) r_j(\zeta, z)$$

has the required properties:

- $r(\zeta, z)$ is analytic for $z \in \Omega$.
- $|r(\zeta, z)| \leq \sum_{j=1}^{n} |\phi_j(\zeta)||r_j(\zeta, z)| \leq \frac{C}{\delta}$.
- Since $\frac{1}{\zeta - z} = \sum_{j=1}^{n} \phi_j(\zeta) \frac{1}{\zeta - z}$, and $\sum_{j=1}^{n} \phi_j(\zeta) = 1$ we get that:

$$\left| r(\zeta, z) - \frac{1}{\zeta - z} \right| = \left| \sum_{j=1}^{n} \phi_j(\zeta) \left(r_j(\zeta, z) - \frac{1}{\zeta - z} \right) \right| \leq \frac{C\delta^2}{|\zeta - z|^3}$$

.

□

Chapter 5

Arakelian's Theorem

Norair Unanovich Arakelian is an Armenian and Soviet mathematician, born in 1936. He studied at the Faculty of Physics and Mathematics in Yerevan State University (YSU) and graduated there in 1958. Four years later, Arakelian received his Ph.D. from YSU with thesis *Uniform and tangential approximation by entire functions in the complex domain*. During this period Mkhitar M. Dzhrbashyan was his advisor. In 1970, he recived his doctorate of Science in the Steklov Institute of Mathematics for the thesis *Some questions of Approximation Theory and the Theory of Entire Functions.*

About his professional responsabilities, Arakelian has been Senior Scientist Researcher in the Institute of Mathematics of Academy of Sciences of Arm Union SSR, member of the editorial board of the international journal "*Analysis*", president of the Armenian Mathematical Union and a docent of the Chair of Function Theory of YSU. In addition, he has taught general and special courses of complex analysis and supervised ten Ph.D. thesis on approximation theory and complex analysis.

Nowadays, N.U.Arakelian is the president of the fund "*Research Mathematics*" in Armenia and the head of Department of complex analysis of the Institute of Mathematics of the National Academy of Sciences of Armenia.

One of his the better known results is the approximation theorem we state next.

The aim of this chapter is to show that Arakelian's theorem (Theorem 1.8) follows very easily from Mergelyan's theorem and from Runge's theorem. In particular, Arakelian's theorem turns out to be really elementary for functions that are holomorphic in a neighborhood of a closed subset of the complex plane.

This proof is given by the "Runge case" and by the "Mergelyan case". The difference between them is just the induction hypothesis over the polynomial that approximates the holomorphic function.

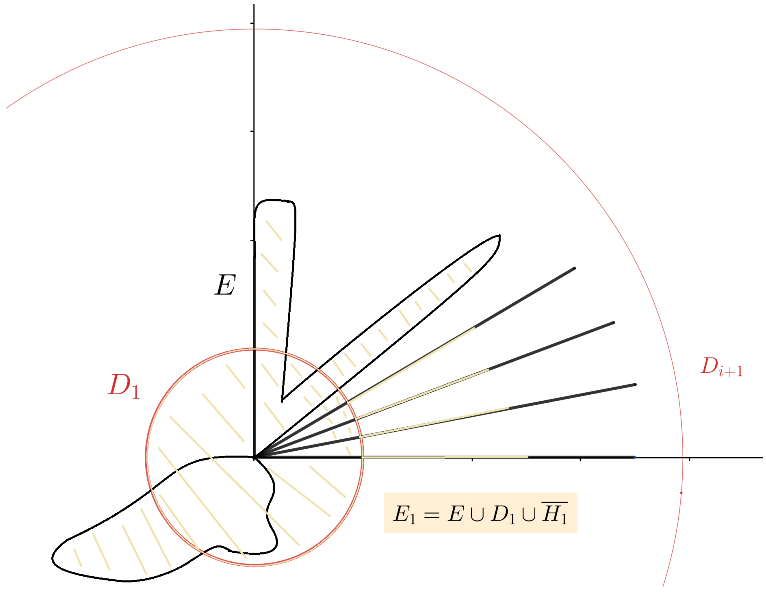


Figure 5.1: Construction of E_1: first we take the union of E and D_1. Then define H_1 as the union of the holes of $E \cup D_1$. Finally take the union $E_1 := E \cup D_1 \cup \overline{H_1}$. Notice that E_1 has no holes.

Proof. Since E is an Arakelian set, there are closed discs $D_i = D(0, r_i)$ for $i = 1, 2, 3...$, whose union is $\mathbb{C}$, so that $D_i \cup \overline{H_i} \subseteq \mathring{D}_{i+1}$ where H_i is the union of the holes of $E \cup D_i$. Put $E_0 = E$ and $E_i = E \cup D_i \cup \overline{H_i}$ for $i \geq 1$. Note that no E_i has holes.

