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THE DE RHAM THEOREM AND AN APPLICATION ON THE LIE GROUP THEORY

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Abstract

The origin of the algebraic topology opened a new way to study the geometric properties through some algebraic invariants. Over the years, mathematicians have been capable of developing different bridges between these two areas. The final goal of this text is to go in depth into one of these connections: the de Rham cohomology. Likewise, we study the main result of this theory: the de Rham theorem. In this text, we obtain this result as a consequence of a more general theorem on sheaf theory. The de Rham theorem plays an essential role in the area of differential geometry where it has many implications. In the final part of this text, we explain a significant application in Lie group theory.

Resumen

El nacimiento de la topología algebraica abrió una nueva manera de estudiar las propiedades geométricas a partir de invariantes algebraicos. Con el curso de los años, los matemáticos han sido capaces de alzar diferentes puentes de conexión entre estas dos áreas. El objetivo último de este texto es adentrarnos en una de estas conexiones: la cohomología de de Rham. A su vez, estudiamos el resultado principal de esta teoría: el teorema de de Rham. En este texto, obtenemos este resultado como consequéncia de un teorema más general sobre teoría de haces. El teorema de de Rham juega un papel fundamental en el ámbito de la geometría diferencial. Entre las diferentes aplicaciones de esta teoría, en la parte final de este texto explicamos una aplición importante en la teoría de los grupos de Lie.

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Content

Preface

Over the past two centuries, mathematicians have developed wondrous theories that enable us to peer ever more deeply into the invisible world of abstract concepts and their interrelationships. In this era, new areas in geometry have sprouted, amongst them topology, differential geometry, and algebraic geometry (as we know it today). However, there were considerable difficulties to fully grasp the new-coming ideas. Namely, the notions were assembled from layer upon layer of the abstract structure producing complex architectures.

In this context, category theory blossomed with the observation that many properties of mathematical systems could be unified and simplified by a presentation with diagrams. Lighting up a way to systematically probing the logical structures, Samuel Eilenberg and Saunders Mac Lane introduced the concepts of categories, functors, and natural transformations in their study of algebraic topology (1942–45). Eventually, the concept of category has been increasingly employed in all branches of mathematics, especially in studies where the relationship between different branches is of importance.

The category theory arose hand in hand with the fundamental group, which was its main inspiration. Indeed, the first homotopy group was also the incentive for several functorial ways of the category of topological spaces. These functors associate to each spaces an algebraic object like a group or a vector space, so that homeomorphic spaces have isomorphic objects. Among them, the reader may be familiar with some types of homologies and cohomologies. To make an overall idea, one can perceive that most of these functors somehow measure the existence of "holes" in different dimensions. In this sense, the de Rham theory of differential forms has notorious importance, since it plays a central role in differential geometry.

The de Rham cohomology is based on differential forms on a smooth manifold using the exterior derivative as a boundary operator (Chapter 3). Against this background, a smooth form is called closed if the differential cancels it. A form contained in the image of the differential is called exact. In this way, all exact forms are closed. In some sense, this cohomology comes up to answer the question of which conditions make closed forms be exact, defining the de Rham cohomology groups as the quotient of closed forms modulo exact forms. In other words, the de Rham groups are a way of understanding a manifold's global topology via the tangent bundle.

Although the de Rham theorem is usually set today using cohomology, it was not the way that Georges de Rham proved the theorem in 1930. The reason is that in those days, the concepts of cohomology, homology, and manifolds had not been defined yet. Oddly enough, reading the paper where he wrote his famous theorem, one can figure out that the de Rham cohomology groups practically stared him in the face. Without too much strife, he could have gone ahead and made the definition, but this probably seemed pointless to him at the time. Instead of using cohomologies, he proved this theorem regarding homology and the integration of differential forms over smooth chains. Apparently, the first modern statement and proof of the De Rham theorem in terms of cohomology was in mimeographed notes of lectures by H. Cartan in 1947.

The principal result of this text is the de Rham theorem (3.4). With this in mind, we shed light onto the sheaf theory. Defining the sheaf cohomology, we prove a fantastic but simple uniqueness theorem for homomorphisms of sheaf cohomology theories (2.5). We are going to exhibit both the de Rham cohomology and the differential singular cohomology as special cases of sheaf cohomology theories. We also prove that the natural homomorphism between the de Rham and differentiable singular theories is an isomorphism. As an added result to this approach, we shall also prove the existence of canonical isomorphism of the continuous singular category to those mentioned above. From these isomorphisms, we conclude that the de Rham cohomology is a topological invariant of a differential manifold.

Knowing which closed forms are exact, brings us important consequences that make this theorem so significant in the differential geometry. For instance, Stokes' theorem implies that if a form is exact, then the integral of it over any compact submanifold without boundary is zero. Another simple consequence is that a smooth 1-form is conservative if, and only if, it is exact. From that, if we combine de Rham theory with Hodge theory, we can find many applications in Quantum Mechanics, too. There is also an interesting application on the Lie group theory that we are going to study in this text.

Lie group theory is named after the Norwegian mathematician Sophus Lie, who was the first person to define this smooth groups. These groups endow smooth manifolds with a group structure in a smart way. In the nineteenth century, Lie tried to simplify problems in partial differential equations using symmetries expressed in the form of group actions. His essential idea was inspired by how the algebraist Evariste Galois had invented group theory and used it to analyze polynomial equations. However, Lie could not have conceived the global objects that we now call Lie groups, for the simple reason that global topological notions such as a manifold (or even topological spaces!) had not yet been formulated. What Lie studied was essentially a local-coordinate version of Lie groups, now called local Lie groups. Despite the limitations imposed by the era in which he lived, he was able to lay much of the groundwork for the current understanding of Lie groups.

The result on Lie group theory that we prove at the end of this text is that the only spheres which admit a Lie group structure are $\mathbf{S}^0$, $\mathbf{S}^1$ and $\mathbf{S}^3$. Since $\mathbf{S}^0$ is constituted by only two points, it has obvious differential and group structures. Regarding the circumference, $\mathbf{S}^1$, one can figure out that this has a group structure induced by complex numbers with the product operation. If someone thinks about the sphere $\mathbf{S}^3$, one will realize that similar phenomenon happens as in $\mathbf{S}^1$, using the quaternions. This sequence ends at that point, due to octonions not having a group structure. The amazing fact is that there was not any way to make up a Lie group structure over the other spheres.

We would want to conclude this introduction with a quick scan of the text that is about to start reading. The core material to prove the de Rham theorem and the proof itself is contained in Chapters 1, 2 and 3, where our approach is mainly based on the proof given by Warner in [1]. To the ideas inspired by this reference, we have added the basic notions of the category theory. The category theory has an essential role in the sheaf theory, since sheaf and presheaf can be defined in a general way using categories and functors. Besides, the cohomology has a particular point of view as a functor application. In this way, the theory of categories does not only simplify the organization and comprehension of the text but also makes possible a more detailed study of the same. We include some illustrative examples to have a better appreciation of the definitions.

Chapter 1 is about the sheaf theory where no previous knowledge of this theory is assumed. It starts with the definition of sheaves and setting the sheaf homomorphisms following by a definition of presheaves and their homomorphisms. By doing so, we have defined the category of sheaves and the category of presheaves. We end this chapter with the construction of the sheafification morphism and exact sequence of sheaves.

Chapter 2 go into the sheaf cohomology theories. We define sheaf cohomology theories and the morphisms, developing a category. It continues defining resolutions of a sheaf which are useful tools that draw a path to build a cohomology theory based on a sheaf. Finally, we proof such an incredible theorem that says that any two sheaf cohomology theories on a manifold are uniquely isomorphic (2.6).

In Chapter 3 we define the cohomologies that are involved in the de Rham theorem namely the de Rham cohomology and the differential singular cohomology. We also define the continuous singular cohomology. We study how to integrate forms over singular simplices, and we prove the Stokes' theorem. We finish it proving that the de Rham homomorphism is, indeed, an isomorphism.

At the second part of this text, we show the application of the de Rham theorem seen above. We have mixed ideas from [1] and [2], giving an exhaustive and pedagogical result. We define the general way to integrate functions over a manifold. In this Chapter, there is a significant component of Lie groups Theory. Lie groups, Lie algebras, the exponential map and the adjoint representation are defined in it. Finally, we prove that the only spheres which admit a Lie group structure are $\mathbf{S}^0$, $\mathbf{S}^1$ and $\mathbf{S}^3$ (4.21).

The images were taken from Warner [1] and Lee [3].

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Chapter 1

Sheaf Theory

In this chapter, we develop fundamental properties of sheaves and presheaves. The principal goal in the first section is to define sheaves. In this context where we have differential manifolds, there are two ways to define the category of sheaves. While the second path is the usual way to define sheaves in any context, the first one is going to be more illustrative. Finally, we are going to see that these two definitions are equivalent.

Simplifying notation, M is going to be taken as a differential manifold despite that in many results we only need M as a second countable, Hausdorff and paracompact topological space. Also, if we do not say the opposite, K will be a Principal Ideal Domain. We assume that the reader knows elementary concepts of smooth manifolds.

1.1 The Sheaf Category

Definition 1.1. *Let $\mathcal{S}$ be a topological space and let $\pi : \mathcal{S} \to M$ be a map. We say that $\mathcal{S}$ with π is a Sheaf of K-modules over M if:*

a) π is a local homeomorphism of $\mathcal{S}$ onto M.

b) For each $m \in M$ the set $\pi^{-1}(m)$ is a k-module.

c) The k-module operations in $\pi^{-1}(m)$ are continuous with the topology of $\mathcal{S}$.

Generally, we use only $\mathcal{S}$ to denote the Sheaf, without mention the application π neither the topology space M.

The map π is called the *projection*; and the K-module $\mathcal{S}_m = \pi^{-1}(m)$ is called *stalk over m*, where $m \in M$. We have defined objects of the category of sheaves. Now we have to establish morphisms. A *sheaf mapping* is a continuous map $\psi : \mathcal{S} \to \mathcal{S}'$ such that $\pi' \circ \psi = \pi$, where $\mathcal{S}$ and $\mathcal{S}'$ are sheaves over M with projections π and π' respectively. It is straightforward that these morphisms map stalks into stalks, and they are local homeomorphism since π is. A sheaf mapping ψ which its restriction on each stalk is a K-module homomorphism is called a *sheaf homomorphism*. A sheaf homomorphism which has an inverse that is also a sheaf homomorphism is a *sheaf isomorphism*. It is trivial to see that these morphisms fulfill the properties in the category definition. Therefore, we have defined the *category of sheaves*.

The most straightforward example of a sheaf over M is that of a so-called *constant sheaf* $\mathcal{G} = M \times G$, where G is a K-module with the discrete topology and $\mathcal{G}$ is given by the product topology. Here the projection is simply $\pi(m, g) = m$. Despite the simplicity of the constant sheaf, it is incredibly useful.

A less trivial example in the *sheaf of germs of* C^∞ *functions on* M. Let $m \in M$ and let f and g functions defined on open sets containing m are said to have the same *germ* at m if they agree on some neighborhood of m. These introduces an equivalent relation on C^∞ functions defined on neighborhoods of m. Namely, two functions being equivalent if, and only if, they have the same germ. The equivalent classes are called *germs*, and we denote the set of germs at m by $\widetilde{F}_m$. Note that $\widetilde{F}_m$ is a real vector space. Let

$$\mathcal{C}^\infty(M) = \bigcup_{m \in M} \widetilde{F}_m \,. \tag{1.1}$$

We define the projection $\pi : \mathcal{C}^\infty \to M$ in the obvious fashion so that $\mathbf{f} \in \widetilde{F}_{\pi(\mathbf{f})}$.

We want to fix a topology in $\mathcal{C}^\infty(M)$ such that π was a homeomorphism, in this way we use open subsets of M to define the topology. Let $U \subset M$ be an open subset. We associate each C^∞ function f on U to the set

$$\bigcup_{m \in U} \mathbf{f}_m = \mathcal{C}^\infty(M), \tag{1.2}$$

where $\mathbf{f}_m$ is the germ of f at m. The collection of these sets forms a basis for a topology on $\mathcal{C}^\infty(M)$ which makes $\mathcal{C}^\infty(M)$ into a sheaf of real vector spaces.

Definition 1.2. *Let $\mathcal{S}$ be a sheaf and let $U \subset M$ an open subset. A continuous map $f : U \to \mathcal{S}$ such that $\pi \circ f = id$ is called a section of $\mathcal{S}$ over U.*

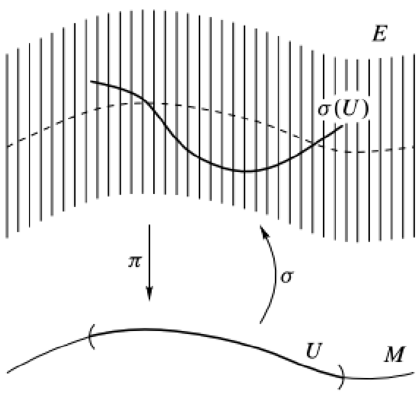


Figure 1.1: A section of a sheaf

Observe that sections are open maps because π is a local homeomorphism. We denote by $\Gamma(\mathcal{S}, U)$ the set of sections of $\mathcal{S}$ over U. If we take $U = M$ we call $\Gamma(\mathcal{S}, U)$ the set of *global sections* or only the set of *sections* which will simply be denoted by $\Gamma(\mathcal{S})$. We will see that this set is a k-module induced by the structure of each $\mathcal{S}_m$.

A special subset of sections is the *0-sections*. The *0-section* is the section f which maps each $m \in M$ into $0_m \in \mathcal{S}_m$, where 0_m is the zero element of the K-module $\mathcal{S}_m$.

We lay down the operations in $\Gamma(\mathcal{S}, U)$ to give it a K-module structure. Let $f, g \in \Gamma(\mathcal{S}, U)$, and let $k \in K$. We define the sum of f and g to be

$$(f + g)(m) = f(m) + g(m), \ \forall m \in M \tag{1.3}$$

and we define the product of f and k to be

$$(kf)(m) = k\big(f(m)\big), \ \forall m \in M. \tag{1.4}$$

It is easy to check that $f + g$ and kf are sections of $\mathcal{S}$ over U. Therefore, $\Gamma(\mathcal{S}, U)$ with these operations is a K-module. Observe that any element in $\Gamma(\mathcal{S}, U)$ can be composed with a sheaf homomorphism $\psi : \mathcal{S} \to \mathcal{S}'$ giving an element of $\Gamma(\mathcal{S}', U)$. It is straightforward that $\Gamma(\psi) : \Gamma(\mathcal{S}, U) \to \Gamma(\mathcal{S}', U)$, defined by the composition of sections with ψ mentioned above, is a k-module homomorphism with the operations $\psi(f + g) = \psi(f) + \psi(g)$ and $\psi(kf) = k\psi(f)$ for any $f, g \in \Gamma(\mathcal{S}, U)$ and $k \in K$. We have that $\Gamma(\psi \circ \varphi) = \Gamma(\psi) \circ \Gamma(\varphi)$ and $\Gamma(Id_{\mathcal{S}}) = Id_{\Gamma(\mathcal{S},U)}$. In other words, for any U open subset of M, we can see $\Gamma(., U)$ as a functor from the category of sheaves to the category of K-modules.

Definition 1.3. *Let $\mathcal{S}$ be a sheaf. An open set $\mathcal{R} \subset \mathcal{S}$ such that $\mathcal{R}_m = \mathcal{R} \cap \mathcal{S}_m$ is a submodule of $\mathcal{S}_m$ for each $m \in M$ is a subsheaf of $\mathcal{S}$. It is clear that subsheaves of sheaves are again sheaves with the natural topology and the restriction of the projection.*

Let $\psi : \mathcal{S} \to \mathcal{S}'$ be a sheaf homomorphism. The *kernel* of ψ is the subset of $\mathcal{S}$ mapped into the 0-section of $\mathcal{S}'$ by ψ, i.e., an $U \subset \mathcal{S}$ open set is a subset of the kernel of ψ if there exists a 0-section f_0 in $\Gamma(\mathcal{S}', U)$ such that,

$$\begin{array}{ccc} U & \xrightarrow{\pi} & M \\ \downarrow{\scriptstyle \psi|_U} & \swarrow{\scriptstyle f_0} & \\ \mathcal{S}'. & & \end{array}$$

The kernel of ψ is, indeed, a subsheaf of $\mathcal{S}$. The subset $im(\psi) = \psi(\mathcal{S}) \subset \mathcal{S}'$ is a subsheaf which we call the *image subsheaf*.

Let $\mathcal{R}$ be a subsheaf of $\mathcal{S}$. For each $m \in M$, we denote by $\mathcal{Z}_m$ the quotient of modules $\mathcal{S}_m/\mathcal{R}_m$, and we take

$$\mathcal{Z} = \bigcup_{m \in M} \mathcal{Z}_m. \tag{1.5}$$

Let $\tau : \mathcal{S} \to \mathcal{Z}$ be the natural projection which associates to each element of $\mathcal{S}_m$ its coset in $\mathcal{Z}_m = \mathcal{S}_m/\mathcal{R}_m$ for each $m \in M$. We give $\mathcal{Z}$ the quotient topology by the function τ, i.e., a set $U \subset \mathcal{Z}$ is an open subset if, and only if $\tau^{-1}(U)$ is open in $\mathcal{S}$. Therefore, the natural projection $\pi_{\mathcal{Z}} : \mathcal{Z} \to M$ which maps each element of $\mathcal{Z}_m$ to m is a local homeomorphism, being $\mathcal{Z}$ a sheaf over M. $\mathcal{Z}$ is called the *quotient sheaf of* $\mathcal{S}$ *module* $\mathcal{R}$. By that construction, we have that $\tau : \mathcal{S} \to \mathcal{Z}$ is a sheaf homomorphism. $\mathcal{Z}$.

Let $\psi : \mathcal{S} \to \mathcal{R}$ be a sheaf homomorphism. Then, we have the usual isomorphism defined by the natural map $\mathcal{S}/ker(\psi) \to im(\psi)$ given by $(s + (ker(\psi)|\mathcal{S}_m)) \mapsto \psi(s)$ for $s \in \mathcal{S}_m$.

1.2 The Presheaf Category

We have already presented sheaves. As we have mentioned, two different ways could be used to define sheaves. Now we are going to proceed with the second one. For this propose we need a few previous notions. We will start introducing the *category of open subsets* of a topological space, which will be used to define a *presheaf*.

Definition 1.4. *Let X be a topological space. We denote by τ_X the category such that:*

a) ***Ob(τ_X)****=$\{U \subset X|$ open subset of $X\}$ is the set of objects.*

b) ***Mor(τ_X)****=$\{\rho_{U,V} \in Hom(V, U) | V \subset U$ and $\rho_{U,V}(x) = x, \forall x \in U\}$ is the set of morphisms, just the usual inclusions.*

It is straightforward that τ_X is a category because if $V \subset U \subset W$ are open subsets we have that the composition is associative, $\rho_{W,U} \circ \rho_{U,V} = \rho_{W,V}$, and $\rho_{U,U} = Id_U$. We call τ_X *the category of open subsets of* X.

We are ready to define the presheaf category. Let us start defining the objects and then we will determine morphisms.

Definition 1.5. *Let X be a topological space, τ_X the category of open subsets of X and $\mathcal{C}$ another category. Then, a presheaf on the category $\mathcal{C}$ is a contravariant functor from τX to $\mathcal{C}$*

$$\begin{aligned} F:\ \tau_X &\to \mathcal{C} \\ U &\mapsto F(U) \end{aligned} \tag{1.6}$$

Let F and G be presheaves on the category $\mathcal{C}$. We have to define morphisms in order to have the *Category of presheaves on the category* $\mathcal{C}$. We define a *presheaf homomorphism of F to G* as a *natural transformation η* of the functors in order to have a category. A natural transformation η from F to G is a family of morphisms satisfying two conditions:

a) The natural transformation associates to every object U of τ_X a morphism $\eta_U : F(U) \to G(U)$ between objects of $\mathcal{C}$.

b) The morphism η_U defined above must be such that for every morphism $\rho_{U,V} \in \mathrm{Mor}(\tau_X)$ we have that the following diagram commutes

$$\begin{array}{ccc} F(V) & \xrightarrow{F(\rho_{U,V})} & F(U) \\ \downarrow{\scriptstyle \eta_V} & & \downarrow{\scriptstyle \eta_U} \\ G(V) & \xrightarrow{G(\rho_{U,V})} & G(U). \end{array}$$

Here we are going to talk only about presheaves on the category of K-modules. Therefore, we can simplify the notation by taking a presheaf $P = \{S_U;\, \rho_{U,V}\}$ consisting of a K-module S_U for each open subset $U \subset M$ and a K-module homomorphism $\rho_{U,V}$ for each inclusion $U \subset V$ of open subsets in M, such that $\rho_{U,U} = Id|_U$, and such that whenever $V \subset U \subset W$, the following diagram commutes:

$$\begin{array}{ccc} S_W & \xrightarrow{\rho_{V,W}} & S_V \\ \downarrow{\scriptstyle \rho_{U,W}} & \swarrow{\scriptstyle \rho_{U,V}} & \\ S_U. & & \end{array}$$

Let $P = \{S_U;\, \rho_{U,V}\}$ and $P' = \{S'_U; \rho'_{U,V}\}$ be presheaves over M. Therefore, by a presheaf homomorphism between P and P' we mean a collection $\{\psi_U\}$ of K-module homomorphisms $\psi_U : S_U \to S'_U$ such that

$$\rho'_{U,V} \circ \psi_V = \psi_U \circ \rho_{U,V}. \tag{1.7}$$

Now, we can see an example of a presheaf related with the sheaf of germs of C^∞ functions on M, which we saw above. For each U open set in M, let $F(U)$ be the set of C^∞ functions over U. $F(U)$ is a real vector space with the operations induced by the structure of $\mathbb{R}$. For each $U \subset V$ open subsets, we can consider the morphism $\rho_{U,V}$ which restricts into U the functions over V. Therefore, we have the presheaf $\{F(U);\, \rho_{U,V}\}$.

At this moment, we can give the second definition of sheaves. We can consider sheaves on any category $\mathcal{C}$, not necessarily the K-modules one. After this definition, we will see that both definitions of sheaves are equivalent in the context where we are.

Definition 1.6. *Let $\mathcal{F}$ be a presheaf on X. We say that $\mathcal{F}$ is a sheaf if whenever the open set $U \subset X$ is expressed as a union $\cup_{\alpha \in I} U_\alpha$ of open sets in X, the following two conditions are satisfied:*

(C_1) Locality: whenever $s, t \in \mathcal{F}(U)$ are such that $\mathcal{F}(\rho_{U_\alpha,U})(s) = \mathcal{F}(\rho_{U_\alpha,U})(t)$ for all $\alpha \in I$ then $s = t$.

(C_2) Gluing: whenever there are elements $s_\alpha \in \mathcal{F}(U_\alpha)$ for each $\alpha \in I$ such that, for each $\beta \in I$, there is a $s_\beta \in \mathcal{F}(U_\beta)$ that $\mathcal{F}(\rho_{U_\alpha \cap U_\beta, U_\alpha})(s_\alpha) = \mathcal{F}(\rho_{U_\alpha \cap U_\beta, U_\beta})(s_\beta)$, then there exists $s \in \mathcal{F}(U)$ such that $s_\alpha = \mathcal{F}(\rho_{U_\alpha,U})(s)$.

It is the usual way to define sheaves. To do not get confused, we will say that a presheaf fulfilling the properties C_1 and C_2 is a *complete presheaf*. Note that the set of complete presheaves is a subset of the presheaf category, it is easy to see that this subset forms a category itself.

The presheaf, $\{F(U); \rho_{U,V}\}$, saw above is an example of a complete presheaf. Note that, whenever f_1 and f_2 are C^∞ functions over U_1 and U_2, open sets in M, respectively where $U_1 \cap U_2 \neq \emptyset$. If we have that $f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$, there exists a unique function f over $U_1 \cup U_2$ such that $f|_{U_1} = f_1$ and $f|_{U_2} = f_2$. Using the existence of f one can prove that $\{F(U); \rho_{U,V}\}$ fulfills C_2, and using the uniqueness of f one can prove that $\{F(U); \rho_{U,V}\}$ fulfills C_1. That example can give an idea to understand properties C_1 and C_2.

Now, we give a similar example to the last one. Taken an open set $U \subset M$, we consider the set of differential k-forms on U which we have to denote by $E^k(U)$. Note that this is a real vector space. Let $\rho_{U,V}$ be the map which restricts a k-form on V to a k-form on U. Yielding the presheaf

$$\{E^k(U); \rho_{U,V}\}. \tag{1.8}$$

Observe that this presheaf fulfills the properties (C_1) and (C_2), so it is a complete presheaf. We will use this presheaf when we talk about the de Rham Cohomology.

