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Finite groups acting on smooth and symplectic 4-manifolds

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Introduction

The main problem in the subject of finite (smooth) transformation groups is to determine which finite groups admit effective and smooth actions on a given smooth manifold X, and to determine the geometry of such actions. In precise terms, this means to give, for any smooth manifold X, a list of finite groups G_i and effective actions $\Phi_{i,j} : G_i \times X \to X$ such that any effective and smooth action of a finite group on X is conjugated to exactly one of the actions $\Phi_{i,j}$, where the conjugation is given by a diffeomorphism of X.

A complete solution for this problem has only been achieved for a few manifolds (the paradigmatic example being S^2, for which any effective action is conjugated to a linear action of a finite subgroup of $SO(3)$). In fact, it is extremely difficult to answer this question in full generality for any given manifold. Therefore, we restrict ourselves to a simpler question, which can be understood as a first step towards the full solution of the problem. We forget about the geometry of the actions and study the problem of determining which finite groups G (up to abstract isomorphism of groups) admit some effective and smooth action on X. This, although simpler than the full classfication problem, is extremely difficult in general, and again the full solution is only known in few cases.

One possibility is to study a coarse version of that problem, by trying to prove theorems that impose restrictions on the finite groups that can act effectively and smoothly on a given manifold. That is, instead of trying to provide a complete list of finite groups that do admit effective and smooth actions on a manifold, we look for negative results that narrow as much as possible the list of finite groups admitting efective actions on a given manifold. Moreover, we want to maximize the class of manifolds for which this negative results apply. Hence, we will be especially interested in theorems applying to all smooth manifolds, or at least to some large class of manifolds, like the closed ones.

A first result along these lines was provided by Mann and Su in 1963. They proved that for any closed smooth manifold X there exists a natural number r (depending only on X) such that, for any prime p, the rank of any p-elementary abelian group acting effectively on X is bounded by r (see Theorem 1.48 for the proof). In particular, there is no closed manifold that admits effective actions of all finite groups.

This is in sharp contrast with the situation for general (not necessarily closed) manifolds. In [60], Popov provides examples of open 4-manifolds admitting effective and smooth actions of all finite groups (more generally, there exists an open 4-manifold in which any finitely presented group acts in a smooth and free way). Therefore, at least in dimensions greater than 3, there is no hope of proving any theorem imposing restrictions on the class of finite groups acting on smooth manifolds that apply to all manifolds. Hence, we will restrict ourselves to the case of closed manifolds, for which we already know that some restrictions exist, by the theorem of Mann and Su.

The first problem we consider in this thesis is related to the so-called Jordan property for diffeomorphism groups. A group $\mathcal{G}$ has the Jordan property (or simply, is Jordan) if there exists a constant C such that any finite subgroup $G \leq \mathcal{G}$ has an abelian subgroup $A \leq G$ satisfying $[G : A] < C$. Hence, the Jordan property is a condition on the finite subgroups of an (infinite) group $\mathcal{G}$. This condition can be thought of as stating that all finite subgroups of $\mathcal{G}$ are "almost abelian". To make this more explicit, one can define for any finite group G

$$\alpha(G) = \min\{[G : A] \mid A \leq G \text{ is abelian}\}.$$

$\alpha(G)$ therefore can be thought of as a complexity measure of G, where the complexity makes reference to how much non-abelian G is (note that $\alpha(G) = 1$ if and only if G is abelian). With this notation, the Jordan property for $\mathcal{G}$ is equivalent to the existence of a constant C (depending only on $\mathcal{G}$) such that $\alpha(G) < C$ for all finite $G \leq \mathcal{G}$. Therefore, the Jordan property asserts that the finite subgroups of $\mathcal{G}$ have bounded complexity. The name of this property was given by Popov, inspired by a classical theorem of C. Jordan, which in modern terminology states that, for any n, the group $\mathrm{GL}(n, \mathbb{C})$ is Jordan (see Theorem 1.50 for a modern proof of this fact).

In the early 90's, É. Ghys attracted attention to this property by asking whether $\mathrm{Diff}(X)$ is Jordan, for any closed manifold X. This can be seen as a huge non-linear generalization of the classical Jordan theorem. Since finite subgroups of $\mathrm{Diff}(X)$ are esentially the same thing as effective and smooth finite group actions on X, the Jordan property for $\mathrm{Diff}(X)$ can be seen as imposing restrictions on the finite groups that can act on X. However, this restriction is in some sense orthogonal to that imposed by the theorem of Mann and Su: the Jordan property for $\mathrm{Diff}(X)$ says that no finite group which is "sufficiently non-abelian" can act effectively on X, while the theorem of Mann and Su says that no abelian group which is large enough (in the sense that cannot be generated by few elements) can act effectively on X.

Since 2010, a lot of progress has been made in the understanding of the Jordan property for diffeomorphism groups. More precisely, there are large classes of smooth manifolds that are now known to have Jordan diffeomorphism group, as the following theorem shows.

Theorem 0.1. *Let X be a closed smooth manifold. Suppose X satisfies one of the following conditions:*

1. $\dim X \leq 3$,

2. *X has non-zero Euler characteristic,*

3. *X is a homology sphere,*

4. *X has a finite unramified covering $\tilde{X}$ such that there are cohomology classes $\alpha_1, \dots, \alpha_n \in H^1(\tilde{X}; \mathbb{R})$ (here n is the dimension of X), such that*

$$0 \neq \alpha_1 \cup \cdots \cup \alpha_n \in H^n(\tilde{X}, \mathbb{R}).$$

Then, $\mathrm{Diff}(X)$ *is Jordan.*

See [79] for the proof of the first statement in the case $\dim X = 3$, Proposition 1.53 for the case $\dim X = 2$, [48] for the proof of the second statement and [47] for the proof of the third statement.

However, there are also closed manifolds with non-Jordan group of diffeomorphisms. The first example of this was found by Csikós, Pyber and Szábo in 2014 (see [13]), when they proved that $\mathrm{Diff}(T^2 \times S^2)$ is not Jordan. This result was later generalized by I. Mundet i Riera in [50], whose main result provides infinitely many examples of smooth manifold with non-Jordan diffeomorphism group. The existence of smooth manifolds with non-Jordan diffeomorphism group gives rise to the following questions.

1. Is there some weakening of the Jordan property that holds for the diffeomorphism group of any closed manifold?

2. Can we determine for which closed manifolds X is $\mathrm{Diff}(X)$ Jordan?

3. If X is a closed manifold such that $\mathrm{Diff}(X)$ is not Jordan, are there geometrically meaningful subgroups of $\mathrm{Diff}(X)$ that are Jordan? (Here geometrically meaningful subgroup of $\mathrm{Diff}(X)$ must be understood as the group of automorphisms of some geometric structure on X, as for example the group of symplectomorphisms of some symplectic structure.)

In this thesis we address these questions in dimension 4.

With respect to the first question, a natural generalization of Ghys' question is to ask whether $\mathrm{Diff}(X)$ has the following property for any closed smooth manifold X: there is a constant C (depending only on X) such that any finite group G acting smoothly and effectively on X has a nilpotent subgroup H with $[G : H] < C$. Recall that a group G is nilpotent if its lower central series reaches the trivial group in a finite number of steps. More precisely, we define $G_1 = G$ and $G_n = [G_{n-1}, G]$, so that G_n is the n-th iterated commutator of G with itself. Then, we say that G is nilpotent of nilpotency class k if $G_k \neq 1$ but $G_{k+1} = 1$. We say that G is nilpotent if it is nilpotent of nilpotency class k for some k. In particular, note that nilpotent of nilpotency class 1 is the same as abelian, so nilpotent groups are indeed a generalization of abelian groups. With this perspective, one of the main results of the thesis is that, while Jordan property fails in general for diffeomorphism groups of closed 4-manifolds, the next best property does hold for all 4-manifolds:

Theorem 0.2. *Let X be a closed smooth 4-manifold. There exists a constant C such that every group G acting in a smooth and effective way on X has a subgroup $G_0 \leq G$ such that $[G : G_0] \leq C$ and:*

1. G_0 *is nilpotent of class at most* 2,

2. $[G_0, G_0]$ *is a (possibly trivial) cyclic group,*

3. $X^{[G_0,G_0]}$ *is either* X *or a disjoint union of embedded tori.*

This is stated in Chapter 4 as Theorem 4.1. The proof is contained in Section 4.10.2. The proof of this theorem uses implicitly the classification of finite simple groups (CFSG), since it uses the main result in [54], whose proof is based in the CFSG.

Using this theorem we can obtain a better understanding of which closed smooth 4-manifolds have Jordan diffeomorphism group. In particular, we obtain some conditions that a 4-manifold must satisfy if it has non-Jordan diffeomorphism group.

Theorem 0.3. *Let* X *be a closed connected oriented smooth* 4*-manifold, and let* $\mathcal{G}$ *be a subgroup of* $\mathrm{Diff}(X)$. *If* $\mathcal{G}$ *is not Jordan then there exists a sequence* $(\phi_i)_{i\in\mathbb{N}}$ *of elements of* $\mathcal{G}$ *such that:*

1. *each* ϕ_i *has finite order* $\mathrm{ord}(\phi_i)$*,*

2. $\mathrm{ord}(\phi_i) \to \infty$ *as* $i \to \infty$*,*

3. *all connected components of* X^{ϕ_i} *are embedded tori,*

4. *for every* $C > 0$ *there is some* i_0 *such that if* $i \geq i_0$ *then any connected component* $\Sigma \subseteq X^{\phi_i}$ *satisfies* $|\Sigma \cdot \Sigma| \geq C$*,*

5. *we may pick for each* i *two connected components* $\Sigma_i^-, \Sigma_i^+ \subseteq X^{\phi_i}$ *in such a way that the resulting homology classes* $[\Sigma_i^{\pm}] \in H_2(X)$ *satisfy* $[\Sigma_i^{\pm}] \cdot [\Sigma_i^{\pm}] \to \pm\infty$ *as* $i \to \infty$.

This is stated and proved in Chapter 4 as Theorem 4.3. This theorem tells us that in dimension 4, the failure of a closed manifold to have Jordan diffeomorphism groups is due to the existence of embedded tori with arbitrarily large positive and negative self-intersection numbers. We also obtain sufficient conditions of a topological and geometrical nature for a manifold to have Jordan diffeomorphism group.

Theorem 0.4. *Let* X *be a connected, closed, oriented and smooth* 4*-manifold. If* X *satisfies any of the following conditions then* $\mathrm{Diff}(X)$ *is Jordan:*

1. *the Euler characteristic of* X *is nonzero:* $\chi(X) \neq 0$*,*

2. *the signature of* X *is nonzero:* $\sigma(X) \neq 0$*,*

3. *the second Betti number of* X *is zero:* $b_2(X) = 0$*,*

4. $b_2^+(X) > 1$ *and* X *has some nonzero Seiberg–Witten invariant (here* $b_2^+(X)$ *denotes the dimension of the space of self-dual harmonic* 2*-forms on* X*),*

5. $b_2^+(X) > 1$ *and* X *has some symplectic structure.*

This is stated and proved in Chapter 4 as Theorem 4.4.

With respect to question 3 above, we consider in this thesis the case of symplectic and almost complex structures. While the group of diffeomorphisms of $T^2 \times S^2$ is not Jordan, I. Mundet i Riera proved in [49] that its group of symplectomorphisms, for any symplectic form, is Jordan (here the Jordan constant C depends on the symplectic form). Moreover, it was proved also by I. Mundet i Riera in [51] that the group of symplectomorphisms of any closed symplectic manifold (in any dimension) satisfies that there is a constant C such that any finite subgroup $G \leq \mathrm{Symp}(X, \omega)$ has a nilpotent subgroup of nilpotency class 2, $H \leq G$ with $[G : H] \leq C$. Moreover, he proves in the same paper that the symplectomorphism group of any closed symplectic manifold (X, ω) with $b_1(X) = 0$ has Jordan symplectomorphism group, and that the group of Hamiltonian diffeomorphisms of any closed symplectic manifold is Jordan. At present, it is not known if there exists some closed symplectic manifold (X, ω) with non-Jordan symplectomorphism group. In this thesis, we prove that such an example does not exist in dimension 4. More precisely, we prove the following.

Theorem 0.5. *For any closed symplectic* 4*-manifold* (X, ω) *we have:*

1. $\mathrm{Symp}(X, \omega)$ *is Jordan.*
2. *If* X *is not an* S^2*-bundle over* T^2*, then a Jordan constant for* $\mathrm{Symp}(X, \omega)$ *can be chosen independently of* ω*.*
3. *If* $b_1(X) \neq 2$*, then* $\mathrm{Diff}(X)$ *is Jordan.*

This is stated in Chapter 4 as Theorem 4.6 and proved in Section 4.13. We also prove that the automorphism group of any closed almost complex 4-manifold is Jordan:

Theorem 0.6. *Let* X *be a closed and connected smooth* 4*-manifold, and let* J *be an almost complex structure on* X*. Let* $\mathrm{Aut}(X, J) \subset \mathrm{Diff}(X)$ *be the group of diffeomorphisms preserving* J*. Then* $\mathrm{Aut}(X, J)$ *is Jordan.*

This is stated in Chapter 4 as Theorem 4.5 and proved in Section 4.12. This result generalizes the result of Y. Prokhorov and C. Shramov, who proved in [62] that the automorphism group of any complex surface is Jordan. However, the proofs are completely different. While the proof of the previous theorem is based on Theorem 0.3 (and hence uses implicitly the CFSG), the proof given in [62] is based on the Enriques–Kodaira classification of compact complex surfaces.

It has also been proved by I. Mundet i Riera, using the results contained in this thesis, that the isometry group of any closed Lorentz 4-manifold is Jordan (see [52]).

The paper [62] is in fact a representative of another problem that has received some attention in later years: the study of the Jordan property for groups of birational automorphisms and for automorphism groups of algebraic varieties. This line started with the proof by Serre in [69] that the group of birational automorphisms of the projective plane over a field of characteristic 0 is Jordan. This result was generalized by Y. Prokhorov and C. Shramov to any dimension (assuming the BAB conjecture for Fano manifolds, which was recently proved by C. Birkar, see [5]), who moreover proved in [61] that $\mathrm{Bir}(X)$ is Jordan for any rationally connected algebraic variety X, as well as for some other classes of algebraic varieties. In the case of (complex) dimension 2, which is the analogue of the problems we treat in this thesis, it was proved by Y. Zarhin in [78] that the product of an elliptic curve with $\mathbb{P}^1$ has non-Jordan group of birational automorphisms, while Popov in [58] proved that the group of birational automorphisms of any *projective* surface X over $\mathbb{C}$ is Jordan if and only if X is not birational to $E \times \mathbb{P}^1$, where E is an elliptic curve. This has been extended for all compact complex surfaces in [62]: the group of birational automorphisms of a compact complex surface X is Jordan if and only if X is not birational to $E \times \mathbb{P}^1$, where E is an elliptic curve. There seems to be an intriguing analogy between diffeomorphism groups of smooth manifolds and birational automorphism groups of algebraic varieties with respect to their behaviour with the Jordan property. Note in particular the analogy between the non-Jordanness of $\mathrm{Bir}(E \times S^2)$ and $\mathrm{Diff}(T^2 \times S^2)$ (in fact, the proof of the latter fact is a reelaboration of the former). However, as far as the author knows, there is at present no precise statement of this analogy.

Another topic we discuss in this thesis is the classification of finite groups acting effectively and symplectically on some particular symplectic 4-manifolds. More concretely, we deal with the case of rational ruled surfaces, which are symplectic manifolds diffeomorphic to S^2-bundles over S^2. The fact that we can achieve for these manifolds a complete classification of the finite groups acting on them is mainly due to the fact that they are one of the few cases where a complete classification of symplectic forms on them is known. In the ruled case, this is due to an important Theorem of Lalonde and McDuff. Up to diffeomorphism, there exist only two oriented S^2-bundles over S^2: the trivial bundle $S^2 \times S^2$ and a twisted bundle that we will denote by X_S. The theorem of Lalonde and McDuff says that $(S^2 \times S^2, \omega_1)$ and $(S^2 \times S^2, \omega_2)$ are symplectomorphic if and only if ω_1 and ω_2 represent the same cohomology class or $[\omega_2] = \tau^*[\omega_1]$, where $\tau(a, b) = (b, a)$ for $(a, b) \in S^2 \times S^2$. Using this fact, we can assume without loss of generality that ω is of the form

$$\omega_{\alpha,\beta} = \alpha\pi_1^*\omega_{S^2} + \beta\pi_2^*\omega_{S^2},$$

where $\alpha, \beta > 0$ are a pair of real numbers, $\pi_i : S^2 \times S^2 \to S^2$ $(i = 1, 2)$ are the two projections, and ω_{S^2} is the standard area form on the unit sphere S^2. With this notation, the main result we prove is:

Theorem 0.7. *The finite groups that act effectively and symplectically on $(S^2 \times S^2, \omega_{\alpha,\beta})$ are:*

1. *If $\alpha \neq \beta$, the groups that are isomorphic to a subgroup of $H_1 \times H_2$, for some finite subgroups H_1, H_2 of $\mathrm{SO}(3)$.*

2. *If $\alpha = \beta$, the groups that are isomorphic to a subgroup of $G_1 \times G_2$, for some finite groups G_1, G_2 of $\mathrm{SO}(3)$, or groups G lying on an exact sequence*

$$1 \to H \times H \to G \to \mathbb{Z}/2 \to 1,$$

for some finite subgroup H of SO(3)*, and where the action by conjugation of a lift $g \in G$ of the non-trival element of $\mathbb{Z}/2$ on $H \times H$ is given by*

$$g(h_1, h_2)g^{-1} = (\phi_1 h_2 \phi_1^{-1}, \phi_2 h_1 \phi_2^{-1}),$$

for some $\phi_1, \phi_2 \in SO(3)$ such that $\phi_1\phi_2 \in H$ and $\phi_2\phi_1 \in H$.

This is stated and proved in Chapter 5 as Theorem 5.17. We also prove a similar result for the non-trivial bundle.

Theorem 0.8. *Let ω be any symplectic form on X_S. Then, G acts effectively and symplectically on (X_S, ω) if and only if G is isomorphic to a finite subgroup of $U(2)$.*

This is stated and proved in Chapter 5 as Theorem 5.24. The proofs of these two theorems are based on the methods of [49], where it is proven that the symplectomorphism groups on $T^2 \times S^2$ are always Jordan. They make heavy use of the theory of J-holomorphic curves on symplectic manifolds.

We finish this introduction by discussing some open questions and proposing some directions for further research in the topics treated in this thesis.

- The most obvious open problem left in this thesis is to completely characterize the closed 4-manifolds with Jordan diffeomorphism group. While Theorems 0.3 and 0.4 impose strong restrictions on the topology of closed 4-manifolds with non-Jordan diffeomorphism group, there are plenty of manifolds that are not covered by those theorems. For instance, the connected sum of $T^2 \times S^2$ with any closed 4-manifold Y with $\chi(Y) = 2$ and $\sigma(Y) = 0$ is not ruled out by those theorems (note that any connected sum has vanishing Seiberg–Witten invariants). Up to now, there are only two known examples of closed 4-manifolds with non-Jordan diffeomorphism group, namely the two orientable S^2-bundles over T^2. The author conjectures that these are the only two examples. A weaker version of this conjecture is the following one: if X is a closed 4-manifold that admits some symplectic structure and X is not diffeomorphic to an S^2-bundle over T^2, then Diff(X) is Jordan. By Theorem 0.5, the only unknown case is when $b_2^+(X) = 1$. This conjecture is supported by the fact that the only case where the Jordan constant C for Symp(X, ω) cannot be chosen independently of the symplectic form is precisely when X is an S^2-bundle over T^2.

- Is there, in dimension higher than 4, an analog of Theorem 0.2? More precisely, is it true that for any closed manifold X (of any dimension) there is a constant $C > 0$ such that any finite group G acting effectively and smoothly on X has a nilpotent subgroup $H \leq G$ with $[G : H] < C$? This has been asked by several people, among which É. Ghys, David Fisher and László Pyber. If this conjecture is true, can one prove a refined version substituting 'nilpotent' by 'nilpotent of nilpotency class k', with k depending only on the dimension? One should remark that at present there is no known manifold where this property is known to be false for $k = 3$. Therefore, although unlikely, it is possible that Theorem 0.2 holds for all closed manifolds of any dimension.

- Is there any closed symplectic manifold with non-Jordan group of symplectomorphisms? In this thesis we prove that the answer is no in dimension 4, so one must study higher dimensional symplectic manifolds. A good first step would be to determine if $T^2 \times \mathbb{C}\mathrm{P}^2$ (with any symplectic structure) has Jordan group of symplectomorphisms. The arguments given in this thesis for dimension 4 rely strongly on Seiberg–Witten theory and on positivity of intersections for two J-holomorphic curves, which are techniques that are not available in higher dimension.

- Generalize the theorems in Chapter 5 to symplectic ruled surfaces over any closed surface. In fact, several of the arguments we give in Chapter 5 for the case of ruled surfaces over S^2 also work in the case of arbitrary genus (some of the techniques used there were first applied to the case of $T^2 \times S^2$ in [49]). The main problem is to deal with the case where the action of a finite group G on $(\Sigma \times S^2, \omega)$ leaves no embedded J-holomorphic curve invariant for any G-invariant almost complex structure J. In the case of $\Sigma = S^2$ one can see that G must preserve the product structure, but for Σ of higher genus there is a priori room for other phenomena. It would be interesting to find out if there can be actions without embedded G-invariant surfaces which do not preserve the product structure.

Contents of the thesis

This thesis is divided in 5 chapters. The first three chapters consist mostly on preliminary results, which we give in order to make the thesis essentially self-contained. The assumed background is that of a master student plus some knowledge of basic homotopy theory, characteristic classes, connections and Chern-Weil theory. No knowledge on the theory of finite transformation groups is assumed.

The first chapter is devoted to the general theory of smooth actions. After some generalities, we prove some fundamental results on smooth actions that will be freely used in the rest of the thesis. We also define what will be one of our main tools in the proof of Theorem 0.2: equivariant cohomology and its associated spectral sequence. Moreover, we provide proofs of some important theorems such as 1.48, adapted to the case of smooth actions. In the final section we provide some general results about the Jordan property, prove the classical Jordan theorem and explain the construction of Popov of an open 4-manifold admitting effective actions of all finite groups. We also include here an example of an open symplectic manifold with non-Jordan symplectomorphism group, and an example of a closed contact manifold with non-Jordan contactomorphism group.

The second chapter starts with some preliminaries in the topology of 4-manifolds. After that, we give a brief introduction to index theory, and state the fundamental G-signature theorem of Atiyah and Singer, which is a crucial ingredient in the proof of Theorems 0.3 and 0.4. The rest of the chapter is devoted to give an introduction to the topic of Seiberg–Witten theory. This plays an important role in the proof of Theorem 0.4 through the use of the generalized adjunction formula, and an even more important role (via Taubes' theorems) in the study of the Jordan property for symplectomorphism groups. The material in Seiberg–Witten theory that we present is by now standard. However, we warn the reader that we make use of it in the case of non-simply connected manifolds with $b_2^+ = 1$, which is not treated in most of the standard reference books on the subject.

The third chapter starts with a discussion of almost complex manifolds and J-holomorphic curves in them. In particular, we discuss the local properties of J-holomorphic curves, which are key for the proof of several of our results in the thesis (both Theorems 0.5 and 0.6, as well as the results in Chapter 5). After that, we turn to a discussion of moduli spaces of J-holomorphic curves and Gromov–Witten invariants in a symplectic manifold. Here, we restrict ourselves to the case of genus 0 J-holomorphic curves, which is all we will need for obtaining the results in Chapter 5.

The fourth chapter contains the main contributions of this thesis. In it, we prove all the theorems concerning the Jordan property in dimension 4. In particular, we prove Theorems 0.2, 0.3, 0.4, 0.6 and 0.5. The results of this chapter are also contained in the article [53].

Finally, in the fifth chapter we give a complete classification of the finite groups acting effectively and symplectically on any symplectic manifold diffeomorphic to an S^2-bundle over S^2. In particular, we prove Theorems 0.7 and 0.8.

Conventions and notation

We end this introduction by collecting for reference some basic terminology and notation used throughout the thesis. Suppose that a group G acts on a space X. We denote by X^G the set of points $x \in X$ such that $gx = x$. If $g \in G$ then we denote by X^g the set of points satisfying $gx = x$. We say that a subspace $Y \subseteq X$ is **invariant** (or **G-invariant**, or **preserved by** G) if for any $y \in Y$ and $g \in G$ we have $gy \in Y$. This is a standard convention (see e.g. Bredon, Chapter I, Section 1; or Brocker–tom Dieck Chap I, Section 4).

For any space X we denote the rational Betti numbers by $b_j(X) = b_j(X;\mathbb{Q}) = \dim_{\mathbb{Q}} H_j(X;\mathbb{Q})$. Integer coefficients will be implicitly assumed in homology and cohomology, so we will denote $H_*(X) = H_*(X;\mathbb{Z})$, and $H^*(X) = H^*(X;\mathbb{Z})$, unless otherwise stated. Following the standard convention we denote by $\chi(X)$ the Euler characteristic of X and by $\sigma(X)$ the signature of X in case X is a closed oriented manifold.

A continuous action of a group G on a manifold X induces an action on $H^*(X)$. We will say that the action is **cohomologically trivial** (**CT** for short), if the induced action on $H^*(X)$ is trivial. If X is orientable and closed, then a CT action is orientation preserving. An action of G on X is **effective** if $g \cdot x = x$ for all $x \in X$ implies that $g = 1$. We will write that an action is **CTE** if it is CT and effective.

For any set S we denote by $\sharp S$ the cardinal of S.

Whenever we say that a group G can be generated by d elements we mean that there is a collection of *non necessarily distinct* elements $g_1, \ldots, g_d$ which generate G.

All manifolds will be assumed by default to be **smooth**. A **closed manifold** means a compact manifold without boundary.

Chapter 1

Preliminaries on group actions

In this chapter we introduce group actions on manifolds together with several constructions and tools that are used repeatedly in the rest of this thesis.

In the first two sections we review the definition of group actions and prove some basic results of smooth actions on smooth manifolds that will be used throughout the rest of the thesis. After that, we digress in order to introduce a very useful technique for computing the cohomology of a fiber bundle: the Serre spectral sequence. Then, in the next sections we introduce classifying spaces and its associated cohomology, which are a crucial ingredient in the definition of one of the most useful algebraic topological tools to study group actions on manifolds: equivariant cohomology. We will then put equivariant cohomology to use in order to obtain some deep theorems about actions of finite groups on closed manifolds. Finally, in the last section of this chapter we introduce the Jordan property, together with some examples of Jordan and non-Jordan groups.

1.1 Group actions

We begin by reviewing basic definitions about group actions, mainly in order to fix notation.

Definition 1.1. *Let X be a set and let G be a group. A (left) action of G on X is a map*

$$\phi : G \times X \to X$$

satisfying the following properties

1. $\phi(1, x) = x$ *for all* $x \in X$,
2. $\phi(g_1, \phi(g_2, x)) = \phi(g_1 g_2, x)$ *for all* $g_1, g_2 \in G$ *and* $x \in X$,

where $1 \in G$ denotes the identity of the group.

In order to simplify the notation, we will write $\phi(g, x) = g \cdot x$ when there is no confusion about the action.

Let X, Y be two sets and suppose G acts both on X and Y. In this situation we can consider the maps that commute with the two actions. More precisely, we say that a map $f : X \to Y$ is G-equivariant if for all $g \in G$, $x \in X$:

$$f(g \cdot x) = g \cdot f(x).$$

Sometimes it is more natural to make the group act on the right of the set. A right action of G on X is a map

$$\phi : X \times G \to X$$

satisfying the following properties

1. $\phi(x, 1) = x$ for all $x \in X$,
2. $\phi(\phi(x, g_1)g_2) = \phi(x, g_1 g_2)$ for all $g_1, g_2 \in G$ and $x \in X$,

In this case, we write the action as $x \cdot g$.

A situation where right actions occur in a natural way is the action of a Lie group G on a principal G-bundle (and the particular case of the monodromy action on a covering space). This example will appear repeatedly in what follows.

