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The Mann-Su theorem

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Abstract

In this text, we give the necessary tools to prove and understand the Mann-Su theorem. In the context of transformation groups theory, the Mann-Su theorem gives a restriction on which finite groups can act effectively on a manifold. Particularly, we will find an upper bound N that only depends on the manifold M such that groups of the form $(\mathbb{Z}_p)^r$ can not act effectively on M if $r > N$. Restricting ourselves to the case of smooth manifolds and actions, we will take a slightly different approach compared to the original paper where L.N Mann and J.C. Su proved the theorem.

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Introduction

While topology studies the properties of topological spaces or differential geometry studies the geometrical properties of differential manifolds, the theory of transformation groups studies the symmetries of some of these objects. The main notion considered is group actions on the space we desire to study X. Roughly speaking, given a group G, we assign to each element of G a homeomorphism (or diffeomorphism) $\theta_g : X \longrightarrow X$, in such a way that the group structure is preserved (so $\theta_e = id_X$ and $\theta_g \circ \theta'_g = \theta_{gg'}$, where e denotes the identity element of G). If the assignation is injective (the only element that is mapped to the identity is e), we say the the group acts effectively on X.

A paradigmatic case that is easy to visualise is the actions of finite groups on the circle S^1. A cyclic group of order n, C_n may act on S^1 in such a way that some fixed generator c acts by rotations of angle $\frac{2\pi}{n}$. This means that we assign each element c^j to the homeomorphism θ_j such that $\theta_j(e^{2\pi ix}) = e^{2\pi i(x+j/n)}$. If we consider also a reflection symmetry on S^1, with the homeomorphism that maps $e^{2\pi ix} \to e^{2\pi i(-x)}$ and we combine it with the previous rotation symmetry, we obtain an action of the dihedral group D_{2n} on S^1.

A natural question that can be formulated is whether there exist more finite groups acting effectively on S^1. The answer is negative. If G be a finite group acting effectively on S^1, then G is a cyclic group or a dihedral group. Although the proof of this statement is not complex, we omit it for the sake of brevity.

Seeing how limited the group actions on S^1 are, one could ask whether there exists a finite group that does not act in any compact manifold. The answer is negative. Given a finite group G, there exists a natural number n (we can choose n to be the cardinality of the group) such that G is a subgroup of the group of permutations of n elements, S_n. Let $T^n = S^1 \times ... \times S^1$ be the n-dimensional torus, S_n acts effectively on T^n by permuting the coordinates, $\sigma(e^{2\pi ix_1}, ..., e^{2\pi ix_n}) = (e^{2\pi ix_{\sigma(1)}}, ..., e^{2\pi ix_{\sigma(n)}})$. Therefore, restricting this action to the subgroup G we obtain an effective action of G on T^n.

If we summarise the previous information, each finite group can act effectively on some manifold, but the set of groups that act on a given manifold could be quite small. An interesting problem is to study the finite groups that can act effectively on a given manifold M.

This text has as main objective to prove one of the theorems that restricts which groups can act effectively on a given compact manifold, in the case where the groups are abelian and finite. In some way, it confirms the intuition suggested by the S^1 case. Given a compact manifold, a finite group acting effectively on it can not be excessively complex.

Theorem 0.1. *Let M be a compact and connected smooth manifold, there exists a natural number N, which only depends on the manifold, such that the groups of the form $(\mathbb{Z}_p)^r$, where p is a prime number, can not act effectively on M if $r > N$.*

Observe that if G is a group acting on M which has $(\mathbb{Z}_p)^r$ as a subgroup, then $(\mathbb{Z}_p)^r$ acts on M. Therefore, if $r > N$, we obtain that $(\mathbb{Z}_p)^r$ does not act effectively (an element $g \neq e$ of $(\mathbb{Z}_p)^r \subset G$ is mapped to the identity map), therefore G does not act effectively on M. This fact shows how versatile the Mann-Su theorem could be.

This theorem is due to L.N. Mann and J. C. Su and was proved in the paper [1], which was published in 1962. In this period, transformation groups theory was an active area of research with the contribution of outstanding mathematicians like A. Borel, P. A. Smith, G. E. Bredon or P. E. Conner amongst others.

The theorem stated in [1] is proved in the general context of cohomology manifolds. In order to reduce the required technicalities, we shall prove the theorem for smooth manifolds (we will assume that compactness and connectedness on the manifolds). Working with smooth manifolds allows us to take a different approach in some parts of the proof and reduces some technical complication (for example, we are able to use singular homology instead of more exotic homology theories).

One of the beauties of the transformation group theory is that it uses a big number of disparate fields of mathematics. This project's topics range from spectral sequences, algebraic topology, fiber bundles theory, equivariant homology to Riemannian geometry or frame bundles. Some of the theorem used throughout the chapters could deserve its own project, like the Serre spectral sequence in the final part of chapter 3. These are some of the reasons why, in mere 50 pages, it is not possible to make a self-contained or even complete exposition on all of these subjects. Some standard topics, like Riemannian geometry or some classic theorems of algebraic topology are not rigorously proved, we rather provide some reference books where they are meticulously exposed. For some statements of fiber bundles, we only show a sketch of their proofs. However, we intend to provide rigorous proofs for the statements that involve group actions and of the claims that are crucial to prove the Mann-Su theorem.

This project contains five different chapters, the firsts four ones explain the different topics required for proving the theorem, while the last one is devoted to explaining the proof of the Mann-Su theorem. The first chapter is an amalgamation of various topics where we provide a concise introduction to the basic definitions of the transformation group theory and fiber bundles, and we state some general results in algebraic topology. The second chapter gives a short presentation of Riemannian geometry and frame bundles. Thereafter, various results regarding smooth group actions are proved. The third chapter is devoted to the construction of spectral sequences, focusing on the case of chain complexes. Moreover, the Serre spectral sequence is explained in the final part of this chapter. The fourth chapter describes classifying spaces and equivariant homology with some of its properties. In the fifth and last chapter, some preliminary lemmas are stated before proving the Mann-Su theorem.

A final remark that shall be made is that the Mann-Su theorem only involves groups of the form $(\mathbb{Z}_p)^r$, which are finite. Assuming the finiteness of the group reduces the complexity of the proof of some of the auxiliary statements announced, like the slice theorem or the construction of classifying spaces. In consequence, we will suppose that the group is finite, although some of the statements could be valid in the more general context of compact Lie groups.

Notation: We will denote by $\mathbb{Z}_p$ the cyclic group of order p, where p is a prime number. Usually, we will use the letter G to denote the group, the letters X or Y to denote a topological space and the letters M or N to denote a manifold. The letters m or n will denote the dimension of the previous manifolds respectively. We will suppose that manifolds do not have boundary. We will say that M is a smooth manifold if all transition maps are C^∞.

We will denote the closed interval $[0, 1]$ by I, while J will denote an open interval. The symbol "$\cong$" will denote an isomorphism, homeomorphism or diffeomorphism, depending on the context. The symbol $\simeq$ will denote homotopy equivalence.

Finally, $\mathcal{M}_{m\times m}$ will denote the group of matrices with m rows and m columns which has real numbers as entries. $Gl(m, \mathbb{R})$ and $O(m, \mathbb{R})$ will denote the general linear group and the orthogonal group of dimension m respectively.

Chapter 1

Preliminaries

In this introductory chapter we will expose basic definitions and results that will be required in subsequent chapters. The first section is dedicated to provide the definition of a transformation group. The second and third section are devoted to explaining twisted products, slices and fiber bundles. These objects will be crucial in subsequent chapters. Finally, in the fourth section two classic theorems of algebraic topology are stated and we define and explain some topological properties about a particularly nice type of topological spaces called CW-complexes.

To elaborate the first part of this chapter, we used the acclaimed book of G. E. Bredon titled *Introduction to compact transformation groups*, [2]. As additional reference, one can consult [3]. For the last section of the chapter, we based on [4], [5] and [6].

1.1 Transformation groups

Definition 1.1. *Let $(G, \cdot)$ be a group, which it is also a Hausdorff topological space. It is called a topological group if the inverse map $\iota : G \longrightarrow G$ ($\iota(g) = g^{-1}$) and the product $\sigma : G \times G \longrightarrow G$ ($\sigma(g,h) = g \cdot h$) are continuous (where $G \times G$ has the product topology).*

The element e will denote the identity element of the group G. To simplify the notation, we will write gh to symbolise the product of elements of G.

Remark 1.2. *Let G be a finite group, we can equip it with the discrete topology. Consequently, it is a topological group because both maps are continuous with this topology. When we talk about a finite group, we shall suppose that it is a topological group with the discrete topology.*

Definition 1.3. *A topological transformation group is a triple (G, X, Θ) where G is a topological group, X is a Hausdorff topological space and Θ is continuous map from $G \times X$ to X, such that:*

- $\Theta(g, \Theta(h, x)) = \Theta(gh, x)$*, for all $g, h \in G$ and $x \in X$.*
- $\Theta(e, x) = x$*, for all $x \in X$.*

The map Θ is called an action of G over X, the space X together with the action is called a G-space or a left G-space. Its obvious counterpart is a right G-space, where

$\Theta(g, \Theta(h, x)) = \Theta(hg, x)$, for all $\in G$ and $x \in X$. Most of the time we will omit the map Θ if the action is clear from the context, and we simply write $g(x)$ or gx to symbolise the action on a left G-space, and xg to symbolise the action on a right G-space. We will always work with left actions unless stated the contrary.

Remark 1.4. *Given $g \in G$, we define the map $\theta_g : X \longrightarrow X$ such that $\theta_g(x) = gx$. Clearly, given $g, h \in G$, we have that $\theta_g \circ \theta_h = \theta_{gh}$, $\theta_e = Id_X$ and $\theta_g \circ \theta_{g^{-1}} = \theta_{gg^{-1}} = Id_X$. Then, θ_g is and homeomorphism of X for all $g \in G$. Therefore, we obtain a group morphism $G \longrightarrow Homeo(X)$ such that the image of g is θ_g.*

Now we can define some natural notions that appear when we start studying group actions.

Definition 1.5. *We define the following subgroups of G:*

- *$ker\Theta = \{g \in G : g(x) = x, \forall x \in X\}$ is called the kernel of the action Θ.*
- *$G_x = \{g \in G : g(x) = x\}$ is called the isotropy group (or stabiliser) of $x \in X$.*

Remark 1.6. *It is clear that $\cap_{x \in X} G_x = ker\Theta$*

Definition 1.7. *An action is said to be effective if $ker\Theta = \{e\}$. An action is said to be free if $G_x = \{e\}$ for all $x \in X$. An action is said to be trivial if $\theta_g = id_X$ for all $g \in G$.*

Example 1.8. *Some examples of topological transformation groups are the following ones:*

1. *Let S^n be the n dimensional sphere and let f be the antipodal map. Let $\mathbb{Z}_2$ be the group with two elements $\{-1, 1\}$. We define a $\mathbb{Z}_2$-action on S^n such that $1(x) = x$ and $-1(x) = f(x)$. Because f is continuous and $f^2 = id$, it fulfills the conditions to be a topological transformation group.*

2. *We can see the sphere S^{2n-1} in $\mathbb{C}^n$, which we can describe by $\{(z_1, ..., z_n) : \Sigma |z_i|^2 = 1\}$. Given an integer m and integer $l_1, ..., l_n$, that are relative prime to m, the cyclic group $\mathbb{Z}_m$ acts on the sphere by rotations. If ρ is a generator of the group, then $\rho(z_1, ..., z_n) = (e^{2\pi i l_1/m} z_1, ..., e^{2\pi i l_n/m} z_n)$. The assumption that those integers are relatively prime is to ensure that the action is free. The rotation action on S^1 explained in the introduction is a particular cases of this example.*

3. *Let S_n be the permutation group of n elements. Let x be a point of the n dimensional torus $T^n = S^1 \times ... \times S^1$, which is the Cartesian product of n copies of S^1. Let x have coordinates $(e^{2\pi i x_1}, ..., e^{2\pi i x_n})$. If $\sigma \in S_n$, we define an action of S_n on T^n such that $\sigma(e^{2\pi i x_1}, ..., e^{2\pi i x_n}) = (e^{2\pi i x_{\sigma(1)}}, ..., e^{2\pi i x_{\sigma(n)}})$. If G is a subgroup of S_n, we obtain an induced action of G on T^n. Since every finite group is subgroup of some permutation group, every finite group acts on T^n for some n.*

Definition 1.9. *Given a set $A \subset X$ and a subgroup $H \subset G$, we define:*

- *$H(A) = \{h(x) : x \in A, h \in H\}$. The set A is said to be invariant under H if $H(A) = A$.*
- *$F(H, A) = \{x \in A : h(x) = x \ \forall \mathrm{h} \in H\}$. If $A = X$, X^H is used to denote $F(H, X)$.*

Definition 1.10. *Let X and Y be two G-spaces, a map $\phi : X \longrightarrow Y$ is said to be equivariant if $\phi(gx) = g\phi(x)$ for all $g \in G$ and $x \in X$. Moreover, if ϕ is an homeomorphism, it is called an equivalence of G-spaces.*

Definition 1.11. *Let x be a point of X, the set $x^* = G(x) = \{g(x) : g \in G\}$ is called the orbit of x.*

Definition 1.12. *We can define an equivalence relation on X such that $x \sim y$ if and only if $G(x) = G(y)$, which means that there exists $g \in G$ such that $y = gx$. If X/G denotes the quotient set of all orbits, we can see it as a topological space with the final topology induced by the quotient map that takes a point into its class. X/G is called the orbit space. The map $\pi : X \longrightarrow X/G$ which maps every element of X to its equivalent class is called the orbit map.*

Lemma 1.13. *The map $\pi : X \longrightarrow X/G$ is an open map.*

Proof. Let U be an open set of X. Then $G(U) = \bigcup_{g \in G} g(U)$ is an open set in X, since $g(U) = \theta_g(U)$ and θ_g is a homeomorphism for all $g \in G$. Hence, $G(U) = \pi^{-1}\pi(U)$ is open and, by definition, $\pi(U)$ is an open set of X/G.

□

Lemma 1.14. *If G is a finite group, then:*

1. *The map $\pi : X \longrightarrow X/G$ is a closed map.*
2. *The space X/G is Hausdorff.*

Proof. For the first part of the lemma, let A be a closed set of X. Then $G(A) = \bigcup_{g \in G} g(A)$ is a closed set in X, because it is the finite union of closed sets since $g(A) = \theta_g(A)$ is the image of a closed set by a homeomorphism. The equality $G(A) = \pi^{-1}\pi(A)$ holds because of the surjectivity of π. If we use the complement of $G(A)$, we obtain that $X - G(A) = \pi^{-1}(X/G - \pi(A))$ and, by definition, the set $X/G - \pi(A)$ is open. In consequence, $\pi(A)$ is closed, as we wanted to prove.

For the second part of the lemma, we take two points $[x], [y] \in X/G$. Their orbits in X are a finite set of points. Using that X is Hausdorff, we can construct two open sets U and V such that $G(x) \subset U$, $G(y) \subset V$ and $U \cap V = \emptyset$. Therefore, the set $X \setminus V$ is closed and $U \subset \bar{U} \subset X \setminus V$, in particular, $G(y) \cap \bar{U} = \emptyset$, hence $[y] \notin \pi(\bar{U})$. Then, the set $\pi(U)$ is open since U is open and $[x] \in \pi(U)$. The set $(X/G) \setminus \pi(\bar{U})$ is open because $\bar{U}$ is closed and $[y] \in (X/G) \setminus \pi(\bar{U})$. Observe that their intersection is the empty set, therefore these are the two desired open sets. □

Remark 1.15. *Since $\pi : X \longrightarrow X/G$ is continuous and surjective, if X is compact or connected, then X/G is compact or connected.*

Example 1.16. *We are interested in the orbit spaces of the first example.*

1. *The orbit space $S^n/\mathbb{Z}_2$ is the real projective space of dimension n, $\mathbb{R}P^n$.*
2. *The orbit space $S^{2n-1}/\mathbb{Z}_m = L_m(l_1, ..., l_n) = L^n_m$ is called lens space.*

1.2 Twisted product and slices

Definition 1.17. *Let X be a right G-space and Y a left G-space, we construct a left G-action on the product $X \times Y$ such that $g(x,y) = (xg^{-1}, gy)$. Its orbit space is called twisted product of X and Y and it is denoted by $X \times_G Y$.*

Remark 1.18. *The orbit of an element (x,y) is denoted by $[x,y]$. Observe that $[xg,y] = [x,gy]$.*

Given a right G-space X, we can construct a left G-action on X, by defining $gx = xg^{-1}$. In this way, we can define the twisted product of two left G-spaces X and Y as the orbit space of the diagonal action of G on $X \times Y$, where $g(x,y) = (gx,gy)$. As a consequence, $X \times_G Y \cong Y \times_G X$.

Remark 1.19. *Let Y and Y' be G-spaces and $f : Y \longrightarrow Y'$ an equivariant map. Then the map $id \times_G f : X \times_G Y \longrightarrow X \times_G Y'$, such that $id \times_G f([x,y]) = [x, f(y)]$ is well defined.*

With the definition and these remarks, we are able to proof some elementary properties of the twisted product of spaces.

Lemma 1.20. *If f is open, then $id \times_G f$ is open.*

Proof. If we consider the following commutative diagram, the statement is an immediate consequence of the orbit maps being open.

$$\begin{array}{ccc} X \times Y & \xrightarrow{id \times f} & X \times Y' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ X \times_G Y & \xrightarrow{id \times_G f} & X \times_G Y' \end{array}$$

□

Lemma 1.21. *1. If $\{pt\}$ is the topological space formed by one point (G acts trivially on it), then $X \times_G \{pt\} \cong X/G$.*

2. $X \times_G G \cong X$.

Proof. The first statement is immediate by observing that $X \times \{pt\}$ is homeomorphic to X and the diagonal action on this space is simply the action of G on X.

To prove the second part of the lemma, we will construct an equivariant homeomorphism. Consider the maps $X \times_G G \longrightarrow X$ such that $[x,g] \to xg$ and $X \longrightarrow X \times_G G$ such that $x \to [x,e]$. Clearly, both map are well defined and one is the inverse of the other. The continuity of the first map is due to the continuity of the transformation map $X \times G \longrightarrow X$ and the openness of the orbit map. The continuity of the second map is due to the continuity of the inclusion $X \longrightarrow X \times G$ such that $x \to (x,e)$ and the continuity of the orbit map.

□

Proposition 1.22. *Let X be a right H-space, Y be a left H-space and a right K-space, and Z be a left K-space. Then $(X \times_H Y) \times_K Z \cong X \times_H (Y \times_K Z)$.*

Proof. We construct a map $(X \times_H Y) \times_K Z \longrightarrow X \times_H (Y \times_K Z)$, such that $[[x,y]z] \to [x,[y,z]]$. It is well defined because

$$[[xh^{-1}, hy]k^{-1}, kz] = [[xh^{-1}, k^{-1}hy], kz] \to [xh^{-1}, [hyk^{-1}, kz]] = [xh^{-1}, h[yk^{-1}, kz]].$$

The continuity is a consequence of the openness of the map

$$(X \times Y) \times Z \longrightarrow (X \times_H Y) \times Z \longrightarrow (X \times_H Y) \times_K Z.$$

The construction and continuity of the inverse is analogous.

□

Corollary 1.23. *Let G be a topological group acting on X and H a subgroup of G. Then $X \times_G G/H \cong X/H$.*

Proof. We have the following chain of homeomorphisms $X \times_G (G/H) \cong X \times_G (G \times_H \{pt\}) \cong (X \times_G G) \times_H \{pt\} \cong X \times_H \{pt\} \cong X/H$.

□

We finish this section by introducing a significant concept that use twisted products in its definition.

Definition 1.24. *Let X be a G-space with G compact, let x be a point of X and let S be a subspace such that $x \in S \subset X$ and $G_x(S) = S$. The subspace S is called a slice at x if the map $G \times_{G_x} S \longrightarrow X$ taking $[g,s] \to g(s)$ is equivariant and a homeomorphism between $G \times_{G_x} S$ and an open neighbourhood of the orbit G(x). This map is called a tube about G(x).*

The slice theorem proves the existence of slices at any point of a space, supposing that the space and the group action satisfy extra conditions. We shall prove the existence of slices in the case that G is a finite group and the space is Hausdorff.