1.3 Relation Between Sheaves and Presheaves

In this section, we are going to see that definitions given above are equivalent. From now on, all sheaves and presheaves are going to be taken in the category of modules, and it is going not to be necessary to specify the category. We start showing that every sheaf given by the first definition leads canonically to a presheaf.

1.3.1 The Sheafification Morphism

Given a sheaf $\mathcal{S}$ over M we define the function $\rho_{U,V} : \Gamma(\mathcal{S}, V) \to \Gamma(\mathcal{S}, U)$, where $V \subset U \subset M$ are open subsets, as the map which carries the sections of $\mathcal{S}$ over V to its restrictions into U. Associating with each open set U in M the K-module $\Gamma(\mathcal{S}, U)$ and the morphisms $\rho_{U,V}$, where $V \subset U$, we have a presheaf $\{\Gamma(\mathcal{S}, U); \rho_{U,V}\}$. In other words, Γ is a contravariant functor, and, thus, a presheaf.

Conversely, we shall now show that each presheaf canonically determines a sheaf. This process is called *sheafification*. In practice, many sheaves will in this way arise naturally from presheaves. The best example to keep in mind during the following construction is the presheaf $\{F(U); \rho_{U,V}\}$; its associated sheaf will be the sheaf of germs of C^∞ functions on M.

Let $P = \{S_U; \rho_{U,V}\}$ be a presheaf of K-modules on M. Let $m \in M$, and let S^*_m be the disjoint union of each of the modules S_U for which $m \in U$. If we set $f \in S_U$ equivalent to $g \in S_V$ if, and only if there is a neighbourhood W of m whith $W \subset U \cap V$ such that $\rho_{W,U}(f) = \rho_{W,V}(g)$, we obtain an equivalence relation on S^*_m. We take $\mathcal{S}_m$ as the set of cosets of elements of S^*_m, which will be the stalk of the associated sheaf over m. If $m \in U$, let $\rho_{m,U} : S_U \to \mathcal{S}_m$ be the natural projection, which assigns to each element of S_U its equivalence class. This projection is called the *sheafification homomorphism*.

Now, we want to give a K-module structure to $\mathcal{S}_m$. In order to do that, let $f \in S_U$ and $g \in S_V$ where $m \in U \cap V$ and let $s_1 = \rho_{m,U}(f)$ and $s_2 = \rho_{m,U}(g)$ be their equivalence classes. There exists a neighborhood $W \subset U \cap V$ of m because U and V are open subsets. Define addition in $\mathcal{S}_m$ by setting

$$s_1 + s_2 = \rho_{p,W}(\rho_{W,U}(f) + \rho_{W,V}(g)), \tag{1.9}$$

and define multiplication by $k \in K$ by setting

$$ks_1 = \rho_{p,U}(kf). \tag{1.10}$$

It is easy to check that these operations are well-defined and give $\mathcal{S}_m$ the structure wanted and the maps $\rho_{p,U}$ are all homomorphisms. $\mathcal{S}_m$ is known as the *direct limit* of the *sheafification homomorphism* of the modules S_U for U containing m. Now let

$$\mathcal{S} = \bigcup_{m \in M} \mathcal{S}_m, \tag{1.11}$$

and let $\pi : \mathcal{S} \to M$ be the obvious projection such that $\pi(\mathcal{S}_m) = m$.

The next step is to topologize $\mathcal{S}$ by taking for a basis of the topology the collection of subsets of $\mathcal{S}$ of the form

$$O_f = \{\rho_{p,U}(f) | p \in U\} \tag{1.12}$$

for the various $f \in S_U$ and all open sets $U \subset M$. We want to see that this collection forms, indeed, a basis for a topology on $\mathcal{S}$. If $s \in O_f \cap O_g$, say $\rho_{p,U}(f) = \rho_{p,V}(g)$, then there is a neighbourhood W of p with $W \subset U \cap V$ for which $\rho_{W,U}(f) = \rho_{W,V}(g)$. Setting $O = \{\rho_{p,W}(\rho_{W,U}(f)) | p \in W\}$, we have that $s \in O \subset O_f \cap O_g$ is an open subset.

Observe that π is a local homeomorphism because it is a homeomorphism on each O_f. So the last thing we have to check is that addition and multiplication defined above are continuous. Consider first multiplication by $k \in K$. Let $s \in \mathcal{S}$, and let O be an open neighbourhood of ks. We set $s = \rho_{p,U}(f)$. Let V be a neighbourhood of p contained in $\pi(O_{kf} \cap O)$. Then $O_{\rho_{V,U}(f)}$ is an open neighbourhood of s which maps, under multiplication by k, into the open neighbourhood $O_{\rho_{V,U}(kf)} \subset O$ of ks. Hence we have the continuity for the multiplication by k. Finally, we show that the addition is also continuous. We define $\mathcal{S} \circ \mathcal{S}$ consisting in all the pairs $(s_1, s_2) \in \mathcal{S} \times \mathcal{S}$ such that $\pi(s_1) = \pi(s_2)$. Let O_f be an open neighbourhood of $s_1 - s_2 = \rho_{p,U}(f)$, and let $s_1 = \rho_{p,V}(g)$ and $s_2 = \rho_{p,W}(h)$, with $p \in U$, $p \in V$, $p \in W$ and $f \in \mathcal{S}_U$, $g \in \mathcal{S}_V$, $h \in \mathcal{S}_W$. Then there exists an open neighbourhood D of p with $D \subset U \cap V \cap W$ such that

$$\rho_{D,U}(f) = \rho_{D,V}(g) - \rho_{D,W}(h). \tag{1.13}$$

It follows that, the open neighbourhood $O_{\rho_{D,V}(g)} \times O_{\rho_{D,W}(h)} \cap (\mathcal{S} \circ \mathcal{S})$ of (s_1, s_2) maps into O_f. Thus the addition is continuous, being $\mathcal{S}$ a sheaf over M with the projection π. We write the sheaf obtained from the presheaf P as $\beta(P)$.

We have seen that the construction of presheaves from sheaves is, actually, a covariant functor which we write as α. We want to see the same thing for the construction of sheaves from presheaves. We have to check how it transforms the homomorphisms. Let $P = \{S_U; \rho_{U,V}\}$ and $P' = \{S'_U; \rho'_{U,V}\}$ be two presheaves on M and let $\{\varphi_U\}$ be a presheaf homomorphism. We associate to this homomorphism a sheaf homomorphism $\varphi : \beta(P) \to \beta(P')$ such that

$$\rho'_{p,U} \circ \varphi_U = \varphi \circ \rho_{p,U} \tag{1.14}$$

for each open set $U \subset M$ and each $p \in U$. Moreover, in going in either direction, from homomorphisms of sheaves to homomorphisms of presheaves, or vice versa, the composition of two homomorphisms induces the composition of the corresponding homomorphisms. Therefore, β is a covariant functor between the category of presheaves and the category of sheaves.

Once seen this, we can also think the sheafification homomorphism, $\rho_{p,U}$, as a contravariant functor between the category of open subsets and the category of modules. The way we have constructed $\rho_{p,U}$ makes it straightforward that $\rho_{p,U}$ seeing as a functor is a complete presheaf.

Having a sheaf $\mathcal{S}$, if we take $\beta\big(\alpha(\mathcal{S})\big)$ we obtain a sheaf which is canonically isomorphic to $\mathcal{S}$. Indeed, let $\xi \in \beta(\alpha(\mathcal{S}))$ be the germ at p of some section f of $\mathcal{S}$ over an open set U containing p, $f \in \Gamma(\mathcal{S}, U)$; that is, $\xi = \rho_{p,U}(f)$. Then, we can consider the map

$$\xi = \rho_{p,U}(f) \mapsto f(p), \tag{1.15}$$

which determines a well-defined map $\beta \circ \alpha : \mathcal{S} \to \mathcal{S}$, which is, in turn, a sheaf isomorphism.

Generally, we do not have an isomorphism between a presheaf $P = \{S_U;\, \rho_{U,V}\}$ and $\alpha(\beta(P))$. For example, let $P = \{S_U;\, \rho_{U,V}\}$ where S_U is the principal ideal domain K for each $U \subset M$ open subset, and the restrictions $\rho_{U,V}$ for $U \subsetneq V$ are zero K-module. Then the presheaf $\beta\big(\alpha(\mathcal{S})\big)$ assigns to each open set U in M the zero K-module.

To have the isomorphism the presheaf has to be complete. Let define a presheaf homomorphism from P to $\alpha\big(\beta(P)\big)$ as follows. For each open subset $U \subset M$ and each $f \in S_U$ we define:

$$\begin{aligned} \Psi_{U,f} &: U \to \beta(P) \\ p &\mapsto \rho_{p,U}(f), \end{aligned} \tag{1.16}$$

which is a section of $\beta(P)$ over U, so $\Psi_{U,f} \in \Gamma\big(\beta(P), U\big)$. We can see Ψ as a presheaf homomorphism consisting of the collection $\{\Psi_U\}$, where $U \subset M$ is an open subset.

$$\begin{aligned} \Psi_U :& S_U \to \Gamma\big(\beta(P), U\big) \\ & f \mapsto \Psi_{U,f}. \end{aligned} \tag{1.17}$$

To prove that this presheaf homomorphism is an isomorphism, we need to see that each of the homomorphisms Ψ_U are isomorphisms. First we prove the injectivity, if $f \in S_U$ maps to the 0-section, then there exists a cover $\{U_\alpha\}_{\alpha\in I}$ of U such that $\rho_{U_\alpha,U}(f) = \rho_{U_\alpha,U}(0)$. Therefore, $f = 0 \in S_U$ for (C_1).

To prove surjectivity, let c be a section of $\beta(P)$ over U. For each $p \in U$, there is a neighbourhood U_p of p and an element $f_p \in S_{U_p}$ such that

$$\rho_{q,U_p}(f_p) = c(q) \tag{1.18}$$

for all $q \in U_p$, due to the construction of $\beta(P)$. It follows from (C_1) that, for each p and q in U,

$$\rho_{U_p \cap U_q, U_p}(f_p) = \rho_{U_p \cap U_q, U_q}(f_q). \tag{1.19}$$

Thus, according to (C_2), there exists $f \in S_U$ such that $f_p = \rho_{U_p \cap U}, U(f_p)$ for each $p \in U$. It follows that $\Psi_{U,f}$ is identity to the section c.

1.3.2 Exact Sequences of Sheaves and Presheaves

Definition 1.7. *An sequence of sheaves and homomorphisms*

$$\dots \xrightarrow{f_i} \mathcal{S}_i \xrightarrow{f_{i+1}} \mathcal{S}_{i+1} \xrightarrow{f_{i+2}} \mathcal{S}_{i+2} \xrightarrow{f_{i+3}} \dots \tag{1.20}$$

is an exact sequence if $ker(f_{i+1}) = im(f_i)$ for each i.

Observe that this definition is equivalent to say that for each $m \in M$ the induce sequence of homomorphisms of the stalks over m, namely,

$$\dots \xrightarrow{\widetilde{f}_i} (\mathcal{S}_i)_m \xrightarrow{\widetilde{f}_{i+1}} (\mathcal{S}_{i+1})_m \xrightarrow{\widetilde{f}_{i+2}} (\mathcal{S}_{i+2})_m \xrightarrow{\widetilde{f}_{i+3}} \dots \tag{1.21}$$

is exact, where $\widetilde{f}_i$ is the restriction $f_i|_{\mathcal{S}_m}$. Observe that this is a sequence of K-modules, we have taken the classical definition of exact sequence in K-modules that are similar to the exactness for the sequence of sheaves.

Observe that if we have $\mathcal{S}$ a sheaf, $\mathcal{R}$ a subsheaf of $\mathcal{S}$ and $\mathcal{Z}$ the quotient sheaf $\mathcal{S}/\mathcal{R}$, then the natural exact sequence of homomorphisms

$$0 \to \mathcal{R} \to \mathcal{S} \to \mathcal{Z} \to 0 \tag{1.22}$$

is exact, where 0 denotes the constant sheaf over M whose stalk over each point is the trivial K-module consisting of only one point. Exact sequences of this form, consisting of only five terms with the first and last being the 0 sheaf, are called *short exact sequence*.

At this point, we want to define tensor products of sheaves. The faster way to do that is to consider first tensor products of presheaves due to they are collections of K-modules and homomorphisms of K-modules. Let $P = \{S_U; \rho_{U,V}\}$ and $P' = \{S'_U; \rho'_{U,V}\}$ be two presheaves on M. Then, their tensor product is the presheaf

$$P \otimes P' = \{S_U \otimes S'_U; \rho_{U,V} \otimes \rho'_{U,V}\}. \tag{1.23}$$

We can also define tensor products of presheaf homomorphisms. Let $\{\varphi_U\} : P \to Q$ and $\{\varphi'_U\} : P' \to Q'$ be presheaf homomorphisms, where Q and Q' are presheaves. Then, their tensor product $\{\varphi_U\} \otimes \{\varphi'_U\}$ is the collection of homomorphisms $\{\varphi_U \otimes \varphi'_U\}$ between $P \otimes P'$ and $Q \otimes Q'$, that is itself a presheaf homomorphism.

If $\mathcal{S}$ and $\mathcal{T}$ are sheaves of K-modules over M, we define their tensor product as

$$\mathcal{S} \otimes \mathcal{T} = \beta\big(\alpha(\mathcal{S}) \otimes \alpha(\mathcal{T})\big). \tag{1.24}$$

Note that this makes sense because $\alpha(\mathcal{S})$ and $\alpha(\mathcal{T})$ are presheaves. We define now tensor products of sheave homomorphisms, let $\varphi : \mathcal{S} \to \mathcal{T}$ and $\phi : \mathcal{S}' \to \mathcal{T}'$ be sheaf homomorphisms, where $\mathcal{S}'$ and $\mathcal{T}'$ are sheaves over M, and let $\{\varphi_U\} : \alpha(\mathcal{S}) \to \alpha(\mathcal{T})$ and $\{\phi_U\} : \alpha(\mathcal{S}') \to \alpha(\mathcal{T}')$ be the corresponding homomorphisms on the presheaves of sections. Then, we define the tensor product $\varphi \otimes \phi$ to be the sheaf homomorphism of $\mathcal{S} \otimes \mathcal{S}'$ into $\mathcal{T} \otimes \mathcal{T}'$ associated with the presheaf homomorphism

$$\{\varphi_U\} \otimes \{\phi_U\} : \alpha(\mathcal{S}) \otimes \alpha(\mathcal{S}') \to \alpha(\mathcal{T}) \otimes \alpha(\mathcal{T}). \tag{1.25}$$

There is another way to define the tensor product of sheaves, but it is canonically isomorphic with the seen above. To define the tensor product in this second way, we identify the stalks of the tensor product of two sheaves with the tensor product of the stalks of the sheaves, note that they are K-modules

$$(\mathcal{S} \otimes \mathcal{T})_m = \mathcal{S}_m \otimes \mathcal{T}_m. \tag{1.26}$$

We define tensor products of two sheaf homomorphisms identifying the sheaf homomorphisms by its restrictions over the stalks in this sense:

$$(\varphi \otimes \phi)|_{(\mathcal{S} \otimes \mathcal{S}')_m} = \varphi|_{\mathcal{S}_m} \otimes \phi|_{\mathcal{S}'_m}. \tag{1.27}$$

Indeed, we could proceed in this way, but it would involve a considerable duplication of previous work, because we should have to prove that $\mathcal{S} \otimes \mathcal{T} = \cup(\mathcal{S} \otimes \mathcal{T})_m$ is, in fact, a sheaf and that $\varphi \otimes \phi = \cup(\varphi \otimes \phi)|_{(\mathcal{S} \otimes \mathcal{S}')_m}$ is a sheaf homomorphism.

Besides that, with the last identifications, the constant sheaf $\mathcal{K} = M \times K$ has the property

$$\mathcal{S} \otimes \mathcal{K} \cong \mathcal{S}. \tag{1.28}$$

We will use this when we talk about cohomology.

Lemma 1.8. *Let B be a K-module, and let*

$$0 \to A' \xrightarrow{g} A \xrightarrow{f} A'' \to 0 \tag{1.29}$$

be a short exact sequence of K-modules. Then the induced sequence

$$A' \otimes B \xrightarrow{g \otimes Id_B} A \otimes B \xrightarrow{f \otimes Id_B} A'' \otimes B \to 0 \tag{1.30}$$

is exact, but $A' \otimes B \xrightarrow{g \otimes Id_B} A \otimes B$ is not necessarily injective. If we have that either A'' or B is torsion-free, then the full sequence

$$0 \to A' \otimes B \xrightarrow{g \otimes Id_B} A \otimes B \xrightarrow{f \otimes Id_B} A'' \otimes B \to 0 \tag{1.31}$$

is exact.

Remember that a K-module X is *torsion-free* if there is no $a \in X \setminus \{0\}$ for which there exists a non-zero element $k \in K$ such that $kx = 0$. We say that a sheaf $\mathcal{S}$ is *torsion-free* if each stalk of $\mathcal{S}$ is a torsion-free K-module. Due to the equivalence between exact sequences of sheaves and the exact sequences of K-modules of their stalks, we have the analogous lemma on sheaves.

Lemma 1.9. *Let $\mathcal{T}$ be a sheaf over M and let*

$$0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0 \tag{1.32}$$

be a short exact sequence of sheaves over M. Then the induce sequence

$$\mathcal{S}' \otimes \mathcal{T} \to \mathcal{S} \otimes \mathcal{T} \to \mathcal{S}'' \otimes \mathcal{T} \to 0 \tag{1.33}$$

is exact, but $\mathcal{S}' \otimes \mathcal{T} \to \mathcal{S} \otimes \mathcal{T}$ is not necessarily injective. If we have that either $\mathcal{S}''$ or $\mathcal{T}$ is torsion-free, then the full sequence

$$0 \to \mathcal{S}' \otimes \mathcal{T} \to \mathcal{S} \otimes \mathcal{T} \to \mathcal{S}'' \otimes \mathcal{T} \to 0 \tag{1.34}$$

is exact.

Definition 1.10. *A sheaf $\mathcal{S}$ over M is said to be fine if for each locally finite open cover $\{U_i\}_{i \in I}$ of M there exists, for each $i \in I$, an endomorphism l_i of $\mathcal{S}$ such that:*

a) $supp(l_i) \subset U_i$

b) $\sum\limits_i l_i = id.$

Here, by $\operatorname{supp}(l_i)$ we mean *the support* of l_i, i.e., the closure of the set of points in M for which $l_i|_{\mathcal{S}_m}$ is not equal to zero. Also observe that b) makes sense because $l_i(s) \in \pi^{-1}(\pi(s))$ for each i and each $s \in \mathcal{S}$. We call $\{l_i\}_{i \in I}$ a *partition of unity subordinate to the cover* $\{U_i\}_{i \in I}$. Observe that if $\mathcal{S}$ and $\mathcal{T}$ are sheaves over M with $\mathcal{S}$ fine, then $\mathcal{S} \otimes \mathcal{T}$ is itself a fine sheaf. Indeed, if $\{l_i\}_{i \in I}$ is a partition of unity fo $\mathcal{S}$ subordinate to the cover $\{U_i\}_{i \in I}$ of M, then $\{l_i \otimes Id|_{\mathcal{T}}\}_{i \in I}$ is a partition of unity for $\mathcal{S} \otimes \mathcal{T}$.

Theorem 1.11. *Let*

$$0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0 \tag{1.35}$$

be a short exact sequence of sheaves over M. Then, it gives rise to an exact sequence

$$0 \to \Gamma(\mathcal{S}') \to \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S}''). \tag{1.36}$$

However, $\Gamma(\mathcal{S}) \to \Gamma(\mathcal{S}'')$ is not necessarily surjective. If we have that $\mathcal{S}'$ is fine, then the full sequence

$$0 \to \Gamma(\mathcal{S}') \to \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S}'') \to 0 \tag{1.37}$$

is exact.

Before seeing the definition, we see an example of the previous short exact sequence where $\Gamma(\mathcal{S}) \to \Gamma(\mathcal{S}'')$ is not surjective. First of all, note that exactness of (1.35) is a purely local property, whereas exactness of (1.36) is a global property. Let M be connected and let p_1 and p_2 be distinct points on M. Let $\mathcal{S}_{p_1,p_2}$ be the "skyscrape" sheaf over M whose stalk is the zero K-module over each point except over p_1 and p_2. Observe that the topology on $\mathcal{S}_{p_1,p_2}$ is uniquely determined by the requirement that $\mathcal{S}_{p_1,p_2}$ be a sheaf. let $\mathcal{K}$ be the constant sheaf with stalk K. There is an obvious homomorphism of $\mathcal{K}$ onto $\mathcal{S}_{p_1,p_2}$, namely, the homomorphism is zero on all stalks of $\mathcal{K}$ except on those over p_1 and p_2 where it is the identity map. However, the associated map $\Gamma(\mathcal{K}) \to \Gamma(\mathcal{S}_{p_1,p_2})$ can not be surjective for $\Gamma(\mathcal{K}) \cong K$, whereas $\Gamma(\mathcal{S}_{p_1,p_2}) \cong K \oplus K$.

Proof. Let $\phi : \mathcal{S}' \to \mathcal{S}$ be a surjective sheaf homomorphism, this induces a homomorphism from the sections of $\mathcal{S}'$ into the sections of $\mathcal{S}$. Let $f \in \Gamma(\mathcal{S})$ be a global section. Due to ϕ is surjective, for each $m \in M$ there exist an $s \in \mathcal{S}'$ such that $\phi(s) = f(m)$. Let $g : M \to \mathcal{S}'$ be a homomorphism such that $g(m) = s$, therefore $\phi \circ g = f$. First, we see that g is continuous let $U \subset \mathcal{S}'$ be an open set, an let s be in U. We have that $f(g^{-1}(U)) = \phi(g(g^{-1}(U))) = \phi(U)$, as ϕ is a local homeomorphism we can take V a neighbourhood of s in U such that ϕ is an homeomorphism over this open. we have that $g^{-1}(V) = f^{-1}(\phi(V))$ is an open subset. Thus, g is continuous. Now we want to see that g is a section of $\mathcal{S}'$. $\pi \circ g = \pi' \circ \phi \circ g = \pi' \circ f = Id|_M$, where the last equality is because of f is a section. Therefore, $\Gamma(\phi)$ is surjective.

Let $\phi : \mathcal{S}' \to \mathcal{S}$ and $\varphi : \mathcal{S} \to \mathcal{S}''$ be sheaf homomorphisms such that ker(ϕ)=im(φ). We have that $\Gamma(\phi) \circ \Gamma(\varphi) = 0$ due to $\phi \circ \varphi = 0$ so $\ker(\Gamma(\phi)) \supset \operatorname{im}(\Gamma(\varphi))$. We want to see the inverse inclusion. Taken f a section of $\mathcal{S}$ such that $\phi(f) = 0$, we have that for each $m \in M$ there exists an $s' \in \mathcal{S}'$ such that $\varphi(s') = f(m)$. We define the homomorphism $g : M \to \mathcal{S}'$ mapping $g(m) = s'$, we can see similarly as before that g is a section. Therefore ker(ϕ)=im(φ), and we have the exactness of the sheaf sequence (1.35).