Let X be a set and let $\mathrm{Bij}(X)$ be the group of bijections of X under composition. A first observation is that given any left action ϕ we can define a map

$$\Phi : G \to \mathrm{Bij}(X)$$

given by $\Phi(g)(x) = \phi(g, x)$. Moreover, the defining properties of the group action imply that Φ is a group morphism. Conversely, given any group morphism $\Phi : G \to X$ we can define an action ϕ of G on X by putting $\phi(g, x) = \Phi(g)(x)$. Therefore, an action of a group G on a set X can be identified with a group morphism $G \to \mathrm{Bij}(X)$. Observe also that any action of G on X induced by restriction an action of any subgroup $H \leq G$ on X.

Next, we introduce some of the basic terminology and notation about group actions.

Definition 1.2. *1. A group action is said to be effective (or faithful) if for every $g \in G$ different from the identity there exists some $x \in X$ such that $g \cdot x \neq x$*

2. A group action is said to be free if for every $g \in G$ different from the identity, and for all $x \in X$, $g \cdot x \neq x$.

3. The orbit of $x \in X$ under the action of G on X is $G \cdot x = \{g \cdot x \mid g \in G\}$.

4. *A group action is said to be transitive if the orbit of any point is X.*

5. *A point $x \in X$ is said to be a fixed point for a subset $A \subseteq G$ if $g \cdot x = x$ for all $g \in A$. The set of all fixed points by A is denoted by X^A. Observe that $X^A = X^{\langle A \rangle}$, where $\langle A \rangle \leq G$ is the subgroup of G generated by A.*

6. *Let $Y \subseteq X$ be a subset, and let $H \leq G$ be a subgroup. We say that Y is H-invariant (or simply invariant, in the case $H = G$) if $g \cdot x \in Y$ for all $g \in H$ and $x \in Y$.*

Therefore, an action is effective if and only if for any $g \in G$ different from the identity we have $X^g \neq X$. Similarly, an action is free if and only if for all $g \in G$ different from the identity we have $X^g = \emptyset$.

If instead of being just a set, X has some additional structure, we may restrict our attention to actions that are compatible with this structure. More generally,

Definition 1.3. *Let $\mathcal{C}$ be a category and $X \in \mathrm{Obj}(\mathcal{C})$. Let G be a group. A $\mathcal{C}$-group action of G on X is a group morphism $G \to \mathrm{Aut}(X)$.*

If the objects of the category $\mathcal{C}$ are sets with some additional structure, one can see, analogously as we have described above for the case of $\mathrm{Bij}(X)$, that a $\mathcal{C}$-group action of G on X is the same as an action of G on X in the usual sense (considering X a set) such that the induced maps

$$\begin{aligned} g : X &\to X \\ x &\mapsto g \cdot x \end{aligned}$$

are automorphisms of X in the given category.

Example 1.4. 1. *A* Set*-group action is the same as a usual group action.*

2. *Let k be a field. Let* k-Vect *be the category of k-vector spaces and linear maps. Then, a* k-Vect*-group action of G on a k-vector space V is the same as an action in which all maps $g : V \to V$ are linear isomorphisms. Equivalently, it is the same as a linear representation of G on V. Such actions are called linear.*

3. *Let* Top *be the category of topological spaces and continuous maps. Then, a* Top*-group action of G on a topological space X is the same as an action in which all maps $g : X \to X$ are homeomorphisms. Such actions are called continuous.*

4. *Let* DiffMan *be the category of smooth manifolds and smooth maps. Then, a* DiffMan*-group action of G on a smooth manifold X is the same as an action in which all maps $g : X \to X$ are diffeomorphisms. Such actions are called smooth.*

5. *Let* SympMan *be the category of symplectic manifolds and symplectic maps. Then, a* SympMan*-group action of G on a symplectic manifold X is the same as an action in which all maps $g : X \to X$ are symplectomorphisms. Such actions are called symplectic.*

If $\mathcal{C}$ is a subcategory of $\mathcal{D}$, it is clear that any $\mathcal{C}$-action on an object $X \in \mathrm{Obj}(\mathcal{C})$ induces a $\mathcal{D}$-action on X in a unique way. The converse is false in general: even if $X \in \mathrm{Obj}(\mathcal{C})$ there may be $\mathcal{D}$-actions on X that do not come from a $\mathcal{C}$-action. In particular, any smooth action on a smooth manifold X is a continuous action by thinking X as a topological space and forgetting the smoothness of maps, and any symplectic action on a symplectic manifold (X, ω) is a smooth action by thinking of X as a smooth manifold and forgetting the symplectic structure.

1.2 Smooth actions

From now on we will focus on smooth actions on smooth manifolds, which is the main topic of interest of this thesis.

The theory of smooth actions becomes much easier if we restrict ourselves to compact Lie groups G. Moreover, since we are only interested in finite group actions, we will only consider finite groups. However, the reader must keep in mind that most of the results proved in this chapter for finite group actions can be easily generalized to compact Lie groups.

General references for this topic can be found in Chapter VI of [8] and in the lecture notes [44]. However, these notes treat the more general case of actions of positive-dimensional compact Lie groups, so some of the proofs are more involved than the ones we give here for finite groups.

Let X be a smooth manifold. Let G be a finite group acting smoothly on X. We say that a riemannian metric h on X is G-invariant if $g^*h = h$ for all $g \in G$.

Proposition 1.5. *Let X be a smooth manifold and let G be a finite group acting smoothly on X. Then, there exists a G-invariant metric on X.*

Proof. Let h be any riemannian metric on X. We define a new metric $\tilde{h}$ by "averaging" h over all elements of G. Specifically, we define $\tilde{h}$ as follows:

$$\tilde{h} = \frac{1}{|G|} \sum_{g \in G} g^* h$$

We claim that $\tilde{h}$ is a riemannian metric. Indeed, this follows from the convexity of the space of riemannian metrics on X and the fact that each g^*h is a riemannian metric (because $g : X \to X$ is a diffeomorphism). It only remains to prove that $\tilde{h}$ is G-invariant. But this is clear, since for any $g \in G$:

$$g^* h = \frac{1}{|G|} \sum_{g' \in G} g^* g'^* h = \frac{1}{|G|} \sum_{g' \in G} (g'g)^* h = \frac{1}{|G|} \sum_{g' \in G} g'^* h = \tilde{h}$$

because when g' ranges over G, $g'g$ also ranges over all of G. □

One of the main advantages of smooth actions over more general continuous actions is the fact that the subset of fixed points of a smooth action is always a (possibly disconnected) smooth submanifold. This follows from the following useful result: smooth actions can always be linearized around a fixed point (more generally, around a G-invariant submanifold). We now turn to a proof of these facts.

We begin with a simple lemma about linear actions on vector spaces.

Lemma 1.6. *Let V be a k-vector space, and let G be a group acting linearly on V. Then, the fixed point set V^G is a linear subspace.*

Proof. If $u, v \in V^G$ and $\lambda, \mu \in k$, then for every $g \in G$,

$$g \cdot (\lambda u + \mu v) = \lambda g \cdot u + \mu g \cdot v = \lambda u + \mu v,$$

since the action is linear. Hence, V^G is a linear subspace of V. □

In the next proposition, and in the rest of this thesis, a vector bundle automorphism of a smooth vector bundle $\pi : E \to X$ means a diffeomorphism $f : E \to E$ such that f sends fibers to fibers and acts as a linear isomorphism restricted to each fiber. That is, writing $E_x = \pi^{-1}(x)$ for the fiber of E over over $x \in X$, f is a vector bundle automorphism if there exists a map $g : X \to X$ such that for all $p \in X$, $f(E_p) = E_{g(p)}$ and $f|_{E_p} : E_p \to E_{g(p)}$ is a linear isomorphism. It follows then that g is a diffeomorphism of the base manifold X. In this situation we say that f lifts g. Note that we do not impose that a vector bundle automorphism lifts the identity on X.

Proposition 1.7. *Let X be a smooth manifold. Let G be a finite group acting smoothly on X. Suppose $Y \subseteq X$ is a G-invariant submanifold. Then there exists a natural action of G on the normal bundle $\pi : N \to Y$ by vector bundle automorphisms. Moreover,*

1. *X^G is a (possibly disconnected) submanifold of X.*
2. *There exists a G-invariant neighbourhood T of Y in X which is diffeomorphic to an open neighbourhood of the zero section of the normal bundle of Y. T is then called a G-invariant tubular neighbourhood of Y.*
3. *If X is connected and the action of G on X is effective, then the action of G on N is also effective.*

Proof. Each diffeomorphism $\phi : X \to X$ induces via the differential a vector bundle automorphism $d\phi : TX \to TX$ lifting ϕ. If G is a group acting on X, the functoriality of the differential implies that the map $G \to \mathrm{Aut}(TX \to X)$ which assigns dg to each $g \in G$ is a morphism of groups. Hence, G acts on TX by vector bundle automorphisms. Therefore, we only need to prove that this action restricts to an action of the normal bundle $N \to Y$. Since Y is G-invariant, the natural action of G on TX restricts to an action of G on $TX|_Y$. Moreover, since TY is invariant for this action, we obtain an induced action of G on the normal bundle $N = TX|_Y / TY$. This finishes the proof of the first part of the statement.

Let h be a G-invariant riemannian metric, which we know to exist by Proposition 1.5. Let exp be the Riemannian exponential map with respect to h. We have a splitting $TX|_Y \simeq TY \oplus (TY)^\perp$. Since h and Y are G-invariant, the action of G on TX preserves this splitting. This implies that the action of G on TX restricts to an action of G on TY, which is the action induced by the restriction of the action of G on X to Y, and to an action of G on $(TY)^\perp$. It is clear then that we have a G-equivariant isomorphism $(TY)^\perp \simeq N$, where the action of G on N is the natural one given in the previous paragraph.

We claim that for all $g \in G$, $p \in X$, $v \in T_pX$,

$$g \cdot \exp_p(tv) = \exp_{g\cdot p}(g \cdot tv) \tag{1.1}$$

holds true for all values of $t \in \mathbb{R}$ in a neigbourhood of 0 small enough so that $\exp_p(tv)$ is defined. In order to prove it, observe that $\gamma(t) := \exp_p(tv)$, defined for t in some neigbourhood of $0 \in \mathbb{R}$ is a geodesic such that $\gamma(0) = p$ and $\gamma'(0) = v$. Since $g : X \to X$ is an isometry of (X, h) (because h is G-invariant), $g \circ \gamma$ is another geodesic of (X, h) with $g \circ \gamma(0) = g \cdot p$ and $(g \circ \gamma)'(0) = g \cdot v$. Therefore, $g \circ \gamma(t) = \exp_{g\cdot p}(t(g \cdot v)) = \exp_{g\cdot p}(g \cdot tv)$.

Note that $p \in X^G$ is a G-invariant submanifold of dimension 0, with $N = T_pX$. So by what we have just proved, there is an induced action of G on T_pX. Using Lemma 1.6, equation (1.1) implies that locally around p, X^G is the image of the linear subspace $(T_pX)^G$. This proves (1).

We now prove part (2). Pick some smooth and strictly positive function $\epsilon : Y \to \mathbb{R}$ such that $\exp_p$ is a diffeomorphism for all $v \in T_pY$ with $||v|| < \epsilon(p)$. Define

$$N_\epsilon = \{(p, v) \in N \mid ||v|| < \epsilon(p)\}.$$

By the choice of ϵ, $\exp|_{N_\epsilon}$ is a diffeomorphism onto its image.

Let $T = \exp(N_\epsilon)$. Then, T is an open neighbourhood of Y, since Y is the image by exp of the zero section of $N \to Y$. Finally, observe that T is G-invariant, since if $x \in T$ we can write $x = \exp(v)$ for some $v \in N_\epsilon$, an equation (1.1) implies that $g \cdot x = g \cdot \exp(v) = \exp(g \cdot v)$, which is again in Y because $||g \cdot v|| = ||v|| < \epsilon$, being g an isometry of (X, h).

Finally, we prove (3). Assume that the action of G on N is not effective. This means that there is some $g \in G$ such that g acts trivially on Y and $g \cdot v = v$ for all $v \in N$. Let $p \in Y$ be any point. Equation (1.1) implies that there is a tubular neigbourhood of Y contained in X^g. This means that there is a connected component of X^g which is an open and closed submanifold of X of the same dimension as X. Since we are assuming X connected, this implies $X^g = X$, so the action of G on X is not effective. □

We have seen that the fixed point set of a smooth action is a disjoint union of connected submanifolds. As the next proposition shows, if the manifold is oriented and the action preserves the orientation, we can say something about the dimensions of these submanifolds in the case of a cyclic group action.

Proposition 1.8. *Let X be an oriented smooth manifold, and let G be a finite cyclic group acting smoothly and effectively on X. Assume moreover that G acts preserving* *orientation. Then, X^G is a disjoint union of connected submanifolds of even codimension.*

Proof. By Proposition 1.7, X^G is a disjoint union of submanifolds. Let $Y \subset X^G$ be a connected component, and let $p \in Y$.

Choosing a G-invariant metric, we may assume that $G \leq SO(n)$, where n is the dimension of X. Since the fixed point subspace of a cyclic subgroup of $SO(n)$ is of even codimension and $T_pX^G = (T_pX)^G$, again by Proposition 1.7, Y is a submanifold of even codimension. □

Note however that this proposition is false if G is not cyclic, as can be seen from the action of the group G of orientation-preserving isometries of a cube in $\mathbb{R}^3$, where we have $(\mathbb{R}^3)^G = \{0\}$.

It is interesting to note that the fact that fixed-point subsets of smooth actions are submanifolds is the reason why smooth actions are much easier than continuous actions on topological manifolds. The next example shows that the analogous statement for continuous actions is false. The proof can be found in [4].

Theorem 1.9 (Bing)**.** *There exists a homeomorphism $i : S^3 \to S^3$ such that $i^2 = id$ (that is, i is an involution), and $(S^3)^i$ is the Alexander's horned sphere.*

Recall that the Alexander's horned sphere A is a subset of S^3 homeomorphic to S^2 but such that $S^3 - A$ is not homeomorphic to two copies of the open disk D^2, thus proving that the Schoenflies theorem is false in dimension 3. The Alexander's horned sphere is not a submanifold of S^3. However, it is true that the fixed-point set of a continuous action on a topological manifold is always a homology manifold, that is, a locally compact topological space Y having the same local homology as a manifold. This means that Y has the property that for some non-negative integer n and every $x \in Y$, $H_k(Y, Y - \{x\}; \mathbb{Z})$ is zero for all $k \neq n$ and isomorphic to $\mathbb{Z}$ for $k = n$. A general theory can be, and has been, developed in order to deal with such actions. For more information on that, see [7] or [8].

We conclude this section with a discussion of cohomologically trivial actions.

Let G be a group acting on a topological space X. Then, there is an induced action of G on $H^*(X)$, by considering

$$g \cdot \alpha = g^*(\alpha)$$

for $g \in G$ and $\alpha \in H^*(X)$, where on the second term we consider g as a homeomorphism $g : X \to X$.

We will say that the action of G on X is cohomologically trivial, and we will use the notation CT, if its induced action on the cohomology of X is trivial. If an action of G on X is CT and effective, we will say that it is CTE.

An important fact is that if the cohomology of a manifold X is finitely generated as an abelian group (e.g. if X is closed), given any action of a finite group G on X we can always obtain a CT action of a bounded index subgroup of G on X.

Proposition 1.10. *Let X be a closed manifold. Then, there exists a constant $C > 0$ such that every finite group G acting on X has a subgroup $G_0 \leq G$ such that $[G : G_0] < C$ and G_0 acts in a CT way on X.*

Proof. Since X is closed, its cohomology is finitely generated. Write $H = H^*(X)$. Let T be the torsion part of H.

A theorem of Minkowski states that given any positive integer k, the group $\mathrm{GL}(k, \mathbb{Z})$ has the following property: there exists a constant C_k such that every finite group $G < \mathrm{GL}(k, \mathbb{Z})$ has order at most C_k (see [45]).

Hence, if G acts on X, there is a subgroup $G' \leq G$ such that G' has bounded index in G and G' acts trivially on H/T. Moreover, there is a subgroup $G'' \leq G'$ of index at most $|\mathrm{Aut}(T)|$ such that G'' acts trivially on T.

Then, putting $F = H/T$ and choosing a decomposition $H = F \oplus T$, we have that G'' acts on H through matrices of the form:

$$\left(\begin{array}{c|c} I & 0 \\ \hline A & I \end{array}\right)$$

for some matrix A, where the I represent the identity matrix of suitable size. Since A represents an element of $\mathrm{Hom}(F, T)$, and we have just seen that there is an injective map of G'' into $\mathrm{Hom}(F, T)$, the subgroup $G_0 \leq G''$ consisting of matrices as above with $A = 0$ has of index at most $|\mathrm{Hom}(F, T)|$ on G'' and acts trivially on H. Hence, G_0 has bounded index on G and the action of G_0 on X is CT. □

1.3 The Serre spectral sequence

In this section we introduce the Serre spectral sequence for computing the cohomology of the total space of a fibration, which we will use extensively later on to compute equivariant cohomologies. Although we will only apply the results of this section to fiber bundles, here we work with the more general notion of (Hurewicz) fibrations, a natural class of maps for which the Serre spectral sequence applies. We will focus on cohomology, since it is the only case we will use. However, an analogous spectral sequence works for homology. For references, the reader can consult any advanced book on algebraic topology, for instance [23] and chapter 9 of [27].

We start by introducing the notion of a fibration and stating some of its properties.

Definition 1.11. *Let E, B, X be topological spaces. Let $i_0 : X \to X \times I$ be defined by $i_0(x) = (x, 0)$. A continuous map $p : E \to B$ has the homotopy lifting property (HLP) with respect to X if for any $f : X \times I \to B$ and $\tilde{f}_0 : X \to E$ making the following diagram commute:*

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}_0} & E \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{f} & B \end{array}$$

there exists a map $\tilde{f} : X \times I \to E$ *making the following diagram commute:*

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}_0} & E \\ {\scriptstyle i_0}\downarrow & \overset{\tilde{f}}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{f} & B \end{array}$$

Definition 1.12. *A (Hurewicz) fibration is a continuous map* $p : E \to B$*, which has the homotopy lifting property with respect to all topological spaces* X*.*

There is also a more general notion of Serre fibration, which is a map that has the HLP with respect to all cubes I^n for $n \geq 0$. While the theory we will introduce in this section works for this more general class of maps, we will stick to Hurewicz fibrations, since they have more pleasant properties and are enough for our purposes.

If $p : E \to B$ is a fibration, we denote its fiber $p^{-1}(b)$ over a point $b \in B$ by E_b.

The main examples of fibrations are fiber bundles. While all fiber bundles are Serre fibrations, not all of them are fibrations. However, we have

Proposition 1.13. *Let* $E \to B$ *be a fiber bundle, and let* B *be a paracompact space. Then,* $E \to B$ *is a fibration.*

See [27, Corollary 6.9] for a proof. The converse is false in general, so a fibration is not always locally a product (for instance, take any affine projection $p : \Delta^2 \to I$, where Δ^2 is the standard 2-simplex and I one of its faces). In particular, fibers of fibrations need not be homeomorphic. However, as the next proposition shows, they are always homotopy equivalent.

Proposition 1.14. *Let* $p : E \to B$ *be a fibration with* B *connected. Given two points* $x, y \in B$*, any homotopy class* $[\gamma]$ *of paths in* B *from* x *to* y *(where the homotopy is relative to the endpoints) induces a map* $E_x \to E_y$*, well-defined up to homotopy. In particular, fibers of a fibration are homotopy equivalent.*

Proof. Let γ be a path in B from x to y. We define a map $\tilde{H} : E_x \times I \to E$ by using the lifting property of p as follows:

$$\begin{array}{ccc} E_x \times \{0\} & \longrightarrow & E \\ \downarrow & \overset{\tilde{H}}{\nearrow} & \downarrow{\scriptstyle p} \\ E_x \times I & \xrightarrow{H} & B \end{array}$$

where $H(e,t) = \gamma(t)$. It follows from the diagram that $\tilde{H}_1$ has image contained in E_y. A further application of the lifting property shows that the map $E_x \to E_y$ we have constructed depends only on the homotopy class of the path γ. By using the path γ^{-1} we obtain in the same way a map $E_y \to E_x$, which can be seen to be a homotopy inverse to the map $E_x \to E_y$ defined above. For full details, see [27, Theorem 6.12]. □

As a corollary, we obtain:

Corollary 1.15. *For any $b \in B$, there is an action of $\pi_1(B,b)$ on the homology and the cohomology of the fiber E_b.*

Proof. Indeed, by the previous Proposition, we have a well-defined morphism of groups from $\pi_1(B,b)$ to the group of self-homotopy equivalences of E_b. Each self-homotopy equivalence of E_b induces an isomorphism of $H_*(E_b)$ and $H^*(E_b)$. Then the functoriality properties of the (co)homology functor imply that we have an action of $\pi_1(B,b)$ on $H_*(E_b)$ and $H^*(E_b)$. □

We next introduce the abstract notion of spectral sequence.

Definition 1.16. *Let R be a commutative ring. A (cohomological) spectral sequence is a pair $(\{E_r^{p,q}\}_{r\geq r_0}, \{d_r\}_{r\geq r_0})$ where $p,q,r,r_0 \in \mathbb{Z}$ and:*

1. *$E_r^{p,q}$ are R-modules ($\{E_r^{p,q}\}_{p,q\in\mathbb{Z}}$ is called the r-th page of the spectral sequence),*
2. *$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ are morphisms of R-modules satisfying $d_r \circ d_r = 0$,*
3. *$E_{r+1} \simeq H^*(E_r)$.*

Usually each page of a spectral sequence is represented by a bidimensional diagram, as follows:

$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$E_r^{0,3}$	$E_r^{1,3}$	$E_r^{2,3}$	$E_r^{3,3}$	$E_r^{4,3}$	$\dots$
$E_r^{0,2}$	$E_r^{1,2}$	$E_r^{2,2}$	$E_r^{3,2}$	$E_r^{4,2}$	$\dots$
$E_r^{0,1}$	$E_r^{1,1}$	$E_r^{2,1}$	$E_r^{3,1}$	$E_r^{4,1}$	$\dots$
$E_r^{0,0}$	$E_r^{1,0}$	$E_r^{2,0}$	$E_r^{3,0}$	$E_r^{4,0}$	$\dots$

Definition 1.17. *Let $E_r^{p,q}$ be a spectral sequence, and suppose that for every $p,q \in \mathbb{Z}$ there is a number $r(p,q) \in \mathbb{Z}$ such that for all $r \geq r(p,q)$, $d_r^{p,q}$ and $d_r^{p-r,q+r-1}$ are zero. In this situation, we write $E_\infty^{p,q} = E_{r(p,q)}^{p,q}$ and say that $E_\infty^{p,q}$ is the limit (or the final page) of the spectral sequence*

In particular, suppose that for some r, $d_r^{p,q} = 0$ for all $p,q \in \mathbb{Z}$. In this situation, we have:

$$E_r^{p,q} \simeq E_{r+1}^{p,q} \simeq E_{r+2}^{p,q} \simeq \dots$$

and hence $E_r^{p,q} \simeq E_\infty^{p,q}$. When this happens, we say that the spectral sequence degenerates at page r.

We now define the notion of convergence of a spectral sequence. Recall that a filtration $F_\bullet A_\bullet$ of a $\mathbb{N}$-graded R-module $A_\bullet$ is a sequence of submodules

$$0 = F_0 A_\bullet \subseteq F_1 A_\bullet \subseteq \cdots \subseteq F_n A_\bullet \subseteq \ldots$$

such that $\varinjlim F_\bullet A_\bullet = A_\bullet$. The associated graded complex of the filtration $\{G_p A_{p+q}\}_{p,q}$ is defined by:

$$G_p A_{p+q} = F_p A_{p+q} / F_{p-1} A_{p+q}.$$

Definition 1.18. *Let $E_r^{p,q}$ be a spectral sequence. We say that it converges to a graded R-module $H_\bullet$, and we denote it by*

$$E_r^{p,q} \Rightarrow H_\bullet$$

if the spectral sequence has a limit $E_\infty^{p,q}$ and there is a filtration in $H^\bullet$ such that

$$G_p H^{p+q} = E_\infty^{p,q}.$$

We can also define morphisms of spectral sequences in the natural way.

Definition 1.19. *Let $E_r^{p,q}$, $E_r'^{p,q}$ be two spectral sequences. A morphism $f : E_r^{p,q} \to E_r'^{p,q}$ is given by a sequence of R-linear maps $f_r^{p,q} : E_r^{p,q} \to E_r^{p,q}$ such that $f_r \circ d_r' = d_r \circ f_r$.*

The final ingredient we need before stating the Serre spectral sequence in full generality is the notion of (co)homology with local coefficients.

Let X be a path-connected and locally path-connected topological space admitting a universal covering $\tilde{X}$, and fix a basepoint $x \in X$. We write $\pi := \pi_1(X, x)$. Let A be a $\mathbb{Z}\pi$-module, that is, an abelian group endowed with a compatible action of π. π acts on $\tilde{X}$ on the right via deck transformations. Therefore, we may define the singular chain complex with values in A:

$$S_*(X; A) = S_*(\tilde{X}) \otimes_{\mathbb{Z}\pi} A,$$

where $S_*(\tilde{X})$ is the usual singular chain complex of $\tilde{X}$. The singular homology of X with local coefficients A is just the homology of the chain complex $S_*(X; A)$. Similarly, the singular cohomology of X with local coefficients A is the cohomology of the cochain complex

$$S^*(X; A) = \mathrm{Hom}_{\mathbb{Z}\pi}(S_*(X; A), A).$$

We are now in position to state the Serre spectral sequence.

Proposition 1.20. *Let $p : E \to B$ be a Serre fibration with B connected. Let $b \in B$ and let $F = p^{-1}(b)$. Then, there exists a spectral sequence $(\{E_r^{p,q}\}_{r \geq 2}, \{d_r\}_{r \geq 2})$ with second page given by:*

$$E_2^{p,q} = H^p(B; \mathcal{H}^q(F; R)),$$

where by $\mathcal{H}^q(F; R)$ we denote the cohomology group $H^q(F; R)$ endowed with the $\pi_1(B, b)$-module structure defined in Corollary 1.15, and the cohomology is with local coefficients.

This spectral sequence converges to the cohomology of E. That is,

$$H^p(B; \mathcal{H}^q(F; \mathbb{Z})) \Rightarrow H^{p+q}(E; \mathbb{Z})$$

Note that if the action of $\pi_1(B, b)$ on $H^*(F; \mathbb{Z})$ is trivial (for instance, if B is simply connected), we can use the usual universal coefficients theorem to compute $H^p(B; H^q(F; \mathbb{Z}))$. Moreover, if k is a field, in this situation we have:

$$H^p(B; H^q(F; k)) \simeq H^p(B; k) \otimes H^q(F; k).$$

The Serre spectral sequence is functorial in the following sense:

Proposition 1.21. *Let $p : E \to B$ and $p' : E' \to B'$ be two fibrations. Let $E_r^{p,q}$ and $E_r'^{p,q}$ the corresponding Serre spectral sequences. Let f, g be a pair of maps making the following diagram commute*

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{g} & B' \end{array}$$

and note that f gives by restriction a mapping of fibers $h : E_b \to E'_{f(b)}$.

Then, f induces a morphism of spectral sequences $\tilde{f} : E_r'^{p,q} \to E_r^{p,q}$ such that on the second page the map $\tilde{f}_2 : H^*(B'; \mathcal{H}^*(F')) \to H^*(B; \mathcal{H}^*(F))$ is induced by the maps $h^* : H^*(B') \to H^*(B)$ and $g^* : H^*(B') \to H^*(B)$. In particular, if the coefficients are fields, $\tilde{f}_2 = g^* \otimes h^*$.

Moreover, this construction is functorial, in the sense that $(g \circ f)_* = f^* \circ g^*$ and $\mathrm{id}^* = \mathrm{id}$.

We end this brief introduction to the Serre spectral sequence by stating a very useful property, the multiplicativity of the Serre spectral sequence.