Proposition 1.25. *Let G be a finite group and X a Hausdorff G-space, then there exists a slice at x for all $x \in X$.*

Proof. Let $G(x) = \{x_1, ..., x_r\}$ be the orbit of the point x, where we choose $x_1 = x$. Since X is Hausdorff, there exists a collection of open sets $\{U_i\}_{i=1,...,r}$ such that $x_i \in U_i$ and $U_i \cap U_j = \emptyset$ if $i \neq j$. For each point in the orbit $G(x)$, we define the set $V_i = \bigcap_{g \in G, gx_j = x_i} gU_j$, which is an open set because it is the intersection of a finite number of open sets. Moreover, observe that $V_i \subset U_i$, hence the sets V_i and V_j are disjoint if $i \neq j$. The set V_1 will be denoted by S and the union of all the sets V_i will be denoted by V, which is a open neighbourhood of $G(x)$. Our objective is to prove that S is a slice at x.

Firstly, let h be an element of G such that $hx_j = x_i$, then $gV_j = V_i$. Using the definition, $hV_j = \bigcap_{g \in G, gx_k = x_j} hgU_k = \bigcap_{g \in G, hgx_k = x_i} hgU_k = V_i$, where the last equality is a consequence of the map $h : G \longrightarrow G$ such that $g \to hg$ being an isomorphism. Particularly, if $g \in G_x$, then $gS = S$. Therefore, the map $G \times_{G_x} S \longrightarrow V$ such that $[g,s] \to gs$ is well defined, exhaustive and equivariant. To prove its injectivity we suppose that there exists $[g,s]$ and $[g', s']$ such that $gs = g's'$. Consequently, $g^{-1}g's' = s \in S$, which means that $g^{-1}g' = h \in G_x$. We can conclude that $g' = gh$ and $s' = h^{-1}s$, which mean that $[g,s] = [g', s']$.

The topological properties which the map requires to be a homeomorphism are induced by the following commutative diagram

$$\begin{array}{ccc} G \times S & \longrightarrow & V \\ \big\downarrow \pi & \nearrow & \\ G \times_{G_x} S & & \end{array}$$

where the top arrow is the map such that $(g, s) \to gs$, which is continuous and open. □

Remark 1.26. *The following facts are direct consequences of the proposition we have just proved.*

- *The above proposition also prove the existence of a tube about each orbit of X.*
- *If the action is free, then $G_x = \{e\}$ for all $x \in M$ and the twisted product becomes the usual Cartesian product.*
- *The open set V_i is a slice at x_i for all i.*
- *If the space X is a smooth manifold, we can suppose that each V_i is inside one chart (W, ϕ) of the differential structure of X, by choosing an open set U_i small enough to be inside one of the charts.*

We are interested in the relation between a slice at $x \in X$ and its image in the orbit space X/G.

Lemma 1.27. *Let G be a finite group and X be a Hausdorff space where G acts freely. Given S a slice at X, the map $\pi_{|S} : S \longrightarrow \pi(S)$ is a homeomorphism.*

Proof. It is injective because S is a slice and the action is free. Suppose that x and y are two points in S such that $[x] = \pi_{|S}(x) = \pi_{|S}(y) = [y]$. In consequence, there exists $h \in G$ such that $y = hx$. Since the action is free, $gS = S$ if and only if $g = e$, thus $h = e$ and $x = y$. It is exhaustive by definition, hence, it is bijective. The continuity and openness are induced by the map $\pi : X \longrightarrow X/G$ and the fact that S is open in M and $\pi(S)$ is open in X/G. □

Remark 1.28. *We can consider the inverse map $\pi_{|S}^{-1} : \pi(S) \longrightarrow S$ of the above homeomorphsim. However, we can construct an homeomorphism between $\pi(S)$ and any of the sets V_i of the tube by composing it with θ_g with $g \in G$. In that sense, we can say that we have chosen a particular slice of the tube to construct this homeomorphism, but we could have chosen any other slice by doing a left translation with an element of the group.*

1.3 Fiber bundles

Definition 1.29. *Let X and B be Hausdorff spaces, G a topological group and F a right G-space where G acts effectively. Then, a fiber bundle over B with total space X, fiber F and structure of group K is a map $p : X \longrightarrow B$ together with a collection of charts (they are usually called local trivializations) $\Phi = \{(U, \phi)\}$, where U is an open set of B and $\phi : F \times U \longrightarrow p^{-1}(U)$ is a homeomorphism that fulfills the following conditions:*

- *For every $x \in F \times U$, we have $(p \circ \phi)(x) = \pi(x)$, where $\pi : F \times U \longrightarrow U$ is the projection map. In other words, the following diagram commutes*

$$\begin{array}{ccc} F \times U & \xrightarrow{\phi} & p^{-1}(U) \\ & \searrow^{\pi} & \downarrow p \\ & & U \end{array}$$

- *Each point of B is contained in a chart.*

- *If (U, ϕ) is a chart and V is an open set such that $V \subset U$, then $(V, \phi_{|V})$ is a chart.*

- *Given charts (U, ϕ) and (U, ψ), there exists a continuous map $\theta : U \longrightarrow K$ such that $\psi(f, u) = \phi(f\theta(u), u)$ for all $f \in F$ and $u \in U$. The map θ is called transition function for the charts ϕ and ψ.*

- *The set of charts Φ is maximal among all sets that satisfy the previous conditions.*

Given a point $x \in B$, $p^{-1}(x) \cong F$ is the fiber at x.

Definition 1.30. *A cross section of a fiber bundle is a map $f : B \longrightarrow X$ such that $p \circ f = id_B$.*

Example 1.31.

1. *Given two Hausdorff spaces X and Y, we consider the projection $\pi : X \times Y \longrightarrow X$. Then it is a fiber bundle with base space X, total space $X \times Y$, fiber Y and trivial group structure.*

2. *Möbius strip is an example of a fiber bundle with group structure that is not trivial. In that case, the base space is S^1, the fiber is an open interval I, and the group structure is $\mathbb{Z}_2$.*

There are two interesting cases, depending on the additional structure that the fiber possesses.

Definition 1.32. *A principal G-bundle is a fiber bundle that has G as a fiber and as structure group. It acts by right translation on the fiber, that is, if $g \in G$ is seen as an element of the structure group and $g' \in G$ is seen as an element of the fiber, then g sends g' to gg' as an element of the fiber.*

Definition 1.33. *A vector bundle is a fiber bundle $p : E \longrightarrow B$ with fiber $\mathbb{R}^n$ an structure of group $Gl(n, \mathbb{R})$. Moreover, if $\phi : \mathbb{R}^n \times U \longrightarrow p^{-1}(U)$ is a local trivialization, then $\mathbb{R}^n \times \{b\} \longrightarrow p^{-1}(b)$ is an isomorphism of vector spaces for all $b \in U$.*

Proposition 1.34. *Let G be a finite group and let X be a Hausdorff G-space. If the action is free, then the map $\pi : X \longrightarrow X/G$ is a principal G-bundle.*

Proof. Let $[x]$ be a point of X/G, its inverse image is the orbit $G(x)$, which is homeomorphic to G because the action is free. To prove that $\pi : M \longrightarrow M/G$ is a principal G-bundle we need to construct the local trivializations. Let S be a slice at x, and let T be the tube such that the map $G \times S \longrightarrow T$ is the homeomorphism $(g, s) \rightarrow gs$ (since the action is free, thus $G_x = \{e\}$, the twisted product becomes the Cartesian product). Note that $T = \pi^{-1}(\pi(S))$ and that we have a homeomorphism between any slice of the tube and $\pi(S)$, by lemma 1.27. With this information, we can obtain the following commutative diagram

$$\begin{array}{ccc} G \times \pi(S) & \xrightarrow{\ f\ } & T \\ & \searrow & \downarrow{\scriptstyle \pi} \\ & & \pi(S) \end{array}$$

where f is a homeomorphism produced by composing $id \times \pi_S^{-1} : G \times \pi(S) \longrightarrow G \times S$ and the tube homeomorphism. The pair $(\pi(S), f)$ is the local trivialization we seek, and if we take all subsets of M/G of that form together with these maps they form a principal G-bundle. It is worth mentioning that the we have chosen a particular slice of the tube in order to construct the homeomorphism, we could have chosen any other slice of the form gS where g is an element of G, generating another pair $(\pi(S), \bar{f})$. Nevertheless, it is clear that $\bar{f}(g', [s]) = g'gs = f(g'g, [s])$. Therefore the transition functions are right translations, as a principal G-bundle requires. □

Proposition 1.35. *Let $p : X \longrightarrow B$ be a principal G-bundle and F a right G-space. Then, there exists a G-bundle with base space B, total space $F \times_K X$ and fiber F. The map is $\pi : F \times_K X \longrightarrow B$ and the image of an element $[f, x]$ is $p(x)$. This new fiber bundle is called the associated bundle to the principal G-bundle $p : X \longrightarrow B$.*

Proof. First of all, we start constructing the local trivializations for the new map with the local trivializations of the principal G-bundle, $\phi : G \times U \longrightarrow p^{-1}(U)$.

Consider the chain of compositions $\bar{\phi} : F \times U \longrightarrow (F \times_G G) \times U \longrightarrow F \times_G (G \times U) \longrightarrow F \times_G p^{-1}(U) \longrightarrow \pi^{-1}(U)$. Observe that all maps are homeomorphism as a consequence of the results proved in the twisted product section. The last map is induced by the inclusion $i : p^{-1}(U) \hookrightarrow X$. Finally, we note that $\bar{\phi}(f, u) = [f, \phi(e, u)]$.

Let ϕ and ψ be charts over U of the principal bundle and $\theta : U \longrightarrow G$ the transition function, we will see that it is the transition function for the charts $\bar{\phi}$ and $\bar{\psi}$. Indeed,

$$\bar{\psi}(f, u) = [f, \psi(e, u)] = [f, \phi(\theta(u)e, u)] = [f, \theta(u)\phi(e, u)] = [f\theta(u), \phi(e, u)] = \bar{\phi}(f\theta(u), u).$$

The remaining properties are directly induced by the principal bundle.

□

Proposition 1.36. *Let $p : X \longrightarrow B$ be a fiber bundle with fiber F and group structure G. Given a Hausdorff space B' and a continuous map $f : B' \longrightarrow B$. We define $f^*X = \{(b', x) \in B' \times X : f(b') = p(x)\}$. We have a new fiber bundle, called the pullback bundle, $\pi : f^*X \longrightarrow B$ such that $\pi(b', x) = b'$. It has the same fiber and the same group structure.*

Given a chart (U, ϕ) of the fiber bundle p, we define a new chart (U', ψ), where $U' = f^{-1}(U)$ and $\psi : F \times U' \longrightarrow \pi^{-1}(U')$ such that $\psi(a, x') = (x', \phi(a, f(x')))$. It is a straightforward process to verify that, with these charts, $\pi : f^*X \longrightarrow B$ satisfies all the properties required to be a fiber bundle.

1.4 Homology preliminaries and CW complexes

Firstly, we recall briefly the construction of singular homology. Let X be a topological space, a singular n-simplex is a continuous map $\sigma : \Delta^n \longrightarrow X$ where Δ^n is the standard n-simplex. We define a n-chain to be a finite formal sum of singular n-simplices, $\Sigma_{i=1}^{r} \lambda_i \sigma_i$, where $\lambda_i \in \mathbb{Z}$ for all i and we consider the abelian group formed by all n-chains, $C_n(X)$. A group morphism is defined $\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$ such that the image of a n-simplex is $\partial_n(\sigma) = \Sigma_{i=0}^{n}(-1)^i \sigma_i$, where σ_i is the singular $(n-1)$-simplex obtained by restricting σ to the i-th face of Δ^n (the simplex with vertices $[v_1, ..., v_{i-1}, v_{i+1}, ..., v_n]$).

It is a straightforward computation to prove that $\partial_{n-1}\partial_n = 0$. Thus, the following chain complex is obtained

$$... \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to ...$$

The n-th homology group, denoted by $H_n(X)$, is $H_n(X) = ker(\partial_n)/Im(\partial_{n+1})$. The subgroups $ker(\partial_n)$ and $Im(\partial_{n+1})$ are usually called cycles and boundaries respectively.

Observe that we have chosen coefficients in $\mathbb{Z}$. However, coefficients could be chosen to be in another group (or if we replace $\mathbb{Z}$ by another principal ideal domain ring R, the coefficients can be in a R-module V). The universal coefficient theorem provides a relation between homologies with different coefficients.

Theorem 1.37. *Let X be a topological space, let R be a principal ideal domain and let V be a module over R. Then, for each q, the short sequence*

$$0 \to H_q(X;R) \otimes_R V \to H_q(X;V) \to Tor^R(H_{q-1}(X;R), V) \to 0$$

is a short exact sequence and splits.

Proof. The proof can be found at [6], theorem 2.34 and corollary 2.35 or [4] section 3A. □

Remark 1.38. *We are concerned about two particular cases. In the first one, $R = \mathbb{Z}$ and $V =$, where p is a prime number. In that cases, H_q are abelian groups, and the short exact sequence is*

$$0 \to H_q(X) \otimes \mathbb{Z}_p \to H_q(X;\mathbb{Z}_p) \to Tor(H_{q-1}(X), \mathbb{Z}_p) \to 0.$$

In the second case, R is the field $\mathbb{Z}_p$, where p is a prime number, and V is a vector space over $\mathbb{Z}_p$. Since $\mathbb{Z}_p$ is a field, $Tor^{\mathbb{Z}_p}(H_{q-1}(X;\mathbb{Z}_p), V) = 0$. Therefore, the exactness of the short sequence implies that $H_q(X) \otimes_{\mathbb{Z}_p} V \cong H_q(X;V)$.

The other that shall be discussed is the Künneth theorem, which give a relation between the homology of two topological spaces X and Y and the homology of $X \times Y$.

Theorem 1.39. *Let X and Y be topological spaces and R a principal ideal domain ring. Then, for each q, the short sequence*

$$0 \to \bigoplus_{j=0}^{q} H_j(X;R) \otimes H_{q-j}(Y;R) \to H_q(X \times Y;R) \to \bigoplus_{j=0}^{q-1} Tor^R(H_j(X;R), H_{q-1-j}(Y;R)) \to 0$$

is a short exact sequence.

Proof. The proof can be found at [6] theorem 3.6. □

Remark 1.40. *We are particularly concerned in the case where* $R = \mathbb{Z}_p$*, which is a field. Since the torsion terms are* 0 *if we work in a field, the following isomorphism holds*

$$\bigoplus_{j=0}^{q} H_j(X;\mathbb{Z}_p) \otimes H_{q-j}(Y;\mathbb{Z}_p) \cong H_q(X \times Y;\mathbb{Z}_p).$$

A CW-complex (or cell complex) is a type of topological space that despite being more general then simplicial complexes, it retains good properties for homology computation because of their construction method.

In this section e^n_α will denote a n-cell, that it is homeomorphic to a open disk of dimension n. This type of space is constructed by attaching various cells in a nice way. The construction method is the following

1. We start with a discrete set X^0, called the 0-skeleton. Each point is called 0-cell.

2. We construct inductively the n-skeleton X^n from X^{n-1}. Given n-cells $e^n_\alpha = \mathring{D}^n_\alpha$, we attach them to the $(n-1)$-skeleton via maps $\phi_\alpha : \partial D^n_\alpha \longrightarrow X^{n-1}$, (observe that $\partial D^n_\alpha \cong S^{n-1}$). If we consider the disjoint union $X^{n-1} \coprod_\alpha D^n_\alpha$, the n-skeleton is the quotient space under the identification $x \sim \phi_\alpha(x)$ for all $x \in \partial D^n_\alpha$. Then, $X^n = X^{n-1} \coprod_\alpha e^n_\alpha$, where e^n_α are the n-cells attached.

3. We can stop the process at a finite stage, then $X^n = X$ and n would be the dimension of the CW-complex. If we keep attaching cells indefinitely, we define the space X to be the union of all n-skeletons, $X = \bigcup_{n\geq 0} X^n$, with the final topology induced by the inclusions. This means that a set $A \subset X$ is open (or closed) if and only if $A \cap X^n$ is open (or closed) in X^n for each n.

Example 1.41.

- *Given a 0-cell and a n-cell we can construct the sphere* S^n *by collapsing the boundary of the n-cell to the 0-cell. Alternatively, we can construct* S^n *starting with a 0-skeleton that consists of two points. Having constructed the (n-1)-skeleton* $X^{n-1} = S^{n-1}$*, we consider it to be the equator of the n-dimensional sphere and we attach two n-cells which are the two hemispheres of the sphere* S^n*. We can consider the sphere* $S^\infty = \bigcup_{n\geq 0} S^n$*.*

- *We can construct a lens space* L^n_m *with one cell for each dimension from 0 to 2n-1 with the appropriate attaching maps. It can be constructed inductively. Given the sphere* $S^{2n-1} \subset \mathbb{C}^n$*, we consider the sphere* S^{2n-3} *described by the* $n-1$ *first coordinates of* S^{2n-1} *and* S^1 *described by the last coordinate of* S^{2n-1}*. We subdivide* S^1 *by taking points of the form* $(0, ..., 0, e^{2\pi i j/m})$ *as vertices, where* $j = 1, .., m$*. We proceed by joining each of these vertices with* S^{2n-3} *via arcs of great circles, obtaining* m $(2n-2)$*-cells. In a similar way, we join the fragments of the subdivided* S^1 *using arcs of great circles, obtaining* m $(2n-1)$*-cells. The group action in* S^{2n-1} *whose orbit space is the lens space maps each constructed* $(2n-1)$*-cell and* $(2n-2)$*-cell to another one of the constructed cells. Therefore, we can identify every* $(2n-1)$*-cell and every* $(2n-2)$*-cell via the group action. Note that the group acting on* S^{2n-3} *produces a lens space* L^{n-1}_m*.*

The following propositions tell us some remarkable facts about the homology of CW-complexes and a way to compute their homology. Because it is not or main goal in the project and it is a standard topic in the majority of algebraic topology texts, we will omit their proof (for example, see [4] for a good exposition about this topic).

Proposition 1.42. *Given a CW-complex X, then:*

- $H_k(X^n, X^{n-1}) = 0$ *if* $k \neq n$, *and when* $k = n$, *it is a free abelian group with basis in one to one correspondence with the n-cells of X.*
- *The homology* $H_k(X^n) = 0$ *if* $k > n$. *In particular, in the case that X is finite dimensional with dimension n, then* $H_k(X) = 0$ *for* $k > n$.
- *The inclusion* $i : X^n \hookrightarrow X$, *induces an isomorphism in the homology groups* $i^* : H_k(X^n) \longrightarrow H_k(X)$ *for* $k < n$.

We will call the group $H_n(X^n, X^{n-1}) = Cell_n(X)$. Observe that given a $(n-1)$-cell e_β^{n-1}, we can take into consideration the quotient map $X^{n-1} \longrightarrow S_\beta^{n-1}$ that results from collapsing $X^{n-1} - e_\beta^{n-1}$ to a point. Then we can compose it with the map that attaches the n-cell e_α^n to the $(n-1)$-skeleton, obtaining a map between spheres of the same dimension, so we can compute its degree. We will denote it by $d_{\alpha\beta}$. With that information, we can build a morphism $d_n : Cell_n(X) \longrightarrow Cell_{n-1}(X)$ such that $d_n(e_\alpha^n) = \Sigma_\beta d_{\alpha\beta} e_\beta^{n-1}$.

Proposition 1.43. *We have the following properties:*

- $(Cell_n(X), d_n)$ *is a chain complex, therefore we can consider its homology* $H_n^{CW}(X)$.
- $H_n^{CW}(X) \cong H_n(X)$.