Let now $\mathcal{S}$ be a sheaf, and let $\varphi : \mathcal{S} \to \mathcal{S}''$ be a surjective sheaf homomorphism with fine kernel $\mathcal{R}$. Now, let t be a global section of $\mathcal{S}''$. We must construct a section s of $\mathcal{S}$ such that $\varphi \circ s = t$. By the continuity of φ and t, and by the fact that π and π'' are local homeomorphisms there exists a covering $\{U_i\}_{i\in I}$ of M by open sets and, for each i, a section s_i of $\mathcal{S}$ over U_i such that

$$\varphi \circ s_i = t_i|_{U_i}. \tag{1.38}$$

Since M is paracompact, we can assume that the cover $\{U_i\}_{i\in I}$ can be taken as a locally finite open cover. The difference

$$s_{ij} = s_i - s_j \tag{1.39}$$

is a section of $\mathcal{R}$ over $U_i \cap U_j$, and on $U_i \cap U_j \cap U_k$ the differences satisfy

$$s_{ij} + s_{jk} = s_{ik}. \tag{1.40}$$

Let $\{l_i\}_{i\in I}$ be a partition of unity for $\mathcal{R}$ subordinate of the cover $\{U_i\}_{i\in I}$ of M. Consider the section $l_i \circ s_{ij}$ of $\mathcal{R}$ over $U_i \cap U_j$. Since the support of l_j lies in U_j, we can extend the section $l_j \circ s_{ij}$ to be a continuous section of $\mathcal{R}$ over U_i by defining it to be zero on points of $U_i \setminus (U_j \cap U_i)$. Let

$$s'_i = \sum_j l_j \circ s_{ij}. \tag{1.41}$$

Then s'_i is the section of $\mathcal{R}$ over U_i, and the difference

$$s'_i - s'_j = \sum_k l_k \circ s_{ik} - \sum_k l_k \circ s_{jk} = \sum_k l_k \circ s_{ij} = s_{ij} = s_i - s_j \tag{1.42}$$

over $U_i \cap U_j$. Thus

$$s_j - s'_j = s_i - s'_i \tag{1.43}$$

on $U_i \cap U_j$. It follows that if we set $s(m) = (s_i - s'_i)(m)$ for each $m \in U_i$, then s is a well-defined global section of $\mathcal{S}$ such that $\varphi \circ s = t$. The proof concludes observing that the image of a fine sheaf by a sheaf homomorphism is a fine sheaf itself. ☐

Theorem 1.12. *Let*

$$0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0 \tag{1.44}$$

be a short exact sequence of sheaves over M, and let $\mathcal{T}$ be a sheaf over M. If either $\mathcal{T}$ or $\mathcal{S}'$ is fine and either of both is torsion-free then the full sequence

$$0 \to \Gamma(\mathcal{S}' \otimes \mathcal{T}) \to \Gamma(\mathcal{S} \otimes \mathcal{T}) \to \Gamma(\mathcal{S}'' \otimes \mathcal{T}) \to 0 \tag{1.45}$$

is exact.

Chapter 2

Sheaf Cohomology Theories

In homology theories context, one sees that these can be studied as covariant functors between the category of topological spaces (or the category of sheaves) and an algebraic category. In the case of cohomology theories, these can be studied as contravariant functors between the same categories than homology theories. Moreover, in many cases, cohomology theories are dual constructions of homology theories, but this fact does not make cohomology ones less interesting. Also, there are many homological algebra results which are used in homology theories and can be used in the same sense of cohomology theories. In our case, we are going to study cohomology theories as functors between the category of sheaves over M and the category of modules.

In this chapter, we assume that the reader is familiarised with some concepts of homological algebra. We are going to use some lemmas that one shall know to follow the reading. Particular importance receives snake lemma. Also, basic notions in homotopy and singular homology can help to a better understanding of this chapter.

Definition 2.1. *A sheaf cohomology theory $\mathcal{H}$ for M with coefficients in sheaves of K-modules over M consists of*

a) a K-module $H^q(M, \mathcal{S})$ for each sheaf $\mathcal{S}$ and for each integer q,

b) a homomorphism $H^q(M, \mathcal{S}) \to H^q(M, \mathcal{S}')$ for each sheaf homomorphism $\mathcal{S} \to \mathcal{S}'$ and for each integer q where $\mathcal{S}$ and $\mathcal{S}'$ are a sheaves over M,

c) a homomorphism $H^q(M, \mathcal{S}'') \to H^{q+1}(M, \mathcal{S}')$ for each short exact sequence $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$ of sheaf over M and for each integer q.

Such that properties (I)-(VI) hold:

(I) $H^q(M, \mathcal{S}) = 0$ for $q < 0$, and there is an isomorphism $H^0(M, \mathcal{S}) \cong \Gamma(\mathcal{S})$ such that for each homomorphism $\mathcal{S} \to \mathcal{S}'$ the diagram

$$\begin{array}{ccc} H^0(M,\mathcal{S}) & \xrightarrow{\cong} & \Gamma(\mathcal{S}) \\ \downarrow & & \downarrow \\ H^0(M,\mathcal{S}') & \xrightarrow{\cong} & \Gamma(\mathcal{S}') \end{array}$$

commutes.

(II) $H^q(M, \mathcal{S}) = 0$ for all $q > 0$ if $\mathcal{S}$ is a fine sheaf.

(III) *If* $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$ *is exact, then the following sequence is exact:*

$$\cdots \to H^q(M, \mathcal{S}') \to H^q(M, \mathcal{S}) \to H^q(M, \mathcal{S}'') \to H^{q+1}(M, \mathcal{S}') \to \ldots \tag{2.1}$$

(IV) *The identity sheaf homomorphism id* : $\mathcal{S} \to \mathcal{S}$ *induces the identity homomorphism id* : $H^q(M, \mathcal{S}) \to H^q(M, \mathcal{S})$.

(V) *If the diagram*

$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & \mathcal{S}' \\ \downarrow & \swarrow & \\ \mathcal{S}'' & & \end{array}$$

commutes, then for each q so does the diagram

$$\begin{array}{ccc} H^q(M, \mathcal{S}) & \longrightarrow & H^q(M, \mathcal{S}') \\ \downarrow & \swarrow & \\ H^q(M, \mathcal{S}'') & & . \end{array}$$

(VI) *For each homomorphism of short exact sequences of sheaves*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S}' & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{S}'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{T}' & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{T}'' & \longrightarrow & 0 \end{array}$$

the following diagram commutes:

$$\begin{array}{ccc} H^q(M, \mathcal{S}'') & \longrightarrow & H^{q+1}(M, \mathcal{S}') \\ \downarrow & & \downarrow \\ H^q(M, \mathcal{T}'') & \longrightarrow & H^{q+1}(M, \mathcal{T}'). \end{array}$$

The module $H^q(M, \mathcal{S})$ *is called the qth cohomology module of M with coefficients in the sheaf* $\mathcal{S}$ *relative to the cohomology theory* $\mathcal{H}$.

2.1 Existence of Sheaf Cohomology Theories

Here, we are going to define an important concept which can be used to give rise some cohomology theories: *fine torsion-free resolution of a sheaf.* When we talk about de Rham Cohomology and singular Cohomology, our first goal will be to find a fine torsion-free resolution of a constant sheaf. This tool is not only useful in cohomology theories, but also in many algebraic constructions because it allows studying properties of an algebra concept taking care of other simpler elements.

Definition 2.2. *An exact sequence of sheaves*

$$0 \to \mathcal{A} \xrightarrow{f^0} \mathcal{C}^0 \xrightarrow{f^1} \mathcal{C}^1 \xrightarrow{f^2} \mathcal{C}^2 \xrightarrow{f^3} \ldots \tag{2.2}$$

is called a resolution of the sheaf $\mathcal{A}$. *The resolution is called fine (respectively torsion-free) if each of the sheaves* $\mathcal{C}^i$ *is fine (respectively torsion-free).*

With each resolution of $\mathcal{A}$ and each sheaf $\mathcal{S}$ we can associate a cochain complex

$$\cdots \to 0 \to \Gamma(\mathcal{C}^0 \otimes \mathcal{S}) \to \Gamma(\mathcal{C}^1 \otimes \mathcal{S}) \to \Gamma(\mathcal{C}^2 \otimes \mathcal{S}) \to \Gamma(\mathcal{C}^3 \otimes \mathcal{S}) \to \dots \tag{2.3}$$

which we shall denote by $\Gamma(\mathcal{C}^* \otimes \mathcal{S})$. For each $q > 0$ the module of the q-cochains is $\Gamma(\mathcal{C}^q \otimes \mathcal{S})$, whereas for $q < 0$ the module of q-cochains is the zero module. Observe that this cochain complex does not contain the module $\Gamma(\mathcal{A} \otimes \mathcal{S})$. The homomorphisms in the sequence (2.3) are those induced by the homomorphisms $\mathcal{C}^q \otimes \mathcal{S} \xrightarrow{f^{q+1} \otimes Id|_{\mathcal{S}}} \mathcal{C}^{q+1} \otimes \mathcal{S}$. Note that the homomorphisms $f^q \otimes Id|_{\mathcal{S}}$ together with the sheaves $\mathcal{C}^q \otimes \mathcal{S}$ make a cochain complex of sheaves due to $(f^{q+1} \otimes Id|_{\mathcal{S}}) \circ (f^q \otimes Id|_{\mathcal{S}}) = (f^{q+1} \circ f^q) \otimes Id|_{\mathcal{S}} = 0$. Therefore, the last composition of maps implies that in

$$\cdots \to 0 \to \mathcal{C}^0 \otimes \mathcal{S} \to \mathcal{C}^1 \otimes \mathcal{S} \to \mathcal{C}^2 \otimes \mathcal{S} \to \mathcal{C}^3 \otimes \mathcal{S} \to \dots \tag{2.4}$$

the image of each homomorphism is contained in the kernel of the next, and this in turn implies that 2.3 is indeed a cochain complex.

A homomorphism $\mathcal{S} \to \mathcal{S}'$ when tensored with the indentity homomorphism of the sheaf $\mathcal{C}^q$, yields a homomorphism $\mathcal{C}^q \otimes \mathcal{S} \to \mathcal{C}^q \otimes \mathcal{S}'$. This, in turn, induces a homomorphism of the corresponding cochain complexes and, in this way, it determines a cochain map $\Gamma(\mathcal{C}^* \otimes \mathcal{S}) \to \Gamma(\mathcal{C}^* \otimes \mathcal{S}')$.

We shall now show that each fine torsion-free resolution of the constant sheaf $\mathcal{K} = M \times K$ canonically determines a cohomology theory for M with coefficients in sheaves of K-modules over M. Notice that we have not to see the existence of such resolution, we will see that when we talk about the de Rham cohomology and the singular cohomology. By the moment, we assume that a fine torsion-free resolution exists.

In order to obtain a cohomology theory, first of all, we set $H^q(M, \mathcal{S}) = H^q(\Gamma(\mathcal{C}^* \otimes \mathcal{S})) = ker(\Gamma(f^q \otimes Id|_{\mathcal{S}}))/im(\Gamma(f^{q+1} \otimes Id|_{\mathcal{S}}))$ for each $q \in \mathbb{Z}$. Keep in mind how the functor Γ acts over homomorphisms. Also, for each homomorphism $\mathcal{S} \to \mathcal{S}'$ and each q we associate the homomorphism $H^q(M, \mathcal{S}) \to H^q(M, \mathcal{S}')$ induced, according to homological algebra, by the cochain map $\Gamma(\mathcal{C}^* \otimes \mathcal{S}) \to \Gamma(\mathcal{C}^* \otimes \mathcal{S}')$. In addition, each short exact sequence of sheaves

$$0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0 \tag{2.5}$$

induces, in view that $\mathcal{C}^q$ is a fine torsion-free sheaf for each q, the short exact sequence

$$0 \to \Gamma(\mathcal{C}^* \otimes \mathcal{S}') \to \Gamma(\mathcal{C}^* \otimes \mathcal{S}) \to \Gamma(\mathcal{C}^* \otimes \mathcal{S}'') \to 0 \tag{2.6}$$

with which there is associated a homomorphism $H^q(\Gamma(\mathcal{C}^* \otimes \mathcal{S}'')) \to H^{q+1}(\Gamma(\mathcal{C}^* \otimes \mathcal{S}'))$ via the snake lemma. This is the homomorphism $H^q(M, \mathcal{S}'') \to H^{q+1}(M, \mathcal{S}')$ that we associate with the short exact sequence $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$ and the integer q in the definition of cohomology theories.

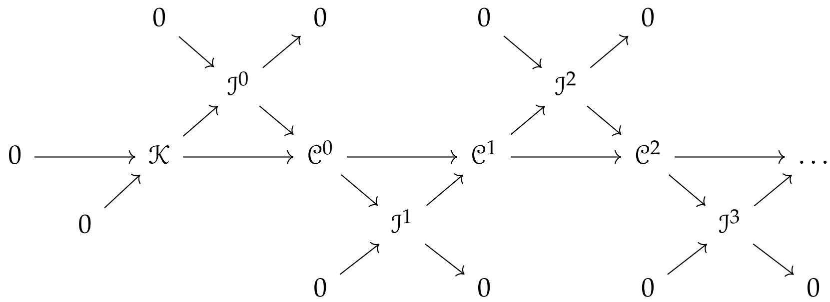
Theorem 2.3. *Let $\mathcal{K}$ be the constant sheaf $M \times K$ and let*

$$0 \to \mathcal{K} \to \mathcal{C}^0 \to \mathcal{C}^1 \to \mathcal{C}^2 \to \dots \tag{2.7}$$

be a fine torsion-free resolution of $\mathcal{K}$. Then, the modules $H^q(M, \mathcal{S})$ and the morphisms $H^q(M, \mathcal{S}) \to H^q(M, \mathcal{S}')$ and $H^q(M, \mathcal{S}'') \to H^{q+1}(M, \mathcal{S}')$ obtained as above, determine a cohomology theory for M with coefficients in sheaves of K-modules over M.

Proof. Note that the axioms **(III)**, **(IV)**, **(V)** and **(VI)** for the definition of cohomology theory holds immediately using homological algebra. We need to see that these fulfill the properties **(I)** and **(II)**. We start with the **(II)**.

Now let $\mathcal{J}^q$ be the kernel of $\mathcal{C}^q \to \mathcal{C}^{q+1}$. Then it follows the next commutative diagram where row sequences and diagonal sequences are exact



Therefore, due to $\mathcal{C}^q$ is a torsion-free sheaf it is $\mathcal{J}^q$ because it could be seen as a subsheaf of $\mathcal{C}^q$. It follows that for any sheaf $\mathcal{S}$ the sequence

$$0 \to \mathcal{J}^q \otimes \mathcal{S} \to \mathcal{C}^q \otimes \mathcal{S} \to \mathcal{J}^{q+1} \otimes \mathcal{S} \to 0 \tag{2.8}$$

is exact. Moreover, we obtain the exact sequence

$$0 \to \Gamma(\mathcal{J}^q \otimes \mathcal{S}) \to \Gamma(\mathcal{C}^q \otimes \mathcal{S}) \to \Gamma(\mathcal{J}^{q+1} \otimes \mathcal{S}) \tag{2.9}$$

The homomorphism $\Gamma(\mathcal{C}^q \otimes \mathcal{S}) \xrightarrow{\gamma_{q+1}} \Gamma(\mathcal{C}^{q+1} \otimes \mathcal{S})$ is the composition $\Gamma(\mathcal{C}^q \otimes \mathcal{S}) \to \Gamma(\mathcal{J}^q \otimes \mathcal{S}) \to \Gamma(\mathcal{C}^{q+1} \otimes \mathcal{S})$. We obtein from the large diagram that the homomorphism $\Gamma(\mathcal{J}^q \otimes \mathcal{S}) \to \Gamma(\mathcal{C}^q \otimes \mathcal{S})$ is an injection. Then, $\ker(\gamma)$ is the submodule $\Gamma(\mathcal{J}^q \otimes \mathcal{S})$ of $\Gamma(\mathcal{C}^q \otimes \mathcal{S})$ for each $q > 0$. Thus,

$$H^q(M, \mathcal{S}) = H^q(\Gamma(\mathcal{C}^* \otimes \mathcal{S})) \cong \Gamma(\mathcal{J}^q \otimes \mathcal{S})/Im(\gamma_q). \tag{2.10}$$

Now, if $\mathcal{S}$ happens to be a fine sheaf, the full sequence

$$0 \to \Gamma(\mathcal{J}^q \otimes \mathcal{S}) \to \Gamma(\mathcal{C}^q \otimes \mathcal{S}) \to \Gamma(\mathcal{J}^{q+1} \otimes \mathcal{S}) \to 0 \tag{2.11}$$

is exact, and we have that

$$H^q(M, \mathcal{S}) \cong \Gamma(\mathcal{J}^q \otimes \mathcal{S})/Im(\gamma_q) = 0 \text{ for } q > 0. \tag{2.12}$$

Thus axiom **(II)** is satisfied.

To see that axiom **(I)** is fulfilled, note that we have that $H^q(M, \mathcal{S}) = 0$ for $q < 0$ due to the chain is right-bounded by zero from $q = 0$. For $q = 0$ the isomorphism $\mathcal{K} \to \mathcal{J}^0 \subset \mathcal{C}^0$ induces an isomorphism

$$\mathcal{S} \cong \mathcal{K} \otimes \mathcal{S} \xrightarrow{\cong} \mathcal{J}^0 \otimes \mathcal{S} \subset \mathcal{C}^0 \otimes \mathcal{S} \tag{2.13}$$

for any sheaf $\mathcal{S}$. Therefore, we have the isomorphism

$$\Gamma(\mathcal{S}) \xrightarrow{\cong} \Gamma(\mathcal{K} \otimes \mathcal{S}) \xrightarrow{\cong} \Gamma(\mathcal{J}^0 \otimes \mathcal{S}) \tag{2.14}$$

which clearly satisfies

$$\begin{array}{ccc} H^0(M, \mathcal{S}) & \xrightarrow{\cong} & \Gamma(\mathcal{S}) \\ \downarrow & & \downarrow \\ H^0(M, \mathcal{S}') & \xrightarrow{\cong} & \Gamma(\mathcal{S}') \end{array}$$

for each homomorphism $\mathcal{S} \to \mathcal{S}'$. Thus axiom **(I)** is satisfied. $\square$

2.2 Homomorphisms Between Sheaf Cohomology Theories

Now, we will define homomorphisms between cohomology theories. With that, we will have the category of sheaf cohomology theories on M with coefficients in sheaves of K-modules over M. Note that, due to cohomology theories could be studied as contravariant functors, these homomorphisms have to be natural transformations, with a few more properties, in order to have a new category. In other words, natural transformations provide a way of transforming one functor into another while respecting the internal structure of categories involved. Hence, a natural transformation can be considered to be a "morphism of functors". Indeed, this intuition can be formalized to define the so-called *functor categories*.

Definition 2.4. *Let $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ be two sheaf cohomology theories on M with coefficients in sheaves of K-modules over M. A homomorphism of the cohomology theory $\mathcal{H}$ to the theory $\widetilde{\mathcal{H}}$ is a collection of K-module homomorphisms*

$$H^q(M, \mathcal{S}) \to \widetilde{H}^q(M, \mathcal{S}) \tag{2.15}$$

for each sheaf $\mathcal{S}$ and for each integer q, such that the following conditions hold:

(I) for $q = 0$, the diagram

$$\begin{array}{ccc} H^0(M, \mathcal{S}) & \xrightarrow{\cong} & \Gamma(\mathcal{S}) \\ \downarrow & & \downarrow \\ \widetilde{H}^0(M, \mathcal{S}') & \xrightarrow{\cong} & \Gamma(\mathcal{S}') \end{array}$$

commutes.

(II) For each homomorphism $\mathcal{S} \to \mathcal{T}$ and each integer q the diagram

$$\begin{array}{ccc} H^q(M, \mathcal{S}) & \longrightarrow & H^q(M, \mathcal{T}) \\ \downarrow & & \downarrow \\ \widetilde{H}^q(M, \mathcal{S}) & \longrightarrow & \widetilde{H}^q(M, \mathcal{T}). \end{array}$$

commutes.

(III) For each short exact sequence of sheaves $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$ the following diagram commute for each q integer:

$$\begin{array}{ccc} H^q(M, \mathcal{S}'') & \longrightarrow & H^{q+1}(M, \mathcal{S}') \\ \downarrow & & \downarrow \\ \widetilde{H}^q(M, \mathcal{S}'') & \longrightarrow & \widetilde{H}^{q+1}(M, \mathcal{S}') \end{array}$$

An *isomorphism of cohomology theories* is a homomorphism of cohomology theories in which each of the K-module homomorphism $H^q(M, \mathcal{S}) \to \widetilde{H}^q(M, \mathcal{S})$ is an isomorphism.

We need to introduce the notion of *sheaf of germs of discontinuous sections of* $\mathcal{S}$ to prove the next theorem. Let $\mathcal{S}$ be a sheaf over M. By a *discontinuous section of* $\mathcal{S}$ over the open subset $U \subset M$ we mean any map $f : U \to \mathcal{S}$, not necessarily continuous, such that $\pi \circ f = Id$. The set of discontinuous sections of $\mathcal{S}$ has a K-module structure built exactly as we constructed the K-module structure over the continuous sections of $\mathcal{S}$. Therefore, we can assign to each open subset $U \subset M$ the K-module of all the discontinuous sections of $\mathcal{S}$ over U hence we have a presheaf where the restriction maps are the trivial ones.

We call to the associated sheaf $\mathcal{S}_0$ of the previous presheaf the *sheaf of germs of discontinuous sections of* $\mathcal{S}$. The most important property of $\mathcal{S}_0$ is that, whenever M were a differential manifold, it would be a fine sheaf (actually it would be enough that M were paracompact, Hausdorff and fulfill the second axiom of countability). Then, it is an incredibly useful sheaf, because each sheaf could be canonically associated with another being fine. For let $\{U_i\}$ be a locally finite open cover of M. Choose a refinement $\{V_i\}$ such that $\overline{V}_i \subset U_i$ for each i, we can do this due to the properties of M mentioned above. Then, associate with each point in M a set V_i containing the point, and then for each i define a function φ_i on M to have the value 1 at all points associated with V_i and to have the value 0 elsewhere. Therefore, $\mathrm{supp}(\varphi_i) \subset U_i$ and $\sum \varphi_i = Id$. Now, we want endomorphisms of the presheaf of discontinuous sections of $\mathcal{S}$. We associate with each φ_i a homomorphism $\widetilde{l}_i$ such that

$$\widetilde{l}_i(s)(m) = \varphi_i(m)s(m), \tag{2.16}$$

for each discontinuous section s of $\mathcal{S}$ over an open subset $U \subset M$ and each $m \in U$. Note that we can do this since we do not need φ to be continuous. The presheaf endomorphisms $\widetilde{l}_i$ induce endomorphisms l_i of the sheaf $\mathcal{S}_0$. It is straightforward that $\{l_i\}$ forms a partition of unity over $\mathcal{S}_0$ subordinated to the locally finite cover $\{U_i\}$ of M. Thus $\mathcal{S}_0$ is fine.

Finally, observe that there is a natural injection of $\mathcal{S}$ into $\mathcal{S}_0$, due to each element of the stalk $\mathcal{S}_m$ is the value at m of a continuous section f of $\mathcal{S}$ defined on some neighbourhood of m. So, we map $f(m)$ to the germ of f at m in $\mathcal{S}_0$. Let $\overline{\mathcal{S}}$ be the quotient sheaf $\mathcal{S}_0/\mathcal{S}$, then for each sheaf $\mathcal{S}$ we have constructed a short exact sequence

$$0 \to \mathcal{S} \to \mathcal{S}_0 \to \overline{\mathcal{S}} \to 0 \tag{2.17}$$

in which the middle sheaf is fine.

Theorem 2.5. *Let $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ be cohomology theories on M with coefficients in sheaves of K-modules over M. Then there exists a unique homomorphism of $\mathcal{H}$ to $\widetilde{\mathcal{H}}$.*

Proof. It is more useful to prove first the uniqueness and then the existence. Suppose that we have a homomorphism of $\mathcal{H}$ to $\widetilde{\mathcal{H}}$, and let $\mathcal{S}$ be a sheaf. Applying the homomorphism to the short exact sequence (2.17), we have the commutative diagram for $q = 1$

$$\begin{array}{ccccccc} \Gamma(\mathcal{S}_0) & \longrightarrow & \Gamma(\overline{\mathcal{S}}) & \longrightarrow & H^1(M,\mathcal{S}) & \longrightarrow & 0 \\ \downarrow_{Id} & & \downarrow_{Id} & & \downarrow & & \\ \Gamma(\mathcal{S}_0) & \longrightarrow & \Gamma(\overline{\mathcal{S}}) & \longrightarrow & \widetilde{H}^1(M,\mathcal{S}) & \longrightarrow & 0, \end{array} \tag{2.18}$$

and for each $q > 1$ we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{q-1}(M,\overline{\mathcal{S}}) & \longrightarrow & H^q(M,\mathcal{S}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{H}^{q-1}(M,\overline{\mathcal{S}}) & \longrightarrow & \widetilde{H}^q(M,\mathcal{S}) & \longrightarrow & 0. \end{array} \tag{2.19}$$

Recall that $\mathcal{S}_0$ is a fine sheaf, so $\widetilde{H}^q(M,\mathcal{S}_0) = 0$ for each $q \geq 1$. Note that the rows of the previous diagrams are exact. Now uniqueness of the homomorphism follows for $q = 0$ from the axiom (I) for the definition (2.4); follows for $q = 1$ from the diagram (2.18) and follows inductively for $q > 1$ from the diagrams (2.19).