Proposition 1.22. *Let $p : E \to B$ be a Serre fibration with B connected. Let $b \in B$ and put $F = p^{-1}(b)$. Let $(\{E_r^{p,q}\}_{r\geq 2}, \{d_r\}_{r\geq 2})$ be the associated Serre spectral sequence with coefficients in a ring R. Then:*

1. *For each $r \geq 2$ and every integers p, q, p', q' there exist maps*

$$E_r^{p,q} \otimes E_r^{p',q'} \to E_r^{p+p',q+q'}$$

such that for $r = 2$ coincide with $(-1)^{qp'}$ times the cup product:

$$\begin{aligned} H^p(B; \mathcal{H}^q(F; R)) \otimes H^{p'}(B; \mathcal{H}^{q'}(F; R)) &\to H^{p+p'}(B; \mathcal{H}^{q+q'}(F; R)) \\ \alpha \otimes \beta &\mapsto (-1)^{p'q} \alpha \cup \beta, \end{aligned}$$

for $\alpha \in H^p(B; \mathcal{H}^q(F; R))$ and $\beta \in H^{p'}(B; \mathcal{H}^{q'}(F; R))$.

2. *For each $r \geq 2$, the differentials $d_r^{p,q}$ are graded derivations, that is,*

$$d_r(xy) = d_r(x)y + (-1)^{p+q} x d_r(y),$$

for all $x \in E_r^{p,q}$ and $y \in E_r^{p',q'}$.

1.4 Classifying spaces and group cohomology

Let G be a group. Under mild conditions on G, which will be always satisfied in the case of finite groups, we can define a topological space canonically associated (up to homotopy equivalence) to the group G. This topological space is called the classifying space of G and will be denoted by BG.

This construction allows us to obtain information about the group G by using algebraic topology techniques to study the space BG. In particular, the homology and cohomology of BG are important invariants associated to the group G, that can also be defined in purely algebraic terms, hence giving us a link between the topological study of BG and the algebraic study of G.

There are several ways to introduce the space BG. We start with the most useful definition for our purposes, since it will allow us to generalize it later to the setting of group actions on a topological space.

Definition 1.23. *Let G be a topological group. A universal G-bundle is a principal G-bundle $EG \to BG$ such that EG is a contractible topological space. We call the base space $BG := EG/G$ a classifying space of G.*

We will see later that we can construct a classifying space BG for any discrete group G (in particular, for any finite group).

The following proposition justifies the name 'classifying space'.

Proposition 1.24. *For any paracompact topological space X, there is a bijection between $[X, BG]$ and $Prin_G(X)$, the set of isomorphism classes of principal G-bundles over X.*

The bijection is given as follows. To every homotopy class of maps $[f] \in [X, BG]$, we associate the principal G-bundle $f^*(EG) \to X$. This bundle is independent of the representative of the homotopy class that we have chosen.

From this proposition, we see that indeed BG (and the bundle $EG \to BG$) is well-defined up to homotopy equivalence. Indeed, suppose $(EG)' \to (BG)'$ is another universal bundle for G. Then, since $[(BG)', BG] \simeq Prin_G((BG)')$, there exists some $f : BG \to (BG)'$ such that $f^*((EG)') = EG$. Similarly, there exists some $g : (BG)' \to BG$ such that $g^*(EG) = (EG)'$. Therefore, $(g \circ f)^*(EG) = EG$. Using now the bijection $[BG, BG] \simeq Prin_G(BG)$, we see that $[g \circ f] = [id_{BG}]$. Analogously, we have $[f \circ g] = [id_{(BG)'}]$. Therefore, f and g are homotopy equivalences, and they induce bundle isomorphisms $EG \simeq (EG)'$.

Recall that an Eilenberg-MacLane space $K(G, n)$, where $n \geq 1$ and G is a group (abelian if $n > 1$), is a topological space X with $\pi_k(X) = 0$ if $k \neq n$ and $\pi_n(X) = G$.

Proposition 1.25. *Let G be a finite group. Then, any $K(G, 1)$ space is a classifying space for G. In particular, $K(G, 1) \simeq BG$.*

Proof. Since G is finite, $EG \to BG$ is a covering map and EG is the universal covering of BG. Therefore, $\pi_1(BG) = G$ and $\pi_k(BG) = 0$ for all $k > 1$ because BG has a contractible covering. Conversely, if X is a $K(G, 1)$ space, its universal covering $\tilde{X}$ is a principal G-bundle that must have all homotopy groups trivial, and hence must be contractible. Moreover, the action of $G = \pi_1(X)$ on $\tilde{X}$ by deck transformations is free and transitive on the fibers. Therefore, X is a classifying space for G. □

Using this characterization, we can prove that BG exists for every finite group G by proving that there exists a $K(G, 1)$ space. Indeed, the following is a well-known theorem in algebraic topology (see for instance Section 1B of [24] for information about the spaces $K(G, 1)$, and specifically Example 1B.7 for a proof of the following proposition).

Proposition 1.26. *There exists a classifying space BG for any finite group G.*

Let $f : G \to H$ be a morphism of groups. Then, there is an induced map $\tilde{f}$: $K(G, 1) \to K(H, 1)$ (see [24, Proposition 1B.9]). Using this fact, one can prove:

Proposition 1.27. *There exists a functor $B : Grp \to Top$ such that BG is a classifying space for G.*

Definition 1.28. *Let G be a group and let A be a G-module. We define the group cohomology (resp. the group homology) of G with coefficients A as $H^*(BG; A)$ (resp. $H_*(BG; A)$).*

1.5 Equivariant cohomology

In the same way as in the previous section we associated to each group G a topological space BG that captures some information about the group, in this section we explain how to associate canonically (up to homotopy equivalence) a topological space to a group action on a topological space. The cohomology of this space is thus an invariant associated to a group action, and it is called the equivariant cohomology.

We will only need some basic facts about equivariant cohomology, and we will restrict ourselves to finite groups. However, if the reader is interested in equivariant cohomology for smooth actions of compact Lie groups on manifolds, and particularly in models via differential forms (analogous to the de Rham cohomology model for singular cohomology with real coefficients) he or she may consult [21] or [33].

Definition 1.29. *Let X, Y be two topological spaces. Let G be a group and suppose that G acts on the right on X and on the left on Y. We define an equivalence relation $\sim$ on $X \times Y$ by $(x, y) \sim (xg, gy)$. We denote $X \times_G Y := (X \times Y)/\sim$.*

Definition 1.30. *Let X be a topological space and assume that G acts on the left on X. Let $p : EG \to BG$ be the universal bundle. Then, the Borel construction (also called homotopy quotient) is the topological space $EG \times_G X$, which is sometimes also denoted X_G. The equivariant cohomology of X with action G with coefficients in an abelian group A is the cohomology of the Borel construction:*

$$H^*_G(X; A) = H^*(EG \times_G X; A)$$

where H^* on the right side denotes singular cohomology.

Observe that the definition of equivariant cohomology generalizes both singular cohomology and cohomology of groups.

Indeed, taking G as the trivial group, we observe that $EG = \{*\}$ and $EG \times_G X = \{*\} \times X$. Hence, $H^*_G(X) = H^*(\{*\} \times X) = H^*(X)$. On the other hand, taking $X = \{*\}$, and the trivial action of G on the one-point space, we have that $EG \times_G X = EG/G = BG$. Hence, $H^*_G(X) = H^*(EG \times_G X) = H^*(BG) = H^*(G)$.

The following two results cover two extreme cases where we can easily compute the equivariant cohomology of a group G on a topological space X: the case of trivial actions and the case of free actions. This two propositions will be used extensively without further mention.

Proposition 1.31. *Let X be a topological space and G a group acting trivially on X. Then,*

$$EG \times_G X \cong BG \times X.$$

In particular, if k is a field,

$$H^*_G(X) \simeq H^*(BG) \otimes H^*(X)$$

Proof. If G acts trivially on X, we have $EG \times_G X = EG/G \times X \cong BG \times X$.

For the second part, since k is a field, we can use Kunneth's theorem to obtain $H^*_G(X; k) \simeq H^*(BG \times X; k) \simeq H^*(BG; k) \otimes H^*(X; k)$. □

Proposition 1.32. *Let X be a topological space and G a group acting freely on X. Then,*

$$EG \times_G X \simeq X/G.$$

In particular, for any abelian group A,

$$H^*_G(X; A) \simeq H^*(X/G; A).$$

Proof. Since G acts freely on X, the projection $EG \times_G X \to X/G$ is a fibration with fiber EG. Since EG is contractible, then $EG \times_G X \simeq X/G$. The second part of the statement follows immediately from the definition of equivariant cohomology. □

Next we study the functorial properties of equivariant cohomology.

Proposition 1.33. *Let G be a group. Let X, Y be two topological spaces, and suppose that G acts on them. Suppose $f : X \to Y$ is a G-equivariant map. Then, there is an induced map:*

$$f^* : H^*_G(Y) \to H^*_G(X).$$

Proof. Define a map $\hat{f} : EG \times_G X \to EG \times_G Y$ by $\hat{f}([e, x]) = [e, f(x)]$. The fact that f is G-equivariant assures us that this map is well-defined. Therefore, we have induced maps on the cohomology:

$$f^* = \hat{f}^* : H^*(EG \times_G Y) \to H^*(EG \times_G X)$$

as wanted. □

Recall that $H_G^*(\{*\}) \simeq H^*(BG)$. Since we always have a G-equivariant map $f : X \to pt$, the previous proposition gives us a map $f^* : H_G^*(pt) \to H_G^*(X)$. Define $\alpha \cdot \beta := f^*(\alpha) \cup \beta$ for $\alpha \in H_G^*(pt)$ and $\beta \in H_G^*(X)$. We have proved:

Proposition 1.34. *Let G be a group acting on a topological space X. Then, the equivariant cohomology $H_G^*(X)$ is a ring with the cup product and has also the structure of a $H_G^*(pt)$-module.*

To conclude this section, we show how to use the Serre spectral sequence in order to compute $H_G^*(X)$ from $H^*(X)$ and $H^*(BG)$.

Proposition 1.35. *If G is a finite group the map:*

$$\begin{aligned} \pi : EG \times_G X &\to BG \\ [e, x] &\mapsto p(e) \end{aligned}$$

where $p : EG \to BG$ is the universal G-bundle, is a fiber bundle with fiber X.

Proof. For G a finite group, $p : EG \to BG$ is a principal G-bundle (since in this case EG is just the universal covering of BG). Choose a trivialization

$$\Phi : p^{-1}(U) \to U \times G$$

over an open set U in BG.

For $b \in U$, let $s = \Phi^{-1}(b, 1)$. Then s is a continuous section of p over 1. Using the fact that the action of G on $p : EG \to BG$ is transitive on each fiber and free, it is easy to see that for every $b \in BG$, any element in $\pi^{-1}(b)$ has a unique representative of the form $(s(b), x)$.

Therefore, the following map is a homeomorphism.

$$\begin{aligned} \tilde{\Phi} : \pi^{-1}(U) &\to U \times X \\ [s(b), x] &\mapsto (b, x) \end{aligned}$$

This shows that $\pi : EG \times_G X \to BG$ is a fiber bundle with fiber homeomorphic to X. □

When we have introduced fibrations, we have seen that there is an induced action of the fundamental group of the base on the cohomology of the fiber. Applying that to the fiber bundle $\pi : EG \times_G X \to BG$, we have an action of $\pi_1(BG) \simeq G$ on $H^*(X; \mathbb{Z})$.

There is also a natural action of G on $H^*(X; \mathbb{Z})$ induced by the action of G on X. This action is defined by:

$$g \cdot \alpha = g^*(\alpha)$$

for $g \in G$ and $\alpha \in H^*(X; \mathbb{Z})$.

We now prove that these two actions coincide.

Proposition 1.36. *Let G be a finite group acting on a topological space X. The action of G on $H^*(X)$ defined by means of the fibration $\pi : EG \times_G X \to BG$ and the natural induced action of G on $H^*(X)$ coincide.*

Proof. Let $b_0 \in BG$. Let $[\gamma] \in \pi_1(BG, b_0)$ represent $g \in G$ under the isomorphism $\pi_1(BG, b_0) \simeq G$. Recall that the action of $\pi_1(BG, b_0) \simeq G$ on $H^*(X)$ is induced by a self-homotopy equivalence of X, which is defined via lifting over a path. We are going to compute the map it induces on cohomology. The action of $[\gamma]$ on $e \in p^{-1}(b_0)$ is given by the endpoint $\tilde{\gamma}(1)$ of a lift $\tilde{\gamma}$ of γ starting at e. However, we have $\tilde{\gamma}(1) = e \cdot g$, by the definition of the isomorphism $\pi_1(B, b_0) \simeq G$. Hence, the map $p^{-1}(b_0) \to p^{-1}(b_0)$ (whose homotopy class is independent of the chosen lift) is given by right multiplication by g.

Using this, it is easy to see that in the bundle $EG \times_G X \to BG$, a lift of γ starting at $[e, x]$ ends at $[e \cdot g, x] = [e, g \cdot x]$. Therefore, under the identification $\pi^{-1}(b_0) \cong X$, the map induced by $[\gamma]$ corresponds to the action of g on X. Hence, the map induced by the fibration on $H^*(X; \mathbb{Z})$ coincides with the one induced by the action of G on X. □

Since $\pi : EG \times_G X \to BG$ is a fiber bundle with fiber X, it is in particular a fibration, so putting together the last two results, we get:

Corollary 1.37. *Let A be an abelian group. Let G be a finite group acting on a topological space X. Then, there exists a spectral sequence $(\{E_r^{p,q}\}_{r\geq 2}, \{d_r\}_{r\geq 2})$ with second page given by:*

$$E_2^{p,q} = H^p(BG; \mathcal{H}^q(X; A)),$$

where $\mathcal{H}^q(X; A)$ is the homology group $H^q(X; A)$ endowed with the G-module structure given by the natural action on the cohomology induced by the action of G on X, and the cohomology is with local coefficients.

This spectral sequence converges to the equivariant cohomology of X. That is,

$$H^p(BG; H^q(X; A)) \Rightarrow H_G^{p+q}(X; A)$$

1.6 Computation of equivariant cohomology

In this section we introduce the Gysin exact sequence and use it to compute the cohomology of some classifying spaces we will need later. A good reference for this material is Section III.2 in [74].

Proposition 1.38. *Let $E \to X$ be an orientable $(k-1)$-sphere bundle $\pi : E \to X$ (i.e. its fibre is homeomorphic to S^{k-1}). Then, there is the following exact sequence, called the Gysin sequence:*

$$\cdots \to H^i(X) \overset{\cup e}{\to} H^{i+k}(X) \overset{\pi^*}{\to} H^{i+k}(E) \to H^{i+1}(X) \to \dots$$

where $e \in H^k(X)$ is called the Euler class and all maps are morphisms of $H^*(X)$-modules.

Proof. See Theorem 13.2 of [9]. □

We will use this result as our main tool for the computation of group cohomologies (including its ring structure).

We start with the computation of $H^*(BS^1)$. Recall that the infinite-dimensional sphere S^∞ is defined as the colimit of the finite-dimensional spheres S^n with respect to the natural inclusions on the equator $S^n \to S^{n+1}$. We can therefore identify points of S^∞ with sequences of real numbers $(x_1, x_2 \dots)$ that are eventually zero (i.e., there exists some n_0 such that $x_n = 0$ for all $n > n_0$) and satisfy $\sum_i x_i^2 = 1$. We claim that S^∞ is contractible. Indeed, if we define the shift map

$$\sigma : S^\infty \to S^\infty$$

by the formula $\sigma(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$, then a homotopy between the identity and the constant map c at $(1, 0, 0, \dots)$ is given by

$$H(x,t) = \frac{tc(x) + (1-t)x + t(1-t)\sigma(x)}{||tc(x) + (1-t)x + t(1-t)\sigma(x)||}.$$

Since the sequence of odd-dimensional spheres is cofinal, we can also define S^∞ as the colimit of the odd-dimensional spheres S^{2n-1} with respect to the inclusions $S^{2n-1} \to S^{2n+1}$. Thinking of S^{2n-1} as being embedded in $\mathbb{C}^n$, we immediately see that there is a free action of S^1 on S^∞ given, for $e^{i\theta} \in S^1$ and $(z_1, \dots, z_n) \in S^{2n-1} \subseteq S^\infty$ by:

$$e^{i\theta} \cdot (z_1, \dots, z_n) = (e^{i\theta} z_1, \dots, e^{i\theta} z_n).$$

The quotient of S^∞ by this action is clearly $\mathbb{C}\mathrm{P}^\infty$, which is defined as the colimit of the complex projective spaces $\mathbb{C}\mathrm{P}^n$ with respect to the inclusions $\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^{n+1}$ given by $[z_1 : \dots : z_n] \mapsto [z_1 : \dots : z_n : 0]$. Therefore, $\mathbb{C}\mathrm{P}^\infty$ is a classifying space for S^1. We now compute its cohomology using the Gysin sequence.

Proposition 1.39. *There is an isomorphism of rings*

$$H^*(\mathbb{C}\mathrm{P}^\infty) \simeq \mathbb{Z}[[\tau]],$$

where τ is a generator of $H^2(\mathbb{C}\mathrm{P}^\infty)$.

Proof. We have a fiber bundle (the infinite-dimensional Hopf fibration)

$$\pi : S^\infty \to \mathbb{C}\mathrm{P}^\infty,$$

defined by $\pi(z_1, \dots, z_n) = [z_1 : \dots : z_n]$, and with fiber S^1. By Proposition 1.38, we have the following exact sequence of $H^*(\mathbb{C}\mathrm{P}^\infty)$-modules:

$$\dots H^{i+1}(S^\infty) \to H^i(\mathbb{C}\mathrm{P}^\infty) \overset{\cup\tau}{\to} H^{i+2}(\mathbb{C}\mathrm{P}^\infty) \to H^{i+2}(S^\infty) \to \dots,$$

where we denote by τ the Euler class of the bundle $\pi : S^\infty \to \mathbb{C}\mathrm{P}^\infty$. Since S^∞ is contractible, the maps

$$H^i(\mathbb{C}\mathrm{P}^\infty) \overset{\cup\tau}{\to} H^{i+2}(\mathbb{C}\mathrm{P}^\infty)$$

are isomorphisms for all $i \geq 0$. Since $\mathbb{CP}^\infty$ is connected, $H^0(\mathbb{CP}^\infty) \simeq \mathbb{Z}$. Hence,

$$H^i(\mathbb{CP}^\infty) \simeq \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

Moreover, we deduce immediately that τ is a generator of $H^2(\mathbb{CP}^\infty)$, and that the ring structure is the one asserted in the statement. □

Using this result, we can now compute the integer cohomology ring of any finite cyclic group.

Proposition 1.40. *There is an isomorphism of rings*

$$H^*(B\,\mathbb{Z}/n) \simeq (\mathbb{Z}/n)[[\tau]]/n\tau,$$

where τ is a generator of $H^2(B\,\mathbb{Z}/n)$ (and is a torsion element of order n).

Proof. Identify $\mathbb{Z}/n$ with the group $\mu_n \subset \mathbb{C}$ of the n-th roots of unity. Clearly, μ_n, being a subgroup of S^1, acts freely on S^∞. Therefore, a model for $B\,\mathbb{Z}/n$ is the infinite lens space S^∞/μ_n. The obvious map

$$\pi : S^\infty/\mu_n \to S^\infty/S^1 \simeq \mathbb{CP}^\infty$$

is a fiber bundle with fiber $S^1/\mu_n \simeq S^1$.

Denote by γ the complex line bundle $ES^1 \times_{S^1} \mathbb{C} \to BS^1$ associated to the universal S^1-principal bundle. Denote by γ^n the $n-th$ tensor power of γ, and by $S(\gamma^n) \to BS^1$ its associated sphere bundle. We claim that the bundle $S^\infty/\mu_n \to BS^1$ is isomorphic to $S(\gamma^n) \to BS^1$. In order to prove it, note that the total space of γ^n is homeomorphic to $ES^1 \times_{S^1} V^n$, where V^n is the representation $S^1 \times \mathbb{C} \to \mathbb{C}$ given by $(e^{i\theta}, z) \mapsto e^{in\theta}z$. Therefore, $S(\gamma^n) = ES^1 \times_{S^1} S(V^n)$. The map $S^1 \to S(V^n)$ given by $z \mapsto z^n$ induces then a homeomorphism $S^1/\mu_n \to S(V^n)$. Hence, we obtain a homeomorphism

$$ES^1 \times_{S^1} S^1/\mu_n \to ES^1 \times_{S^1} S(V^n)$$

over BS^1. This finishes the proof of the claim.

We can now apply the Gysin sequence to the bundle $S(\gamma^n) \to BS^1$. Since the Euler class of $\gamma \to BS^1$ is $\tau \in H^2(BS^1)$, we have that the Euler class of γ^n is $e(\gamma^n) = n\tau \in H^2(BS^1)$. From the Gysin exact sequence we obtain:

$$\cdots \to H^{i-1}(S(\gamma^n)) \to H^{i-2}(\mathbb{CP}^\infty) \stackrel{\cup n\tau}{\to} H^i(S(\gamma^n)) \to \ldots.$$

Combining this with our previous computation of $H^*(\mathbb{CP}^\infty)$, we deduce that $H^{2i+1}(B\,\mathbb{Z}/n)$ vanishes, while $H^{2i}(B\,\mathbb{Z}/n) \simeq \mathbb{Z}/n$ for $i > 0$ and $H^0(B\,\mathbb{Z}/n) \simeq \mathbb{Z}$. To compute the cup product, note that the Gysin exact sequence also imply that the maps

$$q^* : H^{2i}(BS^1) \to H^{2i}(S(\gamma^n))$$

are surjective. Since $q^* : H^*(BS^1) \to H^*(S(\gamma^n))$ preserves the cup product, this fact combined with the knowledge of the ring structure of $\mathbb{CP}^\infty$ implies that the ring structure of $H^*(B\,\mathbb{Z}/n)$ is the one given in the statement. This completes the proof of the proposition. □

Finally, we give, for any prime p, the cohomology ring of $\mathbb{Z}/p$ with coefficients in $\mathbb{Z}/p$.

Proposition 1.41. *For $p \neq 2$, there is an isomorphism of rings*

$$H^*(B\mathbb{Z}/p;\mathbb{Z}/p) \simeq \mathbb{Z}/p[t] \otimes_{\mathbb{Z}/p} \Lambda[s],$$

where t generates $H^2(B\mathbb{Z}/p;\mathbb{Z}/p)$ and s generates $H^1(B\mathbb{Z}/p;\mathbb{Z}/p)$, and $\Lambda[s]$ denotes the exterior algebra on the variable s. For $p = 2$, there is an isomorphism of rings

$$H^*(B\mathbb{Z}/2;\mathbb{Z}/2) \simeq \mathbb{Z}/2[s],$$

where s is a generator of $H^1(B\mathbb{Z}/p;\mathbb{Z}/p)$.

For the proof, we refer the reader to [74, Theorem III.2.5].

1.7 Computing $H^*((\mathbb{Z}/p^r)^d;\mathbb{Z}/n)$

The goal of this section is to provide a proof of the following theorem, which will be used in the study of the Jordan property for groups of diffeomorphisms of 4-manifolds.

Theorem 1.42. *Let a, b be natural numbers and let $c = \min\{a, b\}$. For any natural number d, any nonnegative integer k and any prime p we have*

$$H^k((\mathbb{Z}/p^a)^d;\mathbb{Z}/p^b) \simeq (\mathbb{Z}/p^c)^{\binom{k+d-1}{d-1}},$$

where we consider on the coefficient group $\mathbb{Z}/p^b$ the trivial $(\mathbb{Z}/p^a)^d$-module structure.

If X and Y are topological spaces such that $H^k(X)$ and $H^k(Y)$ are finitely generated abelian groups for every k, then Künneth's formula gives

$$H^k(X \times Y) \simeq \bigoplus_{p+q=k} H^p(X) \otimes H^q(Y) \oplus \bigoplus_{p'+q'=k+1} \operatorname{Tor}(H^{p'}(X), H^{q'}(Y)) \tag{1.2}$$

(see e.g. [15, Chap. VII, Prop. 7.6]). The universal coefficient theorem gives isomorphisms

$$H^k(X) \simeq \operatorname{Hom}(H_k(X),\mathbb{Z}) \oplus \operatorname{Ext}(H_{k-1}(X),\mathbb{Z}). \tag{1.3}$$

and

$$H^k(X;\mathbb{Z}/p^b) \simeq \operatorname{Hom}(H_k(X),\mathbb{Z}/p^b) \oplus \operatorname{Ext}(H_{k-1}(X),\mathbb{Z}/p^b). \tag{1.4}$$

Let a, b be positive integers and let $c = \min\{a, b\}$. Fix a prime p and define for convenience

$$G_a = \mathbb{Z}/p^a, \qquad G_b = \mathbb{Z}/p^b, \qquad G_c = \mathbb{Z}/p^c.$$

There are non canonical isomorphisms

$$\operatorname{Tor}(G_a, G_a) \simeq G_a, \qquad \operatorname{Tor}(\mathbb{Z}, G_a) = \operatorname{Tor}(G_a,\mathbb{Z}) = 0, \tag{1.5}$$

and

$$\operatorname{Ext}(G_a, G_b) \simeq G_c, \qquad \operatorname{Ext}(G_a,\mathbb{Z}) \simeq G_a, \qquad \operatorname{Ext}(\mathbb{Z}, G_b) = 0. \tag{1.6}$$

Lemma 1.43. *There exists a function* $e : \mathbb{Z}_{\geq 0} \times \mathbb{N} \to \mathbb{Z}_{\geq 0}$ *such that*

$$H^k(G_a^d; G_b) \simeq G_c^{e(k,d)} \tag{1.7}$$

for every $(k,d) \in \mathbb{Z}_{\geq 0} \times \mathbb{N}$ *(by convention* $G_c^0 = 0$*).*

The crucial fact here is that $e(k,d)$ is independent of p and a, b, c.

Proof. We first prove that there exists a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}_{\geq 0}$ such that

$$H^k(G_a^d) \simeq G_a^{f(k,d)} \quad \text{for every } k, d \in \mathbb{N}, \text{ and} \qquad H^0(G_a^d) \simeq \mathbb{Z} \tag{1.8}$$

(again we take the convention that $G_a^0 = 0$). We prove the existence of $f(k,d)$ using induction on d. First note that setting

$$f(k,1) = \begin{cases} 1 & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd,} \end{cases}$$

formulas (1.8) hold for $d = 1$. For the inductive step we note that if BG denotes the classifying space of a group G we have

$$BG_a^d \simeq BG_a^{d-1} \times BG_a,$$

so we can relate the cohomology of G_a^d (which coincides with the singular cohomology of BG_a^d) to that of G_a^{d-1} and G_a using (1.2). To be specific, using (1.5) we have, for every $k \in \mathbb{N}$ and every $d \geq 2$,

$$f(k,d) = \sum_{0 \leq 2l \leq k} f(k-2l, d-1) + \sum_{0 < 2l < k+1} f(k+1-2l, d-1),$$

where l takes integer values. The first summation comes from the terms with $\otimes$ in Künneth's formula (more concretely, the summand for each value of l corresponds to $H^{k-2l}(G_a^{d-1}) \otimes H^{2l}(G_a) \simeq H^{k-2l}(G_a^{d-1}) \simeq G_a^{f(k-2l,d-1)}$) and the second summation comes from the terms with Tor (more concretely, the summand for each l corresponds to

$$\mathrm{Tor}(H^{k+1-2l}(G_a^{d-1}), H^{2l}(G_a)) \simeq H^{k+1-2l}(G_a^{d-1}) \simeq f(k+1-2l, d-1);$$

we avoid the extreme values $2l = 0$ and $2l = k+1$ because $\mathrm{Tor}(G_a, \mathbb{Z}) = \mathrm{Tor}(\mathbb{Z}, G_a) = 0$). This proves the existence of a function f satisfying (1.8).