Example 1.44. *If we consider the two previous examples, we have:*

- *We can compute the homology of the infinite dimensional sphere. Using the isomorphism induced by the inclusion and that* $X^n = S^n$, *we have that* $H_k(S^\infty) = 0$ *for k>0 and* $H_0(S^\infty) = \mathbb{Z}$. *Indeed,* S^∞ *is contractible, a fact that will be proved in chapter 4.*
- *We can compute the homology of the lens space* L_m^n *via the cell complex structure previously explained. Observe that* $Cell_i(L_m^n) = \mathbb{Z}$ *for* $i = 0, ..., 2n-1$ *and* 0 *otherwise. Because of the inductive process used to construct the lens space, we need only to compute the maps* d_{2n-1} *and* d_{2n-2} *and proceed inductively. Without providing the details, the map* d_{2n-1} *is* 0, *since it is the result of a reflection fixing* S^{2n-3} *and a rotation. The map* d_{2n-2} *is a multiplication by m, since it is induced by the orbit map* $S^{2n-3} \longrightarrow L_m^{n-1}$. *Therefore, the chain complex* $(Cell_n(X), d_n)$ *has the following form*

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} ... \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

of length $2n-1$. *If we use it to compute the homology of the lens space, we obtain that* $H_k(L_m^n)$ *is*

$$\begin{cases} \mathbb{Z} \quad if \quad k = 0, 2n-1 \\ \mathbb{Z}_m \quad if \quad 0 < k < 2n-1 \quad \text{and } k \text{ is odd} \\ 0 \quad otherwise \end{cases}$$

We can compute the homology of the infinite dimensional lens space, defined in the same way as the case of the sphere, using the last property stated in 1.42 and the homology of the finite dimensional case. If we do the computation, we obtain that $H_k(L_p^\infty)$ *is*

$$\begin{cases} \mathbb{Z} & if \quad k=0 \\ \mathbb{Z}_m & \text{if } k \text{ is odd} \\ 0 & otherwise \end{cases}$$

We will be interested in subsequent chapters in the homology in coefficients over $\mathbb{Z}_p$*, where* p *is a prime, instead of* $\mathbb{Z}$ *of the infinite dimensional lens space* L_p^∞*. In that case, using the universal coefficient theorem, we obtain that*

$$0 \to H_k(L_p^\infty) \otimes \mathbb{Z}_p \to H_k(L_p^\infty; \mathbb{Z}_p) \to Tor(H_{k-1}(L_p^\infty), \mathbb{Z}_p) \to 0.$$

It is known that $Tor(0, \mathbb{Z}_p) = 0$ *and* $Tor(\mathbb{Z}_p, \mathbb{Z}_p) = \mathbb{Z}_p$*. If we use these two equalities in the exact sequence, distinguishing if* k *is odd or even, we reach the conclusion that* $H_k(L_p^\infty) = \mathbb{Z}_p$ *for all* $k \geq 0$*.*

Chapter 2

Smooth actions

The objective of this second chapter is to give a succinct exposition about smooth group actions. Topological actions can be extremely complex. For instance, R. H. Bing proved in 1952 that there exists an involution (a $\mathbb{Z}_2$ action) on the 3-sphere such the fixed point set is an Alexander horned sphere. However, if we restrict ourselves to smooth manifolds and actions, we can use the additional structures in order to avoid this type of cases and simplify some arguments. Moreover, we have at our disposal the powerful tools of differential geometry. In the first section, some basic facts of Riemannian geometry and frame bundles are briefly explained, while in the second sections we utilize them to deduce some useful statements regarding group actions on smooth manifolds.

The bibliography about smooth manifold is extensive, this chapter is basically based on [7] and [9]. The development of frame bundles is based on the notes [10] and the lectures of the subject of differential geometry given at UB during this course 2018-2019.

2.1 Riemannian manifolds

Let M be a smooth manifold, for each point p of M we can construct its tangent space T_pM, which is the vector space that consists of all the tangent vectors of smooth curves γ such that $\gamma(0) = p$, or the space of all derivations of the algebra of smooth real valued functions at p. Then, we can define $TM = \bigcup_{p\in M} T_pM$, which is called the tangent bundle, and the projection $\pi : TM \longrightarrow M$ such that $\pi(p,v) = p$, where p is a point of M and v is a vector of T_pM, is a vector bundle with fiber $\mathbb{R}^m$ and group structure $Gl(m,\mathbb{R}) = \{A \in \mathcal{M}_{m\times m}(\mathbb{R}) : det(A) \neq 0\}$, that is the group of invertible matrices with product of matrices as operation.

Definition 2.1. *Let M be a smooth manifold. A Riemannian metric on the manifold is a tensor field g of type (2,0), (which means that for every $p \in M$ we have a map $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ that is bilinear and smooth) that is symmetric and positive-definite. The pair (M,g) is called a Riemannian manifold.*

A first example of Riemannian metric is the usual inner product on $\mathbb{R}^n$, that we will denote by $\langle\cdot,\cdot\rangle$. The following propositions shows that any manifold can be equipped with a Riemannian metric.

Proposition 2.2. *Let M be a smooth manifold, then it has a Riemannian metric*

Proof. Given an open cover $\{U_\alpha\}$ (we can choose it so (U_α, ϕ_α) is a chart for each α), then it is a well known fact that there exists a partition of the unity $\{f_i\}_{i\in\mathbb{N}}$ subordinated to $\{U_\alpha\}$, so $sup(f_i) \subset U_{\alpha(i)}$ for all $i \in \mathbb{N}$ (see [8], theorem 1.11).

We can construct a Riemannian metric g_α in each U_α, because it is diffeomorphic to an open set of $\mathbb{R}^n$ with diffeomorphism ϕ_α. If $p \in M$ is in U_α, then $g_{\alpha,p}(u,v) = \langle d_p\phi_\alpha(u), d_p\phi_\alpha(v)\rangle$. Then $g = \sum_{i\in\mathbb{N}} f_i g_{\alpha(i)}$ is a Riemannian metric on M.

□

Definition 2.3. *Given two Riemannian manifolds (M,g) and (N,h), an isometry is a diffeomorphism $f : M \longrightarrow N$ such that for every point $p \in M$ and every pair of vectors $v, w \in T_pM$, the relation $g_p(v,w) = h_{f(p)}(d_pf(v), d_pf(w))$ is satisfied.*

The next concept we need to introduce is the exponential map, which requires the use of geodesics. To define rigorously geodesics, we should first define the concepts of covariant derivative and affine connection ∇, and expose their compatibility with a Riemannian metric via the Levi-Civita theorem. Since it is not one of our main objective and it is a standard topic on differential geometry books, we will omit the details. (One can consult [9] for a exposition about those subjects).

Definition 2.4. *Let $\gamma : J \longrightarrow M$ be a parametrised curve in a Riemannian manifold (M,g). Then γ is a geodesic if $\nabla_{\gamma'}\gamma' = 0$ for every point in the curve.*

Given a point $p \in M$ and $v \in T_pM$, it can be proved that there exists a geodesic curve $\gamma : (-\epsilon, \epsilon) \longrightarrow M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$, and this geodesic is unique. This fact is a direct consequence of the existence and uniqueness of solutions of an ODE. It is possible to generalise the concept to a geodesic flow, obtaining the following proposition.

Proposition 2.5. *Given a point $p \in M$, we have an open neighbourhood V of p, an $\epsilon > 0$ and a smooth map $\gamma : (-2,2) \times \Omega \longrightarrow M$, where $\Omega = \{(q,v) \in TV : |v| < \epsilon\}$, such that the curve $t \to \gamma(t,q,v)$ is the only geodesic which fulfills the conditions $\gamma(0,q,v) = q$ and $\gamma'(0,q,v) = v$.*

Definition 2.6. *Let p be a point in M and U an open set TM like the one from the previous proposition. Then, the map $exp : U \longrightarrow M$ given by $exp(q,v) = \gamma(1,q,v)$ is called the exponential map on U.*

It is clear that exp is a smooth map. We can choose a point $p \in M$ and obtain $exp_p : B_\epsilon(0) \subset T_pM \longrightarrow M$ such that $exp_p(v) = exp(p,v)$. Clearly, it is a smooth map. An important property of the exponential map is presented in the next proposition.

Proposition 2.7. *Given a point $p \in M$, then there exists an $\epsilon > 0$ such that $exp_p : B_\epsilon(0) \longrightarrow M$ is a diffeomorphism of $B_\epsilon(0)$ with an open set in M.*

Proof. We will calculate the differential of exp_p at 0 in order to use the inverse function theorem. Given $v \in T_pM$, we have

$$d_0exp_p(v) = \frac{d}{dt}(exp_p(tv))|_{t=0} = \frac{d}{dt}(\gamma(1,p,tv))|_{t=0} = \frac{d}{dt}(\gamma(t,q,v))|_{t=0} = v.$$

This implies that $d_0 exp_p = id_{T_pM}$, so we can apply the inverse function theorem to find the desired result.

□

The last objects we shall study in this section are the frame bundles. Let M be a smooth manifold, we consider the tangent bundle $\pi : TM \longrightarrow M$ which is a vector bundle. Recall that a local trivialization $\phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^m$ restricted to $\pi^{-1}(p)$ for any $p \in U$ gives an isomorphism of vector spaces $\phi_p : \pi^{-1}(p) \longrightarrow \{p\} \times \mathbb{R}^m \cong \mathbb{R}^m$.

Let p be a point in M, we consider the set of basis of T_pM, $F(T_pM) = \{(v_1, ..., v_m) : v_1, ..., v_m \in T_pM \text{ is a basis of } T_pM\}$. We can construct an isomorphism of vector spaces between $\mathbb{R}^m$ and T_pM which maps the canonical basis $\{e_1, ..., e_m\}$ of $\mathbb{R}^m$ to a basis $\{v_1, ..., v_m\}$ of T_pM. Since we can obtain any other basis of T_pM transforming $\{v_1, ..., v_m\}$ with an automorphism of T_pM, the set $F(T_pM) = Iso(\mathbb{R}^m, T_pM) = \{f : \mathbb{R}^m \longrightarrow T_pM : f \text{ is an isomorphism}\}$. Each element of $F(T_pM)$ is called a frame of p.

Definition 2.8. *Let M be a smooth manifold, the set*

$$F(M) = \{(p; v_1, ..., v_m) : p \in M,\ v_1, ..., v_m \in T_pM \text{ form a basis of } T_pM\} = \coprod_{p \in M} Iso(\mathbb{R}^m, T_pM)$$

is called the frame bundle of M.

Our objective is to prove that $F(M)$ is a smooth manifold and the map $\pi' : F(M) \longrightarrow M$ such that $\pi'(p; v_1, ..., v_m) = p$ is principal fiber bundle with group $Gl(m, \mathbb{R}) = \{A \in \mathcal{M}_{m \times m}(\mathbb{R}) : det(A) \neq 0\}$.

Observe that $Iso(\mathbb{R}^m, T_pM) \cong Gl(m, \mathbb{R})$.

Lemma 2.9. *The group $Gl(m; \mathbb{R})$ is a smooth manifold.*

Proof. The group $Gl(m; \mathbb{R})$ is an open set of $\mathcal{M}_{m \times m}(\mathbb{R}) \cong R^{m^2}$ since it is $det^{-1}(\mathbb{R} \setminus \{0\})$ and $det : \mathcal{M}_{m \times m}(\mathbb{R}) \longrightarrow \mathbb{R}$ is a continuous map. Hence, it is a smooth manifold. □

Proposition 2.10. *Let M be a smooth manifold. The set $F(M)$ is a smooth manifold and the map $\pi' : F(M) \longrightarrow M$ such that $\pi'(p; v_1, ..., v_m) = p$ is principal fiber bundle with group $Gl(m, \mathbb{R})$.*

Proof. Let U be an open set of M such that we have the local trivialization of TM, $\phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^m$. Then, the map $\phi' : \pi'^{-1}(U) \longrightarrow U \times Gl(m, \mathbb{R})$ such that $\phi'(p, x) = (p, \phi_p \circ x)$, where $x \in Iso(\mathbb{R}^m, T_pM)$, is bijective (observe that $\phi_p \circ x : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is an isomorphism). Moreover, the following diagram commutes

$$\begin{array}{ccc} \pi'^{-1}(U) & \xrightarrow{\phi'} & U \times Gl(m, \mathbb{R}) \\ \downarrow{\scriptstyle \pi'} & \swarrow & \\ U & & \end{array}$$

The fact that ϕ' is bijective implies that a topology and a differential structure can be given to every $\pi'^{-1}(U)$ induced by the topology and differential structure of $U \times Gl(m, \mathbb{R})$ such that ϕ' becomes a diffeomorphism.

Therefore, the union of all the sets $\pi'^{-1}(U)$ obtained from the local trivialization of $\pi : TM \longrightarrow M$ is $F(M)$. We equip $F(V) = \bigcup \pi'^{-1}(U)$ with the topology where a set W is open if and only if $W \cap \pi'^{-1}(U)$ is open in $\pi'(U)$ for all U. The induced differential structure in each $\pi'^{-1}(U)$ induces a differential structure in $F(M)$. The above diagram also provides the local trivializations for $\pi' : F(M) \longrightarrow M$. The remaining properties of fiber bundles are induced by the properties of the vector bundle $\pi : TM \longrightarrow M$ since its structural group is $Gl(m, \mathbb{R})$. □

Definition 2.11. *Let (M, g) be a Riemannian metric, the set $F_g(M) = \{(p; v_1, ..., v_m) : p \in M \text{ and } v_1, ..., v_m \in T_pM \text{ forms an orthogonal basis of } T_pM\} = \coprod_{p \in M} Isom(\mathbb{R}^m, T_pM)$, where $Isom(\mathbb{R}^m, T_pM)$ denotes the isometries of $\mathbb{R}^m$ to T_pM, is called the orthogonal frame bundle.*

As above, our objective is to show that $F_g(M)$ is a submanifold of $F(M)$.

Lemma 2.12. *The orthogonal group $O(m, \mathbb{R}) = \{A \in \mathcal{M}_{m\times m}(\mathbb{R}) : A^tA = Id\}$ is a submanifold of $Gl(m, \mathbb{R})$.*

Proof. In order to prove the assertion of the lemma, we will use the typical result in differential geometry, the preimage theorem, which claims that if M and N are smooth manifolds and $f : M \longrightarrow N$ is a smooth map, then $f^{-1}(y) \neq \emptyset$ is a submanifold of M if $y \in N$ is a regular value (which means that for all $x \in f^{-1}(y)$ the map $d_xf : T_xM \longrightarrow T_yN$ is exhaustive).

In our case, the manifold $M = \mathcal{M}_{m\times m}(\mathbb{R})$ and the manifold $N = S\mathcal{M}_{m\times m}(\mathbb{R}) = \{A \in \mathcal{M}_{m\times m}(\mathbb{R}) : A \text{ is a symmetric matrix}\}$ and the map f fulfills that $f(A) = A^tA$. If the identity matrix Id was a regular value, the claim would be proved because $f^{-1}(Id) = O(m, \mathbb{R})$.

Firstly, notice that $f^{-1}(Id) \neq \emptyset$ since $Id \in f^{-1}(Id)$. The next step is to compute the differential for a point in $f^{-1}(Id)$. Observe that we have isomorphisms $T_p\mathcal{M}_{m\times m}(\mathbb{R}) \cong \mathcal{M}_{m\times m}(\mathbb{R})$ and $T_qS\mathcal{M}_{m\times m}(\mathbb{R}) \cong S\mathcal{M}_{m\times m}(\mathbb{R})$. Let A be a matrix in $f^{-1}(Id)$ and let $B \in \mathbb{R}_{m\times m}$, consider the smooth curve $\gamma(t) = Bt + A$. Since $\gamma(0) = A$ and $\gamma'(0) = B$, we obtain that $d_Af(B) = (f \circ \gamma)'(0)$. We have that $(f \circ \gamma)(t) = (Bt + A)^t(Bt + a) = Id + (B^tA + A^tB)t + B^tBt^2$. This implies that $(f \circ \gamma)'(0) = B^tA + A^tB$. Let C be a matrix of $S\mathcal{M}_{m\times m}(\mathbb{R})$, then $d_Af(\frac{1}{2}AC) = \frac{1}{2}A^tAC + \frac{1}{2}C^tA^tA = \frac{1}{2}C + \frac{1}{2}C^t = C$. Thus, d_Af is exhaustive for all $A \in f^{-1}(Id)$. In consequence it is a submanifold of $\mathcal{M}_{m\times m}(\mathbb{R})$, and since $O(m; \mathbb{R}) \subset Gl(m, \mathbb{R})$, which is an open submanifold of $\mathcal{M}_{m\times m}(\mathbb{R})$, we obtain that $O(m, \mathbb{R})$ is a submanifold of $Gl(m, \mathbb{R})$. □

We want to repeat the analogous process of the construction of $F(M)$ in the case of $F_g(M)$. However, the maps $\phi_p : \pi^{-1}(p) \longrightarrow \mathbb{R}^m$ should not be necessarily an isometry.

To solve this problem, given a local trivialization $\phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^m$ of the vector bundle $\pi : TM \longrightarrow M$, we construct sections $s_i : U \longrightarrow \pi^{-1}(U)$ such that $s_i(p) = \phi_p^{-1}(e_i)$, $i = 1, ..., m$. For all $p \in U$, the collection of vectors $\{s_1(p), ..., s_m(p)\}$ are a basis of T_pM. Using the Gram-Schmidt algorithm, we can obtain new section $\sigma_i : U \longrightarrow \pi^{-1}(U)$ such that $\{\sigma_1(p), ..., \sigma_m(p)\}$ is an orthogonal basis of T_pM. In that way, we can construct the following local trivialization $\psi : U \times \mathbb{R}^m \longrightarrow \pi^{-1}(U)$ which satisfies that $\psi(p, \sum_{i=1}^m \lambda_ie_i) = \sum_{i=1}^m \lambda_i\sigma_i(p)$. This new map fulfills that $\psi_p : \mathbb{R}^m \longrightarrow T_pM$ is an isometry.

The same process used in proposition 2.1 can be employed replacing $Gl(m,\mathbb{R})$ by $O(m,\mathbb{R})$ (notice that $Isom(\mathbb{R}^m, T_pM) \cong O(m,\mathbb{R})$) to show that $F_g(M)$ is a principal bundle with structural group $O(m,\mathbb{R}^m)$.

Proposition 2.13. *Let M be a smooth manifold, then the frame bundle $F(M)$ is a smooth manifold. If M is a Riemannian manifold with metric g, then $F_g(M)$ is a submanifold of $F(M)$.*

The fact that it is a submanifold is a consequence of $O(m,\mathbb{R})$ being a submanifold of $Gl(m,\mathbb{R})$.

Proposition 2.14. *Let M be a Riemannian manifold, and let g and g' be two Riemannian metrics. Then, the orthogonal frame bundles $F_g(M)$ and $F_{g'}(M)$ are diffeomorphic.*

Proof. We define the map $f : F_g(M) \longrightarrow F_{g'}(M)$ such that $f(p; v_1, ..., v_m) = (p; v'_1, ..., v'_m)$, where $v'_1, ..., v'_m$ are obtained by using the Gram-Schmidt algorithm on $v_1, ..., v_m$ in order to create a g'-orthogonal basis. Our goal is to prove that f is a diffeomorphism.

Recall that Gram-Schmidt algorithm is defined recursively where $v'_1 = v_1$ and $v'_i = v_i - \sum_{j=1}^{i-1} \frac{g'_p(v_i, v'_j)}{g'_p(v_j, v_j)} v'_j$. If we rewrite these equations without recursivity, we obtain that $v'_i = v_i - \sum_{j=1}^{i-1} f_{ji}(g'_p(v_k, v_{k'}))v_j$, where f_{ji} are functions that only depend on $g'_p(v_k, v_{k'})$ for $k, k' = 1, ..., i-1$. We can arrange these information in a triangular matrix

$$A(p, v_1, ..., v_m) = \begin{bmatrix} 1 & f_{12} & \dots & f_{1m} \\ 0 & 1 & \dots & f_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

This matrix sends each basis of T_pM to the g'-orthogonal basis obtained by performing the Gram-Schmidt algorithm.

Let $\{u'_1, ..., u'_m\}$ be a g'-orthogonal basis of T_pM. Given a g'-orthogonal basis $\{v'_1, ..., v'_m\}$ obtained by using Gram-Schmidt algorithm in a g-orthogonal basis $\{v_1, ..., v_m\}$, there exists a matrix $B \in O(m,\mathbb{R})$ such that $u'_i = Bv'_i$. We claim that if the Gram-Schmidt algorithm is used on the basis $\{u_1, ..., u_m\}$ such that $u_i = Bv_i$, we recover the basis $\{u'_1, ..., u'_m\}$. In order to prove this assertion, we need to compute $BA(p, v_1, ..., v_m)B^{-1}$. For one element of the basis u_i,

$$\begin{aligned} BA(p, v_1, ..., v_m)B^{-1}u_i &= BA(p, v_1, ..., v_m)v_i \\ &= B(v_i - \sum_{j=1}^{i-1} f_{ji}(g'_p(v_k, v_{k'}))v_j) \\ &= u_i - \sum_{j=1}^{i-1} f_{ji}(g'_p(v_k, v_{k'}))u_j \\ &= u_i - \sum_{j=1}^{i-1} f_{ji}(g'_p(u_k, u_{k'}))u_j \\ &= u'_i, \end{aligned}$$

where penultimate equality is a consequence of $g'_p(u_k, u_{k'}) = g'_p(Bv_k, Bv_{k'}) = g'_p(v_k, v_{k'})$. The above observation implies the bijectivity of the map $f : F_g(M) \longrightarrow F_{g'}(M)$. Given a g'-orthogonal basis $\{u'_1, ..., u'_m\}$, the basis $\{u_1, ..., u_m\}$ such that $(BA(p, v_1, ..., v_m)B^{-1})^{-1}u'_i = u_i$ satisfies that $f(p, u_1, ..., u_m) = (p; u'_1, ..., u'_m)$. If $f(p, v_1, ..., v_m) = f(q, u_1, ..., u_m)$, then $p = q$ and the matrix that connects the both g'-orthogonal basis is the identity matrix. Since, all the matrices are invertible, the matrix that change the basis $\{v_1, ..., v_m\}$ to $\{u_1, ..., u_m\}$ is the identity, therefore both basis are the same.