At this point, we see the existence of a homomorphism between the two cohomology theories. We define the homomorphism for $q = 0$ by the axiom (I) for the definition (2.4), and it follows immediately that the property (III) for the definition (2.4) is satisfied for $q = 0$. We define the homomorphism (I) in the definition (2.4) by (2.18) for $q = 1$, and inductively for $q > 1$ by (2.19). To prove that this indeed defines a homomorphism of $\mathcal{H}$ to $\widetilde{\mathcal{H}}$, it remains to show that **def** 2.4(III) holds for $q > 0$ and that the property (II) of the definition (2.49 holds for all q.

Let $\mathcal{S} \to \mathcal{T}$ be a sheaf homomorphism where $\mathcal{T}$ is a sheaf over M. We use two copies of the diagram (2.18) to make the following diagram

$$
\begin{array}{ccccccccc}
 & \Gamma(\mathcal{S}_0) & \longrightarrow & \Gamma(\overline{\mathcal{S}}) & \longrightarrow & H^1(M,\mathcal{S}) & \longrightarrow & 0 & \\
\swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & & & \\
\Gamma(\mathcal{T}_0) & \longrightarrow & \Gamma(\overline{\mathcal{T}}) & \longrightarrow & H^1(M,\mathcal{T}) & \longrightarrow & 0 & & \\
\downarrow & \Gamma(\mathcal{S}_0) & \downarrow\longrightarrow & \Gamma(\overline{\mathcal{S}}) & \downarrow\longrightarrow & \widetilde{H}^1(M,\mathcal{S}) & \longrightarrow & 0 & \\
 & \swarrow & & \swarrow & & \swarrow & & & \\
\Gamma(\mathcal{T}_0) & \longrightarrow & \Gamma(\overline{\mathcal{T}}) & \longrightarrow & \widetilde{H}^1(M,\mathcal{T}) & \longrightarrow & 0 & &
\end{array}
\tag{2.20}
$$

In the lattice, commutative follows straightforwardly from the commutativity of the two left faces.

We can build an analogous diagram of the previous one (2.20) from the (2.19) diagrams. With these diagrams we have that the axiom (II) of the definition (2.4) holds.

We want to see that (III) in the definition (2.4) holds. Suppose that we have a short exact sequence

$$0 \to \mathcal{R} \to \mathcal{S} \to \mathcal{T} \to 0 \tag{2.21}$$

where $\mathcal{S}$, $\mathcal{R}$ and $\mathcal{T}$ are sheaves. Let $\mathcal{R}_0$ and $\mathcal{S}_0$ be the sheaves of germs of discontinuous sections of $\mathcal{R}$ and $\mathcal{S}$ respectively, keep in mind that these are fine. Having the injection $\mathcal{R} \to \mathcal{S}$ we can map $\mathcal{R}$ into $\mathcal{S}_0$ by the composition $\mathcal{R} \to \mathcal{S} \to \mathcal{S}_0$ which is injective. Let $\mathcal{P}$ and $\overline{\mathcal{R}}$ be the quotient sheaves $\mathcal{S}_0/\mathcal{R}$ and $\mathcal{R}_0/\mathcal{R}$ respectively. Then, there are uniquely determined homomorphisms $\mathcal{T} \to \mathcal{P}$ and $\overline{\mathcal{R}} \to \mathcal{P}$. The first one holds due to the injection $\mathcal{S} \hookrightarrow \mathcal{S}_0$ and the isomorphism $\mathcal{S} \cong \mathcal{T}/\mathcal{R}$. We get the homomorphism $\overline{\mathcal{R}} \to \mathcal{P}$ from the injection $\mathcal{R} \hookrightarrow \mathcal{S}_0$. Therefore the following diagram commutes, in which the rows are exact:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{T} & \longrightarrow & 0 \\
 & & \downarrow Id & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{S}_0 & \longrightarrow & \mathcal{P} & \longrightarrow & 0 \\
 & & Id\uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{R}_0 & \longrightarrow & \overline{\mathcal{R}} & \longrightarrow & 0
\end{array}
\tag{2.22}
$$

It follows from cohomology axioms that there are commutative diagrams

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & \Gamma(\mathcal{R}) & \longrightarrow & \Gamma(\mathcal{S}) & \longrightarrow & \Gamma(\mathcal{T}) & \longrightarrow & H^1(M,\mathcal{R}) & \longrightarrow & \dots \\
 & & \downarrow Id & & \downarrow & & \downarrow & & \downarrow Id & & \\
0 & \longrightarrow & \Gamma(\mathcal{R}) & \longrightarrow & \Gamma(\mathcal{S}_0) & \xrightarrow{\varphi_1} & \Gamma(\mathcal{P}) & \xrightarrow{F_1} & H^1(M,\mathcal{R}) & \longrightarrow & 0 \\
 & & Id\uparrow & & \uparrow & & \uparrow & & Id\uparrow & & \\
0 & \longrightarrow & \Gamma(\mathcal{R}) & \longrightarrow & \Gamma(\mathcal{R}_0) & \xrightarrow{\varphi_2} & \Gamma(\overline{\mathcal{R}}) & \xrightarrow{F_2} & H^1(M,\mathcal{R}) & \longrightarrow & 0
\end{array}
\tag{2.23}
$$

and, for $q > 0$,

$$
\begin{array}{ccccccc}
\dots \longrightarrow H^q(M,\mathcal{S}) & \longrightarrow & H^q(M,\mathcal{T}) & \longrightarrow & H^{q+1}(M,\mathcal{R}) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow Id & & \\
0 & \longrightarrow & H^q(M,\mathcal{P}) & \xrightarrow{\cong} & H^{q+1}(M,\mathcal{R}) & \longrightarrow & 0 \\
 & & \cong\uparrow & & Id\uparrow & & \\
0 & \longrightarrow & H^q(M,\overline{\mathcal{R}}) & \xrightarrow{\cong} & H^{q+1}(M,\mathcal{R}) & \longrightarrow & 0
\end{array}
\tag{2.24}
$$

In both diagrams rows are exact. Therefore, from the diagram (2.23) follows that the homomorphism $H^0(M,\mathcal{T}) \to H^1(M,\mathcal{R})$ is the composition

$$H^0(M,\mathcal{T}) \xrightarrow{\cong} \Gamma(\mathcal{T}) \to \Gamma(\mathcal{P}) \xrightarrow{\tilde{\pi}} \Gamma(\mathcal{P})/Im(\varphi_1) \xleftarrow{\cong} \Gamma(\overline{\mathcal{R}})/Im(\varphi_2) \xrightarrow{\cong} H^1(M,\mathcal{R}), \tag{2.25}$$

where $\tilde{\pi}$ is the projection into the quotient sheaf. The second isomorphism is obtained from the commutativity of the diagram and the third isomorphism holds from the exactness of the exact sequence $0 \to ker(\Gamma(\overline{\mathcal{R}})) \to \Gamma(\overline{\mathcal{R}}) \to H^1(M,\mathcal{R}) \to 0$. We see with more detail the second isomorphism: let p be an element of Since F_1 and F_2 are surjectives, there exists an element $\overline{r} \in \Gamma(\overline{\mathcal{R}})$ such that $F_2(\overline{r}) = F_1(p)$. We can consider that $\overline{r} = r + a$, where $r \in \Gamma(\overline{\mathcal{R}})$ and $a \in \varphi(\mathcal{R}_0)$ since the third row is exact. Therefore $F_2(r+a) = F_2(r)$. Given $\gamma : \Gamma(\overline{\mathcal{R}}) \to \Gamma(\mathcal{P})$, the map in the diagram, we take the element $p - \gamma(\overline{r}) \in \Gamma(\mathcal{P})$ which is mapped into $F_1(p - \gamma(\overline{r})) = 0$ because of the diagram commutes. So, there exists an $s \in \Gamma(\mathcal{S}_0)$ such that $\varphi_1(s) = p - \gamma(\overline{r})$ and $p - \varphi(s) = \gamma(r+a)$. Here holds the surjectivity. The injectivity can be proved in an analogous way. Finally, the isomophism hols due to $\Gamma(\mathcal{P})/Im(\varphi_1) \cong \Gamma(\mathcal{P})/Ker(F_1) \cong \Gamma(\overline{\mathcal{R}})/Im(F_2) \cong \Gamma(\overline{\mathcal{R}})/Im(\varphi_2)$.

It follows from (2.24) that for each $q > 0$ the homomorphism $H^q(M,\mathcal{T}) \to H^{q+1}(M,\mathcal{R})$ is the composition

$$H^q(M,\mathcal{T}) \to H^q(M,\mathcal{P}) \xleftarrow{\cong} H^q(M,\overline{\mathcal{R}}) \xrightarrow{\cong} H^{q+1}(M,\mathcal{R}), \tag{2.26}$$

which holds directly from the diagram.

From (2.25) we obtain the diagram

$$\begin{array}{ccccccccc} H^0(M,\mathcal{T}) & \xrightarrow{\cong} & \Gamma(\mathcal{T}) & \longrightarrow & \Gamma(\mathcal{P})/Im(\varphi_1) & \underset{\cong}{\longleftarrow} & \Gamma(\overline{\mathcal{R}})/Im(\varphi_2 & \xrightarrow{\cong} & H^1(M,\mathcal{R}) \\ \downarrow & & \downarrow Id & & \downarrow Id & & \downarrow Id & & \downarrow \\ \widetilde{H}^0(M,\mathcal{T}) & \xrightarrow{\cong} & \Gamma(\mathcal{T}) & \longrightarrow & \Gamma(\mathcal{P})/Im(\varphi_1) & \underset{\cong}{\longleftarrow} & \Gamma(\overline{\mathcal{R}})/Im(\varphi_2) & \xrightarrow{\cong} & \widetilde{H}^1(M,\mathcal{R}) \end{array} \tag{2.27}$$

in which the first and last squares commute by the definitions of the K-module homomorphisms $H^0(M,\mathcal{T}) \to \widetilde{H}^0(M,\mathcal{T})$ and $H^1(M,\mathcal{R}) \to \widetilde{H}^1(M,\mathcal{R})$, and the middle two squares trivially commutes. Thus the property (III) in the definition (2.4) is proved for $q = 0$. From (2.26) and the corresponding sequence for the $\widetilde{\mathcal{H}}$ theory we obtain the diagrams

$$\begin{array}{ccccccc} H^q(M,\mathcal{T}) & \longrightarrow & H^q(M,\mathcal{P}) & \underset{\cong}{\longleftarrow} & H^q(M,\overline{\mathcal{R}}) & \xrightarrow{\cong} & H^{q+1}(M,\mathcal{R}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \widetilde{H}^q(M,\mathcal{T}) & \longrightarrow & \widetilde{H}^q(M,\mathcal{P}) & \underset{\cong}{\longleftarrow} & \widetilde{H}^q(M,\overline{\mathcal{R}}) & \xrightarrow{\cong} & \widetilde{H}^{q+1}(M,\mathcal{R}) \end{array} \tag{2.28}$$

in which the first two squares commutes by the axiom (III) for the definition (2.4), and the last square commutes by definition of the homomorphism $H^{q+1}(M,\mathcal{R}) \to \widetilde{H}^{q+1}(M,\mathcal{R})$. Thus (III) follows for $q > 0$, and the proof of the Theorem is complete. □

Corollary 2.6. *Any two cohomology theories on M with coefficients in sheaves of K-modules over M are uniquely isomorphic.*

Proof. We have a unique homomorphism between $\mathcal{H}$ and $\widetilde{\mathcal{H}}$. By the Theorem, there must also exist a homomorphism $\widetilde{\mathcal{H}} \to \mathcal{H}$. Their composition $\mathcal{H} \to \widetilde{\mathcal{H}} \to \mathcal{H}$, must be unique. But we have another endomorphism $\mathcal{H} \xrightarrow{Id} \mathcal{H}$, so they have to be the same. □

This incredible Theorem says that the cohomology depends only on the topology of the space but not on the others structures space has, as the differentiable one.

Chapter 3

The De Rham Theorem

Having told about cohomology theories, now we are ready to study two of them, singular cohomology and de Rham cohomology. Even though we have seen that they are isomorphic, we will give an explicit isomorphism between them. Furthermore, we can define two different kinds of singular cohomologies: continuous and differential one. We call singular cohomology the continuous one only, we will use singular differential cohomology for the differential one.

In this chapter, we also define integration of forms over simpleces, which is necessary to build the de Rham cohomology. One can notice that there is no way to define integration of a function in a coordinate-independent way on a smooth manifold. Differential forms turn out to have just the right properties to defining integrals intrinsically. On the other hand, we are going to prove a version of Stokes' theorem with forms and simplices. It is a generalisation of the fundamental theorem of calculus, as well as of the three great classical theorems of vector analysis: Green's theorem for vector fields in the plane; the divergence theorem for vector fields in the space; and (the classical version of) Stokes' theorem for surface integrals in $\mathbb{R}$.

3.1 The Singular Cohomology

Giving a view about what is to come, singular cohomology is essentially a contravariant version of singular homology. It does not give us more new information about topological spaces, but information is organized differently, which is much more appropriate for some applications. Indeed, there is a way of adjoining two cocycles of degree p and q to form a composite cocycle of degree p + q, making singular cohomology into an algebra structure. The application defined in this way is called the cup product, but we are not going to study it in this text.

From now on, we are going to take $K = \mathbb{R}$; it is not strictly necessary for singular cohomology, where we only need K to be an arbitrary Principal Ideal Domain. We also need to introduce a couple of concepts. For each $p \geq 1$ we take the set

$$\Delta^p = \left\{ (a_1, \ldots, a_p) \in \mathbb{R}^p \mid \sum_{i=1}^{p} a_i \leq 1 \text{ and each } a_i \geq 0 \right\}. \tag{3.1}$$

We call Δ^p as *standard p-simplex in* $\mathbb{R}^p$, taking the point 0 for Δ^0.

A *continuous singular p-simplex σ in U*, where $U \subset M$ is an open subset, is a continuous map $\sigma : \Delta^p \to U$. Usual, we are going to call them as a singular p-simplex in U. We define

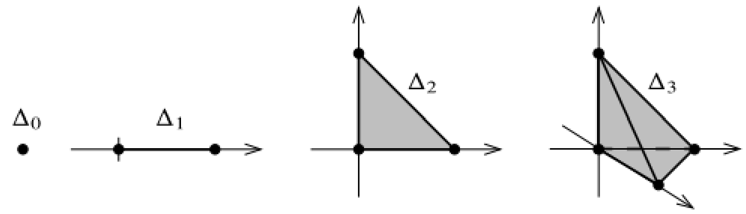


Figure 3.1: Standard p-simplices for $p = 0, 1, 2$ and 3

differential singular p-simplex σ in U as singular p-simplex in U which can be extended to be a $\mathcal{C}^\infty$ (differential) map of a neighbourhood of Δ^p in $\mathbb{R}^p$ into U. We can study both in the same way except for a few particular cases, so we are going to treat them equally and, should we need any explicit details, will be mentioned.

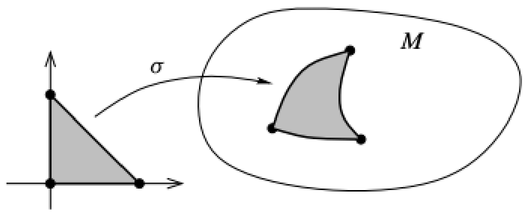


Figure 3.2: Singular 3-simplex

We shall let $S_p(U)$ denote the free abelian group generated by the singular p-simplices in U. Elements of $S_p(U)$ are formal linear combinations with integer coefficients:

$$c = \sum_{i=0}^{n} \lambda_i \sigma_i \quad \text{where } \sigma_i \text{ are } p\text{-simplex in } U \text{ and } \lambda_i \in \mathbb{N}\ . \tag{3.2}$$

Now we shall define a homomorphism $S_p(U) \to S_{p-1}(U)$ which reflects the properties of an oriented boundary algebraically. Since $S_p(U)$ is a free abelian group on the set of singular p-simplices $\sigma : \Delta^p \to U$, it suffices to define a homomrphism on all singular p-simplices σ and then extend by linearity into $S_p(U)$. A moment's thought suggests that boundary operator should be determined somehow by restricting σ to the "faces" of standard simplex and then summing with appropriate signs to handle the issue of orientation. In order to get this right, we define what should be understood by *faces* of a simplex. For each $p \geq 0$ we define the collecion of maps $K_i^p : \Delta^p \to \Delta^{p+1}$ for $0 \leq i \leq p+1$ as follows

$$\text{for } p = 0, \qquad K_0^0(0) = 1 \text{ and } k_1^0(0) = 0;$$

$$\text{for } p \geq 1, \qquad \begin{cases} K_0^p(a_1, \ldots, a_p) = \left(1 - \sum_{i=1}^{p} a_i, a_1, \ldots, a_p\right) \\ K_i^p(a_1, \ldots, a_p) = (a_1, \ldots, a_{i-1}, 0, a_i, \ldots, a_p),\ 1 \leq i \leq p+1. \end{cases}$$

If σ is a p-simplex in M with $p \geq 1$, we difine its *ith face*, $0 \leq i \leq p$, to be the $(p-1)$-simplex

$$\sigma^i = \sigma \circ K_i^{p-1} \tag{3.3}$$

and, finally, we define the *boundary operator* as the homomorpism $S_p(U) \to S_{p-1}(U)$ such that

$$\partial\sigma = \sum_{i=0}^{p} (-1)^i \sigma^i. \tag{3.4}$$

Observe that this map is well defined, that is $\partial\sigma \in S_{p-1}(U)$. At this point one can notice that it is better to write ∂_p but, usually, there is no confusion written out ∂.

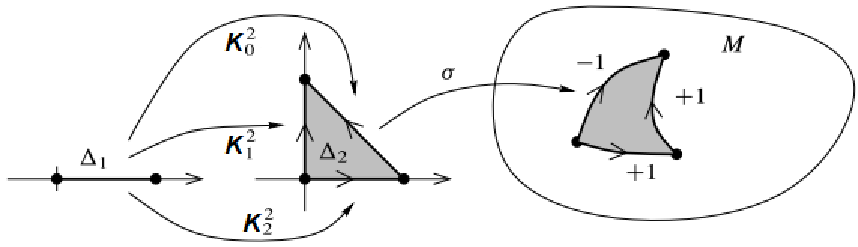


Figure 3.3: The boundary operator

We claim that $\partial \circ \partial = 0$. To proof it, we see $\partial \circ \partial(\sigma) = 0$ for any p-symplex σ, and by linearity it holds for any p-chain. Let us calculate compositions $K_i^{p+1} \circ K_j^p$. For $p = 0$ we have four cases

$$\begin{aligned} i = j = 0 &\quad 0 \mapsto K_0^1(K_0^0(0)) = (1, 0) \\ i = 1,\, j = 0 &\quad 0 \mapsto K_1^1(K_0^0(0)) = (0, 0) \\ i = 0,\, j = 1 &\quad 0 \mapsto K_0^1(K_1^0(0)) = (0, 1) \\ i = j = 1 &\quad 0 \mapsto K_1^1(K_1^0(0)) = (0, 1) \end{aligned}$$

For $p \geq 1$,

$$\begin{aligned} 1 \leq i < j &\quad (a_1, \ldots, a_p) \mapsto (a_1, \ldots, a_{i-1}, 0, a_i, \ldots, a_{j-1}, a_j, 0, \ldots, a_p) \\ 1 \leq i = j &\quad (a_1, \ldots, a_p) \mapsto (a_1, \ldots, a_{i-1}, 0, 0, a_i, \ldots, a_p) \\ 0 = i < j &\quad (a_1, \ldots, a_p) \mapsto \left(1 - \sum_{i=1}^{p} a_i,\, a_1, \ldots, a_{j-1}, 0, a_j, \ldots, a_p\right) \\ 0 = i = j &\quad (a_1, \ldots, a_p) \mapsto \left(0, 1 - \sum_{i=1}^{p} a_i,\, a_1, \ldots, a_p\right). \end{aligned}$$

Thinking for a moment, one can note that in all these cases we have $K_i^{p+1} \circ K_j^p = K_{j+1}^{p+1} \circ K_i^p$. Therefore, $\partial(\partial(\sigma)) = \sum_j (-1)^j \sum_i (-1)^i \sigma \circ K_i^{p-1} \circ K_j^{p-2} = \sum_j (-1)^j \sum_i (-1)^i \sigma \circ K_{j+1}^{p-1} \circ K_i^{p-2} = 0$. So, $\partial \circ \partial = 0$.

Let $S^p(U, K)$ be the K-module $\mathrm{Hom}(S_p(U), K)$. Note that an element f in $S^p(U, K)$ is just a function which assigns to each singular p-simplex in U an element of K. Such an f is called a *singular p-cochain on U*. Scalar multiplication and addiction in these modules are defined by

$$\begin{aligned} (kf)(\sigma) &= k(f(\sigma)) \\ (f + g)(\sigma) &= f(\sigma) + g(\sigma), \end{aligned} \tag{3.5}$$

and extened into homomorphisms of $S_p(U)$ by linearity.

If $V \subset U \subset M$ are open subsets, we can define restriction maps setting

$$\rho_{V,U} : S^p(U, K) \to S^p(V, K) \tag{3.6}$$

which are homomorphisms that assign each element in $f \in S^p(U, K)$ with its restriction into singular p-simplices which lie in V. Then, we have presheaves on M called the presheaves of singular p-cochains

$$\{S^p(U, K);\, \rho_{V,U}\} \tag{3.7}$$

for each $p \in \mathbb{N}$. Observe that these presheaves do not fulfil (C_1) but (C_2) for $p \geq 1$. Here we make a distintion to denote presheaves of differential singular p-cochains by $\{S^p_\infty(U,K);\, \rho_{V,U}\}$. As we have seen, there are sheaves associated by the sheafification homomorphism to these presheaves that we write as $\mathcal{S}^p(M,K)$ and we call the *sheaves of germs of singular p-cochains*.

As in the case of $S_p(U)$, for each $p \geq 0$ we can define a coboundary homomorphism in $S^p(U,K)$

$$d : S^p(U,K) \to S^{p+1}(U,K) \tag{3.8}$$

defined $d(f)$ for each $f \in S^p(U,K)$ by

$$df(\sigma) = f(\partial\sigma) \tag{3.9}$$

for each σ a $p+1$-simplex on U, and extend linearly. Neither we write indices d^p in this homomorphism due to there is not any confusion. Property $\partial \circ \partial = 0$ implies that $d \circ d = 0$. Futhermore, the fact that d commutes with restrictions $\rho_{U,V}$ yields d as a presheaf homomorphism

$$\{S^p(U,K);\, \rho_{V,U}\} \xrightarrow{d} \{S^{p+1}(U,K);\, \rho_{V,U}\}. \tag{3.10}$$

Using this two properties, d makes

$$\cdots \to 0 \to S^0(U,K) \xrightarrow{d} S^1(U,K) \xrightarrow{d} S^2(U,K) \xrightarrow{d} S^3(U,K) \xrightarrow{d} \dots \tag{3.11}$$

into a cochain complex denoted by $S^*(U,K)$.