Now, using the universal coefficients theorem (1.3), the fact that the homology of a finite p-group is a finite p-group in each degree > 0, and (1.6), we deduce that

$$H_k(G_a^d) \simeq G_a^{f(k+1,d)} \quad \text{for every } k, d \in \mathbb{N}, \text{ and} \qquad H_0(G_a^d) \simeq \mathbb{Z}.$$

Combining this formulas with (1.4) it follows that

$$H^1(G_a^d; G_b) \simeq G_c^{f(2,d)}$$

and that for every $k \geq 2$ we have

$$H^k(G_a^d; G_b) \simeq G_c^{f(k+1,d)} \oplus G_c^{f(k,d)} \simeq G_c^{f(k+1,d)+f(k,d)}.$$

Thus setting

$$e(1,d) := f(2,d), \qquad e(k,d) := f(k+1,d) + f(k,d) \text{ for every } k \geq 2$$

we obtain a function $e : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}_{\geq 0}$ which satisfies (1.7). □

In view of the lemma, to compute the function e it suffices to consider the case $a = b = c = 1$, i.e., to compute

$$H^*((\mathbb{Z}/p)^d; \mathbb{Z}/p).$$

But this is much easier than the general case because, $\mathbb{Z}/p$ being a field, we may apply Künneth's formula for fields, which does not contain Tor terms:

$$H^k(X \times Y; \mathbb{Z}/p) \simeq \bigoplus_{p+q=k} H^p(X; \mathbb{Z}/p) \otimes H^q(Y; \mathbb{Z}/p)$$

(again, under finiteness assumptions for $H^*(X; \mathbb{Z}/p)$ and $H^*(Y; \mathbb{Z}/p)$ on each degree). This formula, together with the standard computation

$$H^k(\mathbb{Z}/p; \mathbb{Z}/p) \simeq \mathbb{Z}/p \qquad \text{for every } k \geq 0 \tag{1.9}$$

implies the following recursion formula for $d \geq 2$, which is much easier than the previous ones:

$$e(k,d) = e(0,d-1) + e(1,d-1) + \cdots + e(k,d-1). \tag{1.10}$$

It is now elementary to prove, e.g. using induction on d (with (1.9) at the initial step and (1.10) at the induction step), that

$$e(k,d) = \binom{k+d-1}{d-1}.$$

The proof of the theorem is thus complete.

1.8 Applications to actions on closed manifolds

In this section we use the tools developed in this chapter in order to prove two theorems that we will use repeatedly in this thesis. The first one imposes constraints on the cohomology of the fixed-point submanifold of a smooth action, while the second one gives a bound on the number of generators of a finite abelian group acting effectively on a closed manifold.

We start with some lemmas that will be useful in order to deal with the Serre spectral sequence with local coefficients (that is, with a non-trivial action of the fundamental group of the base in the fiber).

Lemma 1.44. *Let p be a prime. Let V be a finite-dimensional non-zero $\mathbb{Z}/p$-vector space, and let $G = \mathbb{Z}/p$ act linearly on V. Then, V^G is a non-zero subspace of V.*

Proof. Let $n = \dim V$. Then, $|V| = p^n$. Since the action is linear, V^G is a linear subspace of V.

We have:

$$|V| = |V^G| + \sum_i |G \cdot v_i|,$$

where v_i are representatives for the orbits of the action which have more than one element. Since every orbit has cardinality either 1 or p, p divides

$$\sum_i |G \cdot v_i|.$$

Furthermore, $|V| = p^{\dim V}$ is divisible by p. Therefore, p divides $|V^G|$. Since the action is linear, $0 \in V^G$, so V^G is not empty. Hence, V^G is a nontrivial subspace of V. □

Lemma 1.45. *Let p be a prime. Let V be a finite-dimensional $\mathbb{Z}/p$-vector space, and let $G = \mathbb{Z}/p$ act linearly on V. Then, there exists a $\mathbb{Z}/p$-invariant filtration of V*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_r = V$$

such that the induced action of G on V_{i+1}/V_i is trivial for every i.

Proof. Define $V_1 = V^G$. Since the action of G is linear, this is a vector subspace of V. Moreover, it is clearly G-invariant. Since $V_1/V_0 = V_1 = V^G$, the action of G on V_1/V_0 is trivial.

Suppose that we have defined $V_0, V_1, \dots, V_i$ with the required properties, and let us define V_{i+1}. Let $\pi : V \to V/V_i$ be the projection. Consider the induced action of G on V/V_i. By Lemma 1.44, $(V/V_i)^G$ is a nontrivial subspace of V/V_i. Define $V_{i+1} = \pi^{-1}((V/V_i)^G)$.

Since the dimension of V is finite and the inclusions $V_i \subset V_{i+1}$ are strict, there is some r for which $V_r = V$. It is clear by construction that this filtration satisfies the stated properties. □

Lemma 1.46. *Let X be a closed manifold, and suppose we have an action of $G = \mathbb{Z}/p$ on X. Consider the fiber bundle associated to the Borel construction:*

$$EG \times_G X \to BG$$

with fiber X, and the action induced by $\pi_1(BG)$ on the fiber X. Then,

$$\dim H^p(BG; \mathcal{H}^q(X)) \leq (\dim H^p(BG))(\dim H^q(X)),$$

where all cohomologies have $\mathbb{Z}/p$ coefficients.

Proof. In this proof all cohomologies are understood to have $\mathbb{Z}/p$ coefficients.

Consider the Serre spectral sequence for equivariant cohomology with $\mathbb{Z}/p$ coefficients. By Corollary 1.37, we know that:

$$H^*(BG; \mathcal{H}^*(X)) \Rightarrow H^*_G(X),$$

where $\mathcal{H}^*(X)$ is the cohomology of X with a (possibly non-trivial) G-module structure (or, equivalently, a $\mathbb{Z}/p$-vector space endowed with a linear action of G) and the homology on the left is with local coefficients.

We show that we can bound the dimension (as a $\mathbb{Z}/p$-vector space) of each term $H^p(BG; \mathcal{H}^q(X))$.

By Lemma 1.45, there exists a G-invariant filtration of $\mathcal{H}^q(X)$

$$0 = \mathcal{H}_0 \subset \mathcal{H}_1 \subset \cdots \subset \mathcal{H}_r = \mathcal{H}^q(X)$$

with the property that the action of G on $\mathcal{H}_{i+1}/\mathcal{H}_i$ is trivial, for every i.

Consider, for each i, the exact sequence:

$$0 \to \mathcal{H}_i \to \mathcal{H}_{i+1} \to \mathcal{H}_{i+1}/\mathcal{H}_i \to 0$$

This induces a long exact sequence on cohomology:

$$\cdots \to H^p(BG; \mathcal{H}_i) \to H^p(BG; \mathcal{H}_{i+1}) \to H^p(BG; \mathcal{H}_{i+1}/\mathcal{H}_i) \to \ldots$$

Therefore, for each i, we get:

$$\dim H^p(BG; \mathcal{H}_{i+1}) - \dim H^p(BG; \mathcal{H}_i) \leq \dim H^p(BG; \mathcal{H}_{i+1}/\mathcal{H}_i).$$

Summing for all i, we obtain the following bound:

$$\dim H^p(BG; \mathcal{H}^q(X)) \leq \sum_i \dim H^p(BG; \mathcal{H}_{i+1}/\mathcal{H}_i).$$

Since the action of G on $\mathcal{H}_{i+1}/\mathcal{H}_i$ is trivial,

$$H^p(BG; \mathcal{H}_{i+1}/\mathcal{H}_i) \simeq H^p(BG) \otimes \mathcal{H}_{i+1}/\mathcal{H}_i.$$

Combining the last two facts, we finally arrive at:

$$\dim H^p(BG; \mathcal{H}^q(X)) \leq (\dim H^p(BG))(\dim H^q(X)).$$

Thus the proof of the lemma is complete. □

With these preliminaries, we can prove the first main theorem of this section. A proof of this theorem for the case $G = \mathbb{Z}/p$, in a slightly more general form and in a more general context can be found in Theorem III.4.3 of [7].

Theorem 1.47. *Let X be a smooth manifold of dimension n, and let G be a finite p-group acting smoothly on X. Then,*

$$\sum_j b_j(X^G; \mathbb{Z}/p) \leq \sum_j b_j(X; \mathbb{Z}/p)$$

Proof. In this proof all cohomologies will be with $\mathbb{Z}/p$ coefficients. We will write $H^*(X)$ for $H^*(X; \mathbb{Z}/p)$.

Let $k > \dim X$. Consider the Serre spectral sequence for the equivariant cohomology of X. By Lemma 1.46, we have the following bound:

$$\begin{aligned} \dim H^k_G(X) &\leq \sum_{i+j=k} (\dim H^i(BG))(\dim H^j(X)) = \\ &= \sum_{\substack{i+j=k \\ i\geq 0}} b^j(X; \mathbb{Z}/p) = \sum_{j=0}^{k} b^j(X; \mathbb{Z}/p), \end{aligned} \tag{1.11}$$

since $b_j(X) = 0$ if $j > \dim X$ and $\dim H^i(BG) = 1$ for all $i \geq 0$.

Let U be a G-equivariant tubular neighbourhood of X^G, which exist by Proposition 1.7, and let $V = X - X^G$. The associated Mayer-Vietoris sequence for the equivariant cohomology reads:

$$\dots H^k_G(U \cap V) \to H^k_G(U) \oplus H^k_G(V) \to H^k_G(X) \to H^{k+1}_G(U \cap V) \to \dots$$

Since G acts freely on V,

$$H^k_G(V) \simeq H^k(V/G), \quad H^k_G(U \cap V) \simeq H^k((U \cap V)/G).$$

Both V/G and $(U \cap V)/G$ are smooth manifolds of dimension n. Since $k > \dim X$, the cohomology groups $H^K(V/G)$ and $H^k((U \cap V)/G)$ are both zero. Therefore, from the above exact sequence we conclude

$$H^k_G(X) \simeq H^k_G(U).$$

Since U retracts to X^G, and G acts trivially on X^G:

$$H^k_G(X) \simeq H^k_G(X^G) \simeq \bigoplus_{i+j=k} H^i(BG) \otimes H^j(X^G)$$

Since $k > n \geq \dim X^G$ and $\dim H^i(BG) = 1$ for all $i \geq 0$, we have

$$\dim H^k_G(X) = \sum_{j=0}^{\dim X^G} b_j(X^G; \mathbb{Z}/p). \tag{1.12}$$

Combining (1.11) and (1.12), we obtain:

$$\sum_j b_j(X^G; \mathbb{Z}/p) \leq \sum_j b_j(X; \mathbb{Z}/p).$$

This finishes the proof in the case $G \simeq \mathbb{Z}/p$.

We consider now the general case of a p-group G. Let $|G| = p^k$. We proceed by induction on k. We have already proved the case $k = 1$. Assume that the proposition is true for $k' < k$ and let us prove it for k. Since G is a p-group, its center $Z(G)$ is nontrivial. Let $G_0 \leq Z(G)$ be of order p. Applying what we have proved to G_0, we obtain:

$$\sum_j b_j(X^{G_0}; \mathbb{Z}/p) \leq \sum_j b_j(X; \mathbb{Z}/p)$$

Since $G_0 \leq Z(G)$, G/G_0 acts on X^{G_0}. Now G/G_0 is a p-group of order p^{k-1}, and therefore by induction hypothesis:

$$\sum_j b_j((X^{G_0})^{G/G_0}; \mathbb{Z}/p) \leq \sum_j b_j(X^{G_0}; \mathbb{Z}/p)$$

Observe that $(X^{G_0})^{G/G_0} = X^G$. Therefore, combining the last two inequalities, we obtain:

$$\sum_j b_j(X^G; \mathbb{Z}/p) \leq \sum_j b_j(X; \mathbb{Z}/p)$$

Hence, the proof of the proposition is finished. □

The following theorem was first proven by Mann and Su. The theorem actually holds for any continuous action on a closed topological manifold, but here we only give the proof for smooth actions on closed manifolds, which is somewhat simpler. For a proof of the theorem in full generality, we refer the reader to the original paper, [53]. Recall that we say that a group G is an elementary abelian p-group if $G \simeq (\mathbb{Z}/p)^r$ for some natural number r. In this case, r is called the rank of G.

Theorem 1.48 (Mann-Su). *Let X be a closed smooth manifold. There exists a natural number $r > 0$, depending only on $H^*(X)$, such that every elementary abelian p-group that acts effectively on X has rank at most r.*

Proof. Let $n = \dim X$. We prove first that the general case can be reduced to that of free actions. Let G be a finite group acting effectively and smoothly on X. Endow X with a G-invariant riemannian metric g and let $P_g \to X$ be the principal $O(n)$-bundle of orthonormal frames of X. We have $P_g \simeq P_{g'}$ for all pairs g, g' of riemannian metrics on X, since the space of riemannian metrics on X is contractible. Since g is G-invariant, the action of G on X lifts to a smooth action of G on P_g. Since the action of G on X is effective, the action on P_g is free: indeed, otherwise there is some $g \in G$ different from the identity and a point $p \in X$ fixed by g such that g acts trivially on T_pX, contradicting (2) of Proposition 1.7. Moreover, P_g is a closed manifold since both X and $O(n)$ are compact. Finally, we show that we can bound the Betti numbers of P_g by the Betti numbers of $O(n)$ (which obviously only depend on n) and the Betti numbers of X. To see that, realize P_g as an iterated fibration of spheres as follows. We quotient P_g by the action of $O(1) < O(n)$ to obtain a fibration $P_g \to P_g/O(1) =: P_g^1$, with fiber $O(1) \cong S^0$, and a fibration $P_g^1 \to X$ with fiber $O(n)/O(1)$. We can now quotient P_g^1 by the action of $O(2)/O(1) < O(n)/O(1)$ to obtain another fibration $P_g^1 \to P_g^2$ with fiber $O(2)/O(1) \cong S^1$, and a fibration $P_g^2 \to X$ with fiber $O(n)/O(2)$. Proceeding inductively in this way, we obtain a tower of fibrations:

$$P_g = P_g^0 \to P_g^1 \to P_g^2 \to \cdots \to P_g^{n-1} \to X,$$

where each $P_g^i \to P_g^{i+1}$ is an S^i-bundle. Using now the Gysin exact sequence inductively, we obtain that there is a bound on the Betti numbers of P_g that depends only on $H^*(X)$ (and on $n = \dim X$).

From now on, let G be an elementary p-group of rank r and assume that G acts on X freely. By Proposition 1.10, there is a constant $C > 0$ that depends only on X and a subgroup $G_0 \leq G$ with $[G : G_0] < C$ such that the action of G_0 is CTE on X. Therefore, it is enough to prove the theorem for CTE actions, since if G_0 has rank at most r', then G has rank at most Cr'.

We claim that we can take $r = \max_p \sum_j b_j(X; \mathbb{Z}/p)$, where p ranges over all primes. Since X is closed, it has finitely generated cohomology, and this implies that r is finite. For the remainder of the proof, all cohomologies are understood to be with $\mathbb{Z}/p$ coefficients. So we write $H^*(X)$ for $H^*(X; \mathbb{Z}/p)$. Consider the spectral sequence for the equivariant cohomology with $\mathbb{Z}/p$ coefficients

$$H^*(BG; \mathcal{H}^*(X)) \Rightarrow H^*_G(X)$$

Since the action is CTE, by Proposition 1.36 the action of $\pi_1(BG)$ on $H^*(X)$ is trivial. Therefore,

$$H^*(BG; \mathcal{H}^*(X)) \simeq H^*(BG) \otimes H^*(X)$$

Since the action of G on X is free, we know that $H^*_G(X) \simeq H^*(X/G)$. Since X/G is a manifold of dimension n, $H^k_G(X) = 0$ for all $k > n$. On the other hand, from the spectral sequence,

$$\begin{aligned}
&\dim H^k_G(X) \geq \\
&\geq (\dim H^k(BG))(\dim H^0(X)) - \sum_{i=1}^{k} (\dim H^{k-i}(BG))(\dim H^i(X)) = \\
&= \dim H^k(BG) - \sum_{i=1}^{k} b_i(X) \dim H^{k-i}(BG)
\end{aligned}$$

Since $G \simeq (\mathbb{Z}/p)^r$, we have for each i

$$\dim H^i(BG) = \binom{i+r-1}{i} \leq C_i r^i,$$

where C_i is a constant independent of r. Therefore,

$$\dim H^k_G(X) \geq C_k r^k - \sum_{i=1}^{k} b_i(X; \mathbb{Z}/p) C_{k-i} r^{k-i}$$

From this formula, we see that for r big enough the right-hand side is positive, in contradiction with the fact that $H^k_G(X) = 0$. Hence, r must be bounded, and this bound depends only on the sum of the betti numbers of X. This finishes the proof of the theorem. □

1.9 The Jordan property

In this section we present and prove some general facts about the Jordan property for groups. We also provide a proof of the classical theorem of Jordan and give some examples of non-Jordan groups arising in geometric contexts.

Definition 1.49. *Let $\mathcal{G}$ be an (infinite) group. We say that $\mathcal{G}$ has the Jordan property (or that $\mathcal{G}$ is Jordan) if there exists a constant $C > 0$ such that any finite group $G \leq \mathcal{G}$ has an abelian subgroup $A \leq G$ satisfying $[G : A] \leq C$.*

Observe that any finite group $\mathcal{G}$ has trivially the Jordan property, so we will only be interested in cases where $\mathcal{G}$ is infinite. More specifically, we will be interested in studying the Jordan property for the group of automorphisms of various topological and geometric structures, such as diffeomorphism groups of smooth manifolds, symplectomorphisms groups of symplectic manifolds, automorphisms groups of almost complex manifolds, etc.

The terminology comes from the following classical theorem of C. Jordan (see [26]).

Theorem 1.50. *For any n, the group $\mathrm{GL}(n, \mathbb{C})$ is Jordan.*

Given its importance in this thesis, we provide a proof of this theorem. The following proof is due to T. Tao (see [71]).

Before starting the proof, we recall some facts about the Frobenius norm of a matrix.

Definition 1.51. *Let $A \in \mathrm{GL}(n, \mathbb{C})$. The Frobenius norm of A is*

$$||A|| = \sqrt{\mathrm{Tr}(A^* A)},$$

where A^* denotes the conjugate transpose of A.

Lemma 1.52. *The Frobenius norm $||\cdot||$ on $\mathrm{GL}(n, \mathbb{C})$ satisfies:*

1. *For any $A = (a_{ij})_{i,j=1,\dots n} \in \mathrm{GL}(n, \mathbb{C})$,*

$$||A|| = \left(\sum_{i,j} |a_{ij}|^2 \right)^{1/2}.$$

2. *$||\cdot||$ is a norm.*

3. *The Frobenius norm is invariant under unitary tranformations. That is, for all $A \in \mathrm{GL}(n, \mathbb{C})$ and $U \in U(n)$,*

$$||UA|| = ||AU|| = ||A||.$$

Proof. 1. is immediate from the definition. In order to prove that the Frobenius norm is really a norm, we observe that the formula in 1 shows that the Frobenius norm is just the usual 2-norm in $\mathbb{C}^{n^2}$, where we think of a matrix $A \in \mathrm{GL}(n, \mathbb{C})$ as a vector in $\mathbb{C}^{n^2}$.

Finally, we prove the invariance of the Frobenius norm by unitary transformations. If $A \in \mathrm{GL}(n, \mathbb{C})$ and $U \in U(n)$,

$$||UA|| = \sqrt{\mathrm{Tr}((UA)^*(UA))} = \sqrt{\mathrm{Tr}(A^*U^*UA)} = \sqrt{\mathrm{Tr}(A^*A)} = ||A||.$$

The proof that $||AU|| = ||A||$ is completely analogous. □

After these preliminaries, we can prove the theorem.

Proof of Theorem 1.50. First, observe that it is enough to consider finite subgroups of $U(n)$. Indeed, if $G \leq GL(n, \mathbb{C})$ is finite, pick any hermitian metric h in $\mathbb{C}^n$, and consider

$$h'(u, v) = \frac{1}{|G|} \sum_{g \in G} h(gu, gv),$$

for any $u, v \in \mathbb{C}^n$. Then h' is again an hermitian metric on $\mathbb{C}^n$, as is readily verified. Moreover, h' is G-invariant, in the sense that $h'(gu, gv) = h'(u, v)$ for all $u, v \in \mathbb{C}^n$. Hence, the matrix of an element $g \in G$ in a unitary basis for h' is unitary. This shows that G is conjugated in $\mathrm{GL}(n, \mathbb{C})$ to a subgroup of $U(n)$. Therefore, it is enough to consider the case $G \leq U(n)$.

We proceed to prove the theorem for $U(n)$ by induction on n. For $n = 1$, the result is trivial since $U(1)$ is abelian.

Assume $n > 1$. We denote by C_k the Jordan constant for $U(k)$, for every $k < n$. Assume first that there is some $g \in G$ which is in the center of G and is not a multiple of the identity. In this case, by definition, G is contained in the centralizer of g in $U(n)$, that is, $G \leq Z_{U(n)}(g)$. It is a basic fact in linear algebra that unitary matrices can always be diagonalized by unitary matrices, that is, there exists $U \in U(n)$ such that UgU^{-1} is a diagonal matrix. Therefore, in a suitable unitary basis, g is a diagonal matrix which is not a multiple of the identity. This shows that

$$Z_{U(n)}(g) \simeq U(n_1) \times \cdots \times U(n_k),$$

where k is the number of distinct eignevalues of g and the n_i are the dimensions of the associated eigenspaces. In particular, $n_i < n$ for all i. Let G_i be the projection of G to $U(n_i)$, for $i = 1, \ldots, k$. By induction hypothesis, there is an abelian subgroup $A_i \leq G_i$ such that $[G_i : A_i] < C_{n_i}$, where C_{n_i} only depends on n_i and without loss of generality we take $C_{n_i} \geq 1$. Let

$$A := G \cap (A_1 \times \cdots \times A_k).$$

Then, A is abelian, and

$$[G : A] \leq [G_1 \times \cdots \times G_k : A_1 \times \cdots \times A_k] = \prod_{i=1}^{k} [G_i : A_i] \leq \prod_{i=1}^{k} C_{n_i} \leq \prod_{i=1}^{n-1} C_i^n.$$

Therefore, the theorem does hold in this case.

We may now assume that every central element of G is a multiple of the identity. Since $U(n)$ is compact, there exists a constant $D_n > 0$, depending only on n, such that $||A|| < D_n$ for all $A \in U(n)$. Choose

$$\epsilon < \min\left(\frac{1}{2}, \frac{\sqrt{n|e^{2\pi i/n} - 1|^2}}{2D_n}\right).$$

Let G' be the subgroup of G generated by all $g \in G$ such that $||g - I|| < \epsilon$, where $||g|| = \sum_{i,j} |g_{i,j}|$ is the Frobenius norm of a matrix (which is invariant under unitary transformations), and $I \in U(n)$ is the identity. We claim that there is a constant $C_{n,\epsilon}$, depending only on n and ϵ, such that $[G : G'] \leq C_{n,\epsilon}$. Since $U(n)$ is compact, there is a finite $\epsilon/2$-covering, that is, a finite set $\{U_1, \dots, U_N\}$ of elements of $U(n)$ with the property that for any $U \in U(n)$, there exists $i \in \{1, \dots, N\}$ such that $||U - U_i|| < \epsilon/2$. Let $g_1G', \dots g_lG'$ be the left cosets of G' in G. If $l > N$, there is some $k \in \{1, \dots, N\}$ and some $i, j \in \{1, \dots, l\}$ such that $||g_i - U_k|| < \epsilon/2$ and $||g_j - U_k|| < \epsilon/2$. It follows that $||g_i - g_j|| < \epsilon$, and hence $||g_i g_j^{-1} - I|| < \epsilon$, which implies that $g_i g_j^{-1} \in G'$, a contradiction. Hence, we can take $C_{n,\epsilon} := N$ and the claim is proved. It now suffices to find a constant $C' > 0$ and an abelian subgroup $A \leq G'$ with $[G' : A] < C'$, since then

$$[G : A] = [G : G'][G' : A] < C_{n,\epsilon}C'.$$

Arguing exactly as before, we can assume that G' does not have any central element which is not a multiple of the identity. If G' consists only on multiples of the identity, it is abelian and we are done. Otherwise, let $g \in G'$ be an element as close as possible to the identity which is not a multiple of the identity (we can pick such a g since G is finite). Let $h \in G'$ be such that $||h - I|| < \epsilon$. Then, using the unitary invariance of the Frobenius norm and the triangle inequality,

$$\begin{aligned} ||[g, h] - I|| = ||gh - hg|| = ||(g - I)(h - I) - (h - I)(g - I)|| \leq \\ \leq 2||g - I||||h - I|| < 2\epsilon||g - I|| < ||g - I||. \end{aligned}$$

Therefore, $[g, h]$ is closer than g to the identity. By the choice of g, this implies that $[g, h]$ is a multiple of the identity, say $[g, h] = e^{i\alpha}I$ for some $\alpha \in \mathbb{R}$. On the other hand, $[g, h]$ has determinant 1, which means that α is an integral multiple of $2\pi/n$. Since $||e^{i\alpha}I - I|| = \sqrt{n|e^{2\pi i/n} - 1|^2}$, and we have

$$||[g, h] - I|| < 2\epsilon||g - I|| < 2\epsilon D_n < \sqrt{n|e^{2\pi i/n} - 1|^2},$$

we must have $[g, h] = I$. That is, g is central in G'. But this contradicts the fact that all the central elements of G' are multiples of the identity. This finishes the proof of the theorem. □

1.9.1 Diffeomorphism groups

We now prove that every closed surface has Jordan group of diffeomorphisms, and give an example of an open 4-manifold in which all finite groups act (so, in particular, this open 4-manifold has non-Jordan diffeomorphism group).

Proposition 1.53. *Let Σ be a closed surface. Then,* $\mathrm{Diff}(\Sigma)$ *is Jordan.*

Proof. By Lemma 4.8 it suffices to consider the case where Σ is orientable. Choose an orientation of Σ. Let G be a finite group acting effectively and smoothly on Σ. Let g be a G-invariant riemannian metric on Σ. Then, the conformal class of g gives an almost complex structure j on X (see Section 3.1 for definitions). Since in dimension 2 every almost complex structure is integrable (see Corollary 3.3), (Σ, j) is a Riemann surface. Let $G_0 \leq G$ be the subgroup of orientation-preserving elements, and note that $[G : G_0] \leq 2$. Since j is G_0-invariant, we can identify G_0 with a subgroup of $\mathrm{Aut}(\Sigma, j)$. If the genus of Σ is 0, by the uniformization theorem for Riemann surfaces, $(\Sigma, j) \simeq \mathbb{C}\mathrm{P}^1$, so G is a finite subgroup of $\mathrm{PSL}(2, \mathbb{C})$. Since G is finite, it is contained in a compact maximal subgroup of $PSL(2, \mathbb{C})$, which is isomorphic to $\mathrm{SO}(3)$. Using the classification of the finite subgroups of $\mathrm{SO}(3)$ (cyclic groups, dihedral groups and the three polyhedral groups A_4, S_4 and A_5) we see directly that $\mathrm{Diff}(X)$ is Jordan. If $g(\Sigma) = 1$, then (Σ, j) is an elliptic curve. The subgroup $\mathrm{Aut}_0(\Sigma, j) \leq \mathrm{Aut}(\Sigma, j)$ fixing a given point $p \in \Sigma$ satisfies $[\mathrm{Aut}(\Sigma, j) : \mathrm{Aut}_0(\Sigma, j)] \leq 12$, and $\mathrm{Aut}_0(\Sigma, j) \simeq T^2$. Since the finite subgroups of T^2 are products of cyclic groups (hence abelian), we are done. Finally, if $g(\Sigma) > 1$, we may apply Hurwitz's theorem to conclude that any finite subgroup $G \leq \mathrm{Aut}(\Sigma, j)$ satisfy $\sharp G \leq 84(g(\Sigma) - 1)$. □

We also provide here the following theorem of Popov which proves that there is an open smooth 4-manifold admitting effective (in fact, free) actions of all finite groups. In particular, this shows that no non-trivial theorem can be proven on finite group actions that apply to all 4-manifolds.

Recall that a group G is finitely presented if it admits a presentation

$$G = \langle F \mid R \rangle,$$

with both F and R finite. In particular, all finite groups are finitely presented, but the latter class is much wider (for instance, it contains every free group on a finite number of generators). The following theorem is one of the main results in [60].

Theorem 1.54 (Popov)**.** *There is an open connected* 4*-manifold X such that every finitely presented group acts in an effective and free way on X.*

Proof. There exists a universal finitely presented group $\mathcal{U}$, that is, $\mathcal{U}$ is a finitely presented group, and every finitely presented group G is isomorphic to a subgroup of $\mathcal{U}$ (see, for instance, [66, Theorem 12.29]). There exists a closed connected 4-manifold Y with $\pi_1(Y) \simeq \mathcal{U}$ (see for instance [12]). Let X be the universal cover of Y. Then $\mathcal{U}$, and therefore also every finitely presented group, acts effectively and freely on X by deck transformations. The fact that X is non-closed follows from Theorem 1.48, which in particular implies that there is no closed manifold admitting effective actions of all finite groups. $\square$

1.9.2 Symplectic and contact manifolds

In this subsection we provide examples of open symplectic (resp. closed contact) manifolds with non-Jordan group of symplectomorphisms (resp. contactomorphisms).