Local trivialization will be used in order to prove that $f : F_g(M) \longrightarrow F_{g'}(M)$ is a diffeomorphism. Let $\pi : F_g(M) \longrightarrow M$ and $\pi' : F_{g'}(M) \longrightarrow M$ be the principal fiber bundles and let $\psi : U \times O(m, \mathbb{R}) \longrightarrow \pi^{-1}(U)$ and $\psi : U \times O(m, \mathbb{R}) \longrightarrow \pi'^{-1}(U)$ local trivializations over an open set U. Observe that $f(\pi^{-1}(U)) = \pi'^{-1}(U)$. The following commutative diagram holds

$$\begin{array}{ccc} U \times O(m,\mathbb{R}) & \xrightarrow{\psi'^{-1}\circ f \circ \psi} & U \times O(m,\mathbb{R}) \\ \Big\downarrow \psi & & \Big\downarrow \psi' \\ \pi^{-1}(U) & \xrightarrow{\quad f \quad} & \pi'^{-1}(U) \end{array}$$

Since ψ and ψ' are diffeomorphisms by construction, it suffices to show that $\psi'^{-1} \circ f \circ \psi : U \times O(m, \mathbb{R}) \longrightarrow U \times O(m, \mathbb{R})$ is a diffeomorphism. Suppose that $\psi(p, Id) = (p; v_1, ..., v_m)$ and that $\psi(p, Id) = (p; v'_1, ..., v'_m)$. Let $C(p) \in Gl(m, \mathbb{R})$ be the matrix such that $C(p)v_i = v'_i$ for all i. Observe that $\psi(p, B) = (p; u_1, ..., u_m)$ such that $Bv_i = u_i$ for all i. The diagram below shows schematically the changes of basis

$$\begin{array}{ccc} \{v_1, ..., v_m\} & \xrightarrow{C(p)} & \{v'_1, ..., v'_m\} \\ \Big\downarrow B & & \Big\downarrow B' \\ \{u_1, ..., u_m\} & \xrightarrow{A(p,u_1,...,u_m)} & \{u'_1, ..., u'_m\} \end{array}$$

Therefore $\psi'^{-1} \circ f \circ \psi(p, B) = (p, B') = (p, A(p, u_1, ..., u_m)BC(p))$. The fact that the metric is a tensor field and that f_{ji} are smooth implies that the entries of $A(p, u_1, ..., u_m)$ are smooth functions. The entries of $C(p)$ are smooth by construction. Thus, $\psi'^{-1} \circ f \circ \psi(p, B)$ is a smooth map. Since all the matrices are invertible we can construct the inverse of the map, which is smooth. In consequence, $(\psi'^{-1} \circ f \circ \psi)(p, B)$ is a diffeomorphism. □

Remark 2.15. *If M is compact, then $F_g(M)$ is also compact since $O(m, \mathbb{R})$ is compact.*

2.2 Smooth actions

Definition 2.16. *Let G be a smooth manifold which has a group structure. It is called a Lie group if the operations of a group $\sigma : G \times G \longrightarrow G$ such that $\sigma(g, h) = gh$ and $\iota : G \longrightarrow G$ such that $\iota(g) = g^{-1}$ are smooth.*

Remark 2.17. *Observe that a Lie group is a topological group, and we can consider its action on topological spaces.*

Definition 2.18. *Let (G, M, Θ) be a topological transformation group, where M is a smooth manifold and G is a Lie group. The action is said to be smooth if the map $\Theta : G \times M \longrightarrow M$ is smooth.*

Observe that if the action is smooth, for all $h \in G$, the maps $\theta_h : M \longrightarrow M$ are smooth. Therefore, they are diffeomorphisms. In fact, if each $\theta_h : M \longrightarrow M$ is smooth, then the action is smooth. The prove of this statement in the general case is difficult. However if we consider the case of G being a finite group, the proof becomes straightforward.

Lemma 2.19. *Let G be a finite group and let $\Theta : G \times M \longrightarrow M$ an action. If the maps $\theta_g : M \longrightarrow M$ are smooth for all $h \in G$, then the action is smooth.*

Proof. If the group G is finite (and it is equipped with the discrete topology), then for every $h \in G$ the set $\{h\} \times M$ is an open set of $G \times M$. Thus, $G \times M = \bigcup_{h \in G}(\{h\} \times M)$ and two of these sets, $\{h\} \times M$ and $\{h'\} \times M$ are disjoint if $h \neq h'$. The restriction of the action on each one of these sets is $\Theta_{|\{h\}\times M} = \theta_h$, which is smooth by hypothesis. The smoothness of each restriction and the fact that the sets are open and disjoint implies that the map $\Theta : G \times M \longrightarrow M$ is smooth and therefore, the action is smooth too. □

We want to take advantage of Riemannian metrics to study the group action on the manifold. The first step is to construct a suitable metric.

Definition 2.20. *Let (M, g) be a Riemannian manifold and G a Lie group that acts smoothly on M, the metric is said to be invariant if*

$$g_{h(p)}(h(v), h(w)) = g_p(v, w)$$

for all $p \in M$, $v, w \in T_pM$ and for all $h \in G$, where $h(v) = d_p\theta_h(v)$.

Proposition 2.21. *If (M,g) is a Riemannian manifold and G is a compact Lie group, then there exists an invariant Riemannian metric.*

Proof. We are basically concerned about actions of finite groups, so we will assume that the group is finite. Let $\sharp G$ denote the cardinality of the group.

Given the metric g, we define a new one by putting $\bar{g} = \frac{1}{\sharp G}\sum_{h\in G} h^*g$, where if we fix a point $p \in M$, $h^*g(v, w) = g_{h(p)}(h(v), h(w))$ for all $v, w \in T_pM$. It is clear that if f is an element of G, then

$$\bar{g}_{f(p)}(f(v), f(w)) = \frac{1}{\sharp G}\sum_{h\in G} h^* g_{f(p)}(f(v), f(w)) = \frac{1}{\sharp G}\sum_{h\in G}(h \cdot f)^* g_p(v, w) = \bar{g}(v, w)$$

for all $v, w \in T_pM$. The key fact in this chain of equalities is that the map such that $h \to h \cdot f$ is an isomorphism of groups.

The case where the group G is not finite has exactly the same proof, but changing the finite sum for the Haar integral (See [2], chapter VI, section 2).

□

In this case, each map $\theta_g : M \longrightarrow M$ is an isometry, and we shall say that G acts by isometries on M. Using this new metric, we obtain an the following lemma.

Lemma 2.22. *If we have a smooth action on a manifold with an invariant metric, the exponential map is equivariant.*

Proof. We need to specify how acts G in TM. Given $h \in G$ and $(p, v) \in TM$, we define $h(p, v) = (hp, d_p\theta_h(v))$. It is clearly an action because $\theta_h : M \longrightarrow M$ are diffeomorphisms for all $h \in G$.

The crucial fact to prove the lemma is that isometries preserve geodesics, in the sense that if $\gamma : J \longrightarrow M$ is a geodesic and $\phi : M \longrightarrow M$ is an isometry, then $\phi \circ \gamma : J \longrightarrow M$ is a geodesic. Given $p \in M$ and $v \in T_pM$ such that $exp(p, v)$ is well defined, we consider the geodesic used in the exponential map definition $\gamma(t, p, v)$, which is a geodesic $\gamma : J \longrightarrow M$ which fulfills that $\gamma(0, p, v) = p$ and $\gamma'(0, p, v) = v$. For all $h \in G$, the curve $\theta_h \circ \gamma(t, p, v) = h(\gamma(t, p, v)) = \tilde{\gamma}(t)$ is a geodesic such that $\tilde{\gamma}(0) = hp$ and $\tilde{\gamma}'(0) = d_p\theta_h(v)$. Therefore, using the uniqueness of geodesics, we conclude that $\tilde{\gamma}(t) = \gamma(t, hp, d_p\theta_h(v))$, and consequently $h(exp(p, v)) = h(\gamma(1, p, v)) = \gamma(1, hp, d_p\theta_h(v)) = exp(h(p, v))$.

□

Having defined the concept of smooth action, we are interested in the structure of some of the spaces that appear in the context of group actions, such as M^G or M/G.

Proposition 2.23. *If we have a smooth action of a Lie group G on M, the set of fixed points M^G is a submanifold.*

Proof. Let p be a point of M^G, we consider the diffeomorphism induced by the exponential map on that point $exp_p : B_\epsilon(0) \longrightarrow V \subset M$.

Given a fixed point p, we can induce an action of G on T_pM. Given $v \in T_pM$, we define $gv = d\theta_g(v)$. It is clearly well defined because $\theta_g : M \longrightarrow M$ are diffeomorphisms.

Let q be a point in $M^G \cap V$, we consider the vector $v_q = exp_p^{-1}(q)$, which is unique since a diffeomorphism is bijective. Therefore, we can define the subspace of T_pM generated by all these vectors, $F = \langle v_q \rangle_{q \in M^G \cap V}$, which it is a submanifold of T_pM. Because $d_p\theta_g$ is linear for every $g \in G$, every vector in F is a fixed point of the action of G on T_pM, and since the exponential map is equivariant and a diffeomorphism if we restrict it on $B_\epsilon(0)$, we obtain that $exp_p(B_\epsilon(0) \cap F) = M^G \cap V$, which is a submanifold of V. In consequence, M^G is a submanifold of M.

□

If we focus on the space M/G, some glaring problems appear, the most prominent one is that M/G is not usually a topological manifold. If we restrict ourselves to finite groups, we can resolve some of these problems, for instance, we have already seen that if the group G is finite, the orbit space M/G is Hausdorff. However, we need to add an additional property to ensure that M/G is an smooth manifold, the action shall be free.

Proposition 2.24. *Let M be a smooth manifold and let G be a finite group acting smoothly in M. If the action of G is free, then M/G is a smooth manifold.*

Proof. Firstly, we shall prove that M/G is a topological manifold. We need to check that it satisfies the three properties that every topological manifold verifies. First of all, we have already seen in 1.14 that M/G is Hausdorff. Let $\{B_i\}_{i \in \mathbb{N}}$ be a numerable basis of M, then $\{\pi(B_i)\}_{i \in \mathbb{N}}$ is a numerable basis of M/G. Let $[x]$ be point M/G and V an open set such that $[x] \in V$, then $\pi^{-1}(V)$ is an open set of M such that $x \in M$. Therefore, $x \in B_i \subset \pi^{-1}(V)$ for some i. This implies that $[x] \in \pi(B_i) \subset V$, thus $\{\pi(B_i)\}_{i\in\mathbb{N}}$ is a numerable basis of the topology on M/G. Let $[x]$ be in M/G, then $\pi^{-1}([x]) = Gx$ is the orbit of x. Let S be a slice at x such that it is inside a chart (U, ϕ). Recall that $\pi_{|S} : S \longrightarrow \pi(S)$ is a homeomorphism. Thus, the map $\phi_{|S} \circ \pi_{|S}^{-1} : \pi(S) \longrightarrow \phi(S)$ is a homeomorphism between an open set of M/G and an open set of an Euclidean space, therefore M/G is locally Euclidean. We have just proved that M/G is a topological manifold.

Moreover, we shall see that the pair $(\pi(S), \phi_{|S} \circ \pi_{|S}^{-1})$ is a chart and that the collection of all these pairs where S is a slice creates a differential structure on M/G. Suppose that S and S' are slices such that $\pi(S) \cap \pi(S') \neq \emptyset$. It is possible that $S \cap S' = \emptyset$. However, since the action is free, there exists and unique $g \in G$ such that $gS \cap S' \neq \emptyset$. This implies that $g^{-1}S \cap S' \neq \emptyset$. We have the following commutative diagram

$$\begin{array}{ccc} S \cap g^{-1}S' & \xrightarrow{\quad\theta_g\quad} & gS \cap S' \\ & {\scriptstyle \pi_{|S}} \searrow \quad \nearrow {\scriptstyle \pi_{|S'}^{-1}} & \\ & \pi(S) \cap \pi(S') & \end{array}$$

If we have pairs $(\pi(S), \phi_{|S} \circ \pi_{|S}^{-1})$ and $(\pi(S'), \psi_{|S'} \circ \pi_{|S'}^{-1})$, the composition $(\psi_{|S'} \circ \pi_{|S'}^{-1}) \circ (\phi_{|S} \circ \pi_{|S}^{-1})^{-1} = \psi_{|S'} \circ \theta_g \circ \phi_{|S}^{-1}$, which is smooth because is the composition of smooth maps (θ_g is a diffeomorphism since the action is smooth). ☐

Remark 2.25. *Observe that the dimensions of M and M/G are equal, since the group G is finite (thus, its dimension is 0).*

Example 2.26. *The sphere S^n is a smooth manifold for all $n \in \mathbb{N}$, and $\mathbb{Z}_p$ is a finite groups for all $p \in \mathbb{N}$. If we consider the group actions described in the first chapter, then the projective space $\mathbb{R}P^n$ and the lens space L_p^n are both smooth manifolds.*

This chapter's last statement will acquire an important role in the proof of the Mann Su theorem we will provide in the last chapter.

If G acts on a Riemannian manifold and g is a G-invariant metric, then we can define an action on $F_g(M)$ such that $h(p; v_1, ..., v_m) = (hp; d_p\theta_h(v_1), ..., d_p\theta_h(v_m))$. It is well defined because the group acts by isometries on M.

Proposition 2.27. *Let M be a Riemannian manifold, which is compact and connected. Let G be a finite group acting smoothly and effectively on M, and let g be an invariant Riemannian metric. Then, the group G acts freely on $F_g(M)$.*

Proof. Instead of taking an arbitrary point $x \in F_g(M)$ and proving that $G_x = \{e\}$, we will prove that if an element $h \in G$ is in a stabilizer for some $x \in F_g(M)$, then $h = e$. Let h be in G, we define the following set

$$\begin{aligned} S_h &= \{p \in M : \text{there exists a basis of } T_pM \text{ fixed by } d_p\theta_h\} \\ &= \{p \in M : \text{all vectors in } T_pM \text{ are fixed by } d_p\theta_h\}. \end{aligned}$$

The set S_h is closed. If we take a sequence of points $\{p_n\}_{n\in\mathbb{N}}$ in S_h converging to a point p (in the sense that all the points in the sequence are inside a chart (U, ϕ) and the sequence $\{\phi(p_n)\}_{n\in\mathbb{N}}$ converges to $\phi(p)$ in the euclidean space), then the point p is in S_h because θ_h is smooth.

The set S_h is open. Let p be in S_h, there exists a diffeomorphism given by the exponential map $exp_p : B_\epsilon(0) \longrightarrow U \subset M$. Let q be in U, then there exists $v \in B_\epsilon(0) \subset T_pM$ such that $exp_p(v) = q$. Since the exponential map is equivariant and $d_p\theta_h$ leaves each vector of T_pM fixed, we obtain that $hq = hexp_p(v) = exp_p(hv) = exp_p(v) = q$. Therefore, every point in U is fixed by h. Let w be in T_qM and let $\gamma : J \longrightarrow U$ be a smooth curve such that $\gamma(0) = q$ and $\gamma'(0) = w$. In this case, $d_p\theta_h(w) = (\theta_h \circ \gamma)'(0)$. But $\theta_h \circ \gamma = \gamma$ since every point of U is fixed by h. In consequence $d_p\theta_h(w) = \gamma'(0) = w$. We can conclude that $q \in S_h$.

Since $S_h \subset M$ is open and closed and M is connected, then $S_h = M$ or $S_h = \emptyset$. If $S_h = M$, then every point of M is fixed by h, therefore $h = e$, because the action is effective. If $S_h = \emptyset$, then $h(p; v_1, ..., v_m) \neq (p; v_1, ..., v_m)$ for all $p \in M$ and for all basis $\{v_i\}_i$ of T_pM, hence, $h \notin G_x$ for all $x \in F_g(M)$. This concludes the proof of the proposition. □

Chapter 3

Spectral sequences

The aim of this chapter is to introduce an object of the context of homological algebra called spectral sequence, a tool that will prove essential in the results of following chapters. The range of applications spectral sequences have in other fields of mathematics, specially in algebra and topology, give them and interest on their own, and give further reasons to study them carefully. This chapter starts with some basic definitions and remarks that motivate the creation of this object. Thereafter, we provide the definition of spectral sequence and we justify it thoroughly. A brief section is committed to explain when a spectral sequence converges. Finally, we expose one application of spectral sequences, which its usefulness will be evident in the next chapters.

The approach to spectral sequences in this chapter is based on the references [12], [11],[13] and [14], using the notation of Spanier's book. A comprehensive review on spectral sequences can be found in [16]. Other ways to introduce spectral sequences, like exact couples, are not included in this chapter.

Finally, all the chapter is developed using homology, the construction using cohomology is essentially the same, changing only some indices.

3.1 Previous definitions

Definition 3.1. *Let (C_*, d) be a chain complex with differential d, an increasing filtration is a sequence of subcomplex $\{F_pC\}_{p\in\mathbb{Z}}$ such that $F_pC \subset F_{p+1}C$ and it is compatible with the differential d, so $d(F_pC_q) \subset F_pC_{q-1}$.*

Definition 3.2. *Given a chain complex with an increasing filtration, we define the associated graded module $Gr(C)$ to be $\bigoplus_p F_{p+1}C/F_pC$.*

Remark 3.3. *It is not difficult to see that we can treat the associated graded module as a chain complex with a differential $d : \bigoplus_p F_{p+1}C_q/F_pC_q \longrightarrow \bigoplus_p F_{p+1}C_{q-1}/F_pC_{q-1}$ induced by the differential of the original chain complex, so we can compute the homology of that chain complex $H_*(F_{p+1}C/F_pC, d)$.*

On the other hand, we can compute the homology of the original complex $H_*(C, d)$. This new chain complex has an increasing filtration induced by the one in the complex C. Because of the *compatibility of the differential with the filtration, we can consider the homology of the subcomplex* (F_pC, d) *and the induced mapping by the inclusion in the homology* $H(i) : H(F_pC) \longrightarrow H(C)$*. Then, we define* $F_pH = Im\{H(i) : H(F_pC) \longrightarrow H(C)\}$*, and we claim that* $\{F_pH\}_{p\in\mathbb{Z}}$ *is an increasing filtration, so we can obtain its associated graded module.*

At this point, a natural question arises. Given a chain complex with a filtration, which relation exists between the homology of the associated graded module and the associated graded module of the homology of the chain complex. One could expect, quite naively, that both chains need to be isomorphic. Unfortunately, this is not usually true, and we need to search to more sophisticated relations. This is where spectral sequences appears.

3.2 The spectral sequence

We need to emphasize an observation that will motivated part of the spectral sequence definition.

Remark 3.4. *With the previous notation, consider a chain complex with an increasing filtration and the filtration in the homology* F_pH*. We can describe this submodule more explicitly computing the image of the map.* $F_pH = \{[a] \in H(C) : a \subset F_pC\} = \dfrac{F_pC \cap Z}{F_pC \cap B}$*, where* Z *and* B *are the cycles and boundaries of* C*. Then, the associated graded module is given by*

$$\bigoplus_p \frac{F_pC \cap Z}{(F_pC \cap B) + (F_{p-1}C \cap Z)}.$$

Definition 3.5. *Given a chain complex with an increasing filtration, we define four submodules as follows:*

- $Z^r_{p,q} = \{a \in F_pC_{p+q} : da \in F_{p-r}C_{p+q-1}\}$.
- $B^r_{p,q} = \{db \in F_pC_{p+q} : b \in F_{p+r}C_{p+q+1}\}$.
- $Z^\infty_{p,q} = \{a \in F_pC_{p+q} : da = 0\} = F_pC_{p+q} \cap Z$.
- $B^\infty_{p,q} = \{db \in F_pC_{p+q} : b \in C_{p+q+1}\} = F_pC_{p+q} \cap B$.

Remark 3.6. *The following properties are easily verified using tha previous definitions and the fact the the filtration is increasing.*

- $d(Z^r_{p,q}) = B^r_{p-r,q+r-1}$.
- $B^0_{p,q} \subset ... \subset B^{r-1}_{p,q} \subset B^r_{p,q} \subset ... \subset B^\infty_{p,q} \subset Z^\infty_{p,q} \subset ... \subset Z^r_{p,q} \subset Z^{r-1}_{p,q} \subset ... \subset Z^0_{p,q}$. *To prove this sequence of inclusions, we only use the fact that the filtration is increasing. For instance, if* $a \in Z^r_{p,q}$ *then* $da \in F_{p-r}C_{p+q-1} \subset F_{p-r+1}C_{p+q+1}$ *and this implies that* $a \in Z^{r-1}_{p,q}$*, proving the inclusion. If* $a \in Z^\infty_{p,q}$ *then* $da = 0 \in F_{p-r}C_{p+q+1}$ *for all* r *implying that* $a \in Z^r_{p,q}$*. If* $a \in B^\infty_{p,q}$ *then* $a = db$ *with* $b \in C_{p+q+1}$*, in consequence,* $da = d^2b = 0$*, and then* $a \in Z^\infty_{p,q}$*. The two last inclusions remaining to be proven have an analogous proof to the previous ones.*

Definition 3.7. *We define the following module* $E^r_{p,q} = \dfrac{Z^r_{p,q}}{Z^{r-1}_{p-1,q+1} + B^{r-1}_{p,q}}$ *and a map induced by the differential d of the original chain complex,* $d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$.