Coboundary presheaf homomorphisms give rise to sheaf homomorphisms between sheaves of germs of singular p-cochains, d is retained to denote it:

$$\mathcal{S}^p(M,K) \xrightarrow{d} \mathcal{S}^{p+1}(M,K). \tag{3.12}$$

Therefore, we have a cochain complex of sheaves similar to the previous one. We can also add a left-term to this complex. One can note that $\mathcal{S}^0(M,K)$ is simple the sheaf of germs of functions on M with values in K. In this way we can inject canonically the constant sheaf $\mathcal{K} = K \otimes M$ into $\mathcal{S}^0(M,K)$ by sending $k \in \mathcal{K}_m$ to the germ at m of the functions on M with constant value k. Thus, we can write this extended sequence

$$0 \to \mathcal{K} \to \mathcal{S}^0(M,K) \to \mathcal{S}^1(M,K) \to \mathcal{S}^2(M,K) \to \mathcal{S}^3(M,K) \to \dots \tag{3.13}$$

Also, an analogous sequence can be written in same way replacing $\mathcal{S}^p(M,K)$ by $\mathcal{S}^p_\infty(M,K)$. In both, we have a fine torsion-free resolution of the constant sheaf $\mathcal{K}$. That the sheaves $\mathcal{S}^p(M,K)$ are torsion-free holds trivially due to $K = \mathbb{R}$ (same happens when K is a principal ideal domain). A partition of unity in $\mathcal{S}^p(M,K)$ for each locally finite open cover of M can be found to see sheaves $\mathcal{S}^p(M,K)$ are fine. Let $\{U_i\}$ be a locally finite open cover of M and take a partition of unity $\{\varphi_i\}$ in M subordinate to this cover, in which the functions take values 0 or 1 only. For each $U \subset M$ open subset and each $p \geq 0$ we define the endomorphisms of presheaves $\widetilde{l_i}$ in $S^p(U,K)$ wich act for each $f \in S^p(U,K)$ and σ p-simplex by setting

$$\widetilde{l_i}(f)(\sigma) = \varphi_i(\sigma(0))f(\sigma), \tag{3.14}$$

and we extend $\widetilde{l_i}$ linearly. As we have said these are endomorphisms of presheaves due to $\widetilde{l}_{,i}$ commutes trivially with the restrictions maps. Let $l_i : \mathcal{S}^p(M,K) \to \mathcal{S}^p(M,K)$ be the endomorphism of sheaves associated with the previous endomorphism of presheves. It follows straightforward that $\text{supp}(l_i) \subset U_i$ and that $\sum l_i = Id$, so we have a partition of unity. Now we want to see that the sequence is exact. Observe that $im(d) \subset Ker(d)$ due to $d \circ d = 0$, so we only need to see the other inclusion. As we are looking for this property in the sheaves $\mathcal{S}^p(M,K)$ note that we can prove this over $S^p(U,K)$ with U a "sufficiently small" open subset of M and, by the sheafification homomorphism, this holds for the sheaves $\mathcal{S}^p(M,K)$. Here we shall make use a property of the differential manifolds (or topological manifolds in general). It suffices to prove this for an open set of $\mathbb{R}^n$, where n is the dimension of the manifold, and using pull-backs of the chart morphisms we can induce it for M. We take U as the open unit ball in $\mathbb{R}^n$, and we want to prove that: if $df = 0$ for $f \in S^p(M,K)$, there exists a $g \in S^{p-1}(M,K)$ such that $dg = f$. It follows if we can find a homotopy operator for each $p \geq 0$

$$h_p : S^p(U,K) \to S^{p-1}(U,K) \tag{3.15}$$

such that

$$d \circ h_p + h_{p+1} \circ d = Id, \tag{3.16}$$

therefore, if $df = 0$ we have that $d \circ h_p(f) = d \circ h_p(f) + h_{p-1} \circ d(f) = f$ and we can take $g = h_p(f)$. We can define h_p acting into $f \in S^p(U,K)$ as follows

$$h_p(f)(\sigma) = f(\widetilde{h}_p(\sigma)), \tag{3.17}$$

for each $\sigma \in S_p(U)$. Hence, we only need to define functions $\widetilde{h}_p : S_{p-1}(U) \to S_p(U)$. Note that, if it is defined in the way that $Id = \partial \circ \widetilde{h}_{p+1} + \widetilde{h}_p \circ \partial$ then (3.16) holds due to the definition of coboundary operators. At this point, there are different paths to define $\widetilde{h}_p$ for p-simplex and for p-diferential simplex. We are only going to see this construction for p-differential simplex because we can use the same functions to prove the case of p-simplex, but we shall not use opposite affirmation. First, let φ be the real-valued $\mathcal{C}^\infty$ function on the real line:

$$f(t) = \begin{cases} e^{-1/t} & \text{For } t > 0 \\ 0 & \text{For } t \leq 0 \end{cases} \quad \text{and} \quad \varphi(t) = \frac{f(t)}{f(t) + f(1-t)}$$

the most important property of this function is that it takes values between 0 and 1, and has value 1 for $t \geq 1$ and the value 0 for $t \leq 0$.

We define the functions $\widetilde{h}_p$ as $\widetilde{h}_p(\sigma)(0) = 0$ and for $(a_1, \ldots, a_p) \neq (0, \ldots, 0)$ by

$$\begin{aligned} \widetilde{h}_1(\sigma)(a_1) &= a_1 \sigma(0), \\ \widetilde{h}_p(\sigma)(a_1, \ldots, a_p) &= \varphi\left(\sum_{i=1}^{p} a_i\right) \sigma\left(\frac{a_2}{\sum a_i}, \ldots, \frac{a_n}{\sum a_i}\right). \end{aligned} \tag{3.18}$$

Geometrically, $\widetilde{h}_p(\sigma)$ is the cone in U obtained by joining σ radially to the origin:

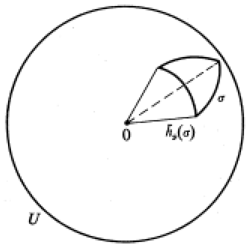


Figure 3.4: Geometry representation of $\widetilde{h}_2$

If σ is a differentiable p-simplex then $\widetilde{h}_p(\sigma)$ is defined on $\mathbb{R}^{p+1}$. Moreover, $\widetilde{h}_p(\sigma)$ are differential homomorphisms on $\mathbb{R}^{p+1}$, the points where problems arise are those for which $\sum a_i = 0$.

But, since the homomorphism $\varphi(t)$ and all of its derivatives vanish faster than any polynomial in t as $t \to 0$, and since σ and each of its derivative is bounded on $\mathbb{R}^{p-1}$, it follows that all derivatives of $\widetilde{h}_p(\sigma)$ of all orders exist and are continuous, and are zero in points where $\sum a_i = 0$. These functions fulfill the properties that we wanted. Finally, observe also that $\widetilde{h}_p(\sigma)$ is continuous for each σ p-simplex, therefore we can use it to prove the case of p-simplex.

$\widetilde{h}_p$ extends to a homomorphism $S_{p-1}(U) \to S_p(U)$. It follows from the definition of boundary operator ∂ that

$$id = \partial \circ \widetilde{h}_{p+1} + \widetilde{h}_p \circ \partial \tag{3.19}$$

on $S_p(U)$ for $p \geq 1$. The case in which $p = 1$ turns out that

$$\sigma = \left(\widetilde{h}_2(\sigma)\right)^2 - \left(\widetilde{h}_2(\sigma)\right)^1 + \left(\widetilde{h}_2(\sigma)\right)^0 + \widetilde{h}_1(\sigma^0) - \widetilde{h}_1(\sigma^1), \tag{3.20}$$

which is easy to prove. Other cases are left for the reader.

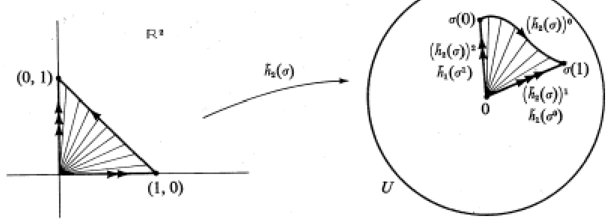


Figure 3.5: Details of application of $\widetilde{h}_p$ over a 1-simplex

These complete the proof that the sequence is fine torsion-free resolution of the constant sheaf $\mathcal{K}$. Then, we have cohomology theories for continuous and the differential cases that we set

$$H^q(M, \mathcal{S}) = H^q(\Gamma(S^*(M,K) \otimes \mathcal{S})),$$
$$H^q_\infty(M, \mathcal{S}) = H^q(\Gamma(S^*_\infty(M,K) \otimes \mathcal{S})).$$

These theories are uniquely isomorphic.

Let G be a K-module. Let $S^p(U,G)$ be the K-module consisting of functions which assign to each singular p-simplex in U an element of G. We may replace K by G and $\mathcal{K}$ by the constant sheaf $\mathcal{G} = G \times M$ in the construction that we have done. Our next goal is to see that *classical singular cohomology groups of M with coefficients in K-module G* are canonically isomorphic with sheaf cohomology modules $H^q(M, \mathcal{G})$. Remember that *classical singular cohomology groups of M* are dual of *classical singular homology groups of M* that are defined in continuous and differential cases by:

$$\begin{aligned} H^q_\Delta(M;G) &= H^q\left(S^*(M,G)\right), \\ H^q_{\Delta\infty}(M;G) &= H^q\left(S^*_\infty(M,G)\right), \end{aligned} \tag{3.21}$$

where $H^q\left(S^*(M,G)\right)$ and $H^q\left(S^*_\infty(M,G)\right)$ are the q cohomology module of the

$$\cdots \to 0 \to S^0(M,G) \xrightarrow{d} S^1(M,G) \xrightarrow{d} S^2(M,G) \xrightarrow{d} S^3(M,G) \xrightarrow{d} \ldots, \tag{3.22}$$

and

$$\cdots \to 0 \to S^0_\infty(M,G) \xrightarrow{d} S^1_\infty(M,G) \xrightarrow{d} S^2_\infty(M,G) \xrightarrow{d} S^3_\infty(M,G) \xrightarrow{d} \ldots \tag{3.23}$$

respectively. We shall show that classical cohomology groups are canonically isomorphic to the cohomology modules $H^q(M, \mathcal{G})$. In order to do that, we see first a proposition.

Proposition 3.1. *Let $\{S_U;\ \rho_{U,V}\}$ be a presheaf on M satisfying the property (C_2) and let $\mathcal{S}$ be the associated sheaf of germs. Let*

$$(S_M)_0 = \{s \in S_M \mid \rho_{p,M}(s) = 0\ \forall p \in M\}. \tag{3.24}$$

Then the sequence

$$0 \to (S_M)_0 \to S_M \xrightarrow{\gamma} \Gamma(\mathcal{S}) \to 0 \tag{3.25}$$

is exact. Where γ is the homomorphism which sends $s \in S$ to the global section $m \mapsto \rho_{m,M}(s)$

Proof. Exactness of the sequence at S_M is the result of the definition of $(S_M)_0$. Thus, we need only to prove that γ is surjective. Let $t \in \Gamma(\mathcal{S})$ be a global section. Then, there exists a locally finite open cover $\{U_\alpha\}$ of M, and there are elements $s_\alpha \in S_{U_\alpha}$ such that

$$\gamma(s_\alpha) = t|_{U_\alpha}. \tag{3.26}$$

Let $\{V_\alpha\}$ be a refinement such that $\overline{V_\alpha} \subset U_\alpha$. Let I_m be the collection of all those indices α for which $m \in \overline{V_\alpha}$, note that I_m is finite. Choose a neighborhood W_m of m such that

(a) $W_m \cap \overline{V_\beta} = \phi$ if $\beta \notin I_m$,

(b) $W_m \subset \bigcup_{\alpha \in I_m} U_\alpha$,

(c) $\rho_{W_m,U_\alpha}(s_\alpha) = \rho_{W_m,U_{\alpha'}}(s_{\alpha'})$ if $\alpha, \alpha' \in I_m$.

Let $s_m \in S_{W_m}$ be the common image of the elements in (c). For all n and m in M we take $p \in W_m \cap W_n$. Then, it follows form (a) that $I_p \subset I_m \cap I_n$. So let $\alpha \in I_p$, and according to (c),

$$s_m = \rho_{W_m,U_\alpha}(s_\alpha) \quad \text{and} \quad s_n = \rho_{W_n,U_\alpha}(s_\alpha), \tag{3.27}$$

so that

$$\rho_{W_m \cap W_n, W_m}(s_m) = \rho_{W_m \cap W_n, U_\alpha}(s_\alpha) = \rho_{W_m \cap W_n, W_n}(s_n). \tag{3.28}$$

which proves

$$\rho_{W_m \cap W_n, W_m}(s_m) = \rho_{W_m \cap W_n, W_n}(s_n). \tag{3.29}$$

Therefore by the property (C_2), there exists an element $s \in S_M$ such that

$$\rho_{W_m,M}(s) = s_m. \tag{3.30}$$

and it follows that $\gamma(s) = t$. □

In the case of singular presheaves, we have that

$$0 \to S_0^*(M,G) \to S^*(M,G) \to \Gamma\big(\mathcal{S}^*(M,G)\big) \to 0 \tag{3.31}$$

is a short exact sequence of cochain complexes. Thus, if we prove that

$$H^p\big((S_0^*(M,G)\big) = 0 \quad \text{for all } q, \tag{3.32}$$

it follows from the long exact sequence associated with the previous one that there are canonical isomorphisms

$$\begin{aligned} H^q\big(S^*(M,G)\big) &\cong H^q\big(\Gamma\big(S^*(M,G)\big)\big), \\ H^q\big(S^*_\infty(M,G)\big) &\cong H^q\big(\Gamma\big(S^*_\infty(M,G)\big)\big). \end{aligned} \tag{3.33}$$

To proof (3.32), observe that this is trivialy for $q < 0$. Module $S_0^0(M,G)$ is also zero, since the presheaf $\{S^0(U,G);\, \rho_{U,V}\}$ is complete. Thus $H^0(S_0^*(M,G)) = 0$. To see that for $q \geq 1$ we use *"$\mathcal{U}$-small" singular p-simpleces*. Remember that, to define these simplices we have to take $\mathcal{U} = \{U_i\}$ an arbitrary open cover of M. So $\mathcal{U}$-small singular p-simplices are those whose ranges lie in elements of $\mathcal{U}$, therefore we can define singular cochain with values in G defined only on $\mathcal{U}$-small singular p-simplices $S_{\mathcal{U}}^p(M,G)$ moreover, we have the cochain complex $S_{\mathcal{U}}^*(M,G)$.

With these definitions, we have that $H^q(S_{\mathcal{U}}^*(M,G)) \cong H^q(S^*(M,G))$, this proof is as identity as in the case of homology groups. There is, also, a restriction homomorphism $j_{\mathcal{U}} : S^p(M,G) \to S_{\mathcal{U}}^p(M,G)$ which yield a surjective cochain map

$$j_{\mathcal{U}} : S^*(M,G) \to S_{\mathcal{U}}^*(M,G). \tag{3.34}$$

The kernel of the homomorphism $j_{\mathcal{U}}$ forms a cochain complex $K_{\mathcal{U}}^*$ such that the sequence of cochain complexes

$$0 \to K_{\mathcal{U}}^* \to S^*(M,G) \to S_{\mathcal{U}}^*(M,G) \to 0 \tag{3.35}$$

is exact. Hence $H^q(K_{\mathcal{U}}^*) = 0$ for all q, due to we have $H^q(S_{\mathcal{U}}^*(M,G)) \cong H^q(S^*(M,G))$. So, let $q \geq 1$, and let f be a cocycle in $S_0^q(M,G)$, that is, $df = 0$. Then, by the definition of $S_0^q(M,G)$, there exists an open cover $\mathcal{U}$ of M consisting of sufficiently small open sets so that $g \in K_{\mathcal{U}}^{q-1} \subset S_0^{q-1}(M,G)$ and $dg = f$. An, therefore, $H^q\big(S_0^*(M,G)\big) \subset H^q(K_{\mathcal{U}}^*) = 0$.

3.2 The Stokes' Theorem and the de Rham Cohomology

A smooth n-form ω is called closed if the differential cancels it $d\omega$. Alternatively, a n-form ω is called exact if there exists an $(n-1)$-form η such that $d\eta = \omega$. In this way, all the exact forms are, in turn, closed. In some sense, de Rham cohomology groups make this dependence quantitative, since they are defining as the set of exact forms module close forms. In other words, these groups formalize the question of which close forms are exact. The final answer is given by the de Rham theorem where it is said that the existence of closed forms is connected with topological properties of the manifold.

In this section, we are going to define de Rham cohomology and specify the homomorphism between de Rham cohomology and singular one. In order to do that, we need to set some concepts that we shall use after. Among them, we start defining the integration of p-forms over differentiable singular p-chains in n-dimensional manifolds. In the next Chapter, we will define a more general way to integrate functions over manifolds.

We start defining *integration of k-forms in* $\mathbb{R}^n$. Let $x_1 \ldots, x_n$ denote the conical coordinate system on $\mathbb{R}^n$. Let ω be an n-form on an open set $D \subset \mathbb{R}^n$. Then there is a unique determined function f on D such that $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_n}$. Let $A \subset D$. We define

$$\int_A \omega = \int_A f dx_{i_1} \ldots dx_{i_n},$$

provided that the latter exists.

Here we need to recall the change of variables theorem. Let ϕ be an diffeomorphism of a bonded open set D in $\mathbb{R}^n$ with a bonded ope set $\phi(D)$. Let $J\phi$ denote the determinant of Jacobian matrix of ϕ, let f be a bounded continuous function on $\phi(D)$, and let A be a subset of D with a Jordan content. Then

$$\int_{\phi(A)} f = \int_A f \circ \phi |J\phi|.$$

Using this theorem to integration of forms we obtain

$$\int_{\phi(A)} \omega = \int_A \pm\delta\phi(\omega),$$

the sing $\pm$ depends on the orientation of the map, but here we do not need to define orientations since the integrals are over simplices. We will talk deeply about orientations in the next Chapter. Therefore, we define integral of the n-form ω over a p-simplex σ by

$$\int_\sigma \omega = \int_{\Delta^p} \delta\sigma(\omega).$$

We extend these integrals linearly to chains, so that if $c = \sum a_i\sigma_i$, then

$$\int_c \omega = \sum a_i \int_{\sigma_i} \omega.$$

Let us continues with the construction of the de Rham cohomology. Here we use the presheaf that we saw in charter 1

$$\{E^k(U); \rho_{U,V}\}, \tag{3.36}$$

where $E^k(U)$ is the set of differential k-forms on U, being U an open subset in M. Remember, that this presheaf fulfils the properties (C_1) and (C_2), so it is a complete presheaf.

Now, we want to build a sequence using this $\mathbb{R}$-modules. To do that, we consider exterior differential. Note that exterior differential is a presheaf homomorphism since it commutes with $\rho_{U,V}$

$$\{E^k_U; \rho_{U,V}\} \xrightarrow{d} \{E^{k+1}_U; \rho_{U,V}\}. \tag{3.37}$$

To continue with an analogous argument as in singular cohomology, we shall set the sheaves associated with the previous presheaves. We denote these sheaves by $\xi^p(M)$, and we shall retain the symbol d for sheaf homomorphisms induced by the exterior differential. The constant sheaf $\mathcal{R} = \mathbb{R} \times M$ can be naturally injected into $\xi^0(M)$ by mapping $a \in \mathcal{R}_m$ to the germ at m of the function with constant value a. Thus we have the sequence

$$0 \to \mathcal{R} \to \xi^0(M) \xrightarrow{d} \xi^1(M) \xrightarrow{d} \xi^2(M) \xrightarrow{d} \xi^3(M) \xrightarrow{d} \dots \tag{3.38}$$

We claim that this sequence is a fine torsion-free resolution. It is clear that this is torsion-free due to $\mathbb{R}$-modules are being taken, and $\mathbb{R}$ is a body. $\xi^p(M)$ is certainly a fine $\mathbb{R}$-modul since a partition of unity can be constructed as follows. If $\{U_i\}$ is a locally finite open cover of M, let $\{\varphi_i\}$ be a partition of unity on M subordinate to this cover. We obtain endomorphisms $\widetilde{l_i}$ of the presheaves $\{E^k_U; \rho_{U,V}\}$ by setting

$$\widetilde{l_i}(f)(m) = (\varphi_i|_U)(m)f(m), \quad \text{for } f \in \mathcal{C}^\infty. \tag{3.39}$$

The associate sheaf endomorphisms l_i on the sheaf E^k_M form a partition of unity subordinate to the cover $\{U_i\}$ of M, and therefore $\xi^p(M)$ is a fine sheaf. To see that this sequence is exact we use an analogous argument as in the case of the singular cohomology taken this lemma:

Lemma 3.2. *(Poincaré lemma) Let $\mathcal{B} \subset \mathbb{R}^n$ be the open unit ball. Then for each $k \geq 1$ there is a linear transformation $h_k : E^k(U) \to E^{k-1}(\mathcal{B})$ such that*

$$h_{k+1} \circ d + d \circ h_k = Id \tag{3.40}$$

Proof. Let us begin with the Cartan formula which expresses the Lie derivative in terms of the exterior differentiation and interior multiplication:

$$L_X = i(X) \circ d + d \circ i(X). \tag{3.41}$$

We shall applied the Cartan formula to the radial vector field

$$X = \sum_{i=1}^{p} x_i \frac{\partial}{\partial x_i} \tag{3.42}$$

on $\mathcal{B}$, where x_i are the canonical coordinates. We define h_k a linear operator α_k on $E^k(\mathcal{B})$ by setting

$$\alpha_k(f dx_{i_1} \wedge \cdots \wedge dx_{i_k})(p) = \left(\int_0^1 t^{k-1} f(tp) dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_k}(p) \tag{3.43}$$

and extending linearly to ll $E^k(\mathcal{B})$. Let us show that

$$\alpha_k \circ L_X = Id \quad \text{on } E^k(\mathcal{B}) \tag{3.44}$$

for that the k-form $f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is applied to $\alpha_k \circ L_X$ for any $p \in \mathcal{B}$

$$\begin{aligned} \alpha_k \circ L_X(f dx_{i_1} \wedge \cdots \wedge dx_{i_k})(p) &= \alpha_k \left\{ \left(kf + \sum x_i \frac{\partial f}{\partial x_i} \right) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right\} (p) \\ &= \left\{ \int_0^1 t^{k-1} \left(kf(tp) + \sum x_i(tp) \frac{\partial f}{\partial x_i}|_{tp} \right) dt \right\} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \left(\int_0^1 \frac{d}{dt} (t^k f(tp)) dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= f dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \end{aligned} \tag{3.45}$$

Where we have used that L_X commutes with d for the first equality, and we have integrated by parts for the third one. Therefore, we obtain

$$Id = \alpha_k \circ i(X) \circ d + \alpha_k \circ d \circ i(X) \tag{3.46}$$

on $E^k(\mathcal{B})$ extended linearly. Now we want to see that α commutes with d

$$\begin{aligned} \alpha_k \circ d(f dx_{i_1} \wedge \cdots \wedge dx_{i_k})(p) &= \alpha_k \left\{ \left(\sum \frac{\partial f}{\partial x_i} \right) dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}(p) \right\} \\ &= \left(\int_0^1 t^{k-1} \sum \frac{\partial f}{\partial x_i}|_{tp} dt \right) dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= d \left(\int_0^1 t^{k-2} f(tp) dt \right) dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= d \circ \alpha_{k-1}(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}). \end{aligned} \tag{3.47}$$

And this expression is extended to $E^k(\mathcal{B})$ linearly. Thus we have that

$$Id = \alpha_k \circ i(X) \circ d + d \circ \alpha_{k-1} \circ i(X) \tag{3.48}$$

on $E^k(U)$. Thus, the desire linear transformation h_k is obtained by

$$h_k = \alpha_{k-1} \circ i \left(\sum x_i \frac{\partial}{\partial x_i} \right) \tag{3.49}$$

□

We have constructed a fine torsion-free resolution of the constant sheaf $\mathcal{R}$. This resolution gives rise to a cohomology theory over M with coefficients in sheaves of real vector spaces by setting

$$H^q(M, \mathcal{T}) = H^q\big(\Gamma(\xi^*(M) \otimes \mathcal{T})\big) \tag{3.50}$$

for $q \geq 0$, where $\mathcal{T}$ is a sheaf of the real vector space over M. Note that we use the same notation that in the case of the singular cohomology, but there is not any confusion because they are isomorphic.

Now, we shall define the classical de Rham cohomology. Let us consider the next cochain complex

$$\dots 0 \to E^0(M) \xrightarrow{d} E^1(M) \xrightarrow{d} E^2(M) \xrightarrow{d} E^3(M) \xrightarrow{d} \dots \tag{3.51}$$

Then, we fix the real vector space $H^q_{deR}(M) = ker(d^q)/Im(d^{q-1})$ for $q \geq 0$ and $H^q_{deR}(M) = 0$ for $q \leq 0$. These are the $q$ph *de Rham classical cohomology group of* M. Futhermore, a p-form ω on M is called *closed* if $d\omega = 0$. It is called *exact* if there exists a $(p-1)$-form ω'such that $\omega = d\omega'$. Therefore, we have

$$H^q_{deR}(M) = \{ \text{ closed } q\text{-forms } \}/\{ \text{ exact } q\text{-forms } \}. \tag{3.52}$$

We want to see that classical de Rham cohomology is, actually, isomorphic to $H^q(M, \mathcal{R})$, where we take $\mathcal{R}$ being the constant sheaf $\mathbb{R} \otimes M$, then

$$H^q(M, \mathcal{R}) = H^q\big(\Gamma(\xi^*(M) \otimes \mathcal{R})\big) \cong H^q\big(\Gamma(\xi^*(M))\big). \tag{3.53}$$

Consider now the cochain complex $\Gamma\big(\xi^*(M)\big)$:

$$\dots 0 \to \Gamma\big(\xi^0(M)\big) \to \Gamma\big(\xi^1(M)\big) \to \Gamma\big(\xi^2(M)\big) \to \dots \tag{3.54}$$

and the (3.51) one. Since we had that the presheaf $\{E^k_U; \rho_{U,V}\}$ is complete, it follows that the natural homomorphisms

$$E^p(M) \to \Gamma\big(\xi^p(M)\big) \tag{3.55}$$

are isomorphisms. Due to these homomorphisms commute with d, they induce a cochain map $E^* \to \Gamma\big(\xi^*(M)\big)$ which is an isomorphism of cochain complexes. Thus there are canonical isomorphisms

$$H^q\big(\Gamma\big(\xi^*(M)\big)\big) \cong H^q\big(E^*(M)\big) = H^q_{\text{de R}}(M). \tag{3.56}$$

As we have seen, all cohomology theories are canonically isomorphic. Therefore classical differential singular cohomology and classical de Rham one are isomorphic. We shall now prove that the explicit homomorphism from de Rham cohomology to differentiable singular one obtained from integration of forms over differentiable singular simplices yields the canonical isomorphism.