Recall that a symplectic manifold (X, ω) is a smooth manifold X endowed with a differential 2-form ω which is non-degenerate (i.e. for any vector field U on X, if $\omega(U, V) = 0$ for all vector fields V on X, then $U = 0$), and closed (i.e. $d\omega = 0$). A symplectomorphism of (X, ω) is a diffeomorphism $f : X \to X$ such that $f^*(\omega) = \omega$. We denote the group of symplectomorphisms of (X, ω) by $\mathrm{Symp}(X, \omega)$.

Definition 1.55. *Let X be a smooth manifold, and $\pi : T^*X \to X$ its cotangent bundle. The Liouville form in T^*X is the 1-form λ on T^*X defined by*

$$\lambda_{(p,\alpha)}(V) = \alpha(d\pi(V)),$$

for all $p \in X, \alpha \in T_p^*X$ and V a vector field on T^*X. The canonical symplectic form on T^*X is $\omega = d\lambda$.

In adapted local coordinates $(q_1, \ldots, q_n, p_1 \ldots, p_n)$ in T^*X, we can write

$$\lambda = \sum_{i=1}^{n} p_i dq_i.$$

Since, locally, we have

$$\omega = d\lambda = \sum_{i=1}^{n} dq_i \wedge dp_i,$$

we see that ω is non-degenerate, and being obviously closed, it is indeed a symplectic form on T^*X.

A very nice property about this symplectic form is that any diffeomorphism of X lifts to a symplectomorphism of (T^*X, ω), as the following proposition shows.

Proposition 1.56. *Let X be a smooth manifold, and let $\pi : T^*X \to X$ be the cotangent bundle. For every diffeomorphism $f : X \to X$, there exists a symplectomorphism $\hat{f} : T^*X \to T^*X$ (with respect to the canonical symplectic form ω on T^*X) with the property that the square*

$$\begin{array}{ccc} T^*X & \xrightarrow{\hat{f}} & T^*X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow{f} & X \end{array}$$

is commutative. Moreover, $\widehat{f \circ g} = \hat{f} \circ \hat{g}$ and $\widehat{f^{-1}} = \hat{f}^{-1}$, so that there exists a monomorphism of $\mathrm{Diff}(X)$ into $\mathrm{Symp}(X, \omega)$.

Proof. Define $\hat{f}(p,\alpha) = (f(p),(f^{-1})^*(\alpha))$ for all $p \in X, \alpha \in T_p^*X$. It is clear that $\hat{f}$ is a diffeomorphism of T^*X with the stated properties, and that it makes the diagram in the statement commute. It only remains to check that it is a symplectomorphism with respect to $\omega = d\lambda$. Indeed, for any $p \in X, \alpha \in T_p^*X$ and V vector field on T^*X we have

$$(\hat{f}^*\lambda)_{(p,\alpha)}(V) = \lambda_{(f(p),(f^{-1})^*(\alpha))}(d\hat{f}(V)) = (f^{-1})^*(\alpha)_{f(p)}(d(\pi\circ\hat{f})(V)) = \\ = (f^{-1})^*(\alpha)_{f(p)}(df\circ d\pi(V)) = \alpha_p(d\pi(V)) = \lambda_{(p,\alpha)}(V),$$

so we have $\hat{f}^*(\lambda) = \lambda$, and finally

$$\hat{f}^*(\omega) = \hat{f}^*(d\lambda) = d\hat{f}^*(\lambda) = d\lambda = \omega.$$

This finishes the proof of the proposition. □

As an immediate corollary we can easily produce for each smooth manifold X such that $\mathrm{Diff}(X)$ is not Jordan, an example of an open symplectic manifold with non-Jordan group of symplectomorphisms, namely (T^*X,ω). As mentioned in theintroduction, it was proved by I. Mundet i Riera that $\mathrm{Diff}(T^2\times S^2)$ is not Jordan (see [49]). Therefore, we obtain an example of an open symplectic manifold with non-Jordan group of symplectomorphisms.

Corollary 1.57. *Let X be a smooth manifold such that* $\mathrm{Diff}(X)$ *is not Jordan. Then,* $\mathrm{Symp}(T^*X,\omega)$ *is not Jordan for the canonical symplectic form ω. In particular,* $\mathrm{Symp}(T^*(T^2\times S^2),\omega)$ *is not Jordan.*

Note that this procedure only produces open symplectic manifolds, so the following question remains: Has every closed symplectic manifold Jordan group of symplectomorphisms?

However, a refinement of the same idea produces closed contact manifolds with non-Jordan groups of contactomorphisms. We next explain this, but before we introduce some basic definitions about contact manifolds.

Definition 1.58. *Let X be a smooth manifold. A contact structure ξ on X is a maximally non-integrable distribution of hyperplanes of TX. That is, ξ is a distribution of hyperplanes of TX, and for every $p \in X$ there is an open neighbourhood $U \subseteq X$ such that if $\xi|_U = \mathrm{Ker}\,\alpha$ for a 1-form α on U, we have*

$$\alpha\wedge d\alpha^n \neq 0.$$

A contact manifold (X,ξ) is a smooth manifold X together with a contact structure ξ on it.

Every contact manifold (X,ξ) has odd dimension because $d\alpha|_\xi$ is a symplectic form. If ξ admits a global expression $\xi = \mathrm{Ker}\,\alpha$ for some (globally defined) 1-form α on X, we will abuse notation and say that (X,α) is a contact manifold. However, one must keep in mind that α is not uniquely determined by ξ.

Definition 1.59. *Let (X, ξ) be a contact manifold. We say that a diffeomorphism $f : X \to X$ is a contactomorphism if $df(\xi) = \xi$. We denote by* $\mathrm{Cont}(X, \xi)$ *the group of contactomorphisms of (X, ξ).*

We now introduce the contact version of the construction we have explained for symplectic manifolds.

Definition 1.60. *Let X be a smooth manifold of dimension n. Define the (oriented) projectivized cotangent space of X by*

$$P_+^*X = (T^*X - 0)/\,\mathbb{R}_+,$$

where 0 stands for the zero section of T^*X and the action of $\mathbb{R}_+$ is by homotheties. The canonical contact structure ξ on P_+^*X is given by $\xi = \mathrm{Ker}\,\tilde{\lambda}$, where $\tilde{\lambda}$ is the 1-form on P_+^*X induced by the Liouville form on T^*X.

Note that P_+^*X is a compact manifold whenever X is compact. In fact, by choosing any riemannian metric g on X, P_+^*X can be identified with the unit cotangent bundle of (X, g). From the expression of λ in local coordinates, it can be seen that ξ is a contact structure on P_+^*X.

Every diffeomorphism $f : X \to X$ induces a contactomorphism $\tilde{f} : P_+^*X \to P_+^*X$ as in the symplectic case: $\tilde{f}$ is just the map induced by $\hat{f} : T^*X \to T^*X$. The fact that $\tilde{f}$ is a contactomorphisms follows from the fact that $\hat{f}$ preserves the Liouville form on T^*X. Finally, as before, we have $\widetilde{fg} = \tilde{f}\tilde{g}$ and $\widetilde{f^{-1}} = \tilde{f}^{-1}$ for all diffeomorphisms $f, g : X \to X$. Therefore, we have obtained:

Proposition 1.61. *Let X be a smooth manifold. Then, there is a monomorphism*

$$\mathrm{Diff}(X) \to \mathrm{Cont}(P_+^*X, \xi),$$

where ξ is the canonical contact structure on P_+^*X.

As a corollary, we get:

Proposition 1.62. *Let X be a closed smooth manifold such that* $\mathrm{Diff}(X)$ *is not Jordan. Then,* $\mathrm{Cont}(P_+^*X, \xi)$ *is not Jordan.*

In particular, $\mathrm{Cont}(P_+^*(T^2 \times S^2), \xi)$ *is not Jordan.*

Since $P_+^*(T^2 \times S^2)$ is closed, we have obtained an example of a closed contact manifold with non-Jordan contactomorphism group.

Chapter 2

Preliminaries on smooth 4-manifolds

In this chapter we provide the facts about smooth 4-manifolds we will need in later chapters. After a first section on topological preliminaries, we devote the next section to the Atiyah–Singer theorem, with focus on the G-signature theorem which will be used later on. The rest of the chapter is devoted to give a quick introduction to Seiberg–Witten theory. Since its discovery in 1995, Seiberg–Witten theory has quickly become one of the most useful tools for the study of smooth 4-manifolds.

2.1 Preliminaries on 4-manifolds

Let X be a closed and oriented 4-manifold. In this section we recall some basic facts and some topological invariants that we will use later on.

Definition 2.1. *The Euler characteristic of X is defined by*

$$\chi(X) = \sum (-1)^i b_i(X;k),$$

where k is any field and $b_i(X;k) = \dim H_i(X;k)$ are the Betti numbers of X.

It is a standard result in algebraic topology that $\chi(X)$ is independent of the field k.

A lot of information about an oriented 4-manifold can be obtained by studying its 2-dimensional submanifolds. This is due to the fact that two 2-dimensional submanifolds of a 4-manifold that are transverse intersect in a finite number of points, and the number of points (taking into account the orientations) is a topological invariant. Let us make all this precise.

Definition 2.2. *Let X be an oriented and closed 4-manifold. The intersection form of X is the symmetric bilinear form on its second cohomology class*

$$Q_X : H^2(X) \times H^2(X) \to \mathbb{Z}$$

defined by

$$Q_X(\alpha, \beta) = \langle \alpha \cup \beta, [X] \rangle,$$

where $[X] \in H_4(X)$ *is the fundamental class of* X*,* $\cup$ *is the cup product and* $\langle \cdot, \cdot \rangle$ *is the Kronecker pairing between* $H^4(X)$ *and* $H_4(X)$*.*

Given two homology classes $A, B \in H_2(X)$*, we define their intersection product by:*

$$A \cdot B = Q_X(PD(A), PD(B)),$$

where $PD(A), PD(B)$ *are the Poincaré duals of* A, B*.*

Note that if $\alpha \in H^2(X)$ is a torsion class (meaning that there is some $n > 0$ such that $n\alpha = 0$), then $Q_X(\alpha, \beta) = 0$ for all $\beta \in H^2(X)$. Indeed, $nQ_X(\alpha, \beta) = Q_X(n\alpha, \beta) = Q_X(0, \beta) = 0$, hence $Q_X(\alpha, \beta) = 0$. This means that the intersection product gives only information about the free part of $H^2(X)$. Therefore, we will usually consider Q_X as a map:

$$Q_X : H^2(X)/T \times H^2(X)/T \to \mathbb{Z},$$

where T is the torsion subgroup of $H^2(X)$.

The following is a consequence of the classical Poincaré duality theorem applied to 4-manifolds:

Proposition 2.3. *For a closed and oriented* 4*-manifold* X*,*

$$Q_X : H^2(X)/T \times H^2(X)/T \to \mathbb{Z}$$

is a perfect pairing, meaning that for every $\alpha \in H_2(X)/T$ *there exists a* $\beta \in H_2(X)$ *such that* $Q_X(\alpha, \beta) = 1$*.*

Any closed oriented 2-submanifold Y of X gives rise to an element of $H_2(X)$. Indeed, let Σ be a closed oriented surface and let $i : \Sigma \to X$ be an embedding such that $i(\Sigma) = Y$ and i identifies the orientation of Σ with that of Y. We define the homology class represented by Y as

$$[Y] = i_*([\Sigma]),$$

where $[\Sigma] \in H_2(\Sigma)$ is the fundamental class of Σ.

With this definition, given an oriented closed manifold X and two closed 2-dimensional submanifolds Y_1, Y_2, we define their intersection number by

$$Y_1 \cdot Y_2 = [Y_1] \cdot [Y_2],$$

where the product on the right is the intersection product of homology classes. In particular, by taking $Y_1 = Y_2 = Y$ we obtain the self-intersection number of a 2-dimensional submanifold Y of X. If X is smooth, we can interpret the intersection number of Y_1 and Y_2 in the following way. We can isotop both submanifolds to another pair of submanifolds Y_1', Y_2' intersecting transversely. The fact that Y_i' is isotopic to Y_i implies that $[Y_i] = [Y_i']$ for $i = 1, 2$. Transversality theory tells us that Y_1' and Y_2' intersect in a finite number of points. If we assign to each intersection point $+1$ or -1 according to their orientation, the sum of these numbers gives us the intersection product $[Y_1] \cdot [Y_2]$. See for instance Section 0.4 of [19] for a detailed discussion.

The next invariant we can define is the signature of a 4-manifold. First, note that we can consider the intersection form Q_X over the reals by tensoring with $\mathbb{R}$. In this way, we get a non-degenerate pairing

$$Q_X^{\mathbb{R}} : H^2(X;\mathbb{R}) \times H^2(X;\mathbb{R}) \to \mathbb{R}\,.$$

Pick two subspaces $H^2_\pm(X;\mathbb{R})$ such that $\pm Q_X^{\mathbb{R}}$ is positive definite on $H^2_\pm(X;\mathbb{R})$ and such that $H^2_\pm(X;\mathbb{R})$ are of maximal dimension with that property. Since $Q_X^{\mathbb{R}}$ is non-degenerate, we have

$$H^2(X;\mathbb{R}) \simeq H^2_+(X;\mathbb{R}) \oplus H^2_-(X;\mathbb{R}).$$

We also define

$$b_2^\pm(X) = \dim H^2_\pm(X;\mathbb{R}).$$

Clearly, $b_2(X) = b_2^+(X) + b_2^-(X)$.

Definition 2.4. *Let X be a closed and oriented 4-manifold. We define the signature of X by*

$$\sigma(X) = b_2^+(X) - b_2^-(X).$$

We end this section with the following proposition.

Proposition 2.5. *Let X be a closed manifold. Then, the map that associates to each complex line bundle over X its first Chern class is a bijection between the isomorphism classes of complex line bundles over X and $H^2(X)$.*

There are several ways to prove this proposition. A rough sketch of a proof using some results of algebraic topology is the following. It is easy (using the standard construction of frame bundles and associated bundles) to see that there is a bijection between the isomorphism classes of complex line bundles over X and the isomorphism classes of $U(1)$-principal bundles over X. Therefore, we know by Proposition 1.24 that there is a bijection between complex line bundles and $[X, BU(1)]$. Noting the isomorphism of groups $U(1) \simeq S^1$, we see that $BU(1) \simeq \mathbb{C}\mathrm{P}^\infty$, since S^1 acts freely on S^∞ and the quotient of the action is $\mathbb{C}\mathrm{P}^\infty$. It can be proved that $\mathbb{C}\mathrm{P}^\infty$ is a $K(\mathbb{Z},2)$ space by using the long exact sequence on homotopy groups of the fibration $S^\infty \to \mathbb{C}\mathrm{P}^\infty$ with fiber S^1. Hence, we obtain a bijection between isomorphism classes of complex line bundles and $[X, K(\mathbb{Z},2)] \simeq H^2(X)$, by the standard fact that $K(A,n)$ represents the n-th cohomology group with coefficients in A.

2.2 The Atiyah–Singer G-signature theorem

In this section we provide a brief introduction to the Atiyah–Singer G-signature theorem, and its applications to 4-manifolds. Good references for this material are the book [31] (which moreover provides lots of applications), the book [6] and the highly readable original series of papers by Atiyah, Singer and Segal (the relevant ones here are [1–3]).

Recall the usual notation of multiindices. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a tuple of non-negative integers, we put $|\alpha| = \sum_k \alpha_k$ and $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$ for all $\xi \in \mathbb{R}^n$. We also define the differentiation operators

$$D^\alpha = \frac{1}{i^{|\alpha|}} \frac{\partial^{|\alpha|}}{\partial x^\alpha}.$$

Definition 2.6. *Let X be a smooth manifold. A differential operator P of order m on X is a linear map*

$$P : \Gamma(E) \to \Gamma(F),$$

where E, F are complex vector bundles over X, that satisfies the following property. Each $x \in X$ has a neighbourhood U with local coordinates $(x_1, \ldots, x_n)$ and local trivializations $E|_U \to U \times \mathbb{C}^p$, $F|_U \to U \times \mathbb{C}^q$, in which P can be written as:

$$P = \sum_{|\alpha| \leq m} A^\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha},$$

where $A^\alpha(x)$ is a $q \times p$ matrix of smooth complex-valued functions, and $A^\alpha(x) \neq 0$ for some α with $|\alpha| = m$.

The reader can check that this definition does not depend on the choice of local coordinates or on the choice of trivializations.

Associated to each differential operator there is the following object.

Definition 2.7. *Let X be a smooth manifold and $P : \Gamma(E) \to \Gamma(F)$ a differential operator of order m on it. We denote by S^*X the unit sphere bundle in T^*X (with respect to some riemannian metric on X), and by $\pi : S^*X \to X$ the projection. Then, the symbol of P is a vector bundle homomorphism*

$$\sigma(P) : \pi^* E \to \pi^* F$$

defined as follows. For a choice of local coordinates $(x_1, \ldots, x_n)$ around $x \in X$ and trivializations of E, F and for each differential 1-form on $S *_x X$ with expression in local coordinates

$$\xi = \sum_k \xi_k dx_k$$

the symbol gives a map

$$\sigma_\xi(P) : E_x \to F_x$$

defined by

$$\sigma_\xi(P) = i^m \sum_{|\alpha| = m} A^\alpha(x) \xi^\alpha.$$

An easy calculation proves that $\sigma_\xi(P)$ is well-defined, i.e., it does not depend on the chosen coordinates and local trivializations. Observe that the definition of the symbol depends only on the terms of highest order (when P is expressed in any coordinate system).

Now we can define a special class of differential operators which will be the focus of the results in this section.

Definition 2.8. *Let X be a smooth manifold and $P : \Gamma(E) \to \Gamma(F)$ a differential operator of order m on X. We say that P is elliptic if for every non-zero 1-form $\xi \in S^*X$, the symbol*

$$\sigma_\xi(P) : E_x \to F_x$$

is invertible.

As we will see later, many geometrically or topologically meaningful differential operators on a manifold are elliptic. For the moment, note that this notion of ellipticity coincides (after expressing P in some local coordinates and trivializations of the vector bundles) with the familiar notion of elliptic partial differential equation of order 2 coming from the classical classification of order 2 PDEs into elliptic, parabolic and hyperbolic PDEs. Hence, in particular, the Laplace-Beltrami operator on X associated to a given riemannian metric

$$\Delta : C^\infty(X) \to C^\infty(X)$$

is an elliptic operator.

The main analytic invariant associated to a differential operator is its index, which we now define. Let H_1, H_2 be two Hilbert spaces, and let $T : H_1 \to H_2$ be a bounded linear operator. T is said to be Fredholm operator if both $\operatorname{Ker} T$ and $\operatorname{Coker} T$ are finite-dimensional. In this case, we define the index of T as

$$\operatorname{ind} T = \dim(\operatorname{Ker} T) - \dim(\operatorname{Coker} T).$$

It turns out that the index of a Fredholm operator is remarkably stable under perturbations. In fact, if $\mathcal{L}(H_1, H_2)$ is the Banach space of bounded linear maps from H_1 to H_2, and we denote by $\mathcal{F} \subset \mathcal{L}(H_1, H_2)$ the subspace of Fredholm operators, the index is locally constant (so that its value depends only on the connected component of T in $\mathcal{F}$. Moreover we have

Proposition 2.9. *The map*

$$\operatorname{ind} : \pi_0(\mathcal{F}) \to \mathbb{Z}$$

induces a bijection between the connected components of $\mathcal{F}$ and the integers $\mathbb{Z}$.

For a proof, see [6, Theorem 3.11].

We now want to apply this to differential operators on a smooth manifold X. One technical difficulty is the fact that the space of smooth sections of a vector bundle, $\Gamma(E)$, is not a Hilbert space. This can be remedied by considering suitable completions of $\Gamma(E)$. In particular, one considers Sobolev completions $L^2_k(E)$, which are Hilbert spaces. Then, a differential operator P of order m can be extended uniquely to a bounded linear map

$$P : L^2_k(E) \to L^2_{k-m}(F),$$

for every $k \geq m$. If P is elliptic, these extensions are Fredholm operators and their index is independent of k. Therefore, we can associate a well-defined (analytic) index to any elliptic differential operator on a manifold.

We say that a family $P_t : \Gamma(E) \to \Gamma(F)$, $0 \leq t \leq 1$, of elliptic operators on X is continuous, if in the local representation of P_t on some coordinates and trivializations of the bundles,

$$P_t = \sum_{|\alpha| \leq m} A^\alpha(x,t) D^\alpha$$

the matrices $A^\alpha(x,t)$ are jointly continuous in x and t. This implies that the order of all the operators in the family must be the same, say m. Moreover, under this hypothesis, the map

$$[0,1] \to \mathcal{F}(L^2_k(E), L^2_{k-m}(F))$$

which sends t to P_t is continuous. Hence, by Proposition 2.9, $\operatorname{ind}(P_t)$ does not depend on t. If $P_0, P_1 : \Gamma(E) \to \Gamma(F)$ are two elliptic operators on X, we say that they are homotopic if they can be joined by a continuous family P_t of elliptic operators. Then, we have

Proposition 2.10. *The index of an elliptic operator on a closed manifold depends only on the homotopy class of the elliptic operator.*

Any two elliptic operators P_0, P_1 with the same symbol can be joined by a continuous path of elliptic operators

$$P_t = tP_0 + (1-t)P_1$$

all having the same symbol. Therefore,

Corollary 2.11. *The index of an elliptic operator on a closed manifold depends only on its symbol.*

Hence, we have seen that the index of an elliptic operator on a closed manifold is a homotopical invariant, depending only on its symbol. The question arises as if there is a way of computing the index of an elliptic operator directly from its symbol, therefore expressing the analytical index of an elliptic operator in terms of topological data of the underlying manifold. The positive answer to this question was given by Atiyah and Singer, which provided its celebrated Atiyah-Singer formula for computing the index of an elliptic operator. We now turn to the statement of such formula.

The most natural way to express the Atiyah-Singer formula is by means of K-theory. K-theory is an extraordinary cohomology theory, meaning that it is a homotopy-invariant functor K^* from the category of topological spaces to the category of graded abelian groups satisfying the usual properties of singular cohomology except the dimension axiom, which is constructed by studying the vector bundles over a topological space. More precisely, if X is any topological space, the isomorphism classes of comple vector bundles over X with $\oplus$ from a commutative monoid. We define $K^0(X)$ as its associated Grothendieck group (recall that given a commutative monoid M, its Grothendieck group is the abelian group whose elements are equivalence classes of pairs of elements $(n,m) \in M \times M$ under the equivalence relation $(n,m) \simeq (n',m')$ if and only if $n+m' = m+n'$). Moreover, $K^0(X)$ endowed with the tensor product of vector bundles is a commutative ring. Given a function $f : X \to Y$, the pullback $f^* : \operatorname{Vect}(Y) \to \operatorname{Vect}(X)$

(where $\mathrm{Vect}(X)$ denotes the set of isomorphism classes of vector bundles over X) induce a group morphism $f^* : K^0(Y) \to K^0(X)$. It is easy to verify that with this definition K^0 is indeed a functor. In particular, since complex vector bundles over a point are classified by their rank, we have $K^0(\mathrm{pt}) \simeq \mathbb{Z}$.

We now define the higher K-theory groups. The inclusion of a point $i : \mathrm{pt} \to X$ induces a map $i^* : K^0(X) \to K^0(\mathrm{pt}) \simeq \mathbb{Z}$. We define the reduced K^0 group of X as $\widetilde{K}^0(X) = \mathrm{Ker}(i^*)$. Note that the map $p : X \to \mathrm{pt}$ that collapses X to a point gives a map $p^* : K^0(\mathrm{pt}) \to K^0(X)$ satisfying $i^* \circ p^* = \mathrm{id}_{\mathrm{pt}}$, which implies that we have a split exact sequence

$$0 \to \mathbb{Z} \to K^0(X) \to \widetilde{K}^0(X) \to 0.$$

Hence, $K^0(X) \simeq \mathbb{Z} \oplus \widetilde{K}^0(X)$. Note that we can recover the K^0 group from the reduced group by $K^0(X) \simeq \widetilde{K}^0(X^+)$, where X^+ denotes the disjoint union of X with a point. We define the K^0 group of a pair (X, Y) as $K^0(X, Y) := \widetilde{K}^0(X/Y)$, where X/Y denotes the quotient topological space of X obtained by collapsing the subspace Y to a point. We define the reduced higher K-theory groups by

$$\widetilde{K}^{-n}(X) = \widetilde{K}^0(\Sigma^n X),$$

for $n > 0$, and where $\Sigma^n X$ denotes the n-th iterated reduced suspension of X. For a pair (X, Y) we define

$$K^{-n}(X, Y) = \widetilde{K}^{-n}(X/Y)$$

and the higher K-theory groups of a space X by

$$K^{-n}(X) := K^{-n}(X, \emptyset) = \widetilde{K}^0(X^+)$$

(here we use the convention $X/\emptyset \cong X^+$). With these definitions, one can check that K-theory groups satisfy all axioms of a generalized cohomology theory. The most notable property of K-theory is the following periodicity theorem due to Bott:

Theorem 2.12 (Bott)**.** *For all $n \geq 0$,*

$$K^{-n}(X) \simeq K^{-n-2}(X).$$

For a proof see for instance [31, Theorem 1.9.19].

Given an elliptic operator P on a closed manifold X, its symbol, $\sigma(P)$ defines in a natural way a class $[\sigma_P] \in K(B^*X/S^*X)$, where B^*X is the unit ball in T^*X, and from this one defines a topological index (interpreted as a class in $K^0(\mathrm{pt}) \simeq \mathbb{Z}$), which Atiyah and Singer proved that coincides with the analytical index of P we have defined before.

However, a careful explanation of this will take us too far from the purposes of this section, and therefore we will content ourselves with giving the statement of the so-called cohomological formulation of the Atiyah-Singer theorem, which expresses the index of P in terms of (singular) cohomological terms. We turn to this now.

Let $H^{ev}(X; \mathbb{Q})$ be the even degree part of $H^*(X; \mathbb{Q})$. There is an isomorphism of rings:

$$\mathrm{ch} : K^0(X) \to H^{ev}(X; \mathbb{Q})$$

called the Chern character, defined by

$$\mathrm{ch}(L_1 \oplus \cdots \oplus L_n) = e^{x_1} + \cdots + e^{x_n}$$

for sums of line bundles, and extending it first to general vector bundles by using the splitting principle, and then to arbitrary elements in $K^0(X)$ by linearity. We also define the Todd class of a complex vector bundle by

$$\mathcal{T}(L_1 \oplus \cdots \oplus L_n) = \prod_{i=1}^{n} \frac{c_1(L_i)}{1 - e^{c_1(L_i)}}$$

for sums of line bundles, and again we extend it to general vector bundles by use of the splitting principle.

We are ready now to state the cohomological form of the Atiyah–Singer theorem. Let

$$\mathcal{T}(X) = \mathcal{T}(TX \otimes \mathbb{C}) \in H^*(X; \mathbb{Q})$$

and define

$$\mathrm{ch}(P) = \phi_*^{-1}(\mathrm{ch}([\sigma(P)]) \in H^*(X; \mathbb{Q})$$

where $\phi_* : H^k(X; \mathbb{Q}) \to H^{k+n}(X; B^*X/S^*X)$ is the Thom isomorphism.

Theorem 2.13. *For any elliptic differential operator P on a closed oriented smooth manifold X, the index* $\mathrm{ind}(P)$ *of P is given by the formula*

$$\mathrm{ind}(P) = (\mathrm{ch}(P) \cup \mathcal{T}(X))[X]$$

where $[X]$ is the fundamental class of X.

As particular cases of this theorem, one may recover several well-known theorems in topology and geometry, as for instance, Gauss-Bonet theorem, the Hirzebruch signature formula, Riemann-Roch theorem, etc. See the references at the beginning of this section for details.

For our purposes we will be interested in an equivariant version of the Atiyah–Singer theorem, which will allow us to compute the index of certain elliptic operators on a manifold X in terms of the fixed-point set of an action of a group G on X.