Remark 3.8. *The module* $E^r_{p,q}$ *and the map* d^r *are well defined. First of all,* $Z^{r-1}_{p-1,q+1} = \{a \in F_{p-1}C_{p+q} : da \in F_{p-r}C_{p+q-1}\}$ *and it is clearly a subset of* $Z^r_{p,q}$ *because* $F_{p-1}C_{p+q} \subset F_pC_{p+q}$. *Then,* $Z^{r-1}_{p-1,q+1} + B^r_{p,q} \subset Z^r_{p,q}$ *and the quotient of these modules makes perfect sense.*

In order to study the map d^r*, we observe that the composition*

$$Z^r_{p,q} \overset{d}{\mapsto} B^r_{p-r,q+r-1} \subset Z^r_{p-r,q+r-1} \overset{\pi}{\mapsto} E^r_{p-r,q+r-1}.$$

sends an element $a \in Z^r_{p,q}$ *to the equivalence class* $[da] \in E^r_{p-r,q+r-1}$. *This composition has the submodule* $Z^{r-1}_{p-1-r,q+r} + B^{r-1}_{p-r,q+r-1}$ *as a kernel. Finally, the image by the map d of the submodule* $Z^{r-1}_{p-1,q+1} + B^{r-1}_{p,q}$ *is* $B^{r-1}_{p-r,q+r-1} \subset Z^{r-1}_{p-1-r,q+r} + B^{r-1}_{p-r,q+r-1}$. *Using the universal property of the quotient, we obtain the map* $d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$ *and it fulfills that* $d^r[a] = [da]$. *Moreover,* $(d^r)^2 = 0$.

Lemma 3.9. $E^{r+1}_{p,q} = H(E^r_{p,q}, d^r)$.

Proof. Recall that

$$H(E^r_{p,q}, d^r) = \frac{ker d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}}{Im d^r : E^r_{p+r,q-r+1} \longrightarrow E^r_{p,q}}.$$

In order to prove this statement, we only need to compute the kernel and the image of these maps.

If $[a] \in ker d^r$ then $[da] = 0$ implying that $da \in Z^{r-1}_{p-1,q+1} + B^{r-1}_{p,q}$. Then

$$\{a \in F_pC_{p+q} : da \in Z^{r-1}_{p-1,q+1} + B^{r-1}_{p,q}\} =$$

$$\{a \in F_pC_{p+q} : da \in F_{p-r-1}C_{p+q-1}\} + \{a \in F_pC_{p+q} : da \in d(Z^{r-1}_{p-1,q+1})\} = Z^{r+1}_{p,q} + Z^{r-1}_{p-1,q+1}.$$

Therefore, we conclude that

$$ker d^r = \frac{Z^{r+1}_{p,q} + Z^{r-1}_{p-1,q+1}}{Z^{r-1}_{p-1,q+1} + B^{r-1}_{p,q}}.$$

To compute the image, we use that $d(Z^r_{p+r,q-r+1}) = B^r_{p,q}$. Therefore, we have the following equality

$$Im d^r = \frac{B^r_{p,q} + Z^{r-1}_{p-1,q+1}}{Z^{r-1}_{p-1,q+1} + B^{r-1}_{p,q}}.$$

To conclude the proof we only need to calculate the quotient using the second and the third isomorphism theorems.

$$H(E^r_{p,q}, d^r) = \frac{Z^{r+1}_{p,q} + Z^{r-1}_{p-1,q+1}}{B^r_{p,q} + Z^{r-1}_{p-1,q+1}} = \frac{Z^{r+1}_{p,q}}{(B^r_{p,q} + Z^{r-1}_{p-1,q+1}) \cap Z^{r+1}_{p,q}} = \frac{Z^{r+1}_{p,q}}{Z^r_{p-1,q+1} + B^r_{p,q}} = E^{r+1}_{p,q}.$$

□

The direct sum of all the modules $\bigoplus_{p,q} E^r_{p,q} = E^r$ has a structure of bigraded module with the differential d^r of bidegree $(-r, r-1)$.

Definition 3.10. *The sequence of bigraded modules together with the differential $\{E^r, d^r\}_{r\geq 0}$ such that $E^{r+1} = H(E^r)$ is called the spectral sequence associated to the filtered complex C.*

Remark 3.11. *Although until now we have been exposing the topic in the context of filtration and chain complex, the definition of the spectral sequence can be generalized, in the sense that the bigraded modules do not need to come from a filtered chain complex. Therefore, a spectral sequence is a collection of bigraded modules, $\{E^r, d^r\}_{r\geq 0}$ with differentials $d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$ such that $H(E^r) = E^{r+1}$. The bigraded module E^r is called the page r of the spectral sequence.*

Remark 3.12. *With the definitions above, we can calculate the*

$$E^\infty_{p,q} = \frac{Z^\infty_{p,q}}{Z^\infty_{p-1,q+1} + B^\infty_{p,q}} = \frac{F_pC_{p+q} \cap Z}{(F_pC_{p+q} \cap B) + (F_{p-1}C_{p+q} \cap Z)}.$$

This imply that the module E^∞ is equal to the associated graded module of the filtration in the homology of C, $Gr(H(C,d))$.

Remark 3.13. *In general, calculating one of the modules E^r can be an arduous labor, but fortuantely, most of the time we will have enough information with the E^0, E^1 and specially E^2 pages of the spectral sequence. This remark is committed to describe E^0 andE^1.*

When $r = 0$, the submodules that are used in the definition of $E^r_{p,q}$ are

$$Z^0_{p,q} = \{a \in F_pC_{p+q} : da \in F_pC_{p+q-1}\} = F_pC_{p+q},$$

$$B^{-1}_{p,q} = \{db \in F_pC_{p+q} : b \in F_{p-1}C_{p+q+1}\} \subset F_{p-1}C_{p+q},$$

$$Z^{-1}_{p-1,q+1} = \{a \in F_{p-1}C_{p+q} : da \in F_pC_{p+q-1}\} = F_{p-1}C_{p+q}.$$

Therefore, using the definition we obtain that $E^0_{p,q} = \dfrac{F_pC_{p+q}}{F_{p-1}C_{p+q}}$. We can conclude that the module E^0 is the associated graded module of the filtration. The differential

$$d^0 : \frac{F_pC_{p+q}}{F_{p-1}C_{p+q}} \longrightarrow \frac{F_pC_{p+q-1}}{F_{p-1}C_{p+q-1}}.$$

is the induced map by the differential of the chain complex (C_*, d) in the associated graded module, and consequently $E^1 = H(Gr(C), d)$.

With this last remark, we start to suspect the answer to the question of the previous section of this chapter. Given a filtered chain complex, the page E^1 is the homology of the graded complex, and the somewhat obscure E^∞ is the graded module associated to the filtration in the homology induced by the original filtration. As someone can expect from the notation, the modules of the spectral sequence connect both modules, in the sense that E^r "approximate successively" to $Gr(H(C,d))$. We may say that the spectral sequence "converge" to $Gr(H(C,d))$. Nonetheless, this last two statements are slightly cryptic and not truly rigorous. In the next section, we will provide a clear notion of convergence and extra conditions on the filtration to ensure its convergence.

3.3 Convergence of spectral sequences

Remark 3.14. *As we said in the remark 3.11, spectral sequences can be defined more generally as a collection of bigraded modules,* $\{E^r, d^r\}_{r\geq 0}$ *with differentials* $d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$ *such that* $H(E^r) = E^{r+1}$*. Using this definition, the module*

$$\bar{E}^{\infty}_{p,q} = \frac{\cap_r ker d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}}{\cup_r Im d^r : E^r_{p+r,q-r+1} \longrightarrow E^r_{p,q}}$$

is said to be the limit of the spectral sequence.

Definition 3.15. *A spectral sequence* $\{E^r, d^r\}_{r\geq 0}$ *is said to converge if for every integers p and q there exists an integer* $r(p,q) \geq 0$ *such that the differential* $d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$ *is trivial for all* $r \geq r(p,q)$*. Then,* $\bar{E}^{\infty}_{p,q}$ *is isomorphic to the direct limit of the sequence*

$$E^{r(p,q)}_{p,q} \to E^{r(p,q)+1}_{p,q} \to ...$$

Remark 3.16. *Frequently, we will require a stronger convergence condition. For every integers p and q, there exists an integer* $r(p,q) \geq 0$ *such that* $E^r_{p,q} \cong \bar{E}^{\infty}_{p,q}$ *for all* $r \geq r(p,q)$*.*

The following proposition provides a first condition for a spectral sequence to converge in the strong sense.

Proposition 3.17. *Let* $\{E^r, d^r\}_{r\geq 0}$ *be a spectral sequence with the property that there exist integer numbers R, N and N′ such that* $E^R_{p,q} = 0$ *for every* $p < N$ *and* $q < N'$*. Then, the spectral sequence converges in the strong sense.*

Proof. We observe that if $E^R_{p,q} = 0$, then $E^{r'}_{p,q} = 0$ for every $r' \geq R$. Then, given p and q, we choose and $r' \geq R$ such that $r' > sup(p-N, q-N'+1)$. We consider the diagram $E^r_{p+r,q-r+1} \longrightarrow E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$, where the two maps are the morphisms d^r from the spectral sequence. The modules from the sides are both 0 for every $r \geq r'$. The first one because $q-r+1 \leq q-r'+1 < N'$ and the second one because $p-r \leq p-r' < N$. Therefore, the maps d^r are 0 and $E^r_{p,q} \cong E^{r+1}_{p,q}$ for every $r \geq r'$. We can conclude that the spectral sequence is convergent and $E^{r'}_{p,q} \cong E^{r'+1}_{p,q} \cong ... \cong \bar{E}^{\infty}_{p,q}$.

□

Remark 3.18. *If R, N and N′ are zero, then the spectral sequence is called a first quadrant spectral sequence.*

To ensure the convergence to the graded module of homology for the case of the spectral sequence of a filtered chain complex, we need additional conditions on the filtration.

Definition 3.19. *An increasing filtration is said to be convergent if* $\cap_p F_p C = 0$ *and* $\cup_p F_p C = C$*. A filtration is said to be bounded below if for any q there exists an integer p(q) such that* $F_{p(q)} C_q = 0$*. A filtration is said to be bounded above if for any q there exists an integer p(q) such that* $F_{p(q)} C_q = C_q$*. A filtration is bounded if it is bounded above and bounded below.*

Theorem 3.20. *Let* (C,d) *be a chain complex with an increasing filtration, which is convergent and bounded below. There exists a convergent spectral sequence such that* $E^1 = H(Gr(C), d)$ *and* $\bar{E}^{\infty}$ *is isomorphic to* $Gr(H(C,d))$*.*

Proof. The existence of the spectral sequence is proved by the construction of the preceding sections. We only need to prove the statements regarding the convergence of the spectral sequence, that is, proving that the spectral sequence converge and $\bar{E}^\infty \cong E^\infty = Gr(H(C,d))$.

In order to prove the first statement, we need to use that $E^r_{p,q} = Z^r_{p,q}/(Z^{r-1}_{p-1,q+1} + B^{r-1}_{p,q})$ and $Z^r_{p,q} = \{a \in F_pC_{p+q} : da \in F_{p-r}C_{p+q-1}\}$. Because the filtration is bounded below, given a $p+q$, there is a p small enough such that $E^r_{p,q} = 0$ for all r. Then, given p and q there exists a r' big enough so $E^{r'}_{p-r',q+r'-1} = 0$, and therefore, the map $d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$ is trivial and the spectral sequence converge to $\bar{E}^\infty$.

Finally, we compute the module $\bar{E}^\infty$. Firstly, we observe that using isomorphism theorems

$$E^r_{p,q} = \frac{Z^r_{p,q}}{Z^{r-1}_{p-1,q+1} + B^{r-1}_{p,q}} \cong \frac{Z^r_{p,q} + F_pC_{p+q}}{F_{p-1}C_{p+q} + d(Z^{r-1}_{p+r-1,q-r+1})}.$$

By the definition, and using the previous computation of $kerd^r$ and Imd^r, we have that

$$\bar{E}^\infty_{p,q} = \frac{\cap_r(Z^r_{p,q} + F_pC_{p+q})}{\cup_r(F_{p-1}C_{p+q} + d(Z_{p+r-1,q-r+1}))} = \frac{\cap_r Z^r_{p,q} + F_pC_{p+q}}{F_{p-1}C_{p+q} + \cup_r d(Z_{p+r-1,q-r+1})}.$$

Given p and q, $\bigcap_r Z^r_{p,q} = Z^\infty_{p,q}$, because for an r sufficiently large $F_{p-r}C_{p+q+1} = 0$, since the filtration is bounded below.

Since $\bigcup_p F_pC = C$, then $\bigcup_r d(Z^{r-1}_{p+r-1,q-r+1}) = B \cap F_pC_{p+q}$, where B are the boundaries of C.

Therefore,

$$\bar{E}^\infty_{p,q} = \frac{\cap_r Z^r_{p,q} + F_pC_{p+q}}{F_{p-1}C_{p+q} + \cup_r d(Z_{p+r-1,q-r+1})} = \frac{Z^\infty_{p,q} + F_pC_{p+q}}{F_{p-1}C_{p+q} + (B \cap F_pC_{p+q})} = \frac{Z^\infty_{p,q}}{B^\infty_{p,q} + Z^\infty_{p-1,q+1}} = E^\infty_{p,q}.$$

□

The next theorem is a corollary of the theorem 3.20.

Theorem 3.21. *Let (C,d) be a chain complex with an increasing filtration which is bounded. There exists a spectral sequence such that $E^1 = H(Gr(C),d)$ and converges to $Gr(H(C,d))$ in the strong sense.*

Proof. First, we observe that the filtration being bounded implies that it is convergent and bounded below, using the theorem 3.20, there exists a convergent spectral sequence such that $E^1 = H(Gr(C),d)$ and $\bar{E}^\infty$ is isomorphic to $Gr(H(C,d)) = E^\infty$. The only remaining part of the theorem is that the convergence is strong.

Given integers p and q, the boundness of the filtration guarantees that there are integers r' and r'' such that $F_{p-r'}C_{p+q-1} = 0$ and $F_{p+r''-1}C_{p+q+1} = C_{p+q+1}$. Let R be an integer such that $R > sup(r', r'')$. Therefore, for every $r \geq R$, we have that $Z^r_{p,q} = Z^\infty_{p,q}$, $B^{r-1}_{p,q} = B^\infty_{p,q}$ and $Z^{r-1}_{p-1,q+1} = Z^\infty_{p-1,q+1}$. Then, $E^r_{p,q} = E^{r+1}_{p,q} = ... = E^\infty_{p,q}$, hence, the spectral sequence converges in the strong sense.

□

Remark 3.22. *It is usual to say that a spectral sequence of a filtered chain complex C converge to the homology $H(C)$, denoted by $E^1_{p,q} \Rightarrow H_{p+q}(C)$, when the page $E^\infty = Gr(H(C,d))$, so $E^\infty_{p,q} = F_p H_{p+q}(C)/F_{p-1}H_{p+q}(C)$. In the majority of the cases, it is not possible to recover all the information of $H_{p+q}(C)$ from the bigraded module $Gr(H(C,d))$. However, this will not become a problem to prove the Mann-Su theorem.*

3.4 The spectral sequence of a fiber bundle

Given a fiber bundle $p : X \longrightarrow B$ with fiber F, we are interested in describing the homology of the total space X using the information we have about the homology of the base space and the fiber. The main tool which allows us to connect these homologies is the Serre spectral sequence (or Leray-Serre spectral sequence). However, it is usually introduced in the wider context of fibrations, which we should define and explain briefly before stating the main results related with Serre spectral sequence.

Definition 3.23. *A mapping $p : E \longrightarrow B$ is said to have the homotopy lifting property respect to a space Y if, given a homotopy $G : Y \times I \longrightarrow B$ and a map $g : Y \times \{0\} \longrightarrow E$ such that $(p \circ g)(y,0) = G(y,0)$, then there exists a homotopy $\bar{G} : Y \times I \longrightarrow E$, such that $\bar{G}(y,0) = g(y,0)$ and $p \circ \bar{G} = G$. We have the following commutative diagram,*

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{\;g\;} & E \\ \big\downarrow & \overset{\bar{G}}{\nearrow} & \big\downarrow p \\ Y \times I & \xrightarrow{\;G\;} & B \end{array}$$

A map satisfying the homotopy lifting property respect to all topological spaces is called a fibration (or a Hurewicz fibration). If it only satisfies the property respect to disks, then it is called a Serre fibration.

Remark 3.24. *Given a map $p : E \longrightarrow B$, being a Serre fibration is equivalent to fulfill the homotopy lifting property respect all CW-complexes.*

The space E is called total space, the space B is called base space and given $b \in B$, $F_b = p^{-1}(b)$ is called the fiber of b over p.

The next proposition shows us the relation between fibrations and fiber bundles.

Proposition 3.25. *A fiber bundle $p : X \longrightarrow B$ with fiber F is a Serre fibration.*

We will give a full proof of this claim omitting some technical details.

Proof. It is sufficient to prove the homotopy lifting property for disks, or equivalently, for cubes I^n. Let $K : I^n \times I \longrightarrow B$ and $k : I^n \times \{0\} \longrightarrow X$ be maps such that $(p \circ k)(x,0) = K(x,0)$. We need to find a map H such that the following diagram commutes,

$$\begin{array}{ccc} I^n \times \{0\} & \xrightarrow{\;k\;} & X \\ \big\downarrow & \overset{H}{\nearrow} & \big\downarrow p \\ I^n \times I & \xrightarrow{\;K\;} & B \end{array}$$

We start choosing an open cover $\{U_\alpha\}$ where (U_α, ϕ_α) are charts. Then $\{K^{-1}(U_\alpha)\}$ is an open cover of a compact set, we can divide the cub in a finite number of tiny cubes that are small enough so that the image under K of one of these cubes lies inside one U_α. By induction on the dimension of the cubes and using the homeomorphism ϕ_α, we can reduce the proof of the proposition to the case of a trivial fibration $p : F \times B \longrightarrow B$.

We have the following diagram

$$\begin{array}{ccc} I^n \times \{0\} & \xrightarrow{k} & F \times B \\ \downarrow i & \overset{H}{\nearrow} & \downarrow p \\ I^n \times I & \xrightarrow{K} & B \end{array}$$

with $H = K \times (\pi_F \circ k \circ \pi_{I^n})$, where π_F and π_{I^n} are the usual projections. □

The last concept we need to introduce to understand the Serre spectral sequence is called system of local coefficients.

Definition 3.26. *A system of of local coefficients on a space B is a collection of groups $\{G_b : b \in B\}$, together with a collection of homomorphisms $h[\alpha] : G_{b_1} \longrightarrow G_{b_0}$, one for each path $\alpha : I \longrightarrow B$ from b_0 to b_1. This collection needs to satisfy the following conditions:*

1. *The constant path c_b has associated the identity homomorphism $h[c_b] = id : G_b \longrightarrow G_b$.*
2. *If two path α and α' satisfy that $\alpha(0) = \alpha'(0)$, $\alpha(1) = \alpha'(1)$ and are homotopic, $\alpha \simeq \alpha'$, then $h[\alpha] = h[\alpha']$.*
3. *If α is a path from b_0 to b_1 and α' is a path from b_1 to b_2 then $h[\alpha * \alpha'] = h[\alpha] \circ h[\alpha']$.*

We will denote it by $\mathcal{G}$.

Definition 3.27. *Let $\mathcal{G}$ and $\mathcal{G}'$ be systems of local coefficients on a space B, a morphism of system of local coefficients is a collection of group morphisms for each $b \in B$, $f_b : G_b \longrightarrow G'_b$, such that for every path $\alpha : I \longrightarrow B$ the following diagram commutes*

$$\begin{array}{ccc} G_{b_1} & \xrightarrow{h[\alpha]} & G_{b_0} \\ \downarrow f_{b_1} & & \downarrow f_{b_0} \\ G'_{b_1} & \xrightarrow{h'[\alpha]} & G'_{b_0} \end{array}$$

Example 3.28.