For each $p \in N$, we define the homomorphism $\kappa_p : E^p(M) \to S^p_\infty(M)$ by setting

$$\kappa_p(\omega)(\sigma) = \int_\sigma \omega \tag{3.57}$$

for each p-simplex σ in M and extended linearly. We want to see that this homomorphism induces a homomorphism of cohomology theories $\kappa^*_p : H^p_{deR}(M) \to H^p_{\Delta\infty}(M)$ which is called *de Rham homomorphism*. First, we shall prove Stokes theorem, which has to be used later.

Perhaps the single most important theorem in integration theory on manifolds is Stokes' theorem. This is a generalisation of Fundamental Theorem of Calculus. Observe that in our context Fundamental Theorem says that if F is a smooth function on the real line, and if σ is a smooth 1-simplex in the real line, then

$$\int_{\partial\sigma} F = \int_{\sigma} dF.$$

We shall present the Stokes' theorem in terms of integration of forms over chains.

Theorem 3.3. *(Stokes' theorem). Let c be a smooth p-chain in M and let ω be a $(p-1)$-form on a neighbourhood of the image of c. Then*

$$\int_{\partial c} \omega = \int_{c} d\omega \tag{3.58}$$

Proof. We will prove it only for simplices, as it juts extends by linearity to chains. Let σ be a singular differentiable p-simplex. Then, by commutativity of pullbacks and differential operators, we get

$$\int_{\sigma} d\omega = \int_{\Delta^p} \delta\sigma(d\omega) = \int_{\Delta^p} d(\delta\sigma(\omega)) = \sum_{i=0}^{p}(-1)^i \int_{\Delta^p} \delta\sigma^i(\omega) = \sum_{i=0}^{p}(-1)^i \int_{\Delta^p} \delta k_i^{p-1} \circ \delta\sigma(\omega). \tag{3.59}$$

We can use an induction argument in p. Case $p = 1$ holds because of the Fundamental Theorem of Calculus since a 0-form is a function $\omega : M \to \mathbb{R}$.

$$\int_{\Delta^1} \frac{d}{dx}(\omega \circ \sigma)dx = \omega\big(\sigma(1)\big) - \omega\big(\sigma(0)\big). \tag{3.60}$$

Continued with the induction argument, let us assume that $p \geq 2$. Then, using that the $(p-1)$-form $\delta\sigma(\omega)$ is taken in $\Delta^p \subset \mathbb{R}^p$, it can be expressed as

$$\delta\sigma(\omega) = \sum_{j=1}^{p} a_j\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_p \tag{3.61}$$

where $x_1, \ldots, x_p$ are canonical coordinate in $\mathbb{R}^p$, circumflex over a term means that the term is not to be omitted and the a_i are $\mathcal{C}^\infty$ on $U \subset \mathbb{R}^p$ a neighborhood of Δ^p. Since integral a is linear map, we may consider the special case in which $\delta\sigma(\omega)$ consists of a single term of the form $a_j\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_p$. In this case, the left-hand side of (3.59) becomes

$$(-1)^{j-1} \int_{\Delta^p} \frac{\partial a_j}{\partial r_j} dx_1 \wedge \cdots \wedge dx_p. \tag{3.62}$$

To evaluate the right-hand side of (3.59), we observe that for $1 \leq i \leq p$,

$$\delta k_i^{p-1}(r_j) = \begin{cases} r_j & \text{if } 1 \leq j \leq i-1 \\ 0 & \text{if } j = i \\ r_{j-1} & \text{if } i+1 \leq j \leq p \end{cases} \quad \text{and} \quad \delta k_0^{p-1}(r_j) = \begin{cases} 1 - \sum_{i=1}^{p-1} r_i & \text{if } j = 1 \\ r_{j-1} & \text{if } j \neq 1. \end{cases} \tag{3.63}$$

Applying these expressions to the right-hand term of (3.59), we obtain

$$\begin{aligned} \sum_{i=0}^{p}(-1)^i \int_{\Delta^p} \delta k_i^{p-1} \circ \delta\sigma(\omega) &= \sum_{i=0}^{p}(-1)^i \int_{\Delta^{p-1}} \delta k_i^{p-1}(a_j\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_p) \\ &= (-1)^{j-1} \int_{\Delta^{p-1}} a_j\big(1 - \sum_{i=1}^{p-1} x_i, x_1, \ldots, x_{p-1}\big) dx_1 \wedge \cdots \wedge dx_p \\ &+ (-1)^j \int_{\Delta^{p-1}} a_j(x_1, \ldots, x_{j-1}, 0, x_j, \ldots, x_{p-1}) dx_1 \ldots dx_p. \end{aligned} \tag{3.64}$$

We shall now apply a change of variables to the first term on the right-hand side of the previous equation. Let φ_j be the diffeomorphism of $\mathbb{R}^{p-1}$ defined by

$$\varphi_j(x_1,\dots,x_{p-1}) = \begin{cases} (x_1,\dots,\,x_{p-1}) & \text{if } j=1 \\ \big(1-\sum_{i=1}^{p-1} x_i,\,x_2,\dots,\,x_{p-1}\big) & \text{if } j=2 \\ \big(x_2,\dots,\,x_{j-1},\,1-\sum_{i=1}^{p-1} x_i,\,x_j,\dots,\,x_{p-1}\big) & \text{if } 3\le j\le p. \end{cases} \tag{3.65}$$

It is straightforward that φ_j is a difeomorphism in U with a neighbourhood of Δ^p, $\varphi_j(\Delta^p)=\Delta^p$ and $det(\varphi_j)=1$, where $J(\varphi_j)$ is the Jacobian matrix of φ_j for each $1\le j\le p$. Then, we have that

$$\begin{aligned} &(-1)^{j-1}\int_{\Delta^{p-1}} a_j(1-\sum_{i=1}^{p-1} x_i\,,x_1,\dots,x_{p-1})dx_1\wedge\cdots\wedge dx_p \\ &= (-1)^{j-1}\int_{\Delta^{p-1}} a_j\big(x_1,\dots,x_{j-1},1-\sum_{i=1}^{p-1} x_i),x_j,\dots,x_{p-1}\big)dx_1\wedge\cdots\wedge dx_{p-1}. \end{aligned} \tag{3.66}$$

Using (3.59), (3.62), (3.64) and (3.66) we see that the proof has been reduced to showing that

$$\begin{aligned} \int_{\Delta^p}\frac{\partial a_j}{\partial x_j}dx_1\dots dx_p = &\int_{\Delta^{p-1}} a_j\big(x_1,\dots,\,x_{j-1},\,1-\sum_{i=1}^{p-1} x_i,\,x_j,\dots,\,x_{p-1}\big)dx_1\wedge\cdots\wedge dx_{p-1} \\ &-\int_{\Delta^{p-1}} a_j(x_1,\dots,\,x_{j-1},\,0,\,x_j\dots,\,x_{p-1})dx_1\wedge\cdots\wedge dx_{p-1}. \end{aligned} \tag{3.67}$$

But this equation is simply the evaluation of the integral of $\partial a_j/\partial x_j$ over Δ^p. Therefore, we can use the hypothesis of induction. □

Now, we see that the homomorphism κ_p induces κ_p^*. Observe that κ_p can be defined for arbitrary open sets in M, and yield presheaf homomorphisms

$$\{E^p(U);\,\rho_{U,V}\} \xrightarrow{\kappa_p} \{S^p_\infty(U);\,\rho_{U,V}\}. \tag{3.68}$$

Here, we want to see that this morphism commutes with coboundary operator $d^p : E^p(U) \to E^{p+1}(U)$

$$\kappa_p(d\omega)(c) = \int_c (d\omega) = \int_{\partial c}\omega = \kappa_p(\omega)(\partial c) = d\kappa_p(\omega)(c), \tag{3.69}$$

where Stokes theorem has been used in the second equality and the definition of coboundary homomorphism in the last one. So we have that κ_p induces a homomorphism which we call, for simplicity, κ_p.

$$\xi^p(M) \xrightarrow{\kappa_p} \mathcal{S}^p_\infty(M). \tag{3.70}$$

Also, this last κ_p induces, in turn, κ_p^*.

Theorem 3.4. ***(The de Rham theorem)*** *The de Rham homomorphism k_p^* is the canonical isomorphism between $H^p_{deR}(M)$ and $H^p_{\Delta\infty}(M;\mathbb{R})$ for each p.*

Proof. The homomorphism $\kappa^p : \xi^p(M) \to \mathcal{S}^p_\infty(M)$ form a commutative diagram:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \mathcal{R} & \longrightarrow & \xi^0(M) & \longrightarrow & \xi^1(M) & \longrightarrow & \xi^2(M) & \longrightarrow & \xi^3(M) & \longrightarrow & \dots \\ & & \downarrow Id & & \downarrow\kappa_0 & & \downarrow\kappa_0 & & \downarrow\kappa_2 & & \downarrow\kappa_3 & & \\ 0 & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{S}^0_\infty(M) & \longrightarrow & \mathcal{S}^1_\infty(M) & \longrightarrow & \mathcal{S}^2_\infty(M) & \longrightarrow & \mathcal{S}^3_\infty(M) & \longrightarrow & \dots, \end{array} \tag{3.71}$$

this diagram shows that κ^* is a homomorphism between two fine torsion-free resolutions of the constant sheaf $\mathcal{R} = M \otimes \mathbb{R}$. Therefore, for each sheaf $\mathcal{J}$ over M the homomorphism κ^* induces a cochain map $\Gamma\big(\xi^*(M) \otimes \mathcal{J}\big) \to \Gamma\big(\mathcal{S}^*_\infty(M) \otimes \mathcal{J}\big)$ from which we obtain straightforward a cohomology theories homomorphism $H^p_{deR}(M, \mathcal{J}) \to H^p_{\Delta\infty}(M, \mathcal{J})$.

Consider now the following commutative diagram of cochain complexes in which rows are exact:

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & E^*(M) & \longrightarrow & \Gamma\big(\xi^*(M)\big) & \longrightarrow & 0 \\ & & & & \downarrow k & & \downarrow & & \\ 0 & \longrightarrow & \big(S^*_\infty(M)\big)_0 & \longrightarrow & S^*_\infty(M) & \longrightarrow & \Gamma\big(\mathcal{S}^*_\infty(M)\big) & \longrightarrow & 0, \end{array} \tag{3.72}$$

where the first row induces the isomorphisms $H^p_{deR}(M) \cong H^p_{deR}(M, \mathcal{R})$. The second row induces the isomorphisms $H^p_{\Delta\infty}(M) \cong H^p(M, \mathcal{R})$. And the last column induces isomorphisms on cohomology theories as we have seen in the previous diagram. Thus from uniqueness of the isomorphism between sheaf cohomology theories, it follows that κ^*_p is the canonical isomorphism $H^p_{deR} \cong H^p_{\Delta\infty}$. □

Maybe, the complexity of the theorem (2.6) makes it difficult to understand which is the essential point of the de Rham theorem. For this reason, the reader who is interested in going depth into the theorem can read a direct proof which is in [2] (chapter V section 9). Also, we think that it is in appealing to look into an other proof of the isomorphism between the singular cohomology and the singular differential cohomology. This implication is also a consequence of Whitney approximation theorem and, in this way, it is more comfortable to understand the essential reason for the isomorphism. A proof for Whitney approximation theorem could be found in the Lee bock [3].

Chapter 4

An Application of the De Rham Theorem in the Lie Group Theory

In the previous chapter, we have proved that there is an isomorphism between singular cohomology and de Rham cohomology. Maybe the importance of the de Rham theorem could be overshadowed for the previous (2.6), where we have proved that any two sheaf cohomology theories for a topological manifold are isomorphic. It is true that this general theorem is amazing, but also the de Rham theorem is. Furthermore, there are other ways to prove this theorem without using sheaf theory.

With the purpose of seeing the interest of the theorem, in this chapter, we wish to derive an interesting consequence of de Rham theorem, namely, the only spheres which are Lie groups are $\mathbf{S}^0$, $\mathbf{S}^1$ and $\mathbf{S}^3$. Here, we need to see some concept about Lie groups, in particular, this chapter can be seen as an introduction to a rather small part of the theory of compact Lie groups.

4.1 Lie Groups and Integration

Lie groups are, without any doubt, the most crucial particular class of differentiable manifolds. Lie groups are differentiable manifolds which are also groups and in which the group operations are smooth. Well-known examples include the general linear group, the unitary group, the orthogonal group and the spheres $\mathbf{S}^0$, $\mathbf{S}^1$ and $\mathbf{S}^3$.

Of central importance for Lie groups theory is the relationship between a Lie group and its Lie algebra of left-invariant vector fields. Here we shall take advantage of this relationship through the use of two of the more original maps around Lie groups: the exponential map and the adjoint representation. Also, we are going to see some characteristics of integration over Lie groups. For this purpose, we take in a more general way the definition of integration given in Chapter 3.

Despite that we will define the principal concepts of the Lie group theory, we assume that the reader has some knowledge about this theory and we will take some propositions as known. Also, we assume that the reader has some knowledge about smooth fields and vector flows.

Definition 4.1. *A Lie group G is a differentiable manifold which is also endowed with a group structure* *such that the map μ is C^∞.*

$$\begin{aligned} \mu : \; G \times G &\to G \\ (\sigma, \tau) &\mapsto \sigma\tau^{-1} \end{aligned} \tag{4.1}$$

Lie groups received the name by the Norwegian mathematician Sophus Lie. One of the principal ideas in this theory is to link the global object, the group, with its local version, which has become known as its Lie algebra. Therefore, many properties of a Lie group are reflected in properties of its Lie algebra. Furthermore, Lie algebras associated with Lie groups are finite dimensional. Throughout this chapter, G and H will denote Lie groups, and we shall use e to denote the identity element of a Lie group.

Definition 4.2. *A Lie algebra $\mathfrak{g}$ is a real vector space $\mathfrak{g}$ together with a bilinear operator $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the bracket, such that for all $x, y, z \in \mathfrak{g}$,*

(a) (anti-commutativity) $[x, y] = -[y, x]$.

(b) (Jacobi identity) $\big[[x, y], z\big] + \big[[y, z], x\big] + \big[[z, x], y\big] = 0.$

We want to associate a Lie group with a Lie algebra using its group and differential structure. For this purpose, for each $\sigma \in G$ we define the diffeomorphisms l_σ and r_σ by

$$\begin{aligned} l_\sigma(\tau) &= \sigma\tau, \\ r_\sigma(\tau) &= \tau\sigma \end{aligned} \tag{4.2}$$

for all $\tau \in G$. Which are called *left translation by σ* and *right translation by σ* respectively. A *vector field X*, i.e, a section of the tangent bundles (not necessarily to assume that is C^∞), is called *left-invariant* if for each $\sigma \in G$, X is l_σ-related, i.e.,

$$dl_\sigma \circ X = X \circ l_\sigma. \tag{4.3}$$

The set of all left-invariant vector fields on Lie groups G and H will be denoted by $\mathfrak{g}$ and $\mathfrak{h}$ respectively. The essential property is that left-invariant vector fields are smooth, and $\mathfrak{g}$ forms a Lie algebra under Lie bracket operation on vector fields. Here, by Lie bracket we denote the map from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$ defined by $[X, Y] = XY - YX$ for all X and Y in gl. Lie bracket is well defined in this way due to $[X, Y]$ is also left-invariant.

Note that if G is a Lie group and $X_e \in T_eG$ is a tangent vector in the tangent space of G at the identity, then $X_{sigma} = dl_\sigma(X_e)$ is a tangent vector at $\sigma \in G$. Therefore, this defines a vector field X on G which is left-invariant and such vector field X is uniquely determined by X_e. Then, we have an isomorphism between $\mathfrak{g}$ and T_eG, consequently, $\dim\mathfrak{g} = \dim G$.

Similarly, if ω is a p-form on G then $\delta l_\sigma \omega$ is another p-form. ω is said to be *left-invariant* if $\delta l_\sigma \omega = \omega$ for all $\sigma \in G$. As it happened with vector fields, left-invariant forms can be seen to be smooth and hence are in one to one correspondence with p-forms on the vector space T_eG.

Remember that whenever we have a p-form ω on M and $X_1, \ldots, X_p$ are vector fields on M, $\omega(X_1, \ldots, X_p)$ makes sense, and it is the function whose value at m is

$$\omega(X_1, \ldots, X_p)(m) = \omega_m(X_1(m), \ldots, X_p(m)). \tag{4.4}$$

Furthermore, differential d acts over $\omega(X_1, \ldots, X_p)$ by

$$\begin{aligned} d\omega(X_0, \ldots, X_p) = &\sum_{i=0}^{p} (-1)^i X_i \omega(X_0, \ldots, \widehat{X_i}, \ldots, X_p) \\ &+ \sum_{i<j} \omega([X_i, X_j], X_0, \ldots, \widetilde{X_i}, \ldots, \widehat{X_j}, \ldots, X_p). \end{aligned} \tag{4.5}$$

If ω is a left-invariant p-form and $X_1, \ldots, X_p$ are left-invariant vector fields, then $\omega(X_0, \ldots, \widetilde{X_i}, \ldots, X_p)$ is a constant function and so $X_i\omega(X_0, \ldots, \widetilde{X_i}, \ldots, X_p) = 0$. Therefore, in this case, the formula for $d\omega$ simplifies to

$$d\omega(X_0, \ldots, X_p) = \sum_{i<j} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_p). \tag{4.6}$$

A map $\varphi : G \to H$ is a *(Lie group) homomorphism* if φ is both C^∞ and a group homomorphism of the abstract groups. We call φ an *isomorphism* if, in addiction, φ is a diffeomorphism. An isomorphism of a Lie group with itself is called an *automorphism*. If $H = Aut(V)$ for some vector space V, or if $H = Gl(n, \mathbb{C})$ or $Gl(n, \mathbb{R})$, then a homomorphism $\varphi : G \to H$ is called a *representation of the Lie group G*. Therefore, the set of Lie groups forms a category with the homomorphisms defined above. We call this category as *Lie group category*.

If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, a map $\psi : \mathfrak{g} \to \mathfrak{h}$ is a *Lie algebra homomorphism* if it is linear and preserves brackets. If, in addiction, ψ is $1:1$ and onto, then ψ is an *isomorphism*. If $\mathfrak{h} = End(V)$ for some vector space V, or if $\mathfrak{h} = \mathfrak{gl}(n, \mathbb{C})$ or $\mathfrak{gl}(n, \mathbb{R})$, then a homomorphism ψ is called a *representation of the Lie algebra* $\mathfrak{g}$. As in the case of Lie groups, the set of Lie algebras also forms a category which we call *Lie algebra category*.

Let $\varphi : G \to H$ be a homomorphism. Then, since φ maps the identity of G to the identity of H, the differential $d\varphi$ is a linear transformation of T_eG into T_eH. Using the natural identifications of the tangent spaces at the identities with Lie algebras, this linear transformation of $\mathfrak{g}$ into $\mathfrak{h}$ which we shall also denote by $d\varphi$. It is easy to see that $d\varphi$ is, indeed, a Lie algebra homomorphism. If $\psi : G \to H$ and $\varphi : H \to T$ are homomorphisms where G, H and T are Lie groups. Then $d(\varphi \circ \psi) = d(\varphi) \circ d(\psi)$, this is called the *chain rule*. In this way, we have defined a covariant functor between the Lie group category and the Lie algebra category.

4.1.1 Integration over a Manifold

At this point, we want to talk about integration over Lie groups, remember that we have already told about integration of p-forms over p-chains. We use a similar construction to define integration over Lie groups. Let ω be an n-form on the open subset $U\mathbb{R}^n$ with compact support and let $x_1, \ldots, x_n$ be the canonical coordinates of R^n. Note that ω extends to all $\mathbb{R}^n$ by 0 with support in some cube. Now ω can be written

$$\omega = f(x_1, \ldots, x_n) dx_1 \wedge \cdots \wedge dx_n \tag{4.7}$$

for a smooth function f which is 0 outside some compact set. Suppose that W is another open set in R^n and let $\varphi : W \to U$ be a diffeomorphism. Then, we have the n-form $\delta\varphi(\omega)$ on W where, by definition,

$$\delta\varphi(\omega) = (f \circ \varphi) d(x_1 \circ \varphi) \wedge \cdots \wedge d(x_n \circ \varphi). \tag{4.8}$$

Now

$$d(x_i \circ \varphi) = \sum_{j=1}^{n} \frac{\partial(x_i \circ \varphi)}{\partial x_j} dx_j = \sum_{j=1}^{n} J_{i,j}(\varphi) dx_j \tag{4.9}$$

where $J_{i,j}(\varphi)$ is the i, j entry of the Jacobian matrix of φ. Thus

$$\begin{aligned} d(x_1 \circ \varphi) \wedge \cdots \wedge d(x_n \circ \varphi) &= \left(\sum_j J_{1,j}(\varphi) dx_j\right) \wedge \cdots \wedge \left(\sum_j J_{n,j}(\varphi) dx_j\right) \\ &= \sum_{s \in \mathfrak{S}_n} \big(J_{1,s_1}(\varphi) \cdots J_{n,s_n}(\varphi)\big) dx_{s_1} \wedge \cdots \wedge dx_{s_n} \\ &= \sum_{s \in \mathfrak{S}_n} sgn(s) \big(J_{1,s_1}(\varphi) \cdots J_{n,s_n}(\varphi)\big) dx_1 \wedge \cdots \wedge dx_n \\ &= det(J(\varphi)) dx_1 \wedge \cdots \wedge dx_n, \end{aligned} \tag{4.10}$$

where $\mathfrak{S}_n$ is the symmetric group. Therefore $\delta\varphi(\omega) = (f \circ \varphi)\, det\big(J(\varphi)\big) dx_1 \wedge \cdots \wedge dx_n$ and so

$$\begin{aligned} \int \delta\varphi(\omega) &= \int_{\mathbb{R}^n} (f \circ \varphi)(x_1, \ldots, x_n)\, det\big(J(\varphi)\big) dx_1 \ldots dx_n \\ &= \pm \int f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n = \pm \int \omega, \end{aligned} \tag{4.11}$$

where the integral defined in chapter 3. Observe, that the last equality holds due to standard Riemann change of variables rule, where the sign is the sign of the $det\big(J(\varphi)\big)$. If U is not connected, then we are assuming here the same sign on all components.

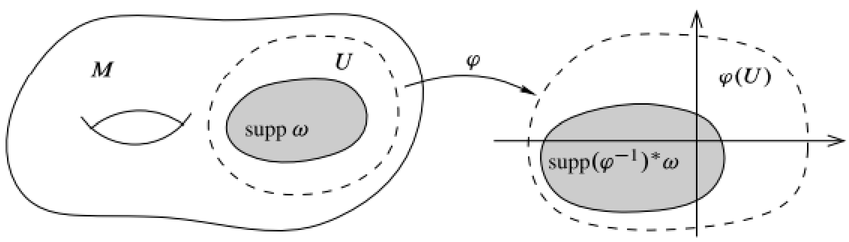


Figure 4.1: The integral of a form over a smooth cart

In order to define the integration over a connected differentiable manifold of dimension n, we have to fix an orientation. We shall call M *orientable* if it is possible to choose in a consistent way an orientation on T_mM^* for each $m \in M$. More precisely, let O be the "0-section" of the exterior n-bundle $\Lambda_n^*(M) = \bigcup_{m \in M} \Lambda_n(T_mM^*)$; that is,

$$O = \cup_{m \in M} \{0 \in \Lambda_n(T_mM^*)\}. \tag{4.12}$$

Then, since each $\Lambda_n(T_mM^*) - \{0\}$ has exactly two components, this follows easily that $\Lambda_n^*(M) - O$ has at most two components. We say that M is *orientable* if $\Lambda_n^*(M) - O$ has two components; and if M is orientable, an *orientation on* M is a choice of the one of the two components of $\Lambda_n^*(M) - O$. A non-connected manifold M is said to be orientable if each component of M is orientable, and an orientation is a choice of orientation on each component. Let M be oriented, and let $v_1, \ldots, v_n$ be a basis of T_mM with dual basis $\delta_1, \ldots, \delta_n$. We say that the ordered basis $v_1, \ldots, v_n$ is *oriented* if $\delta_1 \wedge \cdots \wedge \delta_n$ belongs to the orientation.