Let X be a closed smooth manifold, E, F two complex vector bundles over X and G a Lie group acting on the triple (X, E, F), meaning that there is a smooth action of G on X and smooth actions of G on E and on F by vector bundle automorphisms that lift the action on X. We say that a differential operator

$$P : \Gamma(E) \to \Gamma(F)$$

is a G-operator, if it is equivariant with respecto to the action of G on E and F, that is,

$$P(gs) = gP(s)$$

for all $g \in G$ and $s \in \Gamma(E)$. In this situation, we can define a G-index. Note that if P is an elliptic G-operator, then $\operatorname{Ker} P$ and $\operatorname{Coker} P$ are finite-dimensional representations of G, so we can define the G-index of P as the element

$$\operatorname{ind}_G(P) = [\operatorname{Ker} P] - [\operatorname{Coker} P] \in R(G),$$

where $R(G)$ is the representation ring of G, defined as the Grothendieck group of the finite-dimensional complex representations of G. Given an element $g \in G$, we can also define a g-index by

$$\operatorname{ind}_g(P) = \operatorname{Tr}(g|_{\operatorname{Ker} P}) - \operatorname{Tr}(g|_{\operatorname{Coker} P}),$$

which gives a complex number (which is an integer if $g = \mathrm{id}$).

One can introduce a G-equivariant version of K-theory groups of a space X endowed with a G-action. These cohomology groups are denoted by $K^*_G(X)$, and they are defined in the same way as in the non-equivariant case, but restricting to G-bundles and G-equivariant maps. If the action of G on X is trivial, there is an isomorphism

$$K^0_G(X) \simeq K^0(X) \otimes R(G)$$

(compare with the case of equivariant cohomology discussed in the previous chapter). One can then prove a version of the Atiyah–Singer theorem for elliptic G-operators, much in the same way as in the non-equivariant case. However, in this case, one can prove a localization theorem that allows us to compute the index $\operatorname{ind}_g(P)$ for an element $g \in G$ in terms of the fixed-point set of the action of g on X. We now describe this formula. Let X be a closed and oriented smooth manifold, and let G be a cyclic group with generator g acting smoothly on it. We denote the inclusion of the fixed-point set

$$i : X^G \to X.$$

X^G is a submanifold, possibly disconnected and with components of different dimensions. Let $N \to X$ be its normal bundle. The action of G on N determines a decomposition as a direct sum

$$N \simeq N(-1) \oplus \sum_{0<\theta<\pi} N(\theta),$$

where $N(-1)$ is a real bundle where g acts as multiplication by -1 and $N(\theta)$ for $0 < \theta < \pi$) are complex bundles where g act as complex multiplication by $e^{i\theta}$. Then, we have

Theorem 2.14. *Let X be a closed smooth manifold. Let g be a generator of a cyclic group G acting on X and P an elliptic G-operator. Let $N \to X^g$ be the normal bundle of X^g on X, and $i : X^g \to X$ the inclusion of the fixed-point set. Then, with the previous notation, we have*

$$\operatorname{ind}_g P = (-1)^n \left(\frac{\operatorname{ch} i^*[\sigma(P)](g) \mathcal{R}(N^g(-1)) \prod_{0<\theta<\pi} \mathcal{S}^\theta(N(\theta)) \cdot \mathcal{T}(X^g)}{\det(1 - g|N)} \right) [TX^g],$$

where $\mathcal{R}(N(-1))$ (resp. $\mathcal{S}^\theta(N(\theta))$) is a function of the Pontrjagin classes of $N(-1)$ (resp. on the Chern classes of $N(\theta)$), $\det(1-g|N) \in H^0(X^g;\mathbb{C})$ assigns to each component of X^g containing a point x the value $\det(1-g|N_x)$ and n assigns to each component *of X^g its dimension. Here $[TX^g]$ is the fundamental class of the submanifold TX^g of TX, with twisted coefficients in the non-orientable case.*

We will be specially interested in applying this to obtain a formula relating the data of the action of a cyclic group on X with the signature of X. Assume that X is a closed and oriented manifold of dimension $2l$ with l even. Let G be a compact Lie group acting on X preserving the orientation. By choosing a G-invariant riemannian metric, one can check that G preserves the decomposition

$$H^l(X) \simeq H^l_+(X) \oplus H^l_-(X).$$

We introduce the following notation.

$$\sigma(g, X) = \mathrm{Tr}(g|(H^l_+(X)) - \mathrm{Tr}(g|H^l_-(X)),$$

where we consider the induced action of g on the homology group $H^l(X)$. In this situation, applying Theorem 2.14 to the differential operator $D^+ = (d + d^*)|\Lambda^+(X)$, we obtain the following formula.

Theorem 2.15. *With the previous notation, let $2t = \dim X^g$, $2r = \dim N(-1)$ and $s(\theta) = \dim_{\mathbb{C}} N(\theta)$ (these numbers depend on the connected component of X^g). Then, we have*

$$\sigma(g, X) = \left(2^{t-r} \prod_{0<\theta<\pi} (i \tan\theta/2)^{-s(\theta)} \mathcal{L}(X^g)\mathcal{L}(N(-1))^{-1} e(N(-1)) \prod_{0<\theta<\pi} \mathcal{M}^\theta(N(\theta))\right)[X^g],$$

where $\mathcal{L}$ (resp. $\mathcal{M}^\theta$) are functions that only depend on the Pontrjagin classes of $N(-1)$ (resp. the Chern classes of $N(\theta)$), and $e(N(-1))$ denotes the twisted Euler class of $N(-1)$, and $[X^g]$ denotes the twisted fundamental class of X^g, defined using the local coefficient system of orientations of X^g.

If X has dimension 4, we can give an explicit formula for the case where g is of order greater than 2.

Theorem 2.16. *Let X be a closed connected and oriented 4-manifold. Suppose that $\phi \in \mathrm{Diff}(X)$ has finite order bigger than 2.Then the fixed point set X^ϕ is a disjoint union of isolated points $p_1, \dots, p_m$ and embedded surfaces $S_1 \sqcup \cdots \sqcup S_n$, with each S_i connected. Suppose that the action of ϕ on the normal bundle of S_k is by rotation of angle $\theta_k \in S^1$, and the action of ϕ on $T_{p_i}X$ is given by two rotations of angles α_i and β_i in two orthogonal 2-dimensional subspaces of $T_{p_i}X$. Then all connected components of X^ϕ are orientable and*

$$\sigma(X) = \sum_{i=1}^m -\cot\frac{\alpha_i}{2}\cot\frac{\beta_i}{2} + \sum_{k=1}^n \sin^{-2}(\theta_k/2)\, S_k \cdot S_k.$$

Proof. The orientability of the connected components of X^ϕ is guaranteed by (1) in Lemma 4.21. If the order of ϕ is odd then the formula for $\sigma(X)$ follows from [3, Proposition 6.18]. For the general case note that the proof of [3, Proposition 6.18] works equally well if the order of ϕ is even and bigger than 2. Indeed, in this case the normal bundle N of every embedded connected surface $Y \subseteq X^\phi$ supports an invariant almost complex structure (by Lemma 4.25, because Y is orientable and hence so is N) and ϕ acts on N through multiplication by a complex number different from ± 1 (so in the notation of [3, §6] we have $N^\phi(-1) = 0$). □

2.3 Spin^c-structures and Dirac Operators

In this section we introduce some geometric structures that are needed in order to write the Seiberg–Witten equations. Therefore we will give a quick introduction to Spin^c-structures focusing on the concepts we need for Seiberg–Witten theory, omitting most of the proofs. For a comprehensive reference on Clifford algebras and Dirac operators the interested reader can consult [31]. Another useful references for this material are Chapter 1 of [55], Chapters 2 and 3 of [46] and the first part of [67].

As usual in differential topology, we start by introducing some structures in vector spaces, and then we will globalize them to vector bundles over manifolds.

Definition 2.17. *The Clifford algebra* $\mathrm{Cl}(n)$ *is the unitary and associative algebra over* $\mathbb{R}$ *generated by* $\mathbb{R}^n$ *subject to the relations*

$$v \cdot w + w \cdot v = -2\langle v, w\rangle,$$

where $v, w \in \mathbb{R}^n$ *and* $\langle v, w\rangle$ *denotes the Euclidean inner product of* v, w.

It is easy to see that for any orthonormal basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$, $\mathrm{Cl}(n)$ can also be described as the unitary and associative algebra over $\mathbb{R}$ generated by $e_1, \ldots, e_n$ subject to the relations

$$\begin{aligned} e_i \cdot e_j &= -e_j \cdot e_i \quad \text{for } i \neq j, \\ e_i \cdot e_i &= -1, \end{aligned}$$

for all $i, j \in \{1, \ldots, n\}$.

As the following examples show, both complex numbers and quaternions as particular cases of this construction.

1. $\mathrm{Cl}(1) \simeq \mathbb{C}$, where the isomorphism is given by sending e_1 to i.
2. $\mathrm{Cl}(2) \simeq \mathbb{H}$, the Hamilton quaternions, where the isomorphism is given by sending e_1 to i and e_2 to j.

In fact, there is a complete classification of Clifford algebras. The interested reader can consult Section 1.4 of [31].

Clifford algebras come naturally equipped with a $\mathbb{Z}/2$-grading,

$$\mathrm{Cl}(n) = \mathrm{Cl}^0(n) \oplus \mathrm{Cl}^1(n),$$

where $\mathrm{Cl}^0(n)$ (resp. $\mathrm{Cl}^1(n)$) is the subalgebra generated by elements of the form $v_1 \cdot \ldots \cdot v_k$, for k even (resp. k odd).

There is another decomposition of the Clifford algebra that will be important for us. We define the volume element of the Clifford algebra $\mathrm{Cl}(n)$,

$$\omega_{\mathbb{C}} = i^{\left[\frac{n+1}{2}\right]} e_1 \cdot \ldots \cdot e_n.$$

Observe that $\omega_{\mathbb{C}}^2 = 1$, so that multiplication by $\omega_{\mathbb{C}}$ induces a decomposition in eigenspaces

$$\mathrm{Cl}(n) = \mathrm{Cl}^+(n) \oplus \mathrm{Cl}^-(n),$$

where

$$\mathrm{Cl}^{\pm}(n) = \frac{1 \pm \omega_{\mathbb{C}}}{2} \mathrm{Cl}(n).$$

We now introduce the Spin groups, which are subgroups of Clifford algebras that will play a key role in the sequel.

Definition 2.18. $\mathrm{Spin}(n)$ *is the multiplicative subgroup of* $\mathrm{Cl}(n)$ *generated by the elements* $v_1 \cdot \ldots \cdot v_k$*, where* k *is even and* $||v_i|| = 1$ *for all* $i \in \{1, \ldots k\}$*. In particular,* $\mathrm{Spin}(n) \subset \mathrm{Cl}^0(n)$*. We endow* $\mathrm{Spin}(n)$ *with the topology induced by the Euclidean topology of* $\mathrm{Cl}^0(n)$*.*

The geometric significance of the Spin groups is given by the following proposition.

Proposition 2.19. *There is a* 2*-sheeted covering* $\pi : \mathrm{Spin}(n) \to SO(n)$*. In particular, for* $n \geq 3$*,* $\mathrm{Spin}(n)$ *is a simply-connected Lie group.*

Proof. We define

$$\pi(u_1 \cdot \ldots \cdot u_k)(v) = u_1 \cdot \ldots \cdot u_k \cdot v \cdot u_k \cdot \ldots \cdot u_1,$$

for any unit vectors $u_1, \ldots, u_k \in \mathbb{R}^n$.

We claim that for u a unit vector in $\mathbb{R}^n$, $\pi(u)$ is just the reflection on $u^\perp$, the hyperplane orthogonal to u. Let $v \in \mathbb{R}^n$. If $v \in u^\perp$, using the generating relations for $\mathrm{Cl}(n)$ we can compute (taking an orthonormal basis of $\mathbb{R}^n$ containing u and a unit vector parallel to v),

$$u \cdot v \cdot u = -v \cdot u \cdot u = v,$$

while if $v = \lambda u$ for some non-zero real λ,

$$u \cdot v \cdot u = \lambda u \cdot u \cdot u = -\lambda u = -v.$$

This, together with the linearity of the map $\pi(u)$, proves the claim.

It is a classical theorem that $SO(n)$ coincides with the linear maps obtained as composition of an even number of reflections across hyperplanes. In particular, π has image in $SO(n)$ and is surjective. It is also clear that π is a group homomorphism. Finally, observe that $\pi(\alpha) = \pi(-\alpha)$ for all $\alpha \in \mathrm{Spin}(n)$. It is not difficult to check directly that $\mathrm{Ker}(\alpha) = \{1, -1\}$, showing that π is a 2-sheeted covering.

The last statement follows from the fact that $\pi_1(SO(n)) \simeq \mathbb{Z}/2$ for all $n \geq 3$ and the fact that $\mathrm{Spin}(n)$ is connected, which is a straightforward consequence of its definition. □

We proceed to discuss some facts about representations of Clifford algebras and Spin groups.

Definition 2.20. *A (complex) representation of* Cl(n) *(resp.* Spin(n)*) is a complex finite-dimensional vector space V together with a morphism of algebras*

$$\mu : \mathrm{Cl}(n) \to \mathrm{GL}_{\mathbb{C}}(V)$$

(resp. a morphism of groups $\mu : \mathrm{Spin}(n) \to GL_{\mathbb{C}}(V)$).

A representation can be thought of as a vector space V together with a structure of Cl(n)-module (resp. Spin(n)-module) on it. We will usually abuse notation and say that V is a representation, where its module structure is understood.

A representation V of $Cl(n)$ (resp. $Spin(n)$) is said to be irreducible if it cannot be decomposed as $V = V_1 \oplus V_2$, where V_1, V_2 are subspaces of V invariant under the action of Cl(n) (resp. Spin(n)).

We will just need the following fact about representations.

Proposition 2.21. *Let n be even. Then, there is a unique complex irreducible representation of* Cl(n) *(up to isomorphism), which has (complex) dimension $2^{n/2}$. This representation is denoted by $S(n)$, and is called the complex representation of* Cl(n)*. This representation of* Cl(n) *induces by restriction a representation of* Spin(n)*, which splits as the sum of two irreducible representations of* Spin(n)*,*

$$S(n) = S^+(n) \oplus S^-(n).$$

$S(n)$ is the unique representation of Spin(n) *that extends to a representation of* Cl(n)*.*

When $n = 4$, we have an isomorphism $\mathrm{Spin}(4) \simeq \mathrm{SU}(2)\times\mathrm{SU}(2)$. Note that the vector spaces $S^{\pm}(4)$ have complex dimension 2, so $S^{\pm}(4) \simeq \mathbb{C}^2$. The representation $S^+(4)$ (resp. $S^-(4)$) is given by the composition of the projection of $\mathrm{Spin}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)$ on the first factor (resp. the second factor) with the standard action of SU(2) on $\mathbb{C}^2$.

Next, we introduce the groups Spin^c.

Definition 2.22. *The group* Spin$^c(n)$ *is the group* $\mathrm{Spin}(n) \times_{\mathbb{Z}/2} U(1)$*, where we identify elements (x, y) and $(-x, -y)$ in $Spin(n) \times U(1)$.*

Observe that the group Spin$^c(n)$ can also be seen as the multiplicative subgroup of $\mathrm{Cl}(n) \otimes_{\mathbb{R}} \mathbb{C}$ generated by Spin(n) and S^1.

We will need the following facts about Spin^c groups. They are not difficult to derive from the corresponding facts about Spin groups.

Proposition 2.23. $\mathrm{Spin}^c(n) \simeq \mathrm{Spin}(n) \times_{\mathbb{Z}/2} S^1$ *is a double covering of $SO(n) \times S^1$, which is non-trivial on any factor.*

Proposition 2.24. *Let n be even. There is a unique extension of the complex representation $S(n)$ of* Spin(n) *to* Spin$^c(n)$*. Moreover this representation split as $S(n) = S^+(n)\oplus S^-(n)$, where $S^{\pm}(n)$ are the unique extensions to* Spin$^c(n)$ *of the representations of* Spin(n)*.*

We are now ready to extend the previous definitions from the setting of linear algebra to vector bundles.

Let $E \to X$ be a real n-dimensional bundle over a manifold X. Suppose that this bundle is endowed with a riemannian structure (i.e. an inner product on each fiber that varies continuosly from fiber to fiber). Assume also that the bundle is oriented (i.e. there is a continuously varying choice of orientations on each fiber). Recall that the bundle $E \to X$ is orientable if the first Stiefel-Withney class of the bundle is 0.

Definition 2.25. *Let $n \geq 3$. Let $E \to X$ be an oriented vector bundle of rank n. Let $P_{\mathrm{SO}}(E) \to X$ be the associated principal $SO(n)$-bundle.*

1. *A spin structure $\mathfrak{s}$ on E is a principal $Spin(n)$-bundle $P_{\mathrm{Spin}}(E) \to X$ together with a two-sheeted covering*
$$\xi : P_{\mathrm{Spin}}(E) \to P_{\mathrm{SO}}(E),$$
such that $\xi(pg) = \xi(p)\xi_0(g)$ for all $p \in P_{\mathrm{Spin}}(E)$ and $g \in Spin(n)$, where $\xi_0 : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the covering map.

2. *A Spin^c structure $\mathfrak{s}$ on E is a pair $(P_{\mathrm{Spin}^c}(E), P_{U(1)})$, where $P_{\mathrm{Spin}^c}(E)$ is a principal $\mathrm{Spin}^c(n)$-bundle over X and $P_{U(1)}$ is a principal $U(1)$-bundle over X, together with a two-sheeted covering*
$$\xi : P_{\mathrm{Spin}^c}(E) \to P_{SO}(E) \times P_{U(1)}$$
such that $\xi(pg) = \xi(p)\xi_0(g)$ for all $p \in P_{\mathrm{Spin}^c}(E)$ and $g \in \mathrm{Spin}^c(n)$, where $\xi_0 : \mathrm{Spin}^c(n) \to SO(n) \times U(1)$ is the covering map.

If $\mathfrak{s}, \mathfrak{s}'$ are two Spin or Spin^c structures on E, we say that they are isomorphic if there is an isomorphism of the associated Spin or Spin^c bundle (covering the identity on the SO-bundle). We denote by $\mathrm{Spin}(E)$ (resp. $\mathrm{Spin}^c(E)$) the set of isomorphism classes of Spin (resp. Spin^c) structures on E.

We have the following topological characterizations of Spin and Spin^c structures.

Theorem 2.26. *Let $E \to X$ an oriented vector bundle of rank $n \geq 3$. Then,*

1. *E admits some Spin structure if and only if $w_2(E) = 0$. Moreover, in this case, the set of isomorphism classes of Spin structures on E is a torsor over $H^1(X; \mathbb{Z}_2)$.*

2. *E admits some Spin^c structure if and only if $w_2(E)$ is the reduction $\mod 2$ of some integral class $\alpha \in H^2(X)$. Moreover, in this case, the set of isomorphism classes of $Spin^c$ structures on E is a torsor over $H^2(X)$.*

Definition 2.27. *A Spin manifold (resp. a Spin^c manifold) X is an oriented manifold together with a Spin structure (resp. a Spin^c structure) on its tangent bundle.*

Not all oriented manifolds admit Spin or Spin^c structures. For our purposes, since we are mainly interested in 4-manifolds, it is enough to know the following result.

Theorem 2.28. *Any oriented* 4*-manifold admits a* Spin^c *structure on its tangent bundle.*

For the proof, see [67, Theorem 5.11].

We note that, in contrast, there are 4-manifolds that admit no Spin structure. However, all 3-manifolds admit a Spin structure.

If $\mathfrak{s}$ is a Spin^c structure on X, we denote by $\mathcal{L}$ the complex line bundle associated to $P_{U(1)}$. $\mathcal{L}$ is called the determinant bundle of $\mathfrak{s}$.

Proposition 2.29. *Let X be a* Spin^c *manifold. The set of* Spin^c *structures on a manifold X is a torsor over $H^2(X)$, that is, fixing a* Spin^c *structure $\mathfrak{s}_0$ on X, we have an identification*

$$H^2(X) \simeq \mathrm{Spin}^c(X).$$

Next, we introduce the Spin and Spin^c bundles on a Spin or Spin^c manifold. We start by defining the Clifford bundle, which is well-defined for any smooth manifold.

Definition 2.30. *Let X be an oriented riemannian manifold of dimension n. The (complex) Clifford bundle of E is the bundle*

$$\mathrm{Cl}(E) = P_{SO}(E) \times_{cl(\rho)} (Cl(n) \otimes \mathbb{C})$$

where $cl(\rho) : SO(n) \to \mathrm{Aut}(Cl(n))$ is the representation of $SO(n) \leq \mathrm{Cl}(n)$ on the Clifford algebra $\mathrm{Cl}(\mathbb{R}^n)$ given by Clifford multiplication.

Definition 2.31. *Let X be an oriented riemannian manifold with a* Spin *structure.*

The (complex) spinor bundle of X is the complex vector bundle $S(X) \to X$ such that

$$S(X) = P_{Spin}(X) \times_\mu S(n)$$

where $\mu : \mathrm{Spin}(n) \to S(n)$ is the complex representation of the Spin(n) group.

Definition 2.32. *Let X be an oriented riemannian manifold with a* Spin^c *structure.*

The spinor bundle of X is the complex vector bundle $S(X) \to X$ such that

$$S(X) = P_{Spin^c}(X) \times_\mu S(n)$$

where $\mu : \mathrm{Spin}^c(n) \to S(n)$ is the complex representation of the Spin$^c(n)$ group.

Note that we use the same nomenclature for both Spin and Spin^c bundles. It will be usually clear from the context which spinor bundle we are referring to. In the context of Seiberg–Witten theory, we will always refer to the Spin^c spinor bundle. Note also that Clifford bundles are defined for any smooth manifold, while spinor bundles are only defined for Spin and Spin^c manifolds. Since Spin(n) and Spin$^c(n)$ are compact Lie groups, the spinor bundle $S(X)$ comes equipped with a natural hermitian metric. In case X is Spin, we may also define $\mathrm{Cl}(X)$ as the associated bundle $P_{Spin(n)} \times_{\rho'} (Cl(n) \otimes \mathbb{C})$, where $\rho' : \mathrm{Spin}(n) \to \mathrm{Cl}(n)$ is the representation of $\mathrm{Spin}(n)$ on $\mathrm{Cl}(n)$ given by conjugation. A similar remark applies if X is Spin^c.

Recall that there is an action of $\mathrm{Cl}(n)$ on $S(n)$ given by Clifford multiplication. This action globalizes to an action of $\mathrm{Cl}(X)$ on $S(X)$, given by fiberwise Clifford multiplication:

$$\mathrm{Cl}(X) \times S(X) \to S(X).$$

Recall also that we have defined an element $\omega_{\mathbb{C}} \in \mathrm{Cl}(n) \otimes \mathbb{C}$. This globalizes to a section of the bundle $\mathrm{Cl}(X)$ of norm 1, that we will still denote by $\omega_{\mathbb{C}}$.

Recall that in both the Spin and the Spin^c cases we have a splitting of $S(n)$ into irreducible representations, $S(n) = S^+(n) \oplus S^-(n)$, given by the eigenvalues of multiplication by $\omega_{\mathbb{C}}$. This splitting induces a splitting of bundles

$$S(X) = S^+(X) \oplus S^-(X).$$

To end this section, let us consider connections on spinor bundles.

Let X be an oriented riemannian manifold, and denote by ∇ the Levi-Civita covariant derivative.

There is a unique connection, that we still denote by ∇, on the vector bundle $\mathrm{Cl}(X) \to X$ that satisfies

$$\nabla(\varphi\psi) = (\nabla\varphi)\psi + \varphi\nabla(\psi)$$

and coincides with the Levi-Civita connection on $TX \subset \mathrm{Cl}(X)$. That is, ∇ acts as a derivation with respect to the Clifford multiplication.

In a similar fashion, there is a unique connection, still denoted by ∇, on the Spin spinor bundle $S(X)$ that satisfies

$$\nabla(\varphi\sigma) = \nabla(\varphi)\sigma + \varphi(\nabla\sigma).$$

That is, ∇ acts as a derivative with respect to the Clifford module structure of $S(X)$.

In the Spin^c case, we need to specify a unitary connection A on the determinant line bundle of a Spin^c structure $\mathfrak{s}$ (equivalently, a connection on the principal $U(1)$-bundle $P_{U(1)}$ associated to $\mathfrak{s}$). Then, we have a connection on $P_{SO} \times P_{U(1)}$, which in turn induces a connection on P_{Spin^c} via the covering map

$$P_{\mathrm{Spin}^c} \to P_{SO} \times P_{U(1)}.$$

We also have a connection on the associated spinor bundles $S(X)$ and $S^{\pm}(X)$, denoted by ∇_A, which satisfies:

$$\nabla_A(\varphi\sigma) = \nabla(\varphi)\sigma + \varphi(\nabla_A\sigma).$$

To end this section, we define Dirac operators.

Definition 2.33. *A Dirac operator on a* Spin *spinor bundle $S \to X$ is a first order differential operator $D : \Gamma(S(X)) \to \Gamma(S(X))$ defined at $x \in X$ by:*

$$D\sigma = \sum_{j=1}^{n} e_j \nabla_{e_j} \sigma$$

where $(e_1, \dots, e_n)$ *is an orthonormal basis of* T_xX *and* $\sigma \in S_x$.

Similarly, a Dirac operator on a Spin^c *spinor bundle* $S \to X$ *is a first order differential operator* $D^A : \Gamma(S(X)) \to \Gamma(S(X))$ *defined at* $x \in X$ *by:*

$$D^A\sigma = \sum_{j=1}^{n} e_j (\nabla_A)_{e_j}\sigma$$

where A *is a* $U(1)$*-connection on the determinant line bundle of the* Spin^c *structure, and* $(e_1, \dots, e_n)$ *is an orthonormal basis of* T_xX *and* $\sigma \in S_x$. *This is well-defined (i.e. the operators* D *and* D^A *do not depend on the choice of orthonormal basis).*

In both cases the Dirac operator restricts to operators (also called Dirac operators)

$$D^+ : \Gamma(S^+(X)) \to \Gamma(S^-(X))$$
$$D^- : \Gamma(S^-(X)) \to \Gamma(S^+(X))$$

2.4 The Seiberg-Witten moduli spaces

Let X be a closed oriented 4-manifold, and let $\mathfrak{s}$ be a *Spin^c* structure on X, with determinant bundle $\mathcal{L}$ and associated spinor bundles $S^+(X), S^-(X)$. Let A be a unitary connection on $\mathcal{L}$ and let $\psi \in C^\infty(S^+(X))$.

In dimension 4, the Hodge star operator

$$* : \Omega^2(X) \to \Omega^2(X)$$

satsifies $*^2 = 1$, so it gives rise to a decomposition

$$\Omega^2(X) = \Omega^2_+(X) \oplus \Omega^2_-(X),$$

where $\Omega^2_\pm(X)$ are ± 1 the eigenspaces of $*$. We call a form $\phi \in \Omega^2_+(X)$ self-dual, and a form $\phi \in \Omega^2_-(X)$ anti-self-dual.

Lemma 2.34. *There is a canonical isomorphism*

$$\mathrm{End}^{tr}_{\mathbb{C}}(S^+(X)) \simeq \Omega^2_+(X) \otimes \mathbb{C},$$

where $\mathrm{End}^{tr}_{\mathbb{C}}(S^+(X))$ *stands for the traceless complex linear automorphisms of* $S^+(X)$.

We are now ready to introduce the Seiberg–Witten equations. Let X be an oriented 4-manifold, let g be a riemannian metric on X and let $\mathfrak{s}$ be a Spin^c structure on X, with determinant bundle $\mathcal{L}$. We denote by $\mathcal{A}(\mathcal{L})$ the space of unitary connections on the determinant bundle. Define an isomorphism

$$\begin{aligned} q : C^\infty(S^+(X)) \to \quad & \Omega^2_+(X; i\,\mathbb{R}) \\ \psi \mapsto \quad & \psi \otimes \psi^* - \tfrac{|\psi|^2}{2}\,\mathrm{Id}, \end{aligned}$$

where on the right-hand side we use the identification given by the previous lemma. Note that indeed

$$\psi \otimes \psi^* - \frac{|\psi|^2}{2}\,\mathrm{Id} \in \mathrm{End}^{tr}_{\mathbb{C}}(S^+(X)).$$

It can be proved that the image of q is a purely imaginary self-dual form.

A pair $(\psi, A) \in C^\infty(S^+(X)) \times \mathcal{A}(\mathcal{L})$ satisfies the (perturbed) Seiberg–Witten equations if:

$$D^A(\phi) = 0$$
$$F_A^+ = q(\psi) + ih$$

where $h \in \Omega^2_+(X)$ is a self-dual 2-form on X. In the second equation, F_A^+ represents the self-dual part of the curvature form F_A. The term ih is called the perturbation term, and we will usually think of h as a fixed parameter in the equations, and not as part of the solution.