- *Given a space B and a group G, the trivial system of local coefficients, denoted by G, satisfies that $G_b = G$ for all $b \in B$ and each path has the identity as associated homomorphism.*
- *Given a fibration $p : E \longrightarrow B$ with B path-connected, and let G be a group. We can form a system of local coefficients if we define $G_b = H_n(F_b; G)$, where $F_b = p^{-1}(b)$. Using the homotopy lifting property, given a path $\alpha : I \longrightarrow B$, we can construct homomorphisms that fulfill the aforementioned properties. This system of local coefficients is denoted by $\mathcal{H}_n(F; G)$.*

Our goal is to define the homology of a space B with coefficients in $\mathcal{G}$, in a similar way we define the homology with coefficients in a group G (in fact, the definitions will be the same if we use the trivial system of local coefficients).

Let Δ^p be a simplex with leading vertex $v_0 = (1, 0, ..., 0)$ and $v_1 = (0, 1, 0, ..., 0)$. We define the set of singular p-chains with local coefficients in $\mathcal{G}$ by

$$C_p(B, \mathcal{G}) = \{\text{finite formal sums } \sum_i g_i \otimes T_i : T_i : \Delta^p \to B \text{ are continuous and } g_i \in G_{T(v_0)}\}.$$

The boundary map is defined on the basis by

$$\partial_h(g \otimes T) = h[TL^{v_0}_{v_1}](g) \otimes \partial_0 T + \sum_{j=1}^{p} (-1)^j g \otimes \partial_j T$$

where $TL^{v_0}_{v_1} = T(tv_0 + (1-t)v_1)$ and ∂ is the usual boundary map of singular homology, so ∂_i, where i goes from 0 to p, denotes the component of ∂ related with the i-th face of the simplex.

The process of checking that it is a boundary map ($\partial_h \circ \partial_h = 0$) is similar to the one in the usual singular chains case, although we need to be more careful when we compute the part where we change the local coefficient.

The homology with local coefficients is

$$H_*(B, \mathcal{G}) = H(C_*(B; \mathcal{G}), \partial_h).$$

In a similar way, if B is a CW-complex, we define $Cell_p(B; \mathcal{G}) = H_p(B^p, B^{p-1}; \mathcal{G})$.

Theorem 3.29. *Given an abelian group G and a fibration $F \hookrightarrow X \longrightarrow B$ where B is path connected and F is connected, then there exists a first quadrant spectral sequence $\{E^r_{*,*}, d_r\}$ converging to $H_*(X; G)$ such that the second page $E^2_{p,q} = H_p(B, \mathcal{H}_q(F; G))$.*

The proof is rather complex, so we will provide a rough sketch of it.

Proof. We will prove it in the case that B is a CW-complex (to extend it to a paracompact space the CW-complexes approximation theorem shall be used). In that case, we have a filtration in B given by its skeleton, and we can rise it to X using the map p. We define $J^s = p^{-1}(B^s)$. It is clear that $J^0 \subset J^1 \subset ... \subset X$, because $B^0 \subset B^1 \subset ... \subset B$.

Considering these inclusions, we can define an increasing filtration in the chain complex of the total space E in the following way,

$$F_p C_* = Im\{C_*(J^p; G) \longrightarrow C_*(X; G)\}.$$

We can consider that $C_q(J^p; G) = 0$ if $p, q < 0$. Then, it is clear that it is a first quadrant spectral sequence and bounded below. Using arguments of compactness of the simplicies, we can say that every simplex in E will be contained in a J^n for n large enough, and therefore $C_*(X) = \bigcup_{p \geq 0} F_p C_*$. Using the previous theorems of this chapter, the spectral sequence converges to $H_{p+q}(X)$, and the first page is

$$E^1_{p,q} = H(F_p C_{p+q} / F_{p-1} C_{p+q}; G) = H_{p+q}(J^q, J^{q-1}; G).$$

It only remains to determine the second page of the spectral sequence. Firstly, we observe that the first page of the spectral sequence reassembles to the cell homology definition we have seen in the first chapter. In fact, if we were working with a trivial fibration (that is a fibration $p : B \times F \longrightarrow B$, where p is the projection), then we would have that $J^s = B^s \times F$. Therefore, we would use Künneth formula and we would obtain that $E^1_{p,q} = Cell_p(B) \otimes H_q(F; G)$. Unfortunately, this is not always the case, and we need to use a local system of coefficients $\mathcal{H}_*(F; G)$ to describe the second page.

The main result states that $E^1_{p,q} \cong Cell_p(B, \mathcal{H}_q(F; G))$ and the map d_1 is the map of this chain complex. We want to reach a similar situation as the above case. To achieve this objective, we shall use the CW-structure.

The first key observation is that a p-cell e^p_α is contractible, therefore the local system of coefficients $\mathcal{G}$ can be replaced by a trivial local system of coefficients that will be denoted by G_α, which is one of the groups of the local coefficient system $\mathcal{G}$ for an arbitrary point inside e^p_α. Using the excision theorem, it is possible to find the following isomorphism $\bigoplus_\alpha H_p(e^p_\alpha, \partial e^p_\alpha; G_\alpha) \longrightarrow H_p(B^p, B^{p-1}; \mathcal{G})$, where α runs through all the p-cells of the CW-complex. This isomorphism enables us to work with a p-cell and its boundary (that are homeomorphic to a ball and its boundary sphere), reducing the difficulty of the problem. In consequence, $Cell_p(B; \mathcal{H}_q(F; G)) = H_p(B^p, B^{p-1}; \mathcal{H}_q(F; G)) \cong \bigoplus_\alpha H_p(e^p_\alpha, \partial e^p_\alpha; F_\alpha)$.

The other key observation will help us to connect the previous information about the homology in local coefficients with the information about the filtration induced by the skeleton of the CW-complex on the total space E. Observe that $H_p(e^p_\alpha, \partial e^p_\alpha; F_\alpha) = H_p(e^p_\alpha, \partial e^p_\alpha) \otimes H_q(F; G)$ which is isomorphic to $H_{p+q}(e^p_\alpha \times F, \partial e^p_\alpha \times F)$ by using the Künneth formula. Finally, a homotopy equivalence induced by the fibration induces an isomorphism $H_{p+q}(e^p_\alpha \times F, \partial e^p_\alpha \times F; G) \cong H_{p+q}(p^{-1}(e^p_\alpha), p^{-1}(\partial e^p_\alpha); G)$.

Combining this two claims, we obtain that $Cell_p(B; \mathcal{H}_q(F; G)) \cong \bigoplus_\alpha H_p(e^p_\alpha, \partial e^p_\alpha; F_\alpha) \cong \bigoplus_\alpha H_{p+q}(p^{-1}(e^p_\alpha), p^{-1}(\partial e^p_\alpha); G) \cong H_{p+q}(J^p, J^{p-1}; G) = E^1_{p,q}$.

These isomorphisms are natural, giving us the following commutative diagram

$$\begin{array}{ccc} Cell_p(B; \mathcal{H}_q(F; G)) & \xrightarrow{\partial} & Cell_{p-1}(B; \mathcal{H}_q(F; G)) \\ \downarrow \cong & & \downarrow \cong \\ E^1_{p,q} & \xrightarrow{d_1} & E^1_{p-1,q} \end{array}$$

and therefore, the second page of the spectral sequence is $E^2_{p,q} = H_p(B; \mathcal{H}_q(F; G))$.

□

Chapter 4

Equivariant homology

Or main objective is to use the tools from homological algebra developed in the previous chapters to study the group actions on smooth manifolds. Obviously, given a G-space X, applying one of the homology or cohomology theories on the space can give us plenty of information about the space, but it does not provide information about the group action, so we need to find a better candidate that reflects both the structure of the space and the group action. The first candidate that comes to mind is the orbit space X/G, but it may not retain the nice topological properties G-space X has, and it may not contain all the information about the action. Therefore, it is not a suitable choice. Instead, a contractible topological free G-space is constructed, and we link it via a twisted product with the G-space X. The fact that the space is contractible and the action is free will help to retain the information about the G-space X we want to study, while giving us invaluable information about the action. In addition, the computations that are required are manageable with the tools of homological algebra.

The first section of this chapter is devoted to giving some notions about that contractible space and the second section define and provide some of the results of that construction.

The majority of authors expose this topic using cohomology, since cohomology possesses supplementary multiplicative properties to develop the theory. Nevertheless, using homology is sufficient to achieve our main goal. In this chapter, we used [17] and [18]. An interesting article that provides a concise introduction to equivariant cohomology is [19].

4.1 Classifying spaces

We want to construct a contractible free G-space, that we will call EG. The next theorem shows its existence for topological groups and a general method to build it.

Theorem 4.1. *Let G be a topological group, then there exists a contractible free G-space EG.*

A general proof of this fact is due to Milnor, that construct EG as an infinite join of G, $G * G * G * \ldots$. We will show a different proof in the case that G is a finite group.

Proof. Let G be a finite group and $\sharp G = n$. We consider the set $\mathbb{R}[G] = \{\phi : G \to \mathbb{R}\}$ which is a vector space over $\mathbb{R}$. In fact, we have the following isomorphism $\mathbb{R}^n \cong \mathbb{R}[G]$. For all $g \in G$, we define $\phi_g : G \longrightarrow \mathbb{R}$ such that $\phi_g(g') = 1$ if $g = g'$ and $\phi_g(g') = 0$ if $g \neq g'$. Then, $\{\phi_g\}_{g \in G}$ is clearly a basis of $\mathbb{R}[G]$ and we can construct the isomorphism by sending bijectively each vector of the basis to the usual basis of $\mathbb{R}^n$. Moreover, we can construct a left action of G on $\mathbb{R}[G]$. Given $\phi \in \mathbb{R}[G]$, we define $g\phi$ such that for all $g' \in G$ we have that $(g\phi)(g') = \phi(gg')$.

The next step is to define the set

$$Inj(\mathbb{R}[G], \mathbb{R}^k) = \{f : \mathbb{R}[G] \longrightarrow \mathbb{R}^k : f \text{ is linear and injective}\} = A_k.$$

We have a left action induced by the above action, such that if $f \in I_k$, then $gf(\phi) = f(g\phi)$ for all $\phi \in \mathbb{R}[G]$. We claim that $EG = \bigcup_{k \in \mathbb{N}} A_{nk} = A$. We need to prove that the action of G is free and that A is contractible. Given $f \in A$, then $f \in A_{nk}$ for some $k \in \mathbb{N}$. We want to see that $G_f = \{g \in G : gf = f\} = \{e\}$. Given $g \in G_f$, it is clear that $f(\phi) = gf(\phi) = f(g\phi)$ for all $\phi \in \mathbb{R}[G]$. Using the injectivity of f, we have that $g\phi = \phi$ for all $\phi \in \mathbb{R}[G]$. In particular, for each element of the basis ϕ_h, where $h \in G$, we have that $g\phi_h = \phi_{g^{-1}h} = \phi_h$. In consequence, $g^{-1}h = h$ for all $h \in G$. If $h = g$ we obtain that $e = g^{-1}g = g$.

Since $\mathbb{R}^n \cong \mathbb{R}[G]$, we can describe

$$A_{nk} = \{(M_1, ..., M_k) : M_i \in End(\mathbb{R}^n) \text{ and } \forall v \in \mathbb{R}^n \; v \neq 0 \; \exists j \text{ such that } M_j v \neq 0\}$$

where M_j is thought as $n \times n$ matrix. The topology of A fulfills that U is an open set if and only if $U \cap A_{nk}$ are open sets for all $k \in \mathbb{N}$. Observe that we have the inclusions $A_{nk} \hookrightarrow A_{n(k+r)}$ such that $(M_1, ..., M_k) \to (M_1, ..., M_k, 0, ..., 0)$. Therefore, we can express every element of A as a sequence of matrix $M = (M_1, M_2, ..., M_k, 0, ...)$. Finally, we denote $M_{Id} = (Id, 0, ...)$ and we define a shift function $\sigma : A \longrightarrow A$ such that $\sigma(M_1, ..., M_k, 0, ...) = (0, M_1, ..., M_k, 0, ...)$ that is clearly a continuous map. We have all the pieces to prove that the identity in A is homotopic to the constant map that sends every M to M_{Id}.

We define the homotopy $H : A \times I \longrightarrow A$ such that

$$H(M, t) = tM_{Id} + (1-t)M + t(1-t)\sigma(M).$$

It is clear that for $t = 0$ we obtain the identity map, and for $t = 1$ we obtain the constant map. First of all, we need to prove that H is well defined, which means that $H(M, t) \in A$ for all $M \in A$ and $t \in I$, and we can suppose that $t \neq 0, 1$. If $M = (M_1, ..., M_k, 0, ...)$, then $H(M, t)$ is equal to

$$(tId + (1-t)M_1, (1-t)M_2 + t(1-t)M_1, ..., (1-t)M_k + t(1-t)M_{k-1}, t(1-t)M_k, 0, ...).$$

Suppose that we have a vector $v \in \mathbb{R}^n$, such that $H(M, t)_j v = 0$ for all j, where $H(M, t)_j$ denotes the component j of $H(M, t)$. Then, we have that $t(1-t)M_k v = 0$, and consequently $M_k v = 0$. Replacing this information in the previous component, we find that $t(1-t)M_{k-1} v = 0$. Repeating the same argument we can arrive to the first component and find that $tId(v) = tv = 0$ and because $t \neq 0$ we have that $v = 0$. Therefore, $H(M, t) \in A$ for all $M \in A$ and $t \in I$.

In order to prove the continuity, we take an open set $U \subset A$, then the sets $U \cap A_{nk}$ are open in A_{nk}. This allows us to restrict the problem to the finite case, $H_{nk} : A_{nk} \times I \longrightarrow A_{n(k+1)}$, where all maps used are continuous, and therefore, H_{nk} are continuous for all k. From here we can deduce the continuity of H.

□

The orbit space $EG/G = BG$ is the called classifying space of G, and together with the orbit map $\pi : EG \longrightarrow BG$ we can obtain a principal G-bundle.

Proposition 4.2. *The map $\pi : EG \longrightarrow BG$ is G-principal bundle with total space EG and base space BG.*

Proof. The statement is a corollary of proposition 1.34. The total space is Hausdorff since $Inj(\mathbb{R}[G], \mathbb{R}^k) = A_k$ for all $k \in \mathbb{N}$. Let M and N be in EG, if M and N are in the same A_{nk}, we can take open neighbourhoods U and V of M and N respectively which fulfill that $U \cap V = \emptyset$. The sets U and V are open in EG. If M and N are in different A_{nk}, $M \in A_{nk}$ and $N \in A_{nk'}$, we take as disjoint open neighbourhoods the sets A_{nk} and $A_{nk'}$. □

This principal G-bundle is called the universal G-bundle, which satisfies the following property. If B is a paracompact space, there exists a bijection between the homotopy classes of maps $f : B \longrightarrow BG$ and the G-principal bundles with base space B. This bijection sends the map f to the pullback bundle $f^*(EG) \longrightarrow B$. It should be remarked that the spaces EG and BG are unique up to homotopy equivalence.

Finally, an important remark should be made regarding the homology (or cohomology) of a group G and the homology (or cohomology) of its classifying space. The homology of a group G can be defined algebraically, using chain complexes in a similar way we construct the homology of a topological space, (one could see [11] for the construction of group cohomology). However it can be proved that we have an isomorphism between the group homology and the homology of BG, $H_*(BG) \cong H_*(G)$, thus, we will define the group homology of G to be $H_*(BG)$. The same can be said about cohomology. Observe that if G is a topological group, the group homology of G differs from the homology of G seen as a topological space.

In particular, we are concerned about the classifying space of the groups of the form $\mathbb{Z}_p$ where p is a prime number. In that case, we can describe it explicitly in a easier way.

Lemma 4.3. *The infinite dimensional sphere S^∞ is contractible.*

Proof. Recall that we defined $S^\infty = \bigcup_{n \geq 0} S^n$ with the topology that a set A is an open set in S^∞ if and only if $A \cap S^n$ is open for all $n \geq 0$.

Let x be a point of S^∞, then it is inside one of the finite dimensional spheres S^n and we can consider its coordinates $(x_0, ..., x_n)$ such that $\Sigma_i |x_i|^2 = 1$.

On the other hand, we have the continuous inclusion map $i_{n,m} : S^n \longrightarrow S^m$ such that $i_{n,m}((x_0, ..., x_n)) = (x_0, ..., x_n, 0, ..., 0)$, where $n \leq m$. Therefore, we can consider that a point p of S^∞ is a sequence $(x_0, ..., x_n, 0, ...)$ and it has norm one.

We consider the map $\iota : S^\infty \longrightarrow S^\infty$ such that $\iota((x_0, ..., x_n, 0, ...)) = (0, x_0, ..., x_n, 0, ...)$. Using the characterization of the open sets in S^∞ is easy to restrict this map to the case of inclusions of finite dimensional spheres that are continuous. This fact implies that the map ι is continuous.

In order to prove that the space is contractible, we need to see that the identity map is homotopic to the constant map (we will choose the map $c : S^\infty \longrightarrow S^\infty$ that sends S^∞ to the point $(1, 0, 0, ...)$). We will do it in two steps, proving that $id \simeq \iota \simeq c$. We will construct the homotopies explicitly.

To construct the first homotopy $H : S^\infty \times I \longrightarrow S^\infty$ is defined in the following way.

$$H(p, t) = \frac{t\iota(p) + (1-t)p}{|t\iota(p) + (1-t)p|}$$

It is well defined because all the points only have a finite number of coordinates not null and the sum is never zero. Suppose that $t\iota(p) + (1-t)p = 0$ for some (p, t). If p has coordinates $(x_0, ..., x_n, 0....)$, this means that $((1-t)x_0, x_0 t + (1-t)x_1, ..., tx_n, 0....) = (0, 0, ...)$. If $x_0 \neq 0$, then $t = 1$, implying that $(0, x_0, ..., x_n, 0, ...)$, which is not true. If $x_0 = 0$, then the second coordinate has the form of $x_1(1-t) = 0$ and we can repeat the same reasoning. Because the norm is always one, we will eventually find a coordinate $x_i \neq 0$, reaching a contradiction.

Using the characterization of open sets in S^∞, it suffices to prove the continuity on each restriction to S^n. But, in the finite case, all the maps involved are already continuous, hence, the H is continuous.

The second homotopy is defined similarly. It is a map $G : S^\infty \times I \longrightarrow S^\infty$ such that

$$G(p, t) = \frac{tc(p) + (1-t)\iota(p)}{|tc(p) + (1-t)\iota(p)|}$$

Checking that it is well defined and continuous is analogous to the first homotopy.

□

In a slighly different approach, we can describe $S^\infty = \bigcup_{n\geq 1} S^{2n-1}$, where $S^{2n-1} \subset \mathbb{C}^n$. Observe that $\mathbb{Z}_p$ acts freely on it by rotating each sphere as we showed in the first example of chapter 1. In consequence, its classifying space is $B\mathbb{Z}_p = S^\infty/\mathbb{Z}_p = \bigcup_{n\geq 0} S^{2n-1}/\mathbb{Z}_p = L_p^\infty$.

Finally, we need to compute the classifying space of groups of the form $(\mathbb{Z}_p)^r$. In general, if G is a topological group and we consider the space EG, we can construct a free action of G^r to EG^r, such that if $(g_1, ..., g_r) \in G^r$ and $(x_1, ..., x_r) \in EG^r$ then $(g_1, ..., g_r)(x_1, ..., x_r) = (g_1 x_1, ..., g_r x_r)$. Clearly, it is a free action and EG^r is contractible. In consequence, the classifying space of G^r is $B(G^r) = EG^r/G^r \simeq (BG)^r$.

4.2 Borel construction

Let G be a topological group acting on X, and let $\pi : EG \longrightarrow BG$ be the universal G-bundle. Then, we consider the associated bundle $p : EG \times_G X \longrightarrow BG$ constructed in the proposition 1.35.

Definition 4.4. *We define the equivariant homology of space X, $H_*^G(X)$, to be the homology of the total space $H_*(EG \times_G X)$.*

This construction is due to Borel (hence, it also has the name of Borel construction) and it will reflect how acts the group G on X.

Remark 4.5. *The examples below are the most simple cases of equivariant homology computations, and are helpful for the purpose of understanding equivariant homology.*

- *If X is the topological space that only consist of a point, $X = \{pt\}$, then $EG \times_G X \cong EG/G \cong BG$. Therefore $H^G_*(\{pt\}) = H_*(BG) = H_*(G)$.*

- *More generally, if X is a G-space and the action is trivial, then $X_G \cong X \times BG$, so $H^G_*(X) = H_*(X \times BG)$. If we are working with coefficients in a field, then $H^G_*(X) = H_*(BG) \otimes H_*(X)$.*

- *Given a subgroup H of G, then $G/H \times_G EG \cong (G \times EG)/H = EG/H = BH$. Therefore, $H^G_*(G/H) = H_*(BH)$.*

On the other hand, we have the following result if the action is free.