Proposition 4.3. *Let M be a differential manifold of dimension n. Then if there is a nowhere-vanishing n-form on M, then M is orientable.*

Proof. Let ω be a nowhere-vanishing n-form on M, and let

$$\begin{aligned} \Lambda^+ &= \cup_{m \in M} \{a\omega(m) |\, a \in \mathbb{R},\, a > 0\}, \\ \Lambda^- &= \cup_{m \in M} \{a\omega(m) |\, a \in \mathbb{R},\, a < 0\}. \end{aligned}$$

Then $\Lambda_n^*(M) - O$ is the disjoint union of the two open subsets Λ^+ and Λ^-, so $\Lambda_n^*(M) - O$ is disconnected, and M is orientable. □

Therefore we have that every Lie group G is orientable, for if $\omega_1, \dots, \omega_n$ is a basis for the left invariant 1-forms on G, then $\omega_1 \wedge \cdots \wedge \omega_n$ is a global nowhere-vanishing n-form on G. One can prove the other implication of the proposition easily, but we are not going to use it.

Let M and N be oriented n-dimensional manifolds, and let $\varphi : M \to N$ be a differential map. We say that φ *preserves orientations* if the induced map $\delta\varphi : \Lambda_n^*(N) \to \Lambda_n^*(M)$ maps the component of $\Lambda_n^*(N) - O$ determining the orientation on N into the component $\Lambda_n^*(M) - O$ determining the orientation on M. Hence, we can consider only charts which preserve orientations. Let ω be an n-form on M whose support is contained in the open set U where U is the domain of a chart $\varphi : U \to W \subset \mathbb{R}^n$.

Then $\delta\varphi^{-1}(\omega)$ is an n-form on $W \subset \mathbb{R}^n$. Thus we define

$$\int_M \omega = \int_{\mathbb{R}^n} \delta\varphi^{-1}(\omega). \tag{4.13}$$

To show that this is independent of the choice of φ, let $\widetilde{\varphi} : \widetilde{U} \to \mathbb{R}^n$ be another such map and let $\theta = \widetilde{\varphi} \circ \varphi^{-1}$. Then, $\widetilde{\varphi}^{-1} \circ \theta = \varphi^{-1}$ so $\delta\varphi^{-1} = \delta\theta \circ \delta\widetilde{\varphi}^{-1}$. Thus

$$\int \delta\varphi^{-1}(\omega) = \int \delta\theta \circ \delta\widetilde{\varphi}^{-1}(\omega) = \int \delta\widetilde{\varphi}^{-1}(\omega). \tag{4.14}$$

which proves this independence.

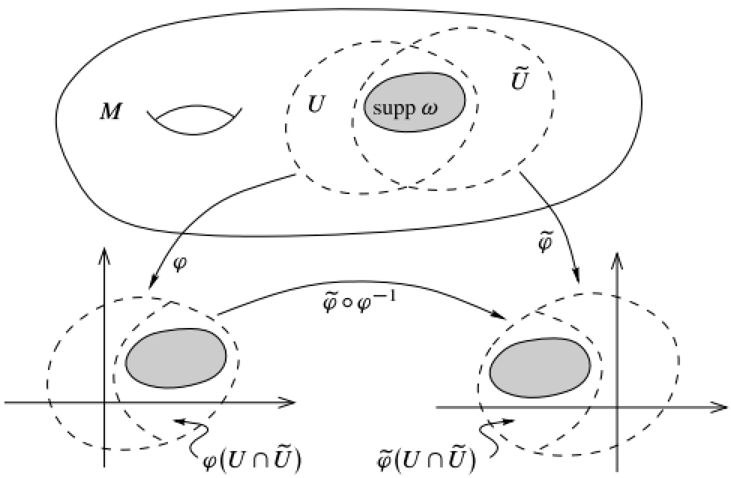


Figure 4.2: Coordinate independence of the integral

Now let ω be an arbitrary n-form on M with a compact support K. Let be $\{U_i, \psi_i\}_{i\in I}$ and atlas of M, such that $K \subset U_1 \cup \cdots \cup U_s$ due to K is compact. Let $U = M - K$, and let $f, f_1, \dots, f_s : M \to \mathbb{R}$ give a smooth partition of unity subordinate to the cover $U, U_1, \dots, U_s$ of M. we define the integral of ω over M by

$$\int_M \omega = \sum_{i=1}^{k} \int f_i \omega. \tag{4.15}$$

We must show that this is well defined, i.e., that this definition is independent of the cover and the partition of unity chosen. Let $V, V_1, \dots, V_l$ and $g, g_1, \dots, g_l$ be another such cover and another such partition of unity respectively, with V_j associated with the oriented atlas $\{V_j, \varphi_j\}_{j\in J}$. Since $g = 0$ on K, it follows that $\sum_j g_j = 1$ there, so that

$$\sum_{i=1}^{k} \int f_i\omega = \sum_{i=1}^{k} \int \sum_{j=1}^{l} g_j f_i\omega = \sum_{i,j} \int g_j f_i\omega = \sum_{i=1}^{k} f_i \sum_{j=1}^{l} \int g_j\omega = \sum_{j=1}^{l} \int g_j\omega. \tag{4.16}$$

Observed that if γ is a diffeomorphism of $\gamma : M \to N$, then, for each ω a form over N with compact support

$$\int_N \omega = \pm \int_M \delta\gamma(\omega) \tag{4.17}$$

with "$+$" if and only if γ is orientation-preserving.

Let G be an n-dimensional Lie group. We have observed that G is orientable. We now fix once and for all an orientation on G.

Consider the left invariant n-forms on G. Since such a form is uniquely determined by its value at one point, and since the nth exterior power of an n-dimensional vector space is one-dimensional, there is exactly a one -dimensional space of left invariant n-forms on G. Choose a non-zero left invariant n-form ω consistent with the fixed orientation on G.

Since G is oriented, the integral of compactly supported n-forms in defined on G as so far. We now define, with respect to ω, the integral of a compactly supported continuous function f on G by setting

$$\int_G f = \int_G f\omega, \tag{4.18}$$

which depends, of course, on the choice of the non-zero left-invariant n-form ω consistent with the orientation on G. Bur since such forms are uniquely determined up to a positive constant multiple, so is the integral (4.18). In the case of a compact group G, we can and always will fix the choice of ω by requiring the normalization

$$\int_G \omega = 1. \tag{4.19}$$

Consider the diffeomorphism l_σ for $\sigma \in G$. By the left-invariance of ω, we have

$$\int_G f = \int_G f\omega = \int_G \delta l_\sigma(f\omega) = \int_G (f \circ l_\sigma)\omega = \int_G f \circ l_\sigma. \tag{4.20}$$

Therefore, the integral is left-invariant, and for all $\tau \in G$ we can write as

$$\int_G f(\tau\sigma) = \int_G f(\sigma). \tag{4.21}$$

Now we ask to what extent the integral (4.18) is also right invariant. That is, when do we have

$$\int_G f = \int_G f \circ r_\sigma \tag{4.22}$$

for each $\sigma \in G$? Note tha the form $\delta r_\sigma(\omega)$ is still left invariance, since

$$\delta l_\tau \delta r_\sigma(\omega) = \delta r_\sigma \delta l_\tau(\omega) = \delta r_\sigma(\omega). \tag{4.23}$$

Thus $\delta r_\sigma(\omega)$ is some constant multiple of ω. Thus there is defined a function $\widetilde{\lambda}$ of G into the non-zero real numbers such that

$$\delta r_\sigma(\omega) = \widetilde{\lambda}\omega. \tag{4.24}$$

It is easily checked that $\widetilde{\lambda}$ is C^∞. We let

$$\lambda(\sigma) = |\widetilde{\lambda}(\sigma)|. \tag{4.25}$$

Observed that

$$\lambda(\tau\sigma) = \lambda(\tau)\lambda(\sigma). \tag{4.26}$$

As $\widetilde{\lambda}(\sigma) \neq 0$ for each $\sigma \in G$, we have that λ is a Lie group homomorphism of G into the multiplicative group of positive real numbers. λ is called the *modular function*. Now since foe each $\sigma \in G$

$$\int_G f\omega = \int_G (f \circ r_\sigma)\lambda(\sigma)\omega, \tag{4.27}$$

it follows that the integral 4.18 is right invariant if and only if $\lambda \equiv 1$ on G. A Lie group G for which $\lambda \equiv 1$ is called *unimodular*. We observe that *each compact Lie group G is unimodular* since for each $\sigma \in G$

$$1 = \int_G \omega = \lambda(\sigma)\int_G \omega = \lambda(\sigma). \tag{4.28}$$

Thus the integral on a compact Lie group is both left and right invariant.

4.2 The Exponential Map and the Adjoint Representation

Here we shall define the exponential map and the adjoint representation. The first is a map which relates a Lie group with its Lie algebra. The second is a representation of a Lie group that will be used in the central theorem to prove that the only spheres which are Lie groups are $\mathbf{S}^0$, $\mathbf{S}^1$ and $\mathbf{S}^3$. Both have significant consequences in the Lie group theory because they allow linking properties of Lie groups with its Lie algebras.

4.2.1 The Exponential Map

Before defining the exponential map, we enunciate a theorem that we will not prove. One can find a proof in [1] (Chapter 3, point 27).

Theorem 4.4. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively and with G simply connected. Let $\psi : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism. Then there exists a unique homomorphism $\varphi : G \to H$ such that $d\varphi = psi$.*

Let G be a Lie group, and let $\mathfrak{g}$ be its Lie algebra. Let $X \in \mathfrak{g}$ and let r be de canonical coordinate on $\mathbb{R}$. Then

$$\beta\frac{d}{dr} \mapsto \beta X \tag{4.29}$$

is a homomorphism of the Lie algebra of $\mathbb{R}$ into $\mathfrak{g}$. Since the real lien is simply connected, there exists, by the previous theorem, a unique homomorphism

$$exp_X : \mathbb{R} \to G \tag{4.30}$$

such that

$$d\, exp_X\left(\beta\frac{d}{dr}\right) = \beta X. \tag{4.31}$$

In other words, $t \mapsto exp_X(t)$ is the unique homomorphism from $\mathbb{R}$ to G whose tangent vector at 0 is $X(e)$. We define the *exponential map* by setting

$$\begin{aligned} exp :&\mathfrak{g} \to G \\ &X \mapsto exp_X(1). \end{aligned} \tag{4.32}$$

The reason for this terminology is that one can show that the exponential map for the general linear group is given by exponentiation of matrices. Now we will prove three properties of the

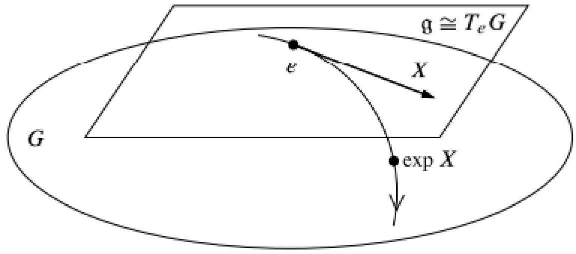


Figure 4.3: The exponential map

exponential map which we will use in the next section. There are also three more properties that can be proved as a result of the three in the proposition, namely:

(a) $exp\big((t_1+t_2)X\big) = \big(exp(t_1X)\big)\big(exp(t_2X)\big)$ for all $t_1, t_2 \in \mathbb{R}$.

(b) $exp(-tX) = \big(exp(X)\big)^{-1}$ for each $t \in \mathbb{R}$.

(c) The 1-parameter group of diffeomorphisms X_t associated with the left invariant vector field X is gien by

$$X_t = r_{exp_X(t)}. \tag{4.33}$$

We will neither prove nor use these properties.

Proposition 4.5. *Let X belong to the Lie algebra $\mathfrak{g}$ of the Lie group G. Then*

(a) $exp(tX) = exp_X(t)$ for each $t \in \mathbb{R}$.

(b) $l_\sigma \circ exp_X$ is the unique integral curve of X which takes the value σ at 0.

(c) $exp : \mathfrak{g} \to G$ is C^∞ and $d\,exp : T_0\mathfrak{g} \to T_eG$ is the indentity map (with the identifications $\mathfrak{g} = T_eG$), so exp gives a diffeomorphism of a neighbourhood of 0 in $\mathfrak{g}$ onto a neighbourhood of e in G.

Proof. First of all, note that d/dr and $d\,exp_X(d/dr)$, which is X, are exp_X related, i.e.

$$d\,exp_X \circ (d/dr) = d\,exp_X(d/dr) \circ exp_X. \tag{4.34}$$

This is true due to, for since exp_X is a homomorphism, $l_{exp_X(t)} \circ \exp_X = \exp_X \circ l_t$; hence

$$\begin{aligned} d\,exp_X\left(\tfrac{d}{dr}\right)(exp_X(t)) &= dl_{exp_X(t)} d\,exp_X\left(\tfrac{d}{dr}\right)(0) = dl_{exp_X(t)} d\,exp_X\left(\tfrac{d}{dr}\Big|_{r=0}\right) \\ &= d(l_{exp_X(t)} \circ exp_X)\left(\tfrac{d}{dr}\Big|_{r=0}\right) = d(\exp_X \circ l_t)\left(\tfrac{d}{dr}\Big|_{r=0}\right) = d(exp_X)\left(\tfrac{d}{dr}\Big|_{r=t}\right). \end{aligned} \tag{4.35}$$

Thus exp_X is an integral curve of X and is the unique one for which $exp_X(0) = e$. Since X is left-invariant, $l_\sigma \circ exp_X$ is also an integral curve of X and is the unique one taking the value σ at 0. Thus part (b) is proved. Now, we define maps φ and ψ of $\mathbb{R}$ into G by setting

$$\varphi(t) = exp_X(st) \quad \text{and} \quad \psi(t) = exp_{sX}(t), \tag{4.36}$$

where $s \in R$. We have observed that ψ is the unique integral curve of sX such that $\psi(0) = e$. Now,

$$d\varphi\left(\frac{d}{dr}\Big|_t\right) = d\,exp_X\left(s\,\frac{d}{dr}\Big|_{st}\right) = sX|_{exp_X(st)}. \tag{4.37}$$

Thus φ also is an integral curve of sX such that $\varphi(0) = e$. By the uniqueness of the integral curves, $\varphi = \psi$. Thus

$$exp_{sX}(t) = exp_X(st), \tag{4.38}$$

where $s, t \in \mathbb{R}$ and $X \in \mathfrak{g}$. Setting $t = 1$, we obtain part (a). Here, we define a vector field V on $G \times \mathfrak{g}$ by setting

$$V(\sigma, X) = \big(X(\sigma), 0\big) \in T_\sigma \oplus T_X\mathfrak{g}. \tag{4.39}$$

Then V is a smooth vector field, and according to part (b), the integral curve of V through (σ, X) is

$$t \mapsto (\sigma exp(tX), X), \tag{4.40}$$

or in other words, the homomorphism of $\mathbb{R}$ into $G \oplus \mathfrak{g}$ associated with the vector field V is given by

$$V_{(\sigma,X)}(t) = \big(\sigma exp(tX), X\big). \tag{4.41}$$

In particular, V is complete, i.e., the domain of $V_{(\sigma,X)}$ is $\mathbb{R}$ foe each $(\sigma, X) \in G \oplus \mathfrak{g}$. Hence

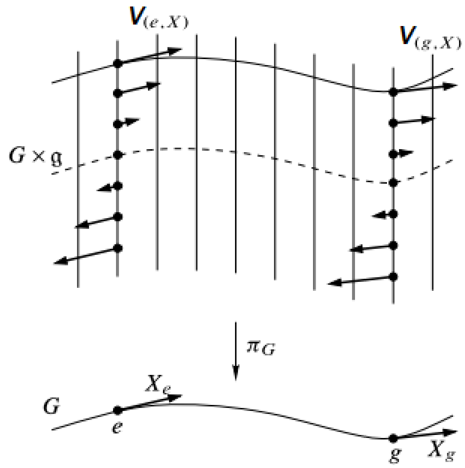


Figure 4.4: Vector field used to proof that the exponential map is smooth

$V_1 = \pi_1 \circ V$ is defined and smooth on all of $G \oplus \mathfrak{g}$. Now let $\pi : G \oplus \mathfrak{g} \to G$ be the projection. Then

$$exp(X) = \pi \circ V_1(e, X). \tag{4.42}$$

Thus we have exhibited exp as the composition of $C\infty$ mappings, so exp is C^∞ That $d\,exp : T_0\mathfrak{g} \to T_eG$ is the identity map is immediate, for tX is a curve in $\mathfrak{g}$ whose tangent vector at $t = 0$ is X, and by part (a), $exp(tX)$ is a curve in G whose tangent vector at $t = 0$ is $X(e)$. □

The next theorem is the main reason we have introduced the exponential map in this text. This theorem allows to thing out the exponential map as a natural transformation between the functor that maps Lie groups into their Lie algebras and the "identity" functor over Lie groups. We will use this theorem to prove a significant result we will talk about the adjoint representation.

Theorem 4.6. *Let $\varphi : H \to G$ be a homomorphism. Then the following diagram is commutative:*

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & G \\ \big\downarrow{\scriptstyle exp} & & \big\downarrow{\scriptstyle exp} \\ \mathfrak{h} & \xrightarrow{d\varphi} & \mathfrak{g}. \end{array}$$

Proof. Let $X \in \mathfrak{h}$. Then $t \mapsto \varphi\big(exp(tX)\big)$ is a smooth curve in G whose tangent vector at 0 is $d\varphi\big(X(e)\big)$. This is also a homomorphism of $\mathbb{R}$ into G since φ is a homomorphism. But $t \mapsto exp\, t\big(d\varphi(X)\big)$ is the unique homomorphism between $\mathbb{R}$ and G whose tangent at 0 is $\big(d\varphi(X)\big)(e)$. Thus

$$\varphi\big(exp(tX)\big) = exp\, t\big(d\varphi(X)\big), \tag{4.43}$$

whence

$$\varphi\big(exp(X)\big) = exp\,\big(d\varphi(X)\big). \tag{4.44}$$

□

4.2.2 The Adjoint Representation

At this point, we are going to introduce the adjoint representation. This map is a result of the action of a Lie group on a manifold. As later we will study some actions as that, first we introduce the definition of this kind of actions. After that, we will talk about the adjoint representation.

Definition 4.7. *Let M be a manifold, and let G be a Lie group. A C^∞ map $\mu : G \times M \to M$ such that*

$$\mu(\sigma\tau, m) = \mu\big(\sigma, \mu(\tau, m)\big), \quad \textit{and} \quad \mu(e, m) = m \tag{4.45}$$

for all $\sigma, \tau \in G$ and $m \in M$ is called an action of G on M on the left.

Note that if $\mu : G \times M \to M$ is an action of G on M on the left, then for a fixed $\sigma \in G$ the map $m \mapsto \mu(\sigma, m)$ is a diffeomorphism of M we shall denote by μ_σ. Similarly, a c^∞ map $\mu : M \times G \to M$ such that

$$\mu(m, \sigma\tau) = \mu\big(\mu(m, \sigma), \tau\big), \quad \text{and} \quad \mu(m, e) = m \tag{4.46}$$

for all $\sigma, \tau \in G$ and $m \in M$ is called an *action of G on M on the right*.

Actions are essential elements in the Lie group theory. To make an idea, whenever a Lie group acts on a geometric object, such as a Riemannian or a symplectic manifold, this action provides a measure of rigidity and yields a rich algebraic structure. The presence of continuous symmetries expressed via a Lie group action on manifolds placed strong constraints on its geometry and facilitated analysis on the manifold. Furthermore, linear actions of Lie groups are especially relevant and are studied in representation theory.

Theorem 4.8. *Let $\mu : G \times M \to M$ be an action of G on M on the left. Assume that $m_0 \in M$ is a fixed point, that is $\mu_\sigma(m_0) = m_0$ for each $\sigma \in G$. Then, the map*

$$\begin{aligned} \psi :& G \to Aut(T_{m_0}M) \\ &\sigma \mapsto d\mu_\sigma|_{T_{m_0}M} \end{aligned} \tag{4.47}$$

is a representation of G.

Proof. ψ is a homomorphism for

$$\psi(\sigma\tau) = d\mu_{\sigma\tau}|_{T_{m_0}M} = d(\mu_\sigma \circ \mu_\tau)|_{T_{m_0}M} = \psi(\sigma) \circ \psi(\tau). \tag{4.48}$$

It remains only to prove that ψ is C^∞. For this, it suffices to prove that ψ composed with an arbitrary coordinate function on $Aut(T_{m_0}M)$ is C^∞. Now, one gets a coordinate system on $Aut(T_{m_0}M)$ by choosing a basis for $T_{m_0}M$ and then by using this basis to identify $Aut(T_{m_0}M)$

with non-singular matrices. One gets the matrix associated with an element of $Aut(T_{m_0}M)$ by applying this element to the basis of $T_{m_0}M$ and then applying the dual basis. So it suffices to prove that if $v_0 \in T_{m_0}M$ and if $\alpha \in (T_{m_0}M)^*$,

$$\sigma \mapsto \alpha\big(d\mu_\sigma(v_0)\big) \tag{4.49}$$

is a C^∞ function on G. For (4.49), it is suffices to prove that

$$\sigma \mapsto d\mu_\sigma(v_0) \tag{4.50}$$

is a C^∞ map of G into $T_{m_0}M$, or equivalently that (4.50) is a C^∞ map of G into $T(M)$, the tangent bundle of M. But (4.50) is exactly the composition of C^∞ maps

$$G \to T(G) \times T(M) \to T(G \times M) \xrightarrow{d\mu} T(M) \tag{4.51}$$

in which the first map sends σ into $\big((\sigma, 0), (m_0, v_0)\big)$, the second map is the canonical diffeomorphism of $T(G) \times T(M)$ with $T(G \times M)$, and the third map is d_μ. Thus ψ is C^∞. □

A Lie group G acts on itself on the left inner automorphisms:

$$a : G \times G \to G, \quad a(\sigma, \tau) = \sigma\tau\sigma^{-1} = a_\sigma(\tau). \tag{4.52}$$

Observe that the identity is a fixed point of this action. Hence, by the theorem, the map

$$\sigma \mapsto da_\sigma|_{T_eG} \tag{4.53}$$

is a representation of G into $Aut(\mathfrak{g})$, using the identification $T_eG \cong \mathfrak{g}$. This is called the *adjoint representation* and is denoted by

$$Ad : G \to Aut(\mathfrak{g}). \tag{4.54}$$

We let the differential of the adjoint representation be denoted by ad,

$$d(Ad) = ad, \tag{4.55}$$

and we denote $Ad(\sigma)$ by Ad_σ and $ad(X)$ by ad_X. Thus, we can use the exponential map to relate Ad with $d(Ad) = ad$:

$$\begin{CD} G @>{Ad}>> Aut(\mathfrak{g}) \\ @V{exp}VV @VV{exp}V \\ \mathfrak{g} @>{ad}>> End(\mathfrak{g}). \end{CD}$$

Also applying the exponential map to the automorphism a_σ of G, we obtain the commutative diagram

$$\begin{CD} G @>{a_\sigma}>> G \\ @V{exp}VV @VV{exp}V \\ \mathfrak{g} @>{Ad_\sigma}>> \mathfrak{g}. \end{CD}$$

In other words,

$$exp\big(tAd_{sigma}(X)\big) = \sigma\big(exp(tX)\big)\sigma^{-1}. \tag{4.56}$$

Proposition 4.9. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $X, Y \in \mathfrak{g}$. Then*

$$ad_X Y = [X, Y]. \tag{4.57}$$

Proof. Using the property (b) on the proposition 4.5 of the exponential map, and fixing $e_{\mathfrak{g}}$ the identity element in $\mathfrak{g}$, we have that

$$\begin{aligned} ad_X Y_{e_{\mathfrak{g}}} &= \big(d_e Ad(X)\big)(Y)_{e_{\mathfrak{g}}} = \frac{d}{dt}\left[Ad_{exp(tv)}\right]\Big|_{t=0}(Y) \\ &= \frac{d}{dt}\left[Ad_{exp(tv)}(Y)\right]\Big|_{t=0} = \frac{d}{dt}\left[d_{e_{\mathfrak{g}}} a_{exp(tv)}(Y)\right]\Big|_{t=0}, \end{aligned} \tag{4.58}$$

for each $X, Y \in \mathfrak{g}$, where $v = X(e)$. Let ϕ_v be the unique integral curve of X with $\phi_v(0) = e$. Fixing $\sigma \in G$, we define

$$\begin{aligned} \phi_{v,\sigma} :& \mathbb{R} \to G \\ & t \mapsto \sigma\, exp(tv), \end{aligned} \tag{4.59}$$

which is the unique integral curve of X which takes the value σ at 0. Therefore,

$$\begin{aligned} ad_X Y_{e_{\mathfrak{g}}} &= \frac{d}{dt}\left[d_{e_{\mathfrak{g}}} a_{exp(tv)}(Y)\right]\Big|_{t=0} = \frac{d}{dt}\left[d_{e_{\mathfrak{g}}}(r_{exp(tv)} \circ l_{exp(tv)})(Y)\right]\Big|_{t=0} \\ &= \frac{d}{dt}\left[d_{exp(tv)} r_{exp(tv)}\big(Y\, exp(tv)\big)\right]\Big|_{t=0} = \frac{d}{dt}\left[d\phi_{-t}\big(Y\, exp(tv)\big)\right]\big|_{t=0} \\ &= \lim_{t\to 0}\left[d\phi_{-t}(Y_{\phi_t(e)}) - Y_e\right] = [X, Y](e). \end{aligned} \tag{4.60}$$

Using that X, Y and $[X, Y]$ are left-invariant we prove the proposition. □

4.3 Lie Group Structure over the Spheres

In this section, if we do not say the opposite, we assume that G is a compact connected Lie group.