We are interested in studying the space formed by its solutions. For that purpose, we define:

$$\mathcal{C}(\mathfrak{s}) = \mathcal{A}(L) \times C^\infty(S^+(X))$$

where $\mathfrak{s}$ is a $Spin^c$ structure on X, $\mathcal{A}(L)$ is the space of (smooth) connections on L and $C^\infty(S^+(X))$ is the space of smooth sections on the spinor bundle associated to $\mathfrak{s}$.

For technical reasons, when studying these spaces it is convenient to work on Banach or Hilbert spaces, in order to apply theorems from functional analysis. However, spaces of smooth sections are Fréchet spaces which are not Banach. For that reason, one works with suitable Sobolev completions of these spaces, which are Banach spaces that allows us to keep track of the regularity of the objects. Recall that Sobolev spaces L^p_k form a filtration:

$$L^p = L^p_0 \supset L^p_1 \supset L^9_2 \supset \dots$$

with the property that $\bigcap_k L^p_k = C^\infty$.

This is why we need to consider the following refined version of the configuration space:

$$\mathcal{C}(\mathfrak{s}) = \mathcal{A}_{L^2_2}(L) \times L^2_2(S^+(\mathfrak{s}))$$

where $\mathcal{A}_{L^2_2}(L)$ are the L^2_2-connections on L and $L^2_2(S^+(\mathfrak{s}))$ is the space of L^2_2 sections of $S^+(\mathfrak{s})$. Then, one develops the theory with these Sobolev-completed version of the configuration space, and at the end one checks, using elliptic regularity (here it is key the fact that Seiberg–Witten equations are elliptic), that all objects of interest are in fact smooth. In this survey of Seiberg–Witten theory we will not be concerned about these technicalities, and will content ourselves with a description of the main steps in stablishing the properties of Seiberg–Witten moduli spaces using the C^∞ objects.

We now introduce the group of bundle automorphisms (also called the group of changes of gauge) into play. Let $\mathfrak{s}$ be a $Spin^c$ structure on X, and let P be the associated principal $Spin^c(4)$-bundle. We define $\mathcal{G}(P)$ as the group of principal bundle automorphisms of P covering the identity on the oriented frame bundle of TX (that is, a smooth map $\sigma : P \to P$ such that $\sigma(eg) = \sigma(e)g$ for all $e \in P, g \in Spin^c(4)$ and such that $\xi(\sigma(e)) = \xi(e)$ for all $e \in P$).

Observe that an element $\sigma \in \mathcal{G}(P)$ can be identified with a map $\tilde{\sigma} : X \to S^1$. Indeed, $e, \sigma(e) \in P$ are both lifts of $\xi(e) \in P_{SO(4)}$. Therefore, both must differ by an element in $Spin^c(4)/SO(4) \simeq S^1$. Since σ is a principal bundle automorphism, this element depends only on $x = \pi(e)$ and not on e, so it is well-defined. Conversely, given $\tilde{\sigma} : X \to S^1$, we can define a bundle automorphism of P as $\sigma(e) = e \cdot \tilde{\sigma}(e)$ for all $e \in P$. Moreover, if $\sigma_1, \sigma_2 \in \mathcal{G}(P)$, then

$$\widetilde{\sigma_1\sigma_2} = \widetilde{\sigma_1}\widetilde{\sigma_2},$$

where on the right hand side, the product is pointwise.

With this definition, $\mathcal{G}(P)$ can be seen to be an infinite-dimensional Lie group. Its Lie algebra is $C^\infty(X; i\,\mathbb{R})$ with trivial Lie bracket.

Every bundle map $\sigma : P \to P$ induces a map

$$S^{\pm}(\sigma) : S^{\pm}(X) \to S^{\pm}(X),$$

defined as follows. If $[e, v] \in P \times_\mu S^{\pm}(4) = S^{\pm}(X)$, then $S^{\pm}(\sigma)([e, v]) = [\sigma(e), v]$. Using the fact that σ is a bundle map one can easily check that this is well-defined and that $S^{\pm}(\sigma)$ is a vector bundle map. In a similar fashion, σ also induces a line bundle map

$$\det \sigma : \mathcal{L} \to \mathcal{L}.$$

Indeed, $\sigma : P \to P$ induces a principal $U(1)$-automorphism by passing to the quotient principal bundle $P/P_{SO(4)}$ (here the fact that σ lifts the identity in $P_{SO(4)}$ is crucial). In turn, this induces a line bundle map in the line bundle $\mathcal{L}$.

Now we can define an action of $\mathcal{G}(P)$ in $\mathcal{C}(\mathfrak{s})$ by the formula:

$$(A, \psi) \cdot \sigma = ((\det \sigma)^* A, S^+(\sigma^{-1})(\psi))$$

This gives a smooth action of $\mathcal{G}(P)$ in $\mathcal{C}(\mathfrak{s})$.

We can study the effect of this group action on the Seiberg-Witten equations. First of all, we define the (perturbed) Seiberg-Witten operator:

$$F^{SW} : \mathcal{C}(\mathfrak{s}) \times C^\infty(\Omega^2_+ X \otimes \mathbb{R}) \to C^\infty((\Omega^2_+ X \otimes i\,\mathbb{R}) \oplus S^-(\mathfrak{s}))$$

given by:

$$F^{SW}(A, \psi, h) = (F_A^+ - q(\psi) - ih, D^A(\psi)).$$

Therefore, a pair (A, ψ) is a solution of the Seiberg-Witten equations perturbed by h if and only if $F^{SW}(A, \psi, h) = 0$.

It can be proved that the set of solutions of the Seiberg–Witten solutions is invariant under the action of the gauge group $\mathcal{G}(P)$. We will consider the quotient $\mathcal{B}(\mathfrak{s}) = \mathcal{C}(\mathfrak{s})/\mathcal{G}(P)$. But before doing that, we study stabilizers of points in $\mathcal{C}(\mathfrak{s})$ under this action.

Lemma 2.35. *Let $(A, \psi) \in \mathcal{C}(\mathfrak{s})$. Then, its stabilizer under the action of $\mathcal{G}(P)$ is trivial unless $\psi = 0$. If $\psi = 0$, identifying the elements of $\mathcal{G}(P)$ with maps in $C^\infty(X, S^1)$, the stabilizer consists of all constant maps. Therefore, this stabilizer can be identified with S^1.*

This lemma tells us that there are two kinds of points in the configuration space, those with trivial stabilizer and those with stabilizer isomorphic to S^1. We call the former irreducible points, and the latter reducible points. We will denote by $\mathcal{C}^*(\mathfrak{s}) \subset \mathcal{C}(\mathfrak{s})$ the subset of irreducible points. Observe that it is an open subset of $\mathcal{C}(\mathfrak{s})$ since it consists of the points (A, ψ) with $\psi \neq 0$. We can now study the moduli spaces of solutions of the (perturbed) Seiberg–Witten equations. This space, denoted by $\mathcal{M}^h(\mathfrak{s})$ is the subset of $\mathcal{B}(\mathfrak{s})$ consisting of (equivalence classes of) solutions to the Seiberg–Witten equations perturbed by h, considered as a topological space with the subspace topology.

The next step in order to set up Seiberg–Witten invariants is to show that with a suitable perturbation h this space is a closed smooth manifold. We start with the following theorem.

Theorem 2.36. *Let $b_2^+(X) > 0$. Fix a metric on X. Then, for a generic self-dual C^∞ 2-form h the following holds. For any Spin^c structure $\mathfrak{s}$ on X, the moduli space $\mathcal{M}^h(\mathfrak{s})$ is a smooth submanifold of $\mathcal{B}^*(\mathfrak{s})$ of dimension*

$$\frac{c_1(L)^2 - (2\chi(X) + 3\sigma(X))}{4},$$

where $\chi(X)$ and $\sigma(X)$ denote the Euler characteristic and signature of X, respectively. If this dimension is negative, there are no solutions to the Seiberg-Witten equations perturbed by h.

In the statement of this theorem, the term 'generic' means that the theorem holds true for every h in a subset of $\Omega^2_+(X)$ of Baire second category.

We will sketch the proof of this theorem, avoiding technical details. In order to begin, we introduce a parametrized moduli space, that takes into account all possible perturbations at once, and we show that it is a smooth manifold.

We define the parametrized moduli space of irreducible solutions $\mathcal{PM}^*(\mathfrak{s}) \subset \mathcal{B}(\mathfrak{s}) \times \Omega^2_+ X$ consisting of all pairs $([A, \psi], h)$ such that $F^{SW}(A, \psi, h) = 0$ and $\psi \neq 0$.

Proposition 2.37. *$\mathcal{PM}^*(\mathfrak{s})$ is a smooth submanifold of $\mathcal{B}^*(\mathfrak{s}) \times \Omega^2_+ X$.*

Consider the projection map $\pi : \mathcal{PM}^*(\mathfrak{s}) \to \Omega^2_+ X$. Then, $\mathcal{M}^h(\mathfrak{s}) = \pi^{-1}(h)$.

Moreover, π is smooth, and its differential is Fredholm with index

$$d(L) = \frac{c_1(L)^2 - (2\chi(X) + 3\sigma(X))}{4}$$

Proof. Let $([A, \psi], h) \in \mathcal{PM}^*(\mathfrak{s})$. One can show that $DF_{(A,\psi,h)}$ is surjective. Therefore, by the inverse mapping theorem, the space $\widetilde{\mathcal{PM}}^*(\mathfrak{s})$ consisting of all triples $(A, \psi, h) \in \mathcal{C}^*(\mathfrak{s}) \times \Omega^2_+ X$ such that $F^{SW}(A, \psi, h) = 0$ is a smooth submanifold of $\mathcal{C}^*(\mathfrak{s}) \times \Omega^2_+ X$. Since all points are irreducible (and hence have trivial stabilizer) dividing out by the action of $\mathcal{G}(\mathfrak{s})$ we see that $\mathcal{PM}^*(\mathfrak{s})$ is a smooth submanifold.

It is not difficult to see that π is smooth and that $d\pi$ is Fredholm. We can compute the index of $d\pi$ by using the Atiyah-Singer index theorem, and obtain the formula in the statement. □

Using the Smale-Sard theorem, we obtain that there is a subset of Ω^2_+X of Baire second category of regular values for the map $\pi : \mathcal{PM}^*(\mathfrak{s}) \to \Omega^2_+X$. We call such a regular value a generic perturbation. Therefore, we obtain the following corollary.

Corollary 2.38. *For a generic $h \in \Omega^2_+X$ the space $\mathcal{M}^{h*}(\mathfrak{s})$ of irreducible solutions of the Seiberg-Witten equations perturbed by h is a smooth submanifold of $\mathcal{B}(\mathfrak{s})$ of dimension $d(L)$.*

The next step is to prove that the moduli spaces $\mathcal{M}^h(\mathfrak{s})$ are always compact. It is important to note that this moduli space can have reducible points, and therefore it is not necessarily a smooth manifold.

Proposition 2.39. *For any $h \in \Omega^2_+X$, $\mathcal{M}^h(\mathfrak{s})$ is compact.*

The proof of this proposition uses crucially the fact that the equations are elliptic.

Now we will show that, if $b_2^+(X) > 0$, by picking a generic pertubation, we can obtain moduli spaces that do not contain any reducible points. Therefore, combining the previous theorems, this moduli spaces will be closed smooth manifolds.

Proposition 2.40. *Suppose $b_2^+(X) > 0$. Then, there is a subset of the space of riemannian metrics on X of Baire second category such that there are no reducible solutions to the (unperturbed) Seiberg-Witten equations with respect to the given metric (we will call such metrics generic). Moreover, for any Spin^c structure on X, if there are no reducible solutions to the unperturbed Seiberg-Witten equations, then there are no reducible solutions to the perturbed Seiberg-Witten equations for all sufficiently small perturbations h. For any metric and a generic perturbation there are no reducible solutions.*

Proof. For any metric, a reducible solution to the perturbed Seiberg-Witten equations satisfy $F_A^+ = ih$. By Chern-Weil theory, F_A^+ represents the cohomology class $2\pi/ic_1(L)$. Therefore, the orthogonal projection of h into the self-dual harmonic 2-forms (denoted by h^+) must be equal to $-2\pi\alpha^+$, where α^+ is the self-dual part of the harmonic representative of $c_1(L)$, and hence depends only on the Spin^c structure $\mathfrak{s}$. Self-dual harmonic forms form a linear subspace of $H^2(X;\mathbb{R})$ of dimension $b_2^+(X)$. Hence, the space of perturbations h such that $-2\pi\alpha^+ = h^+$ is a linear subspace of codimension b_2^+. So, if $b_2^+ > 0$, a generic h satisfies $-2\pi\alpha^+ \neq h^+$. Therefore, for a generic h there are no reducible solutions to the perturbed Seiberg-Witten equations. □

Moduli spaces of Seiberg–Witten theory are orientable. For a proof of this fact and discussion about how to choose such an orientation, we refer the reader to [55, Section 2.2.4]. Here we only remark that such an orientation is uniquely defined given orientations of $H^1(X;\mathbb{R})$ and $H^2_+(X;\mathbb{R})$.

Suming up, we have obtained the following result.

Corollary 2.41. *Let X be an oriented closed manifold with $b_2^+(X) > 0$. Then, for a generic perturbation $h \in \Omega^2_+ X$, and for any Spin^c structure $\mathfrak{s}$ on X, the moduli space $\mathcal{M}^h(\mathfrak{s})$ is a compact oriented smooth manifold of dimension $d(L)$.*

If we want to construct invariants of the underlying smooth 4-manifold from the moduli spaces of perturbed Seiberg-Witten equations, we need to be sure that the invariants that we obtain are independent of the chosen metric and perturbation. That is, we need to ensure that the obtained invariant depends exclusively on the smooth manifold X.

We will see that this is only possible if $b_2^+(X) > 1$. In the case $b_2^+(X) = 1$, we don't have a well-defined invariant, but we will see later how to define a pair of invariants, depending on the manifold X, and the metric g and the perturbation h used to define the Seiberg-Witten equations.

The next proposition gives us a way of relating moduli spaces (via a smooth cobordism) in the case $b_2^+(X) > 1$.

Given a smooth path of metrics γ on X, and a smooth path of self-dual 2-forms η, where $\eta(t)$ is self-dual with respect to the metric $\gamma(t)$, for all t, we define the parametrized moduli space $\mathcal{M}(\mathfrak{s}, \eta)$ as the set of all elements $([A, \psi], t) \in \mathcal{B}(\mathfrak{s}) \times [0, 1]$ satisfying the equations:

$$F_A^{+_t} = q(\psi) + i\eta(t)$$
$$D_{A,\gamma(t)}\psi = 0$$

where $+_t$ means the self-dual part of F_A with respect to the metric $\gamma(t)$, and $D_{A,\gamma(t)}$ is the Dirac operator obtained using the Levi-Civita connection associated with the metric $\gamma(t)$ and the connection A on L.

Theorem 2.42. *Let X be a closed oriented 4-manifold such that $b_2^+(X) > 1$. Let g_0, g_1 be two metrics on X, and let h_0, h_1 be two generic self-dual 2-forms on X with respect to g_0, g_1 respectively.*

Then, for a generic path of metrics γ with $\gamma(0) = g_0$ and $\gamma(1) = g_1$, and for a generic path of self-dual 2-forms η such that $\eta(0) = h_0$ and $\eta(1) = h_1$, the parametrized moduli space $\mathcal{M}(\mathfrak{s}, \eta)$ as defined above consists only of irreducible points and is a smooth compact submanifold with boundary of $\mathcal{B}(\mathfrak{s}) \times [0, 1]$. Moreover, its boundary is the disjoint union of the moduli spaces $\mathcal{M}(\mathfrak{s}, h_0)$ with metric g_0, and $\mathcal{M}(\mathfrak{s}, h_1)$ with metric g_1.

The proof of this theorem is analogous to the proof of Corollary 2.41. However, now the condition $b_2^+(X) > 1$ appears (compare with the condition $b_2^+(X) > 0$ in the non-parametric case). Since this condition will be playing a major role later, let us discuss it briefly.

Let $\mathcal{S}_\mathfrak{s}$ be the space of all pairs (metric, perturbation) (g, η) such that its corresponding moduli space $\mathcal{M}(\mathfrak{s}, (g, \eta))$ does not contain reducible points.

Proposition 2.43. *The space $\mathcal{S}_\mathfrak{s}$ is connected if and only if $b_2^+(X) > 1$. If $b_2^+(X) = 1$, $\mathcal{S}_\mathfrak{s}$ has two connected components.*

See Proposition 7.10 and the following discussion in [67].

Therefore, in the case $b_2^+(X) = 1$, there exist pairs $(g_1, \eta_1), (g_2, \eta_2)$ such that any path between them contains a pair (g, h) such that its associated moduli space contains reducible solutions. This is why we need to impose the condition $b_2^+(X) > 1$ in Theorem 2.42 in order to ensure that two generic moduli spaces can be joined with a smooth cobordism.

2.5 SW invariants for $b_2^+(X) > 1$

For this section, assume that X is a smooth oriented 4-manifold with $b_2^+(X) > 1$. We will construct an invariant of X.

Fix a riemannian metric g on X. Fix also a $Spin^c$ structure $\mathfrak{s}$ on X with determinant bundle L. Choose orientations for $H^1(X;\mathbb{R})$ and $H^2_+(X;\mathbb{R})$. Then, for a generic self-dual 2-form h, $\mathcal{M}(\mathfrak{s}, h)$ is a compact oriented smooth submanifold of $\mathcal{B}^*(\mathfrak{s})$ of dimension $d(L)$.

We define now a principal $U(1)$-bundle over $\mathcal{B}^*(\mathfrak{s})$. Recall that $\mathcal{B}^*(\mathfrak{s}) = \mathcal{C}^*(\mathfrak{s})/\mathcal{G}(\mathfrak{s})$, where $\mathcal{G}(\mathfrak{s})$ is the group of changes of gauge, and can be thought as the group of all maps $\sigma : X \to S^1$. Fix a base point $x_0 \in X$. We define $\mathcal{G}^0(\mathfrak{s})$ as the subgroup of $\mathcal{G}(\mathfrak{s})$ consisting of all maps such that $\sigma(x_0) = 1$. Let $\mathcal{B}^0(\mathfrak{s}) = \mathcal{C}^*(\mathfrak{s})/\mathcal{G}^0(\mathfrak{s})$. Then, $\mathcal{B}^0(\mathfrak{s}) \to \mathcal{B}^*(\mathfrak{s})$ is a principal $U(1)$-bundle. Let $\mu \in H^2(\mathcal{B}^*(\mathfrak{s}); \mathbb{Z})$ be its first Chern class.

Definition 2.44. *We define the Seiberg-Witten invariant of the $Spin^c$ structure $\mathfrak{s}$ as follows. Let $d(L) = (c_1(L)^2 - (2\,\mathfrak{s}(X) + 3\sigma(X))/4$ be the dimension of $\mathcal{M}(\mathfrak{s}, h)$. If $d(L)$ is even, we define:*

$$SW(\mathfrak{s}) = \int_{\mathcal{M}(\mathfrak{s},h)} \mu^{d(L)/2} \tag{2.1}$$

If $d(L)$ is odd, define $SW(\mathfrak{s}) = 0$.

Remark 2.45. *$d(L)$ is even if and only if $b_1(X) - b_2^+(X)$ is odd.*

Proposition 2.46. *Assume $b_2^+(X) > 1$. Then, $SW(\mathfrak{s})$ does not depend on the choice of metric nor on the choice of perturbation.*

Proof. Let g_0, g_1 be two metrics on X, and let h_0, h_1 be two self-dual 2-forms on X (with respect to g_0, g_1 respectively). By Theorem 2.42, we can choose paths γ of metrics and η of 2-forms such that $\eta(t)$ is self-dual with respect to $\gamma(t)$, and $\gamma(0) = g_0, \gamma(1) = g_1, \eta(0) = h_0, \eta(1) = h_1$, such that the parametrized moduli space $\mathcal{M}(\mathfrak{s}, \eta)$ is a closed smooth manifold with boundary consisting only of irreducible points, where the boundary consists of the disjoint union of $\mathcal{M}(\mathfrak{s}, h_0)$ and $\mathcal{M}(\mathfrak{s}, h_1)$.

Moreover, the choice of orientations of the cohomology spaces of X and the canonical orientation of I induce an orientation of $\mathcal{M}(\mathfrak{s}, \eta)$, such that its boundary, as an oriented manifold, is $\mathcal{M}(\mathfrak{s}, h_1) - \mathcal{M}(\mathfrak{s}, h_0)$.

By Stokes' theorem,

$$0 = \int_{\mathcal{M}(\mathfrak{s},\eta)} d(\mu^{d(L)/2}) = \int_{\mathcal{M}(\mathfrak{s},h_1)} \mu^{d(L)/2} - \int_{\mathcal{M}(\mathfrak{s},h_0)} \mu^{d(L)/2}.$$

So,

$$\int_{\mathcal{M}(\mathfrak{s},h_1)} \mu^{d(L)/2} = \int_{\mathcal{M}(\mathfrak{s},h_0)} \mu^{d(L)/2}$$

and $SW(\mathfrak{s})$ does not depend on the metric nor the perturbation chosen. □

2.6 SW invariants for $b_2^+ = 1$

In this section we treat the case where $b_2^+(X) = 1$. For an in-depth discussion we refer the reader to Section 7.4 of [67] and references therein. The difference with the case when $b_2^+(X) > 1$ is that there might exist generic pairs of (metric, perturbation) (g_1, η_1), (g_2, η_2) whose moduli spaces cannot be connected by any smooth cobordism. More precisely, for any fixed Spin^c structure $\mathfrak{s}$ the space $\mathcal{S}_{\mathfrak{s}}$ of all pairs (metric, perturbation) whose moduli space of Seiberg–Witten solutions contain no reducible solution (that is, solutions (A, ψ) with $\psi = 0$) has two connected components. The Seiberg–Witten moduli spaces associated to two generic elements of $\mathcal{S}_{\mathfrak{s}}$ can be connected by a smooth cobordism if the two elements belong to the same connected component of $\mathcal{S}_{\mathfrak{s}}$, but there is no reason to expect the existence of such a cobordism if they belong to different connected components. Hence, we should consider two possibly different Seiberg–Witten invariants, one for each connected component of $\mathcal{S}_{\mathfrak{s}}$.

Fix an orientation of the one-dimensional vector space $H^2_+(X;\mathbb{R})$, i.e., a connected component of $H^2_+(X;\mathbb{R}) \setminus \{0\}$, and call it the positive orientation of $H^2_+(X;\mathbb{R})$. One can prove that it is possible to label the connected components of $\mathcal{S}_{\mathfrak{s}}$ as $\mathcal{S}^+_{\mathfrak{s}}$ and $\mathcal{S}^-_{\mathfrak{s}}$ in such a way that the following holds. For any metric g on X let us denote by ω_g the unique self-dual g-harmonic 2-form of L^2-norm 1 whose cohomology class induces the positive orientation in $H^2_+(X;\mathbb{R})$. Then $(g, \pm \mathbf{i}\lambda\omega_g) \in \mathcal{S}^{\pm}_{\mathfrak{s}}$ for $\lambda > 0$ sufficiently big.

Definition 2.47. *Let X be a closed orientable smooth 4-manifold. Fix an orientation of $H_2^+(X;\mathbb{R})$. We define the (positive and negative) Seiberg–Witten invariants of X*

$$SW^{\pm} : \mathrm{Spin}^c(X) \to \mathbb{Z},$$

where $SW^{\pm}(\mathfrak{s})$ is the invariant obtained from the formula (2.1) by using a generic pair belonging to $\mathcal{S}^{\pm}_{\mathfrak{s}}$.

Define

$$w(\mathfrak{s}) = SW^+(\mathfrak{s}) - SW^-(\mathfrak{s}).$$

This difference $w(\mathfrak{s})$ can be computed by means of a so-called wall-crossing formula. As we will see, this wall-crossing formula depends exclusively on topological information of X. Therefore, even if in this case there is no unique diffeomorphism invariant of X (since a diffeomorphism of X can interchange the components of $\mathcal{S}_{\mathfrak{s}}$), the pair of invariants $SW^{\pm}$ depend only on the diffeomorphism class of X.

We now decribe the general wall-crossing formula. We start with some definitions. For proofs and a more detailed discussion we refer the reader to Section 9.2 of [67]. Fix a *Spin*^{c} structure $\mathfrak{s}$ on X, and denote by $\mathcal{L}$ its determinant bundle. Let $h \in H^2_+(X;\mathbb{R})$. The set

$$\widetilde{\mathcal{T}}(h) = \{A \in \mathcal{A}(\mathcal{L}_{\mathfrak{s}}) \mid F_A^+ + ih = 0, d^*(A - A_0) = 0\}$$

is an affine space parallel to $H^1(X; i\,\mathbb{R})$. Quotienting out by the action of the group

$$\mathcal{G}_0(x_0) = \{u : X \to S^1 \mid d^*(u^{-1}du) = 0, u(x_0) = 1\}$$

we obtain a torus

$$\mathcal{T} = \widetilde{\mathcal{T}}/\mathcal{G}_0 \simeq H^1(X; i\,\mathbb{R})/H^1(X; 2\pi i\,\mathbb{Z}).$$

We call $\mathcal{T}$ the torus of reducible solutions.

There is a universal complex line bundle

$$E \to X \times \mathcal{T}.$$

This bundle satisfies the following property. For any $\rho \in \mathcal{T}$ the restriction of the bundle

$$S(X) \otimes E \to X \times \mathcal{T}$$

to $X \times \rho$ is a bundle $S(X)_\rho \to X$ equipped with a natural Spin^c connection ∇_ρ in the same gauge equivalence class of ρ. The first Chern class of this universal bundle, can be described as the cohomology class associated to the 2-form Ω given by

$$\Omega_{x,A}((v,\alpha),(w,\beta)) = \frac{1}{2\pi i}(\beta(v) - \alpha(w)),$$

for any $v, w \in T_xX$ and $\alpha, \beta \in T_A\mathcal{T}$. Now we can state the wall-crossing formula.

Theorem 2.48. *Let X be a closed, connected and oriented smooth 4-manifold with $b_2^+(X) = 1$ and $b_1(X) = 2k$.Let $\mathfrak{s}$ be a* Spin^c *structure on X, and denote by $c = c_1(\mathcal{L}_{\mathfrak{s}})$ the first Chern class of its determinant bundle. Suppose also that $d(\mathfrak{s}) = c^2 - 2\chi(X) - 3\sigma(X) \geq 0$. Then,*

$$w(\mathfrak{s}) = SW^+(\mathfrak{s}) - SW^-(\mathfrak{s}) = \frac{1}{k!}\int_{\mathcal{T}}\left(-\frac{1}{4}\int_X \Omega \wedge \Omega \wedge c\right)^k.$$

2.7 The generalized adjunction formula

One of the early successes of Seiberg-Witten theory was to provide a proof of the Thom conjecture, which states that the minimal genus of a closed smooth surface embedded in $\mathbb{CP}^2$ representing a given positive homology class is attained by an algebraic curve. This was proved by Kronheimer and Mrowka in [29], where they introduced a special case of the generalized adjunction inequality, which is a smooth generalization of the usual adjunction formula for almost complex 4-manifolds. This has been generalized later by several people, and it is one of the most powerful tools for the study of embedded surfaces on a smooth 4-manifold. We present it here together with some consequences that we will use later on.

We say that a Spin^c structure $\mathfrak{s}$ is a Seiberg–Witten basic class if $SW(\mathfrak{s}) \neq 0$ (here we only consider the $b_2^+(X) > 1$ case).

Theorem 2.49. *Let X be a closed oriented 4-manifold with $b_2^+(X) > 1$. Let Σ be a connected surface embedded in X. Assume $[\Sigma] \cdot [\Sigma] \geq 0$. Then, for every basic class $\mathfrak{s}$ the following inequality holds:*

$$|\langle c_1(\mathfrak{s}), [\Sigma]\rangle| + [\Sigma] \cdot [\Sigma] \leq 2g(\Sigma) - 2,$$

where $c_1(\mathfrak{s})$ denotes the first Chern class of the determinant bundle of $\mathfrak{s}$.

Observe that this is indeed a generalization of the classical adjunction formula for embedded complex surfaces in an almost complex manifold which we next recall (for definitions and background on almost complex manifolds see Section 3.1).