Proposition 4.6. *Let X be a free G-space, then $H^G_*(X) \cong H_*(X/G)$.*

Although the above proposition is a general fact, we will prove it when the space $X = N$ is a compact smooth manifold and G is a finite group.

Proof. Firstly, since the action on N is free, we have the G-principal bundle $\pi : N \longrightarrow N/G$. We can construct its associated bundle with fiber EG, $p : EG \times_G N \longrightarrow N/G$, such that $p([M, x]) = [x]$. Our goal is to find a global section $s : N/G \longrightarrow EG \times_G N$ and to show that $s \circ \pi \simeq id_{EG \times_G N}$ in order to obtain a homotopy equivalence $N/G \simeq EG \times_G N$.

To construct the section we shall use the concrete structure of EG for a finite group we developed in the above section and the fact that N/G is a manifold. Recall that a point of EG can be expressed as a sequence of $n \times n$ matrices $(M_1, M_2, ...)$ such that there exists a number j_0 such that $M_j = 0$ for $j > j_0$. Let $M^{(i)}_{Id} \in EG$ be the point of the form $(0, ..., 0, Id, 0, ...)$, where the identity is in the i-th position.

We have seen in the second chapter that the collection $\{\pi(S)\}_{S \text{ slice}}$ of the images by π of all the slices we can construct in N, is an open cover of N/G. Since N is compact, it follows that N/G is compact. In consequence, a finite open cover $\{\pi(S_i)\}_{i=1,..,k}$ can be obtained. Therefore, there exists a finite partition of the unity $\{\psi_i\}_{i=1,..,k}$ subordinated to the previous finite open cover. Since the action is free, we have a homeomorphism $\pi^{-1}_{|S_i} : \pi(S_i) \longrightarrow S_i$. Let T_i be the tube $G \times S_i \longrightarrow T_i$.

Given $x \in T_i$, there exists an unique $g \in G$ such that $g^{-1}x \in S_i$. We define an equivariant map $f_i : T_i \longrightarrow EG$ such that $f_i(x) = gM^{(i)}_{Id}$. If we use the partition of the unity, we can construct an equivariant map $f : X \longrightarrow EG$ such that $f(x) = \sum \psi_i([x]) f_i(x) = (\psi_1 g_1 Id, \psi_2 g_2 Id, ...)$. Observe that since each $x \in N$ is inside one of the tubes T_i, there is always a component different than 0, and that for $j > k$, the j-th component of $f_i(x)$ is 0 because of the finiteness of the partition of the unity. Moreover, note that if $x \notin T_j$ for some j and then $\psi_j = 0$. Thus, f is well defined and equivariant. In consequence, it follows that the map $N \longrightarrow EG \times N$ such that $x \to (f(x), x)$ is equivariant and it induces a map $s : N/G \longrightarrow EG \times_G N$ which satisfies that $s[x] = [f(x), x]$. Then, it is clear that $\pi \circ s = id_{N/G}$. Therefore, it is a global section.

In order to prove that $s \circ \pi \simeq id_{EG \times_G N}$, we will construct an equivariant homotopy on $EG \times N$. Observe that $id_{EG \times N}$ induces the map $id_{EG \times_G N}$ on the orbit space and the map $\phi : EG \times N \longrightarrow EG \times N$ such that $\phi(M, x) = (f(x), x)$ induces the map $s \circ \pi$ on the orbit space. If we construct an equivariant homotopy between these maps, it will induce a homotopy between $id_{EG\times_G N}$ and $s \circ \pi$ and the assertion of the proposition will be proved.

The homotopy reassembles closely to the ones stated in the previous subsection. We define $H : (EG \times N) \times I \longrightarrow EG \times N$ such that $H((M, x), t) = tM + (1 - t)f(x) + t(1 - t)\sigma^k(M), x)$, where σ^k is the shift function composed with itself k times (therefore, it moves each matrix k positions to the right). Using an analogous process to the ones made in the preceding subsection, it can be seen that the homotopy is well defined. Since every map involved is equivariant, the homotopy H is also equivariant. Therefore H is the homotopy we seek. In consequence, the proposition is proved.

□

One of the most useful aspects of the Borel construction is that we have a Serre spectral sequence that arises from the associated fiber bundle.

Remark 4.7. *Let X be a G-space, then there exists a spectral sequence $\{E^r_{*,*}, d^r\}$ such that $E^2_{p,q} = H_p(BG, \mathcal{H}_q(X; R))$ and it converges to $H^G_*(X)$.*

At this point, we show a slightly different approach of systems of local coefficients on a space B. The idea of assigning a group to each point of b closely resembles the fiber bundle construction, where each point has its fiber. Indeed, systems of local coefficients on B can be understood as a fiber bundle with base space B, fiber an abelian group A and structural group $G \subset Aut(A)$. If we restrict a local trivialization $\phi : U \times A \longrightarrow p^{-1}(U)$ to $\{x\} \times G$, we obtain a group isomorphism between $\{x\} \times G$ and $p^{-1}(X)$. The collection of homomorphism of the system of local coefficients and its properties are consequence of the lifting property of fiber bundles.

Consider the system of local coefficients which appears in the Serre spectral sequence for the Borel construction, $\mathcal{H}_q(X; R)$. The action of G on X induces an action on $H_q(X; R)$ for all q. Therefore, the system of local coefficients can be described by the associated fiber bundle $p : \mathcal{H}_q(X; R) = EG \times_G H_q(X; R) \longrightarrow BG$.

Let x be a point in BG. We consider its fiber $p^{-1}(x) \cong H_q(X; R)$. Then, each homotopy class of a loop with base point x (in other words, each element of $\pi_1(BG, x)$) has associated an automorphism of $p^{-1}(x)$. Since $\pi_1(BG, x) \cong G$, we obtain another action of G on $H_q(X; R)$. Nevertheless, this action coincides with the action of G induced in $H_q(X; R)$ by the action of G on X. This fact is a direct consequence of the construction of the fiber bundle associated to the principal G-bundle $\pi : EG \longrightarrow BG$.

Chapter 5

Elementary p-groups smooth actions on smooth manifolds

In the previous chapters, we studied properties of some objects, like orbits or orbit spaces, where we had an effective action of a group on a manifold together with some additional conditions, like the group acting freely or the action being smooth, as hypothesis. In this last chapter, we change our viewpoint to answer a different type of questions. We are concerned with the existence of an effective action. Given a compact manifold M, which finite groups can act effectively on M?

The Mann Su theorem, [1], answers the question for a concrete type of group, acting on any cohomological manifold. Since the machinery used in that paper is rather complex and we have not developed it in this project, we will restrict ourselves to the case that the manifold is compact, connected and smooth and the action is smooth.

In the first section of this chapter, some preliminary lemmas are proved before stating and proving the Mann Su theorem.

5.1 Preliminary lemmas

Definition 5.1. *An elementary p-group of rank r is a group isomorphic to $(\mathbb{Z}_p)^r$, which is the direct sum of r copies of $\mathbb{Z}_p$, where p is a prime number.*

In this chapter, G always denotes an elementary p-group of rank r, so $G \cong (\mathbb{Z}_p)^r$. With these groups, it is convenient to use the homology with coefficients in $\mathbb{Z}_p$. Therefore, we can consider them to be $\mathbb{Z}_p$ vector spaces and use concepts as the dimension of a vector space.

Definition 5.2. *Let X be a topological space. The value of $dim(H_i(X;\mathbb{Z}_p)) = b_i$ is called the i-th Betti number mod p. We denote by B the sum $\sum_i b_i$.*

Lemma 5.3. *Let BG be the classifying space of G, then the dimension of its homology with coefficients in $\mathbb{Z}_p$ is*

$$dim(H_k(BG;\mathbb{Z}_p)) = \binom{k+r-1}{r-1}.$$

Proof. Since $G \cong (\mathbb{Z}_p)^r$, we have that $BG \simeq (B\mathbb{Z}_p)^r$. We proceed by induction on the rank of the group.

- If $r = 1$, then $BG \simeq B\mathbb{Z}_p = L_p^\infty$. We have proved in chapter 1 that $H_k(L_p^\infty; \mathbb{Z}_p) = \mathbb{Z}_p$ for all $k \geq 0$. Thus, we obtain that

$$dim(H_k(BG; \mathbb{Z}_p)) = 1 = \binom{k+0}{0}.$$

- Suppose we have proved the statement for $r' < r$. We have that $H_k(BG; \mathbb{Z}_p) = H_k((B\mathbb{Z}_p)^r; \mathbb{Z}_p)$. Therefore, we can use the Künneth formula to obtain that

$$H_k(BG; \mathbb{Z}_p) = \bigoplus_{n+m=k} (H_n(B\mathbb{Z}_p; \mathbb{Z}_p) \otimes H_m((B\mathbb{Z}_p)^{r-1}; \mathbb{Z}_p)).$$

If we compute its dimension, we obtain that

$$dim(H_k(BG; \mathbb{Z}_p)) = \sum_{m=0}^{k} dim(H_{k-m}(B\mathbb{Z}_p; \mathbb{Z}_p)) \cdot dim(H_m((B\mathbb{Z}_p)^{r-1}; \mathbb{Z}_p)).$$

The dimension of the first term is always 1. Using the inductive hypothesis, we obtain that

$$dim(H_k(BG; \mathbb{Z}_p)) = \sum_{m=0}^{k} \binom{k+r-2}{r-2} = \sum_{j=r-2}^{k+r-2} \binom{j}{r-2} = \binom{k+r-1}{r-1}.$$

□

Lemma 5.4. *Let V be a vector space over $\mathbb{Z}_p$ with dimension d. Suppose that the group G acts on V linearly ($\theta_g : V \longrightarrow V$ is a lineal map for all $g \in G$). Then, we have a chain of vector subspaces of length d, $0 = V_0 \subset V_1 \subset .... \subset V_d = V$ such that it is G-invariant and the action of G on V_{i+1}/V_i is trivial.*

Proof. We have that $V \cong (\mathbb{Z}_p)^d$ (it is an isomorphism of vector spaces), therefore the cardinality of V is p^d. Since $G \cong (\mathbb{Z}_p)^r$, the cardinality of the orbits of the action of G on V could be 1,p,p^2,...,p^r.

Because the action is linear, the orbit of the 0 vector is $G(0) = \{0\}$. In consequence, the sum of the cardinalities of the rest of the orbits shall be $p^d - 1$. This implies that there exists another fixed vector. Otherwise, the cardinality of each orbit would be a multiple of p, hence, their sum would be a multiple of p. However, their sum is equal to $p^d - 1$ which is not a multiple of p. We have reach a contradiction. Thus, there exists a vector $v_1 \neq 0$ which is a fixed point.

We define $V_1 = \langle v_1 \rangle$, which is clearly G-invariant. We proceed similarly on the quotient space V/V_1. A vector of the quotient space $[v_2] \neq [0]$ can be found such that it is a fixed point of the action of G in V/V_1 (this action is induced by the action on V). Therefore, we define $V_2 = \langle v_1, v_2 \rangle$, which is clearly G-invariant and $V_2/V_1 = \langle [v_2] \rangle$. In consequence, the action of G on V_2/V_1 is trivial.

If we repeat that process, we obtain an ascending chain of vector subspaces, where the dimension of one of those subspaces is the dimension of the preceding subspaces plus one. Therefore, the length of the chain is $dim(V) = d$ and the chain is $0 = V_0 \subset V_1 \subset .... \subset V_d = V$. Because of the construction process, it is clear that the chain is G-invariant and that the action of G on V_{i+1}/V_i is trivial. □

Lemma 5.5. *Let $\mathcal{L}$,$\mathcal{L}'$ and $\mathcal{L}''$ be systems of local coefficients on a space X such that we have the following short exact sequence $0 \to \mathcal{L}' \xrightarrow{f} \mathcal{L} \xrightarrow{g} \mathcal{L}'' \to 0$. Then, there exists an induced long exact sequence on the homology*

$$... \to H_{k+1}(X;\mathcal{L}'') \to H_k(X;\mathcal{L}') \to H_k(X;\mathcal{L}) \to H_k(X;\mathcal{L}'') \to H_{k-1}(X;\mathcal{L}') \to ...$$

Proof. Recall that for every $x \in X$ we have group morphisms $f_x : \mathcal{L}'_x \longrightarrow \mathcal{L}_x$ and $g_x : \mathcal{L}_x \longrightarrow \mathcal{L}''_x$. Therefore, the short exact sequence of local coefficients on X induces a short exact sequence of group for all $x \in X$, $0 \to \mathcal{L}'_x \xrightarrow{f_x} \mathcal{L}_x \xrightarrow{g_x} \mathcal{L}''_x \to 0$.

We will prove that these short exact sequences induce a short exact sequence of chain complexes $0 \to C_*(X;\mathcal{L}') \xrightarrow{f_*} C_*(X;\mathcal{L}) \xrightarrow{g_*} C_*(X;\mathcal{L}'') \to 0$, hence, the lemma will be a direct consequence of the theorem from homological algebra that constructs a long exact sequences of homology from a short exact sequence of chain complexes.

For every natural number k, we define a linear map $f_k : C_k(X;\mathcal{L}') \longrightarrow C_k(X;\mathcal{L})$ such that $f_k(g \otimes T) = f_{T(v_0)}(g) \otimes T$. It clearly commutes with the differential of the chain complex $\partial_k f_k = f_{k-1}\partial'_k$.

$$\begin{aligned} f(\partial_h(g \otimes T)) &= f(h[TL^{v_0}_{v_1}](g)) \otimes \partial_0 T + \sum_{j=1}^{p}(-1)^j f(g) \otimes \partial_j T \\ &= (h[TL^{v_0}_{v_1}](f(g))) \otimes \partial_0 T + \sum_{j=1}^{p}(-1)^j f(g) \otimes \partial_j T \\ &= \partial_h(f(g \otimes T)). \end{aligned}$$

The process of defining the map g_* is analogous. Finally, observe that we have a short exact sequence of complexes $0 \to C_k(X;\mathcal{L}') \xrightarrow{f_k} C_k(X;\mathcal{L}) \xrightarrow{g_k} C_k(X;\mathcal{L}'') \to 0$ for all $k \in \mathbb{N}$, since $0 \to \mathcal{L}'_x \xrightarrow{f_x} \mathcal{L}_x \xrightarrow{g_x} \mathcal{L}''_x \to 0$ is a short exact sequence for all $x \in X$.

Therefore, we can construct the short exact sequence $0 \to C_*(X;\mathcal{L}') \xrightarrow{f_*} C_*(X;\mathcal{L}) \xrightarrow{g_*} C_*(X;\mathcal{L}'') \to 0$ using the previous statements, proving the lemma. □

Proposition 5.6. *Let X be a G-space. Then, the following inequality holds*

$$dim(H_k(BG;\mathcal{H}_q(X;\mathbb{Z}_p)) \leq \binom{k+r-1}{r-1} dim(H_q(X;\mathbb{Z}_p)).$$

Proof. Since G acts on X, a lineal action is induced on $H_q(X;\mathbb{Z}_p)$, which is a vector space over $\mathbb{Z}_p$. Let d be its dimension. Therefore, there exists an ascending chain of length d, $0 = V_0 \subset V_1 \subset ... \subset V_d = H_q(X;\mathbb{Z}_p)$, because of the lemma 5.4. with this chain, we can construct an ascending chain of systems of local coefficients $0 = \mathcal{H}_0 \subset \mathcal{H}_1 \subset ... \subset \mathcal{H}_d = \mathcal{H}_q(X;\mathbb{Z}_p)$, where $\mathcal{H}_i = EG \times_G V_i$. Since the group G acts trivially on the quotients

V_{i+1}/V_i, the system of local coefficients $\mathcal{H}_{i+1}/\mathcal{H}_i$ is trivial ($\mathcal{H}_{i+1}/\mathcal{H}_i = EG \times_G V_{i+1}/V_i = BG \times V_{i+1}/V_i$).

For each i from 0 to $d-1$, there exists a short exact sequence of systems of local coefficients $0 \to \mathcal{H}_i \xrightarrow{\iota} \mathcal{H}_{i+1} \xrightarrow{\pi} \mathcal{H}_{i+1}/\mathcal{H}_i \to 0$, induced by the inclusion and the projection. Thus, using lemma 5.5, we obtain the following long exact sequence of homology for each i.

$$... \to H_k(BG; \mathcal{H}_i) \xrightarrow{\iota_k} H_k(BG; \mathcal{H}_{i+1}) \xrightarrow{\pi_k} H_k(BG; \mathcal{H}_{i+1}/\mathcal{H}_i) \to H_{k-1}(BG; \mathcal{H}_i) \to ...$$

From this long exact sequence the following short exact sequence can be extracted,

$$0 \to Im(\iota_k) \hookrightarrow H_k(BG; \mathcal{H}_{i+1}) \to Im(\pi_k) \to 0.$$

Thus, the following equality regarding their dimensions holds

$$dim(H_k(BG; \mathcal{H}_{i+1})) = dim(Im(\iota_k)) + dim(Im(\pi_k)).$$

Since $dim(H_k(BG; \mathcal{H}_i)) = dim(ker(\iota_k)) + dim(Im(\iota_k))$, we obtain that

$$dim(Im(\iota_k)) \leq dim(H_k(BG; \mathcal{H}_i)).$$

Since $Im(\pi_k) \subset H_k(BG; \mathcal{H}_{i+1}/\mathcal{H}_i)$, we have that

$$dim(Im(\pi_k)) \leq dim(H_k(BG; \mathcal{H}_{i+1}/\mathcal{H}_i)).$$

Combining these statements, we obtain that

$$dim(H_k(BG; \mathcal{H}_{i+1})) - dim(H_k(BG; \mathcal{H}_i)) \leq dim(H_k(BG; \mathcal{H}_{i+1}/\mathcal{H}_i))$$

for all k and i.

If the number k is fixed and we sum all the inequalities varying i, most of the terms on the left of the inequality will cancel each other, reaching the below inequality

$$dim(H_k(BG; \mathcal{H}_q(X; \mathbb{Z}_p))) - dim(H_k(BG; \mathcal{H}_0)) \leq \sum_{i=0}^{d-1} dim(H_k(BG; \mathcal{H}_{i+1}/\mathcal{H}_i)).$$

Since $V_0 = 0$, then $\mathcal{H}_0 = 0$, hence, the second term on the inequality disappears. Because $\mathcal{H}_{i+1}/\mathcal{H}_i = BG \times V_{i+1}/V_i$ and $V_{i+1}/V_i \cong \mathbb{Z}_p$, the above inequality becomes

$$dim(H_k(BG; \mathcal{H}_q(X; \mathbb{Z}_p))) \leq \sum_{i=0}^{d-1} dim(H_k(BG; \mathbb{Z}_p)).$$

Using lemma 5.3 and that $d = dim(H_q(X; \mathbb{Z}_p))$, we obtain the inequality we sought,

$$dim(H_k(BG; \mathcal{H}_q(X; \mathbb{Z}_p)) \leq \binom{k+r-1}{r-1} dim(H_q(X; \mathbb{Z}_p)).$$

□

Remark 5.7. *Observe that the equality holds if the action of G on $H_q(X;\mathbb{Z}_p)$ is trivial. If we suppose that the action is trivial, $H_k(BG;\mathcal{H}_q(X;\mathbb{Z}_p)) = H_k(BG;H_q(X;\mathbb{Z}_p))$. Using the universal coefficients theorem with the field $R = \mathbb{Z}_p$, we obtain that $H_k(BG;H_q(X;\mathbb{Z}_p)) = H_k(BG;\mathbb{Z}_p) \otimes H_q(X;\mathbb{Z}_p)$. Thus, $dim(H_k(BG;H_q(X;\mathbb{Z}_p)) = dim(H_k(BG;\mathbb{Z}_p))dim(H_q(X;\mathbb{Z}_p))$.*

Remark 5.8. *In order to prove the Mann-Su theorem, we will use the upper bounds of lemma 5.6 when X is the frame orthogonal bundle of the given smooth connected manifold M, $F_g(M)$, where g is the G-invariant metric constructed in chapter 2. In this remark we explain how the previous remark will be used in the case where $q = 0$. Although M is connected, $F_g(M)$ is not necessarily connected, since $O(m,\mathbb{R})$ has two connected components (one consists of matrices with determinant equal to 1, and the other consists of matrices with determinant -1). Indeed, it can be proved that if M is non-orientable, then $F_g(M)$ is connected. Hence, $H_0(F_g(M);\mathbb{Z}_p) \cong \mathbb{Z}_p$. On the other hand, if M is orientable, then $F_g(M)$ has two connected components. In consequence $H_0(F_g(M);\mathbb{Z}_p) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$.*

If $p \neq 2$, then the action of the group G on $F_g(M)$ can not map points of one connected component to the other. Therefore, the action of G on $H_0(F_g(M);\mathbb{Z}_p)$ is trivial in both cases. By using the lemma 5.8, we obtain that $dim(H_k(BG;H_0(F_g(M);\mathbb{Z}_p)) = dim(H_k(BG;\mathbb{Z}_p))dim(H_0(F_g(M);\mathbb{Z}_p))$. In the non-orientable case, $dim(H_0(F_g(M);\mathbb{Z}_p)) = 1$. In the orientable case, $dim(H_0(F_g(M);\mathbb{Z}_p)) = 2$. In conclusion, we obtain that $dim(H_k(BG;H_0(F_g(M);\mathbb{Z}_p)) \geq dim(H_k(BG;\mathbb{Z}_p))$.