Let $\mu : G \times M \to M$ an action of G on M on the left. Then a p-form ω on M is said to be μ-invariant if $\delta\mu_\sigma(\omega) = \omega$ for each $\sigma \in G$. Let $E(M)^\mu$ be the set of μ-invariant forms on M. Define

$$I : E(M) \to E(M) \tag{4.61}$$

by

$$I(\omega)(X_1, \dots, X_p) = \int_G \delta\mu_\sigma(\omega)(X_1, \dots, X_n)d\sigma = \int_G \omega\big(d\mu_\sigma(X_1), \dots, d\mu_\sigma(X_n)\big)d\sigma, \tag{4.62}$$

where $X_1 \dots, X_p$ are vector fields over M. Then

$$\begin{aligned} \delta\mu_\tau(\omega)(X_1, \dots, X_p) &= I(\omega)(d\mu_\tau(X_1), \dots, d\mu_\tau(X_n) \\ &= \int_G \omega\big(d\mu_\sigma d\mu_\tau(X_1), \dots, d\mu_\sigma d\mu_\tau(X_n)\big)d\sigma \\ &= \int_G \omega\big(d\mu_{\sigma\tau}(X_1), \dots, d\mu_{\sigma\tau}(X_n)\big)d\sigma \\ &= \int_G \omega\big(d\mu_\sigma(X_1), \dots, d\mu_\sigma(X_n)\big)d\sigma = I(\omega), \end{aligned} \tag{4.63}$$

by the right-invariance of the integral, due to G is compact. Here we have used $d\sigma$ to denote that we are considering the integral as a function of σ in G Thus $I(\omega) \in E(M)^\mu$ for all $\omega \in E(M)$, and therefore

$$I : E(M) \to E(M)^\mu. \tag{4.64}$$

Suppose that $\omega \in E(M)^\mu$. Then

$$I(\omega)(X_1, \dots, X_p) = \int_G \delta\mu_\sigma(\omega)(X_1, \dots, X_n)d\sigma = \int_G \omega(X_1, \dots, X_n)d\sigma = \omega(X_1, \dots, X_n). \tag{4.65}$$

by the normalisation of the integral.

Let $J : E(M)^{\mu} \hookrightarrow E(M)$ be the inclusion. Then we have just shown that

$$I \circ J = Id_{E(M)^{\mu}} \tag{4.66}$$

Lemma 4.10. *The map I commute with the differential morphism d, i.e., $dI = Id$.*

Proof. Using the shorthand notation $X^{\sigma} = d\mu_{sigma}(X)$, we have

$$\begin{aligned} d\big(I\omega)\big)(X_0, \dots, X_p) &= \sum_{i=0}^{p}(-1)^i X_i\big(I(\omega)(X_0, \dots, \widehat{X_i}, \dots, X_p)\\ &+ \sum_{i<j}(-1)^{i+j} I(\omega)([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_p)\\ &= \sum_{i=0}^{p}(-1)^i X_i \int_G \omega(X_0^{\sigma}, \dots, \widehat{X_i^{\sigma}}, \dots, X_p^{\sigma}) d\sigma\\ &+ \sum_{i<j}(-1)^{i+j} \int_G \omega([X_i^{\sigma}, X_j^{\sigma}], X_0^{\sigma}, \dots, \widehat{X_i^{\sigma}}, \dots, \widehat{X_j^{\sigma}}, \dots, X_p^{\sigma}) d\sigma\\ &= \int_G d\omega(X_0^{\sigma}, \dots, X_p^{\sigma})\\ &= I(d\omega)(X_0, \dots, X_p). \end{aligned} \tag{4.67}$$

□

Now G acts as a group of automorphisms on $H_{deR}(M)$, which is the classical de Rham cohomology. Let $H_{deR}(M)^{\mu}$ denote the fixed point set of this action. We want to prove that $H_{deR}(M)^{\mu}$ is all $H_{deR}(M)$ if G is connected. To prove this, we need a previous proposition.

Proposition 4.11. *Let G be a connected Lie group (not necessarily compact), and let U be a neighborhood of e. Then*

$$G = \cup_{n=1}^{\infty} U^n, \tag{4.68}$$

where U^n consists of all n-fold products of elements of U.

Proof. Let V be an open subset of U containing e such that $V = V^{-1}$, where $V^{-1} = \{\sigma^{-1} \in G | \sigma \in V\}$ For example, $V = U \cap U^{-1}$ will do. Let

$$H = \cup_{n=1}^{\infty} V^n \subset \cup_{n=1}^{\infty} U^n. \tag{4.69}$$

Then H is an abstract subgroup of G and is an open subset of G since $\sigma \in H$ implies $\sigma V \subset H$. Thus each coset mod H is open in G. Now, H is the complement in H of the union of all the cosets mod H different from H itself. Therefore H is also a closed subset of G. Since G is connected, and H is also non-empty, H must be all of G. This together with (4.69) prove the proposition. □

To proof that if G is connected then $H^p_{deR}(M) = H^p_{deR}(M)^{\mu}$, we must prove that $\mu_{\sigma} \backsimeq id_M$ for each $\sigma \in G$

Proof. Let U be a be a neighborhood of e. Using the property (c) on the proposition 4.5, we can take U such that there exists an open subset V in $\mathfrak{g}$ where $exp : V \to U$ is a isomorphism. Then

$$\sigma \in G = \bigcup_{n=1}^{\infty} U^n. \tag{4.70}$$

Hence, there exists an n such that $\sigma \in U^n$. Therefore, there exist $a_1, \ldots, a_n$ pints in U such that $\sigma = a_1 \ldots a_n$. As $exp|_V$ is an isomorphism, we take $v_1, \ldots, v_n$ in V such that $exp(v_i) = a_i$. Then, the map

$$\begin{aligned} H : & [0,1] \times M \to M \\ & (t, m) \mapsto \mu_{exp(tv_1)\ldots exp(tv_n)}(m) \end{aligned} \tag{4.71}$$

is a a homotopy between μ_σ and id_M. □

We define $E^p(M)^\mu$ as the set of μ-invariant p-forms over M, which is a real vector space, $E^p(M)^\mu \subset E^p(M)$. As I commutes with the differential d, we can define a cochain complex of real vector spaces where the spaces are subspaces of $E^*(M)$.

$$0 \to E^0(M)^\mu \xrightarrow{\widetilde{d^0}} E^1(M)^\mu \xrightarrow{\widetilde{d^1}} E^2(M)^\mu \xrightarrow{\widetilde{d^2}} E^3(M)^\mu \xrightarrow{\widetilde{d^3}} E^4(M)^\mu \xrightarrow{\widetilde{d^4}} \ldots \tag{4.72}$$

where $\widetilde{d^i}$ is the restriction $d^i|_{E^i(M)^\mu}$. Therefore, we can define the $\mathbb{R}$-modules $H^n\big(E(M)^\mu\big) = ker(\widetilde{d^n})/im(\widetilde{d^{n-1}})$.

Theorem 4.12. *The inclusion $J : E(M)^\mu \hookrightarrow E(M)$ induces an isomorphism*

$$J^* : H^*\big(E(M)^\mu\big) \to H^*_{deR}(M)^\mu. \tag{4.73}$$

Proof. We have $J^* I^* = 1$ since $I \circ J = id$. Therefore, I^* is onto and J^* is an injection. We must show that the image of J^* is all of $H^*_{deR}(M)^\mu$.

If $\alpha = [\omega] \in H^*_{deR}(M)^\mu$ then $I(\omega)$ represent the same class in $H^*_{deR}(M)$. Let $\sigma \in G$. Then $\omega - \delta\mu_\sigma(\omega) = d\eta$ for some $(p-1)$-form η depending on σ, since $[\omega]$ is invariant under G. Therefore, for a smooth p-cycle $c \in S_p(M)$, we have

$$\int_c \omega - \int_c \delta_{\mu_\sigma}(\omega) = \int_c d\eta = \int_{\partial c} \eta = 0. \tag{4.74}$$

Thus

$$\begin{aligned} \int_c I(\omega) &= \int_c \left(\int_G \delta\mu_\sigma(\omega) d\sigma \right) = \int_G \left(\int_c \delta\mu_\sigma(\omega) \right) d\sigma \\ &= \int_G \left(\int_c \omega \right) d\sigma = \left(\int_c \omega \right) \left(\int_G 1 \right) = \int_c \omega. \end{aligned} \tag{4.75}$$

Hence $\int_c \big(I(\omega) - \omega\big) = 0$ for every p-cycle c. In other words, the de Rham isomorphism $H^*\big(E(M)^\mu\big) \to H^*_{deR}(M) \cong H^*_{\Delta\infty}(M;\mathbb{R})$ kills $\big[I(\omega) - \omega\big]$. Therefore, $\big[I(\omega)\big] = [\omega]$ in $H^*_{deR}(M)$. □

Using this theorem, we have that $H^*\big(E(G)^\mu\big) \cong H^*_{deR}(G)^\mu$. As G is a connected Lie group, we have that $H^*\big(E(G)^\mu\big) \cong H^*_{deR}(G)$. If we fix μ as the left-translation and we set L^p_G the set of left-invariant p-forms, we have the next corollary.

Corollary 4.13. *Let G be a compact connected Lie group. Then $H^p_{deR}(G)$ is isomorphic to $H^p(L^*_G)$, where L^*_G is the chain complex of forms ω on L_G with differential given by*

$$d\omega(X_0,\ldots, X_p) = \sum_{i<j} \omega([X_i, X_j], X_0,\ldots, \widehat{X_i},\ldots, \widehat{X_j},\ldots, X_p). \tag{4.76}$$

We can do better than this by applying the theorem to the action $a : G \times G \to G$ given by

$$(\sigma, \tau) \mapsto \sigma\tau\sigma^{-1} \tag{4.77}$$

which we have use to define the adjoint representation. Then, the cohomology of G is given by the a-invariant forms. However, this is the same as the space of left-invariant forms which are also invariant under the conjugation. Therefore the adjoint representation acts over the forms by

$$Ad_\sigma(\omega)(X_1,\ldots, X_p) = \omega\big(Ad_\sigma(X^1),\ldots, Ad_\sigma(X^p)\big). \tag{4.78}$$

Here, we need to make an observation. We call a p-form for a C^∞ mapping of G into $\Lambda^*_p(G)$, which is alternating. A C^∞ mapping of G into $T_{r,s}(G)$, is called a tensor field of type (r, s). We mention one of them by a not necessary alternating form, either a tensor field or an alternating form.

Theorem 4.14. *Let G be a compact connected Lie group. Let ω be a not necessarily alternating on L_G. The adjoint action of G leaves ω invariant if, and only if, the following identity holds:*

$$\sum_{i=1}^{p} \omega(X_1,\ldots, X_{i-1}, [Y, X_i], X_{i+1},\ldots, Xp) = 0 \tag{4.79}$$

for all $Y, X_1,\ldots, X_p \in \mathfrak{g}$.

Proof. We claim that $Ad_\sigma(X) = X$ for each $\sigma \in G$ with $X \in \mathfrak{g}$ if, and only if, $ad_Y(X) = 0$ for all $Y \in \mathfrak{g}$. Note that the exponential map for $Gl(n, \mathbb{R})$ is given by exponentiation of matrices:

$$exp(A) = e^A \tag{4.80}$$

Because, $t \mapsto e^{tA}$ is the unique 1-parameter group of $Gl(n, \mathbb{R})$ whose tangent vector at 0 is A. Remember that

$$e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \frac{A^4}{4!} + \frac{A^5}{5!} + \ldots \tag{4.81}$$

As $Ad\big(exp_X(t)\big) \in Aut(\mathfrak{g})$ and $\mathfrak{g}$ is a vector space, is a $Ad\big(exp_X(t)\big)$ is one.parameter group in $Gl(n, \mathbb{R})$ and therefore has the form $Ad\big(exp_X(t)\big) = e^{tA}$ for some $A \in End(n, \mathbb{R})$. Therefore,

$$ad_X = ad\left(dexp_X\left(\frac{d}{dr}\right)\right) = \left(\frac{d}{dr}e^{tA}\right)\bigg|_{t=0} = A \tag{4.82}$$

Here, $v \in \mathbb{R}^n$, where v is the coordinate vector of a vector field on $\mathfrak{g}$, then $e^{tA}v = v + tAc + (tA)^2 v/2 + \ldots$ If v is fixed by the ad, then this is v and so

$$Av = \left(\frac{d}{dr}\big(ad_{exp_X(t)}(v)\big)\right)\bigg|_{t=0}. \tag{4.83}$$

Conversely, if $Av = 0$ then $e^{tA}v = v$ and so v is fixed under G since the 1-parameter group generate G. Therefore, ω is invariant if, and only if, $ad_Y(\omega) = 0$ for all $Y \in \mathfrak{g}$.

Let Y be a vector field in $\mathfrak{g}$, remember that exp_Y is the unique integral curve of Y with tangent Y. Then,

$$Ad_{exp_Y(t)}(\omega)(X_1,\ldots, X_p) = \omega\big(Ad_{exp_Y(t)}(X^1),\ldots, Ad_{exp_Y(t)}(X^p)\big). \tag{4.84}$$

And so, putting $B_t = Ad_{exp_{-Y}(t)}$, we have

$$
\begin{aligned}
0 = ad(Y)\big[(\omega)(X_1,\ldots,X_p)\big] &= adexp_Y(t)\left(\frac{d}{dt}\right)\Big|_{t=0}\omega(X_1,\ldots,X_p)\\
&= \frac{d}{dt}Ad_{exp_Y(t)}\,(\omega)(X_1,\ldots,X_p)\big|_{t=0}\\
&= \frac{d}{dt}\,\big(\omega(B_tX_1,\ldots,B_tX_p)\big)\big|_{t=0}\\
&= \lim_{t\to 0}\frac{1}{t}\left\{\omega(B_tX_1,\ldots,B_tX_p)-\omega(X_1,\ldots,X_p)\right\}\\
&= \lim_{t\to 0}\frac{1}{t}\left\{\omega(B_tX_1-X_1,\ldots,B_tX_p)+\ldots\right.\\
&+\omega(X_1,B_tX_2-X_2,\ldots,B_tX_p)\\
&\left.+\ \omega(X_1,\ldots,X_{p-1},B_tX_p-X^p)\right\}\\
&= \omega\left(\lim_{t\to 0}\frac{1}{t}(B_tX_1-X_1),\ldots,\lim_{t\to 0}\frac{1}{t}(B_tX_p)\right)\\
&+\omega\left(X_1,\lim_{t\to 0}\frac{1}{t}(B_tX_2-X_2),\ldots,\lim_{t\to 0}\frac{1}{t}(B_tX_p)\right)+\ldots\\
&+\omega\left(X_1,\ldots,X_{p-1},\lim_{t\to 0}\frac{1}{t}(B_tX_p-X_p)\right)\\
&= \omega\big(ad_Y(X_1),\ldots,X_p\big)\\
&+\omega\big(X_1,ad_Y(X_2(\ldots,X_p\big)+\ldots\\
&+\omega\big(X_1,\ldots,ad_Y(X_p)\big)\\
&= \sum_{i=1}^{p}\omega(X_1,\ldots,X_{i-1},[Y,X_i],X_{i+1},\ldots,Xp).
\end{aligned}
\tag{4.85}
$$

□

Proposition 4.15. *Every Ad-invariant form on L_G is closed.*

Proof. Putting $\epsilon_{i,i}=0$, $\epsilon_{i,j}=(-1)^j$ if $i<j$ and $\epsilon_{i,j}=(-1)^{j+1}$ if $i>j$, we have

$$
\begin{aligned}
d\omega(X_0,\ldots,X_p) &= \frac{1}{2}\sum_{i\neq j}(-1)^i\epsilon_{i,j}\omega\big([X_i,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_p\big)\\
&= \frac{1}{2}\sum_{i}(-1)^i\sum_{j}\epsilon_{i,j}\omega\big([X_i,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_p\big)\\
&= 0,
\end{aligned}
\tag{4.86}
$$

since the inner sum is zero by the invariance formula of theorem 4.14. □

Using this proposition over L_G^{*Ad}, i.e., the chain complex of forms ω on L_G which are *Ad*-invariant, we obtain

$$0 \to L_G^{0\,Ad} \xrightarrow{d} L_G^{1\,Ad} \xrightarrow{d} L_G^{2\,Ad} \xrightarrow{d} L_G^{3\,Ad} \xrightarrow{d} \ldots \tag{4.87}$$

But, $d(L_G^{p\,Ad}) = 0$ for each $p \in \mathbb{N}$, because of the proposition. Therefore, $H^p(L_G^*)^{Ad} \cong H^p(L_G^{*Ad}) = ker(d)/img(d) = L_G^{p\,Ad}$. Using the corollary 4.13, we obtain the next result.

Corollary 4.16. *For a compact connected Lie group G, $H^p_{deR}(G)$ is isomorphic to the vector space of invariant alternating p-forms on L_G.*

For the remainder of this section, we assume that G is a compact connected Lie group. Let $[L_G, L_G]$ denote the span of the elements of L_G of the form $[X, Y]$ for $X, Y \in \mathfrak{g}$.

Corollary 4.17.

$$[L_G, L_G] = L_G \quad \textit{if, and only if,} \quad H^1(G;\mathbb{R}) = 0. \tag{4.88}$$

Proof. By the de Rham theorem we have that $H^p(G;\mathbb{R}) \cong H^p_{deR}(G)$ for each $p \in \mathbb{N}$. If $[L_G, L_G] \neq L_G$, then there exists a nonzero 1-form ω vanishing on $[L_G, L_G]$, and conversely. Such a form is invariant since that just means that $\omega\big([L_G, L_G]\big) = 0$ for all $X, Y \in \mathfrak{g}$. □

Corollary 4.18.

$$H^1(G;\mathbb{R}) = 0 \quad \Rightarrow \quad H^2(G;\mathbb{R}) = 0. \tag{4.89}$$

Proof. Let ω be an invariant 2-form on L_G. Then

$$\begin{aligned} 0 = d\omega(X,Y,Z) &= -\omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X) \\ &= -\omega([X,Y],Z) - \big(\omega([Z,X],Y) + \omega(x,[Z,Y]\big) \\ &= -\omega([X,Y],Z) \end{aligned} \tag{4.90}$$

where we have use the Ad-invariance in the last equality. Since $[L_G, L_G] = L_G$ by corollary 4.17, $\omega \equiv 0$. □

Theorem 4.19. *If $H^1(G;\mathbb{R}) = 0$ then the assignment $\eta \mapsto \omega$, where $\omega(X,Y,Z) = \eta\big([X,Y],Z\big)$ is a one-one correspondence from the space of invariant symmetric 2-forms η on L_G to that of invariant alternating 3-forms ω on L_G.*

Proof. Given a 3-form ω, define a 2-form ω_Z on L_G by $\omega_Z(X,Y) = \omega(X,Y,Z)$. We claim that ω_Z is closed. We compute

$$\begin{aligned} 0 &= d\omega(X_0, X_1, X_2, Z) \\ &= -\omega\big([X_0,X_1],X_2,Z\big) + \omega\big([X_0,X_2],X_1,Z\big) - \omega\big([X_1,X_2],X_0,Z\big) \\ &\quad - \omega\big([X_0,Z],X_1,X_2\big) + \omega\big([X_1,Z],X_0,X_2\big) - \omega\big([X_2,Z],X_0,X_1\big) \\ &= \omega_Z(X_0,X_1,X_2), \end{aligned} \tag{4.91}$$

because the last three terms cancel by the invariance of ω. Since $H^2(G;\mathbb{R}) = 0$ by corollary 4.18, we conclude that $\omega_Z = d\rho_Z$ for some 1-form ρ_Z. That is,

$$\omega(X,Y,Z) = \omega_Z(X,Y) = d\rho_Z(X,Y) = \rho_Z\big([X,Y]\big). \tag{4.92}$$

Put $\eta(S,T) = \rho_T(S)$. This is linear in T. Since $[L_G, L_G] = L_G$, $\eta(S,T)$ is also linear in S. For $\sigma \in G$ and $X \in \mathfrak{g}$, denote $da_\sigma(X)$ by X^σ, where a_σ is the map of G which $\tau \mapsto \sigma\tau\sigma^{-1}$. Then

$$\begin{aligned} \eta\big([X,Y],Z\big) &= \omega(X,Y,Z) = \omega(X^\sigma, Y^\sigma, Z^{\sigma)} \\ &= \eta\big([X^\sigma,Y^\sigma],Z^\sigma\big) = \eta\big([X,Y]^\sigma, Z^\sigma\big). \end{aligned} \tag{4.93}$$

Since $[L_G, L_G] = L_G$, it follows that η is invariant.

Now η decomposes uniquely as $\eta = \eta_{sym} + \eta_{skew}$, and both terms must be invariant by the uniqueness of the decomposition. Since $H^2(G;\mathbb{R}) = 0$, we have that $\eta_{skew} = 0$. Thus η is symmetric.

Conversely, if η is given and ω is defined by $\omega(X,Y,Z) = \eta\big([X,Y],Z\big)$ then ω is invariant by the same argument. By invariance of η we have $\eta\big([X,Y],Z\big) = -\eta\big(Y,[X,Z]\big)$. Thus an interchange of X and Y or of X and Z changes the sing of ω. It follows that ω is alternating. □

We will see that there always exists a nontrivial Ad-invariant symmetric 2-form on L_G, for G compact. Let $\{\,,\,\}$ be any inner product on $\mathfrak{g}$. Note that $Ad_\sigma \in Aut(\mathfrak{g})$ for all $\sigma \in G$ We set

$$\langle X, Y\rangle = \int_G \{Ad_\sigma(X), Ad_\sigma(Y)\} d\sigma, \tag{4.94}$$

where $X, Y \in \mathfrak{g}$, and we use $d\sigma$ to denote that we are considering the integral as a function of σ in G. It is immediate that $\langle\,.\,\rangle$ is again an inner product. Therefore

$$\begin{aligned}\langle Ad_\tau(X), Ad_\tau(Y)\rangle &= \int_G \{Ad_\sigma Ad_\tau(X), Ad_\sigma Ad_\tau(Y)\} d\sigma \\ &= \int_G \{Ad_{\sigma\tau}(X), Ad_{\sigma\tau}(Y)\} d\sigma = \int_G \{Ad_\sigma(X), Ad_\sigma(Y)\} d\sigma = \langle X, Y\rangle\end{aligned} \tag{4.95}$$

Hence η defined by

$$\eta(X, Y) = \langle X, Y\rangle = \int_G \{Ad_\sigma(X), Ad_\sigma(Y)\} d\sigma \tag{4.96}$$

is a nontrivial Ad-invariant symmetric 2-form on L_G. Consequently we have the following result:

Corollary 4.20. *If G is nontrivial and $H^1(G;\mathbb{R}) = 0$ then $H^3(G;\mathbb{R}) \neq 0$.*

Corollary 4.21. *The only spheres which are Lie group are $\boldsymbol{S}^0$, $\boldsymbol{S}^1$ and $\boldsymbol{S}^3$.*

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