We shall construct a function that approximates f recursively, so that approximates it successively in each $\widehat{E}_i$. Let's see how can we construct h_i in the "Runge case", i.e, when $f \in H(\widehat{E})$.

Put $h_0 = f$, fix $i \geq 1$ and assume that we have a function $h_{i-1} \in H(\widehat{E_{i-1}})$. There is an open disc Δ that contains $D_i \cup \overline{H_i}$ and whose closure $\overline{\Delta}$ lies in the interior of D_{i+1}.

Choose a continuously differentiable function ψ on $\mathbb{C}$ so that $0 \leq \psi \leq 1$, $\psi = 1$ in Δ and $\psi = 0$ outside D_{i+1}.

Since E_{i-1} has no holes, the same is true of $E_{i-1} \cap D_{i+1}$. Runge's theorem therefore furnishes a polynomial P so that

$$|h_{i-1} - P| < \frac{\epsilon}{2^{i+1}} \quad on \quad E_{i-1} \cap D_{i+1}. \tag{5.1}$$

We shall see next that for some $C > 0$

$$\frac{1}{\pi} \int_{E_{i-1}} \left| (h_{i-1} - P)(w) \frac{\partial \psi}{\partial \bar{z}}(w) \right| \frac{d\lambda(w)}{|z - w|} < \frac{\epsilon}{2^{i+1}} C \tag{5.2}$$

for all $z \in \mathbb{C}$. Note that the integrand vanishes outside D_{i+1}.
In particular, if we take $\Delta = D(0, R)$ and $D_{i+1} = D(0, R_{i+1})$ with $R_{i+1} = 2R$, then

$$\left|\frac{\partial\psi}{\partial\bar{z}}(w)\right| \lesssim \frac{1}{R_{i+1} - R} = \frac{1}{R}$$

Now we claim that

$$\sup_{z\in\mathbb{C}} \int_{D_{i+1}} \frac{d\lambda(w)}{|z-w|} = \int_{D_{i+1}} \frac{d\lambda(w)}{|w|} = 2\pi R_{i+1} = 4\pi R$$

Then, there exists $C > 0$ such that

$$\frac{1}{\pi}\int_{E_{i-1}} \left|\frac{\partial\psi}{\partial\bar{z}}(w)\right| \frac{d\lambda(w)}{|z-w|} \leq \int_{supp(\bar{\partial}\psi)} \left|\frac{\partial\psi}{\partial\bar{z}}(w)\right| \frac{d\lambda(w)}{|z-w|} \leq \frac{1}{R}\int_{D_{i+1}} \frac{d\lambda(w)}{|z-w|} \leq C\,. \tag{5.3}$$

for all $z \in \mathbb{C}$.
Therefore, by Cauchy's formula, (5.1) and (5.3) we get

$$\begin{aligned}
|h_{i-1}(z) - P(z)| &= \left| -\frac{1}{\pi}\int_{E_{i-1}} (h_{i-1} - P)(w)\frac{\partial\psi}{\partial\bar{z}}(w)(w)\frac{d\lambda(w)}{z-w} \right| \\
&\leq \frac{1}{\pi}\int_{E_{i-1}} |(h_{i-1} - P)(w)\frac{\partial\psi}{\partial\bar{z}}(w)| \frac{d\lambda(w)}{|z-w|} \\
&\leq \frac{\epsilon}{2^{i+1}}\frac{1}{\pi}\int_{E_{i-1}} \left|\frac{\partial\psi}{\partial\bar{z}}(w)\right| \frac{d\lambda(w)}{|z-w|} \\
&\leq \frac{\epsilon}{2^{i+1}}\frac{1}{\pi}\int_{supp(\bar{\partial}\psi)} \left|\frac{\partial\psi}{\partial\bar{z}}(w)\right| \frac{d\lambda(w)}{|z-w|} \\
&\leq \frac{\epsilon}{2^{i+1}}C\,.
\end{aligned}$$

Now let V be a neighborhood of E_{i-1} in which h_{i-1} is holomorphic, and which is so close to E_{i-1} that

$$\frac{1}{\pi}\int_V |(h_{i-1} - P)(w)\frac{\partial\psi}{\partial\bar{z}}(w)| \frac{d\lambda(w)}{|z-w|} < \frac{\epsilon}{2^{i+1}} \tag{5.4}$$