If $p = 2$ and M is orientable, the action could map points of one of the connected component of $F_g(M)$ to the other. If it was the case, the action of G on $H_0(F_g(M);\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ would not be trivial. However, with a similar reasoning from the lemma 5.4, there would exist a subgroup G' of G of index 2 (therefore $G' \cong (\mathbb{Z}_p)^{r-1}$) such that G' acts trivially on $H_0(F_g(M);\mathbb{Z}_2)$. In this case, we can work with G' instead of G and we can assume that the action is trivial on $H_0(F_g(M);\mathbb{Z}_2)$. Hence, we can use lemma 5.8 to obtain that $dim(H_k(BG';H_0(F_g(M);\mathbb{Z}_2)) \geq dim(H_k(BG';\mathbb{Z}_2))$. In consequence, the Mann-Su theorem will provide an upper bound N such that $r - 1 < N$. We can deduce that $r < N + 1$, obtaining an upper bound for the rank of G. For the sake of simplicity and brevity, we will not explicitly state this case in the proof the theorem (we will only need to recall to add 1 to the upper bound in this particular case).

The remaining lemmas are related with the homology of the manifolds, and they can be found in [20]. Although the underlying concepts are basically the same, we proceed in a different way in some steps of the proofs.

Lemma 5.9. *Let M be a compact topological manifold which has dimension m. Then $H_k(M) = 0$ for all $k > m$.*

Proof. Since M is a compact manifold, there exists a finite open covering $\{U_i\}_{i=1,\dots,r}$ such that each U_i is homeomorphic to an open set of $\mathbb{R}^m$. Firstly, we will show that an open set $U \subset \mathbb{R}^m$ satisfies that $H_k(M) = 0$ for all $k \geq m$.

Let $\sigma = \sum_{j=1}^{n} \lambda_j \sigma_j$ be a k-cycle (where $\sigma_j : \Delta^k \longrightarrow U$ with $k \geq m$ for all j). Observe that $K = \bigcup_{j=1}^{n} \sigma_j(\Delta^k) \subset U$ is compact, hence, the distance between K and $\mathbb{R}^m \setminus U$, which is a closed set, is positive, $d(K, \mathbb{R}^m \setminus U) = \delta > 0$. Our next step is to divide the Euclidean space in cubes that are sufficiently small so a cube which intersects with K is inside U. We choose cubes whose edges have length $l = \frac{1}{\sqrt{m}} \frac{\delta}{2}$.

Observe that if a and b are points of this cube

$$d(a,b) = \sqrt{\sum_{i=1}^{m}(a_i - b_i)^2} \leq \sqrt{\sum_{i=1}^{m}(\tfrac{1}{\sqrt{m}}\tfrac{\delta}{2})^2} = \tfrac{\delta}{2} < \delta.$$

We divide $\mathbb{R}^m$ in cubes whose edges have length equal to l and their vertices are in positions of the form $(n_1, ..., n_m)$ where $n_1, ..., n_m \in \mathbb{Z}$. More precisely, we will denote by $C_{(n_1,...,n_m)}$ the cube whose coordinates satisfy that $n_j l \leq x_j \leq (n_j + 1)l$ for each $j = 1, ..., m$. Since K is compact, there exists an open ball which has the origin as a center and a radius R such that $K \subset B_R(0)$. Let r denote an integer such that $R < rl$. We construct a cube T with edge's length $2rl$ and centred at the origin. Note that the cubes $C_{(n_1,...,n_m)}$ divide T.

Consider the space C, which is the union of all the cubes $C_{(n_1,...,n_m)}$ which intersect with K (note that because of the compactness of K there is only a finite number of cubes which has non-empty intersection with K). Because of the length of the edge we chose, $K \subset C \subset U$. On the other hand, observe that C and T have a CW-structure induced by the cube's division (their 0-skeleton is the set of vertices of all squares that form these spaces, to construct the 1-skeleton we attach the edges, and we proceed until obtaining the sets C and T). Observe that $C \subset T$ (in fact, C is a subcomplex of T and (T, C) is a called a *CW* pair). We can consider a portion of the relative homology exact sequence

$$... \to H_{k+1}(T, C) \to H_k(C) \to H_k(T) \to ...$$

Since T is a cube, we have that $H_k(T) = 0$. Since T and C do not posses $(k+1)$-cells, it follows that $H_{k+1}(T, C) = 0$. Therefore, $H_k(C) = 0$.

In consequence, σ is a boundary in C, and since $C \subset U$, it is a boundary in U. Thus, its class in the homology is 0. Because the chosen k-cycle was arbitrary, we conclude that all k-cycles are boundaries for $k \geq m$, hence, $H_k(U) = 0$ for $k \geq m$.

In order to prove the lemma, we shall use induction on a finite open cover of M and use the Mayer-Vietoris long exact sequence. The initial case has already been proved in the above discussion. Suppose we have proved that $H_k(\bigcup_{i=1}^{r-1} U_i) = 0$ for all $k > m$. Consider the Mayer-Vietoris long exact sequence

$$... \to H_{k+1}(\bigcup_{i=1}^{r-1} U_i) \oplus H_{k+1}(U_r) \to H_{k+1}(\bigcup_{i=1}^{r} U_i) \to H_k((\bigcup_{i=1}^{r-1} U_i) \cap U_r) \to ...$$

Since $(\bigcup_{i=1}^{r-1} U_i) \cap U_r \subset U_r$, the intersection is homeomorphic to an open subset of $\mathbb{R}^m$. Therefore, $H_k((\bigcup_{i=1}^{r-1} U_i) \cap U_r)$ and $H_k(U_r)$ are trivial for all $k \geq m$. By induction hypothesis, $H_k(\bigcup_{i=1}^{r-1} U_i) = 0$ for all $k > m$. The exactness of the Mayer-Vietoris sequence implies that $H_k(\bigcup_{i=1}^{r} U_i) = 0$ for all $k > m$. Since $M = \bigcup_{i=1}^{r} U_i$, the assertion of the lemma is proved. □

The last lemma we prove in this section provides an upper bound to the Betti numbers mod p that is independent of p.

Lemma 5.10. *Let M be a compact manifold of dimension m. Then, $H_*(M)$ is finitely generated.*

Proof. The first part involve embedding the manifold in a high-dimensional euclidean space $\mathbb{R}^n$.

Since M is a compact manifold, there exists a finite open covering $\{U_i, \phi_i\}_{i=1,...,k}$ such that $\phi_i : U_i \xrightarrow{\cong} B^m$, where B^m is the ball of dimension m which has radius equal to 1 and the origin as a center. Consider the homeomorphism $h : B^m \longrightarrow S^m \setminus \{N\}$, where N is the north pole of S^{m+1}. We construct maps $h_i : U_i \longrightarrow S^m$ such that $f_i(x) = (h \circ \phi_i)(x)$ if $x \in U_i$ and $f_i(x) = N$ if $x \notin U_i$ for each i. Let $f : M \longrightarrow S^m \times S^m \times .... \times S^m$ such that $f = (f_1, f_2, ..., f_k)$. This new map is continuous since the maps f_i are continuous and it is injective. Suppose that x and y are points of M which satisfy that $f(x) = f(y)$. Since the open sets U_i cover M, there exists a j such that $x \in U_j$, therefore $f_j(x) \neq N$. It follows from this fact that $f_j(y) \neq N$ and consequently $y \in U_j$, thus $(h \circ \phi_j)(x) = (h \circ \phi_j)(y)$. Since ϕ_j and h are homeomorphisms (in particular, they are bijective), we obtain that $x = y$. If we consider the standard embedding $S^m \subset \mathbb{R}^{m+1}$ and we compose it with each component f_i, we obtain an embedding of M in $\mathbb{R}^{(m+1)k}$. From this point on, we will suppose that the manifold is embedded in a high dimensional euclidean space $\mathbb{R}^n$.

The next step in the proof is to construct an open neighbourhood $U \subset \mathbb{R}^n$ of M and a map $r : U \longrightarrow M$ which is a retraction. Since the proof is quite long and technical, we will not provide them in this text. The proof of this statement can be found at the appendix 2 of [20]. A proof for a stronger property that has as a corollary the assertion we will not prove can be found at the appendix $\mathbb{A}$ of [4] and at the appendix E of [5].

We comment a basic outline of the proof. Let s be a simplex such that $M \subset s$ (it exists because the compactness of M). The idea is to construct a triangulation of $s \setminus M$ using barycentric subdivision and to use it to construct U and r inductively. The initial case starts by defining a map in the vertices of the simplicies of the triangulation. The inductive process involves extending continuously the map r from the $i-1$-simplices to some of the i-simplices of the triangulation where r is defined on their boundary.

It is interesting to remark that if M is a smooth manifold, a different approach to the proof of the previous statement can be used. The main statement used is called the tubular neighbourhood theorem (see e.g. [14], theorem 10.19 and proposition 10.20).

We proceed in a similar way to the previous lemma. Since M is compact, U is open and $M \subset U$. We can divide $\mathbb{R}^n$ in cubes which are small enough so the ones that intersect M are inside U. Let C denote the union of all cubes which have non-empty intersection with M. Note that C is the union of a finite number of cubes because of the compactness of M, therefore, C is a CW-complex. The retraction r can be restricted to a retraction $r_{|C} : C \longrightarrow M$. The equality $r_{|C} \circ \iota = id_M$, where ι denotes the inclusion, induces the following commutative diagram on the homology for every s:

$$\begin{array}{ccc} H_s(M) & \xrightarrow{\quad id_* \quad} & H_s(M) \\ & {\scriptstyle \iota_*} \searrow \quad \nearrow {\scriptstyle r_*} & \\ & H_s(C) & \end{array}$$

Since C is a CW-complex with a finite number of cells, $H_s(C)$ is finitely generated for all s. From the diagram, it follows that ι_* is injective for all s, therefore $H_s(M)$ is finitely generated. $\square$

Remark 5.11. *Observe that the previous lemma is valid if we work with coefficients in a field of the form $\mathbb{Z}_p$ instead of $\mathbb{Z}$. Thus, we obtain a upper bound for the s-th Betti number $b_s = dim(H_s(M;\mathbb{Z}_p)) \leq dim(H_s(C;\mathbb{Z}_p))$. In addition, $dim(H_s(C;\mathbb{Z}_p))$ is bounded by the number of s-cells of C, a number which does not depend of p. Let β_s denote the number of s-cells of C and let β be the sum of β_s for all s. Then, for all prime number p, the s-th Betti number mod p satisfies that $b_s \leq \beta_s$ and $B = \sum b_s \leq \beta$. Observe that β does not depend on p, but it is not optimal in any way, since it depends on the number of cubes (therefore, it depends on the size of the cubes) and the dimension n of the euclidean space, which could be high if we use the method explained at 5.10.*

An alternative way to justify that B possesses an upper bound is using the universal coefficients theorem. Since $H_s(M)$ is a finitely generated abelian group, we know that $H_s(M) \cong \mathbb{Z}^{r(s)} \oplus \mathbb{Z}_{k_1} \oplus ... \oplus \mathbb{Z}_{k_{n(s)}}$, where each k_i is a power of a prime number. From the universal coefficients theorem, it follows that $H_s(M;\mathbb{Z}_p) \cong (H_s(M) \otimes \mathbb{Z}_p) \oplus Tor(H_{s-1}(M),\mathbb{Z}_p)$, since the short exact sequence splits. Since the tensor product and the Tor functor are additive, $H_s(M;\mathbb{Z}_p)$ can be computed with the decomposition of $H_s(M;\mathbb{Z})$ and $H_{s-1}(M;\mathbb{Z})$. It is known that $\mathbb{Z} \otimes \mathbb{Z}_p \cong \mathbb{Z}_p$, $\mathbb{Z}_{k_i} \otimes Z_p$ is isomorphic to $\mathbb{Z}_p$ if p divides k_i and 0 otherwise, $Tor(\mathbb{Z},\mathbb{Z}_p) = 0$ and $Tor(\mathbb{Z}_{k_i},\mathbb{Z}_p) = \mathbb{Z}_{gcd(k_i,p)}$, where gcd denotes the greatest common divisor. With the aforementioned information, $H_s(M;\mathbb{Z}_p)$ can be computed easily. In particular, if α_s denotes the number of addends of the decomposition of $H_s(M)$ ($\alpha_s = r(s) + n(s)$), the inequality $dim(H_s(M;\mathbb{Z}_p) \leq \alpha_{s-1} + \alpha_s$ holds. Therefore, we can find and upper bound for B, namely, $B \leq 2\sum \alpha_s = \alpha$.

5.2 The Mann Su theorem

Theorem 5.12. *Let M be a compact, connected and smooth manifold. There exists a natural number N such that any elementary p-group of rank $r > N$ can not act effectively on M.*

Before starting the theorem's proof, an essential remark shall be made. We are searching a upper bound that only depends on parameters of the manifold. Indeed, we shall be extremely careful that each step we make towards finding the upper bound does not depend on the group.

Proof. We suppose that an elementary p-group of rank r, G, acts effectively on M, and we will find a upper bound for r that only depends on parameters related with M.

Since M is a compact and connected smooth manifold and the action is effective, then $F_g(M)$ is compact (where g denotes an invariant metric). Therefore, we can suppose that the action is free if we work with $F_g(M)$ instead of M, by proposition 2.27. Note that because of proposition 2.14, the homological properties of the space $F_g(M)$ do not depend on group action, thus, we can replace M for $F_g(M)$ and the bound found will continue depending only on the parameters of the manifold. Committing an abuse of notation, we will write M instead $F_g(M)$. Let m denote its dimension.

By remark 4.7, there exists a first quadrant spectral sequence $\{E^q, d^q\}$, such that $E^2_{t,s} = H_t(BG;\mathcal{H}_s(M;\mathbb{Z}_p))$ and converges to $H^G_{t+s}(M;\mathbb{Z}_p)$. Since M is compact, by lemma 5.9 we know that $H_q(M;\mathbb{Z}_p) = 0$ for all $q > m$. Therefore, $E^2_{t,s} = 0$ for all $s > m$, hence, $E^q_{t,s} = 0$ for all $s > m$ and $q \geq 2$. Recall that $d^q_{t,s} : E^q_{t,s} \longrightarrow E^q_{t-q,t+q-1}$. This implies that for $q \geq m+2$, all differentials are 0. Thus $E^{m+2} = E^{m+3} = ... = E^{\infty}$.

We have a first quadrant spectral sequence, hence, $E^{q+1}_{t+1,0} = kerd^{q+1} : E^{q}_{t+1,0} \longrightarrow E^{q}_{t+1-q,q-1}$. Therefore,

$$dim(E^{q+1}_{t+1,0}) = dim(E^{q}_{t+1,0}) - dim(Imd^{q+1}) \geq dim(E^{q}_{t+1,0}) - dim(E^{q}_{t+1-q,q-1}).$$

Using that $dim(E^{q}_{t+1-q,q-1}) \leq dim(E^{2}_{t+1-q,q-1})$, the following inequality holds,

$$dim(E^{q+1}_{t+1,0}) - dim(E^{q}_{t+1,0}) \leq dim(E^{2}_{t+1-q,q-1}).$$

If we denote $q-1=j$, we have

$$\sum_{j=1}^{m} dim(E^{2}_{t-j,j}) \geq \sum_{j=1}^{m}(dim(E^{j+1}_{t+1,0}) - dim(E^{j+2}_{t+1,0})) = dim(E^{2}_{t+1,0}) - dim(E^{m+2}_{t+1,0}).$$

Rearranging the terms of the inequality and recalling that $E^{m+2} = E^{\infty}$, we obtain that

$$dim(E^{2}_{t+1,0}) \leq \sum_{j=1}^{m} dim(E^{2}_{t-j,j}) + dim(E^{\infty}_{t+1,0}).$$

Since the action on M is free, $H^{G}_{t+1}(M;\mathbb{Z}_p) = H_{t+1}(M/G;\mathbb{Z}_p)$. Because M/G is compact and has dimension m, we know that $H_{t+1}(M/G;\mathbb{Z}_p) = 0$ for all $t \geq m$ (by lemma 5.9). Thus, since the spectral sequence converge to $H^{G}(M)$, we obtain that $E^{\infty}_{t+1,0} = 0$ for all $t \geq m$. If we impose that $t = m$ and we use the bounds found in proposition 5.6 and take into account the remark 5.7 and 5.8, the above inequality transforms into

$$\binom{m+r}{r-1} \leq \sum_{j=1}^{m} \binom{m-j+r-1}{r-1} dim(H_j(M;\mathbb{Z}_p)) = \sum_{j=1}^{m} \binom{m-j+r-1}{r-1} b_j.$$

Recall that $dim(H_j(M;\mathbb{Z}_p)) = b_j$, which are the Betti numbers mod p. If we denote $i = m-j$, the previous inequality becomes

$$\binom{m+r}{r-1} \leq \left(\sum_{i=0}^{m-1} \binom{i+r-1}{r-1}\right) max_i\{b_i\} = \binom{m+r-1}{r} max_i\{b_i\}.$$

If we define $B = \sum_{i=0}^{m} b_i$ and we use that $\binom{m}{n} = \frac{m!}{n!(m-n)!}$, we obtain the inequality

$$\frac{(m+r)!}{(r-1)!(m+1)!} \leq \frac{(m+r-1)!}{r!(m-1)!} max_i\{b_i\} \leq \frac{(m+r-1)!}{r!(m-1)!} B.$$

If we rearrange the factorial terms, we reach the following inequality,

$$r^2 + mr \leq m(m+1)B.$$

The bound that the theorem asserts is found by solving the second grade equation on r provided by the above inequality.

$$r \leq f(m,B) = \frac{\sqrt{m^2 + 4m(m+1)B} - m}{2}.$$

Remark 5.13. If we replace B by one of the upper bounds α or β found in the remark 5.11, we obtain and upper bound for k which is also independent of p. Hence, $f(m,\alpha)$ or $f(m,\beta)$ provides and upper bound for the groups of the form $(\mathbb{Z}_p)^k$ which can act effectively on M for any prime number p.

If we make $N = f(m,\alpha)$ or $N = f(m,\beta)$, this last remark finishes the proof of the theorem.

□

If the action is free the upper bound depends only on the dimension of the manifold M and the sum of its Betti numbers mod p. If the actions is effective but not free, it depends on the dimension of the orthogonal frame bundle $F_g(M)$ and its Betti numbers mod p, which only depend on the manifold M. Albeit it is not extremely difficult to compute the dimension and the Betti numbers of $F_g(M)$, we do not present these computations in this text.

We could ask ourselves whether this bound is optimal. It is a remarkable fact that this bound was computed for an arbitrary manifold, therefore it would not be optimal most of the times, since the specific properties of a manifold are not used. For example, P.A. Smith proved that if $M = S^n$, then $r \leq \frac{n+1}{2}$ if $p \neq 2$ and $r \leq n+1$ in any case (see [21]). Mann and Su provide bounds for elementary 2-groups acting effectively on $\mathbb{R}P^n$ and for elementary p-groups acting effectively on L_p^n in [1].

An easy example where we can compute the upper bound $f(m,B)$ is the case where $M = S^1$ and the action is free. Using the universal coefficients theorem, we obtain that $H_n(S^1;\mathbb{Z}_p) = \mathbb{Z}_p$ if $n = 0,1$ and $H_n(S^1;\mathbb{Z}_p) = 0$ if $n > 1$. Thus, $m = 1$ and $B = 2$. Consequently, the value of $f(m,B) = \frac{\sqrt{17}-1}{2} < 2$. Since r shall be a natural number, this implies that the only elementary p-groups that acts freely on S^1 is the cyclic group $\mathbb{Z}_p$. As we showed in the preface, the cyclic group $\mathbb{Z}_p$ acts freely on S^1 by performing rotations. Therefore, the theorem provides an "optimal" bound in that cases (in the sense that the theorem states that the the maximum rank of an elementary p-group acting on S^1 is 1, although $f(m,B) \neq 1$).

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