Define

$$r(z) = \frac{1}{\pi}\int_V (h_{i-1} - P)(w)\frac{\partial\psi}{\partial\bar{z}}(w)\frac{d\lambda(w)}{(z-w)} \tag{5.5}$$

and

$$h_i = P\psi + (1-\psi)h_{i-1} + r \quad in\ \Delta \cup V. \tag{5.6}$$

This is well defined because $1 - \psi = 0$ in Δ.
The fact that $(h_{i-1} - P)\bar{\partial}\psi$ is continuously differentiable in V and (2.4) show that

$$\bar{\partial}r = (h_{i-1} - P)\bar{\partial}\psi$$

in V, since $\overline{\partial} h_{i-1} = 0$ in V. We deduce that

$$\overline{\partial} h_i = P\overline{\partial}\psi - h_{i-1}\overline{\partial}\psi + \overline{\partial} r = (P - h_{i-1})\overline{\partial}\psi + \overline{\partial} r = 0$$

in V.
In Δ, $\overline{\partial}\psi = 0$. The integral (5.5) extends therefore only over $V\backslash\Delta$, so that $r \in H(\Delta)$. The same is true for h_i because $h_i = P + r$ in Δ. So h_i is holomorphic in the neighborhood $\Delta \cup V$ of E_i and

$$|h_i - h_{i-1}| = |(P - h_{i-1})\psi + r| \leq \psi|P - h_{i-1}| + |r| < \frac{\epsilon}{2^i} \quad on \quad E_{i-1} \tag{5.7}$$

by (5.1); note that $\psi = 0$ outside D_{i+1} and that (5.4) holds .
The sets E_{i-1} contain the discs D_{i-1}, and these expand to cover $\mathbb{C}$. Finally, since

$$h_n - h_0 = h_n - h_{n-1} + h_{n-1} - \cdots - h_1 + h_1 - h_0$$

and

$$|h_n - h_0| \leq \sum_{i=1}^{\infty} |h_i - h_{i-1}| < \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon,$$

letting $n \to \infty$, we obtain $|h - h_0| = |h - f| < \epsilon$. So we get the conclusion of the theorem.
Another way to construct the function that approximates f is by the "Mergelyan case". In this case we assume that $h_{i-1} \in \mathscr{C}(E_{i-1}) \cap H(\mathring{E}_{i-1})$, so we get P in (5.1) from Mergelyan's theorem. Define $r(z)$ as above, replacing E_{i-1} by V in (5.5). So, from now on, we have that

$$r(z) = \frac{1}{\pi}\int_{E_{i-1}} (h_{i-1} - P)(w)\frac{\partial\psi}{\partial\overline{z}}(w)\frac{d\lambda(w)}{z - w}$$

We conclude that h_i (defined by (5.6)) is continuous on E_i, because h_i is continuous on $\Delta \cup E_{i-1}$ and holomorphic in the interior of E_i:

$$\frac{\partial h_i}{\partial\overline{z}} = P\frac{\partial\psi}{\partial\overline{z}} - h_{i-1}\frac{\partial\psi}{\partial\overline{z}} + \frac{\partial r}{\partial\overline{z}} = 0 \quad on \quad E_i.$$

As before, it also satisfies (5.7) on E_{i-1}. □

Remark 5.1. On sets without interior, a considerably stronger version of the theorem can be derived without any extra effort.

Corollary 5.2. *Let E be an Arakelian set with empty interior. Let $\omega : E \longrightarrow \mathbb{R}_+$ be a continuous function. Then for all continuous function f there exists an entire function h such that*

$$|h(z) - f(z)| < \omega(z) \quad z \in E.$$

Proof. By Arakelian's theorem there is an entire funcion g_1 such that

$$|g_1(z) - \log \omega(z)| < 1\,, \quad z \in E.$$

Let $g_2(z) = g_1(z) - 1$; we have

$$Reg_2 = Reg_1 - 1 < \log \omega(z) \quad z \in E.$$

By the same theorem we can find another entire function g_3 such that

$$|g_3(z) - f(z)e^{-g_2(z)}| < 1\,, \quad z \in E.$$

Hence, for all $z \in E$

$$|h(z) - f(z)| = |g_3(z)e^{g_2(z)} - f(z)| \leq |e^{g_2(z)}||g_3(z) - f(z)e^{-g_2(z)}| < |e^{g_2(z)}| < \omega(z)$$

which concludes the proof. $\square$

Conclusions

In this work, we have studied three essential theorems of uniform approximation by entire functions in the complex plane. Runge's theorem maybe is better known as the others, however, we have explained all of them with basic properties of analytic functions, conformal mappings, and some functional analysis.
We have elaborated in detail the proofs of the theorems of Runge, Mergelyan, and Arakelian. Moreover, we have introduced two types of proofs of Runge's theorem using different arguments.
Although some of these results have already been seen in the subject of Complex Analysis, during these months I have understood how to make use of them and I have also realized that some of them turn out to be really useful in the proofs of this work.

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