Proposition 2.50. *Let (X, J) be a 4-dimensional almost complex manifold. Let $\Sigma \subset X$ be an embedded surface such that $JT\Sigma = T\Sigma$. Let $K \in H_2(X)$ be the canonical class of X (recall that $K = PD(-c_1(TX, J))$). Then,*

$$K \cdot [\Sigma] + [\Sigma] \cdot [\Sigma] = 2g(\Sigma) - 2.$$

Proof. Since J preserves $T\Sigma$, it also preserves $N\Sigma$. Therefore, both $T\Sigma \to \Sigma$ and $N\Sigma \to \Sigma$ are complex line bundles. Since $TX|_\Sigma = T\Sigma \oplus N\Sigma$, we obtain $c_1(TX|_\Sigma) = c_1(T\Sigma) + c_1(N\Sigma)$. Using

$$\langle c_1(TX|_\Sigma), [\Sigma]\rangle = \chi(\Sigma) = 2 - 2g(\Sigma),$$

$$\langle c_1(N\Sigma), [\Sigma]\rangle = [\Sigma] \cdot [\Sigma],$$

we get:

$$K \cdot [\Sigma] = \langle -c_1(TX|_\Sigma), [\Sigma]\rangle = 2g(\Sigma) - 2 - [\Sigma] \cdot [\Sigma].$$

□

We will see in Theorem 3.36 that if (X, J) is a Kähler manifold (in particular symplectic), it has a canonical Spin^c structure $\mathfrak{s}_{can}$ which is a basic class with determinant bundle $c_1(\mathfrak{s}_{can}) = c_1(TX, J)$. Hence, we can see the generalized adjunction inequality as a vast generalization of the adjunction formula for J-holomorphic curves in Kähler manifolds.

As an immediate corollary of Theorem 2.49, we get the following vanishing result.

Corollary 2.51. *Let X be an oriented smooth 4-manifold with $b_2^+(X) > 1$. Let Σ be a smoothly embedded surface of genus $g(\Sigma) > 0$. If $[\Sigma] \cdot [\Sigma] > 2g(\Sigma) - 2$, then all Seiberg–Witten invariants of X are zero.*

Proof. Suppose that $\mathfrak{s}$ is a *Spin^c* structure such that $SW(\mathfrak{s}) \neq 0$. By the generalized adjunction formula we then obtain

$$|\langle c_1(\mathfrak{s}), [\Sigma]\rangle| \leq 2g(\Sigma) - 2 - [\Sigma] \cdot [\Sigma] < 0,$$

a contradiction. □

Observe that the restriction $b_2^+(X) > 1$ in Theorem 2.49 is necessary. Indeed, $T^2 \times S^2$ is a 4-manifold with $b_2^+(T^2 \times S^2) = 1$ which has some non-zero Seiberg–Witten invariant (since it admits symplectic structures), but has embedded tori of arbitrarily large positive (and negative) self-intersection number. There is an analogous generalized adjunction inequality for the case $b_2^+(X) = 1$. We only give here the statement for the case of a Kähler manifold with its canonical Spin^c structure. For a general statement see [56, Theorem 1.4].

Theorem 2.52. *Let X be a closed oriented Kähler 4-manifold with $b_2^+(X) = 1$. Let ω be its Kähler form and let K_X be its canonical bundle. Let Σ be a connected surface embedded in X, with genus $g(\Sigma) > 0$. Assume $[\Sigma] \cdot [\Sigma] \geq 0$ and*

$$\int_{\Sigma} \omega > 0.$$

Then:

$$2g(\Sigma) - 2 \geq \langle c_1(K_X), [\Sigma] \rangle + [\Sigma] \cdot [\Sigma].$$

Chapter 3

Preliminaries on J-holomorphic curves

In this chapter we introduce the necessary material on J-holomorphic curves necessary to understand the later chapters of the thesis. We start by a review of almost complex manifolds and J-holomorphic curves on them. Then, we study the local behaviour of J-holomorphic curves in almost complex manifolds, which as we will see is very similar to that of honest holomorphic curves on Kähler manifolds. We devote the rest of the chapter to the study of moduli spaces of J-holomorphic curves in a symplectic manifol and Gromov–Witten invariants, which play a prominent role in the last chapter of the thesis. In the final section, we briefly explain the Seiberg–Witten–Taubes theory, which allows us to relate the theory of J-holomorphic curves in symplectic 4-manifolds with the theory of Seiberg–Witten invariants.

3.1 Almost complex manifolds and J-holomorphic curves

Let X be a smooth manifold. An almost complex structure J on X is a tangent bundle automorphism $J : TX \to TX$ lifting the identity on X such that $J^2 = -id$. A pair (X, J) where X is a smooth manifold and J an almost complex structure on X is called an almost complex manifold. If $(X, J), (Y, I)$ are two almost complex manifolds, we say that a smooth map $f : X \to Y$ is holomorphic (with respect to to J and I) if

$$df \circ J = I \circ df.$$

We say that a holomorphic map $f : (X, J) \to (Y, I)$ is an isomorphism (or a biholomorphism) if it is a diffeomorphism (and in this case f^{-1} is also holomorphic). Given (X, J), we denote by $\mathrm{Aut}(X, J)$ the group of automorphisms of (X, J), that is, the group of all isomorphisms $f : (X, J) \to (X, J)$.

Note that the existence of J implies that the dimension of X must be even. Indeed, this follows from the fact that if $J \in \mathrm{GL}(n, \mathbb{R})$ satisfies $J^2 = -id$, then its eigenvalues must be $i, -i$, so J being real, they must appear in pairs. Moreover, any almost complex manifold carries a natural orientation, so in particular the underlying smooth manifold must be orientable. Indeed, locally we can choose a frame in TX of the form $u_1, \ldots, u_n, Ju_1, \ldots, Ju_n$. The orientation is then defined by $u_1 \wedge Ju_1 \wedge \cdots \wedge u_n \wedge Ju_n$.

For any almost complex manifold X, there is a natural decomposition:

$$TX \otimes \mathbb{C} \simeq T'X \oplus T''X,$$

where $T'X$ (resp. $T''X$) is the the holomorphic tangent bundle (resp. the antiholomorphic tangent bundle), defined as the subbundle of $TX \otimes \mathbb{C}$ of eigenvalue i (resp. $-i$) of J. Moreover, the composition of inclusion and projection

$$TX \to TX \otimes \mathbb{C} \to T'X$$

is an isomorphism $TX \simeq T'X$. Therefore, we can naturally identify the holomorphic tangent bundle of (X, J) with the (real) tangent bundle of X.

The basic example of almost complex manifolds are, of course, complex manifolds. Recall that a complex manifold of dimension n is a smooth manifold X together with an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$, where U_α are open sets in X and $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ are homeomorphisms with the property that for all α, β:

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

are holomorphic maps. Therefore, in this case for each $p \in X$, T_pX has a natural structure of complex vector space of dimension n. There is a canonically defined almost complex structure J on X which is given, in each tangent space, by multiplication with i. Hence, a necessary condition for a smooth manifold in order to admit a complex structure is that it admits an almost complex structure. We say that an almost complex structure J is integrable if it is induced by a complex structure.

Definition 3.1. *The Nijenhuis tensor associated to a tensor A of rank $(1,1)$ on X is defined by:*

$$N_A(U, V) = -A^2[U, V] + A([AU, V] + [U, AV]) - [AU, AV],$$

for all U, V vector fields on X.

A simple check shows that N_A is $C^\infty(X)$-linear, and hence it defines a skew-symmetric tensor field of rank $(1,2)$. Observe also that if J is an almost complex structure on X, then $J \in \mathrm{End}(TX) \simeq TX \otimes T^*X$, so we can think of J as a rank $(1,1)$ tensor field on X, and hence its associated Nijenhuis tensor N_J is defined. The following theorem gives a necessary and sufficient condition for an almost complex structure to be integrable.

Theorem 3.2 (Newlander-Nirenberg)**.** *Let X be a smooth manifold and let J be an almost complex structure on X. Then, J is integrable if and only if $N_J = 0$.*

Suppose the dimension of X is 2. Then, we may take for each point $x \in X$ a non-vanishing vector field V in a neighbourhood of x. Then, (V, JV) are two linearly independent vector fields. Since N_J is skew-symmetric and

$$N_J(V, JV) = [V, JV] + J([JV, JV] + [V, -V]) - [JV, -V] = 0,$$

we have that $N_J = 0$. So by the previous theorem we get:

Corollary 3.3. *In a smooth manifold of dimension* 2 *all almost complex structures are integrable.*

Definition 3.4. *A Riemann surface* (Σ, j) *is a complex manifold of dimension* 1*.*

Now we come to the main object of study in this chapter, J-holomorphic curves.

Definition 3.5. *Let* (Σ, j) *be a Riemann surface. Let* (X, J) *be an almost complex manifold. A* J*-holomorphic curve in* X *is a map*

$$u : \Sigma \to X$$

satisfying $du \circ j = J \circ du$*.*

The last condition is equivalent to $du + J \circ du \circ j = 0$. Introducing the operator:

$$\bar{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j)$$

the condition $du \circ j = J \circ du$ can be stated as the following equation for u

$$\bar{\partial}_J u = 0. \tag{3.1}$$

We are going to describe the equation $\bar{\partial}_J u = 0$ in local coordinates. Pick $p \in \Sigma$. Let $z = (x, y)$ be a complex local coordinate in Σ around p, and let $(v_1, \ldots, v_{2n})$ be local coordinates in X around $u(z)$.

We can always choose the local coordinates of X in such a way that at $u(z)$, J is given by the matrix:

$$J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

In general, we can write $J = J_0 + J'$. We think of J' as a perturbation of J_0 near $u(z)$. Also note that if J is integrable, then by choosing complex local coordinates around $u(z)$, we have $J = J_0$. Write equation (3.1) as

$$du + J_0 du j = -J' du j$$

Then, the left-hand side are the usual Cauchy-Riemann equations, while the right-hand side can be thought as a perturbation term to these equations (since it is zero at z). Hence, equation (3.1) is a non-linear version of the usual Cauchy-Riemann equation, and if J is integrable we recover the usual Cauchy-Riemann equations.

We end this section with the definition of the canonical bundle and the canonical class of an almost commplex manifold.

Definition 3.6. *Let (X, J) be an almost complex manifold of dimension $2n$. The canonical bundle K_X of X is the complex line bundle*

$$K_X = \bigwedge^n T^*X.$$

The canonical class of X is the homology class $K \in H_2(X)$ that is the Poincaré dual of $c_1(K_X)$.

3.2 Local properties

In this section we will describe a series of useful theorems describing the local behaviour of J-holomorphic curves. The recurrent theme is that since the equation they satisfy is a non-linear version of the Cauchy-Riemann equations, J-holomorphic curves in an almost complex manifold behave locally in many respects as honest holomorphic curves in complex manifolds. However, the proofs are usually more involved than in the integrable case.

We start by studying unique continuation of J-holomorphic curves. While in the integrable case this is an easy consequence of the fact that holomotphic functions are given locally by a power series expansion, in the general case this will be more difficult to obtain.

Let $B(\epsilon)$ be the ball of radius ϵ with center 0 in $\mathbb{C}$. Since in this section we only study local propertis of J-holomorphic curves, we may assume $u : B(\epsilon) \to \mathbb{R}^{2n}$. The fact that u is J-holomorphic means that u satisfies:

$$\partial_s u + J(u)\partial_t u = 0.$$

Recall that the ∞-jet of a function $f : B(\epsilon) \to \mathbb{R}^{2n}$ at $z = 0$ is

$$j_0^\infty f = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k,$$

where the right-hand side is a formal power series in z.

We say that a function $f : B(\epsilon) \to \mathbb{R}^{2n}$ vanishes at infinite order at $z = 0$ if $j_0^\infty f = 0$. Equivalently, if $f^{(k)}(0) = 0$ for all $k \geq 0$.

Observe that if f is analytic, then the vanishing of $j_0^\infty f$ implies that $f = 0$ in some neighbourhood of $z = 0$. However, if f is not analytic this is not always true. The following proposition shows that, if u is J-holomorphic, then the vanishing of its ∞-jet at 0 implies the vanishing of u in some neighbourhood of 0, even if u is not analytic.

Proposition 3.7 (Unique continuation). *Let $u, v : B(\epsilon) \to \mathbb{R}^{2n}$ be J-holomorphic. If $u - v$ vanishes at 0 at infinite order, then $u = v$ at some neighbourhood of 0.*

The proof of this proposition can be found in [40, Theorem 2.3.2].

Corollary 3.8. *Let (Σ, j) be a Riemann surface, let (X, J) be an almost complex manifold and let $u, v : \Sigma \to X$ be J-holomorphic maps. If u and v agree at some point $z \in \Sigma$ to infinite order, then $u = v$.*

Proof. u and v agree at infinite order at some point $z \in \Sigma$ means that $u - v$ vanishes to infinite order at 0 for some local coordinate in Σ centered at 0. It is easy to see that this notion is independent of the chosen local coordinate. Let $S \in \Sigma$ be the set of points where u and v agree. By hypothesis $z \in S$, so it is non-empty. Moreover, S is clearly closed, and the previous proposition implies that it is also open. Since Σ is connected, it follows that $S = \Sigma$, so $u = v$. □

Next, we study critical points of J-holomorphic curves. If $u : \Sigma \to X$ is a J-holomorphic curve, a point $z \in \Sigma$ is a critical point of u if $du(z) = 0$. A point $p \in X$ is a critical value if $p = u(z)$ for some critical point z.

Proposition 3.9. *Let $u : \Sigma \to X$ be a non-constant J-holomorphic curve. Then, the set of the preimages of critical values of u,*

$$X = u^{-1}(\{u(z) : z \in \Sigma, du(z) = 0\}),$$

is finite. Moreover, $u^{-1}(x)$ is a finite set for every $p \in X$.

Therefore, like in the holomorphic case, singularities of J-holomorphic curves are always isolated. The proof of this proposition can be found in [40, Lemma 2.4.1].

We will later be interested in studying the intersection of two unparametrized J-holomorphic curves (that is, two subsets $C, C' \subseteq X$ such that $C = u(\Sigma)$ and $C' = v(\Sigma')$ for some J-holomorphic maps u, v). In this situation, we cannot apply Corollary 3.8 in order to conclude that if C and C' are distinct their intersection consists of isolated points. Indeed, suppose $p \in C \cap C'$, and $C \neq C'$, and pick parametrizations $u, v : \Sigma \to X$ such that $u(\Sigma) = C$ and $v(\Sigma) = C'$. We conclude that there exist points $z, z' \in \Sigma$ such that $u(z) = v(z')$. However, since it could happen that $z \neq z'$, we cannot apply directly Corollary 3.8 to conclude that this intersection point is isolated. In spite of this, the next proposition shows that, at least if one of the curves is embedded, intersections of two distinct unparametrized J-holomorphic curves consist of isolated points.

Proposition 3.10. *Let $u, v : B(\epsilon) \to X$ be two J-holomorphic curves. Suppose that $u(0) = v(0)$ and $du(0) \neq 0$. Moreover, assume that there are sequences of distinct points $z_n, w_n \in B(\epsilon)$ such that $\lim_n z_n = \lim_n w_n = 0$ and $u(z_n) = v(w_n)$ for all n. Then, there exists some holomorphic map $\phi : B(\delta) \to B(\epsilon)$ for some $\delta > 0$, such that $\phi(0) = 0$, and*

$$u = v \circ \phi.$$

The proof of this proposition can be found in [40, Lemma 2.4.3].

As a corollary, we obtain a global version of this result:

Corollary 3.11. *Let Σ_0, Σ_1 be two connected compact Riemann surfaces without boundary. Let $u : \Sigma_0 \to X$ and $v : \Sigma_1 \to X$ be two J-holomorphic maps. Suppose that $u(\Sigma_0) \neq v(\Sigma_1)$, and u is non-constant. Then, the set $u^{-1}(v(\Sigma_1))$ is at most countable and can accumulate only at critical points of u. Therefore, if u has no critical points, $u(\Sigma_0) \cap v(\Sigma_1)$ consists of a finite collection of isolated points.*

If $u : B(\epsilon) \to \mathbb{C}^n$ is a holomorphic map, a well-known theorem in complex analysis tells us that locally around 0, u is a branched covering of an injective holomorphic map, that is, there is some $\psi : B(\delta) \to \mathbb{C}^n$ holomorphic and injective such that $\phi(z) = \psi(z^n)$ for some $n \geq 1$ and all $z \in B(\delta)$. We want now to extend this result to the J-holomorphic setting.

Let $u : \Sigma \to X$ be a J-holomorphic curve with Σ closed. We say that u is multiply covered if there exists some closed Riemann surface (Σ', j'), a holomorphic branched covering $\phi : (\Sigma, j) \to (\Sigma', j')$ with $\deg(\phi) > 1$ and a J-holomorphic curve $u' : \Sigma' \to X$ such that

$$u = u' \circ \phi.$$

A curve that is not multiply covered is called a simple curve. A J-holomorphic curve $u : \Sigma \to X$ is said to be somewhere injective if for some point $z \in \Sigma$, $du(z) \neq 0$ and $u^{-1}(u(z)) = \{z\}$. Such a point is then called an injective point. Note that a multiply covered curve cannot have injective points. Indeed, let $u = \phi \circ v$ be a multiply covered curve, with ϕ a branched covering of degree greater than 1 and v a simple curve. If $z \in \Sigma$ satisfy $u^{-1}(u(z)) = \{z\}$ it means that z belongs to the branch locus of ϕ, hence $d\phi(z) = 0$ and $du(z) = 0$.

Conversely, we have:

Proposition 3.12. *Let Σ be a closed Riemann surface and let $u : \Sigma \to X$ be a simple J-holomorphic curve. Then, u is somewhere injective. Moreover, if $Z(u)$ denote the set of points of Σ which are not injective, then $Z(u)$ is at most countable and can only accumulate in the critical points of u.*

The proof of this proposition can be found in [40, Proposition 2.5.2].

As a corollary, we get the following important fact:

Corollary 3.13. *Let $u : \Sigma \to X$ and $v : \Sigma' \to X$ be two simple J-holomorphic curves such that $u(\Sigma) = v(\Sigma')$. Then, there exists a biholomorphism $\phi : \Sigma \to \Sigma'$ such that $u = v \circ \phi$.*

Therefore, if two simple curves are the same as unparametrized curves (i.e., they have the same image in X), one must be a reparametrization of the other. The existence of multiply covered curves imply that this is false for general J-holomorphic curves.

We conclude this section with some important facts about J-holomorphic curves in dimension 4.

Proposition 3.14 (The Adjunction Formula)**.** *Let Σ be a closed connected Riemann surface, and let (X, J) be an almost complex manifold. Let $u : \Sigma \to X$ be a simple J-holomorphic curve. Define $[u] := u_*([\Sigma]) \in H_*(X)$, and $c_1([u]) := \langle c_1(TX), [u] \rangle$. Then,* *the following inequality holds:*

$$[u] \cdot [u] \leq c_1([u]) - \chi(\Sigma)$$

Moreover, u is an embedding if and only if equality holds.

The proof of this proposition can be found in [40, Lemma 2.6.4].

Another important property of J-holomorphic curves in dimension 4 that we will use repeatedly is positivity of intersections. This generalizes to the almost complex setting the well-known fact that two complex curves C, C' inside a complex surface intersecting only in isolated points satisfy $C \cdot C' \geq 0$.

Proposition 3.15 (Positivity of intersections)**.** *Let (X, J) be a 4-dimensional almost complex manifold. Let Σ_1, Σ_2 be closed connected Riemann surfaces and let $\phi_1, \phi_2 : \Sigma_i \to X$ be two distinct simple J-holomorphic curves. Let $S = \{(p, q) \in \Sigma_1 \times \Sigma_2 \mid \phi_1(p) = \phi_2(q)\}$. Then, S is a finite set and:*

$$[\phi_1] \cdot [\phi_2] = \sum_{(p,q)\in S} \delta(p, q).$$

In particular, $[\phi_1] \cdot [\phi_2] \geq 0$ with equality if and only if u_1, u_2 have disjoint images.

This was first stated by Gromov in [20, 2.1.C_2] and a detailed proof was given by McDuff in [37, Theorem 2.1.1] (see page 36 in [40] for some comments on earlier proofs). We next give a detailed proof when at least one curve is immersed. The general case involves substantially more analysis and a complete proof can be found in [37].

Proof. Assume that ϕ_1 is an immersion. Let $I = \{(x, y) \in \Sigma_1 \times \Sigma_2 \mid \phi_1(x) = \phi_2(y)\}$. Let $I^* \subseteq I$ be the set of isolated points. Clearly, $I \setminus I^*$ is closed in $\Sigma_1 \times \Sigma_2$, and hence is compact. Let $\pi_i : \Sigma_1 \times \Sigma_2 \to \Sigma_i$ denote the projection. Then $\pi_i(I \setminus I^*) \subseteq \Sigma_i$ is closed for $i = 1, 2$.

We next prove that $\pi_i(I \setminus I^*)$ is open for $i = 1, 2$. Assume that $(x, y) \in I \setminus I^*$ and that (x_i, y_i) is a sequence in I converging to (x, y) and satisfying $(x_i, y_i) \neq (x, y)$ for every i. Choose charts $f : \Omega \to \Sigma_1$ and $g : \Omega \to \Sigma_2$ with $f(0) = x_1$, $g(0) = x_2$, where $\Omega \subset \mathbb{C}$ is an open subset containing the origin, and define $u = \phi_1 \circ f$, $v = \phi_2 \circ g$. Ignoring if necessary some of the initial points in the sequence (x_i, y_i), we may assume that $x_i = f(z_i)$ and $y_i = g(\zeta_i)$ for $z_i, \zeta_i \in \Omega$. We claim that $\zeta_i \neq 0$ for infinitely many indices i. Otherwise for every i we would have $y_i = y$ and hence $\phi_1(x_i) = \phi_2(y_i) = \phi_2(y) = \phi_1(x)$, which would imply that $\phi_1^{-1}(\phi_1(x)) \supset \{x_1, x_2, \dots\}$ is infinite; but this, by [40, Lemma 2.4.1], is incompatible with the assumption that Σ_1 is compact and $d\phi_1(x) \neq 0$. Hence the claim is proved. We are thus in a position to apply [40, Lemma 2.4.3] and conclude that $\pi_1(I \setminus I^*)$ (resp. $\pi_2(I \setminus I^*)$) contains x (resp. y) in its interior.

To finish the proof we distinguish two possibilities. If $I \setminus I^* \neq \emptyset$ then both projections $\pi_1(I \setminus I^*)$ and $\pi_2(I \setminus I^*)$ are nonempty. Since these projections are open and closed and both Σ_1 and Σ_2 are connected, we have $\pi_i(I \setminus I^*) = \Sigma_i$ for $i = 1, 2$. The equality $\pi_1(I \setminus I^*) = \Sigma_1$ means that for each $x \in \Sigma_1$ there is some $y \in \Sigma_2$ such that $\phi_1(x) = \phi_2(y)$, so $\phi_1(\Sigma_1) \subseteq \phi_2(\Sigma_2)$. Similarly (exchanging Σ_1 and Σ_2) we have $\phi_2(\Sigma_2) \subseteq \phi_1(\Sigma_1)$. Consequently in this case we have $\phi_1(\Sigma_1) = \phi_2(\Sigma_2)$.

The other possibility is that $I \setminus I^* = \emptyset$, so $I = I^*$ and hence I is finite. Then [40, Theorem 2.6.3] implies that $(\phi_1)_*[\Sigma_1] \cdot (\phi_2)_*[\Sigma_2] \geq 0$. □

3.3 J-holomorphic curves in symplectic manifolds

Until now, we have studied J-holomorphic curves on almost complex manifolds. However, there is only a good global theory of J-holomorphic curves in case J is compatible with some symplectic form on X. In fact, we will now be interested in the study of a symplectic manifold (X, ω) by studying its moduli spaces of J-holomorphic curves for some J compatible with ω.

We start by explaining how we can define an almost complex structure compatible with a symplectic form on a symplectic manifold.

Definition 3.16. *Let (X, ω) be a symplectic manifold. Let J be an almost complex structure on X. We say that J is ω-tame if $\omega(u, Ju) > 0$ for all non-zero vector field u on X. We say that J is ω-compatible if it is ω-tame and moreover $\omega(Ju, Jv) = \omega(u, v)$ for all vector fields u, v on X.*

Observe that if we define $g_J(u, v) = \omega(u, Jv)$ for all vector fields u, v on X, then g_J is a riemannian metric on X if and only if J is ω-compatible. Moreover, $g_J(Ju, Jv) = g_J(u, v)$ for all u, v.

If we start with a symplectic manifold (X, ω), we can choose more than one almost complex structure compatible with ω. However, if we are interested in the study of the symplectic manifold, we need to make sure that the choice of J is immaterial, so that the information obtained about (X, ω) via J-holomorphic curves only depends on ω and not on the chosen J. The following theorem, due to Gromov, assures us that this is indeed the case.

Theorem 3.17. *Let (X, ω) be a symplectic manifold. Let $\mathcal{J}$ be the space of all ω-compatible almost complex structures on X endowed with the C^∞-toplogy. Then, $\mathcal{J}$ is contractible.*

The proof of this theorem can be found in [39, Proposition 4.1.1].

Therefore, for our purposes it will not matter which almost compatible structure we choose, as far as it is ω-compatible.

Later on we will be interested in actions of finite groups G on a symplectic manifold (X, ω) by symplectomorphisms. As we will see, a very useful tool for studying such actions is the study of J-holomorphic curves. However, if J is an arbitrary almost complex structure the action of G does not necessarily preserve J-holomorphic curves. Therefore, we need to be able to pick an almost complex structure J on X which is not only compatible with ω, but also with the action of G. That this can be always done is the content of the following result.

Definition 3.18. *Let (X, J) be an almost complex manifold. Let G be a group acting smoothly on X. We say that J is G-invariant if the action is by biholomorphisms. In other terms, we say that J is G-invariant if for every $g \in G$*

$$J \circ dg = dg \circ J.$$

Proposition 3.19. *Let (X, ω) be a symplectic manifold. Let G be any finite group acting symplectically on (X, ω). Then, there exists an almost complex structure J on X which is ω-compatible and G-invariant.*

Proof. See [40, Lemma 5.5.6]. □

We end this section with the following definition.

Definition 3.20. *Let (X, ω) be a closed symplectic manifold, J an ω-compatible almost complex manifold on X and $u : \Sigma \to X$ a J-holomorphic curve. Then we define the energy of u as*

$$E(u) = \int_\Sigma u^*\omega$$

Note that the energy of a J-holomorphic curve is a topological invariant: if $[u] := u_*([\Sigma]) \in H_2(X; \mathbb{Z})$, then

$$E(u) = \langle \omega, [u] \rangle.$$

3.4 Moduli spaces of J-holomorphic curves

In this section we study the properties of moduli spaces of J-holomorphic curves on a symplectic manifold and introduce Gromov–Witten invariants. This parallels closely the study of Seiberg–Witten moduli spaces. However, there is one major difference: Seiberg–Witten moduli spaces are always compact, but this is no longer the case for J-holomorphic curve moduli spaces. Therefore, in order to define invariants by integration over the moduli space, we will need to introduce a suitable compactification of the moduli spaces. This leads to further complications, since in general these compactified moduli spaces are not smooth manifolds. This problem has now been solved in general by introducing the virtual fundamental class of the moduli space, which is a generalization of the usual fundamental class of a closed smooth manifold to more general geometric objects. In particular, one can assign a virtual fundamental class to any compactified moduli space of J-holomorphic curves and use it to define Gromov–Witten invariants. However, we will not care much about this problem, since in this thesis we will only use Gromov–Witten invariants in cases where the moduli space of J-holomorphic maps is already compact.

Since we will only use in this thesis moduli spaces of genus 0 J-holomorphic curves (that is, J-holomorphic curves whose domain is $\mathbb{CP}^1$), we will restrict to this case. We will use the terminology 'J-holomorphic spheres' to refer to genus 0 J-holomorphic curves.

Let (X, ω) be a symplectic manifold, and let J be an ω-compatible almost complex structure on X.

We define the moduli space of (parametrized) J-holomorphic spheres representing the homology class $A \in H_2(X)$ by

$$\mathcal{M}(A, J) = \{u : \mathbb{C}\mathrm{P}^1 \to X \mid u_*([S^2]) = A, \bar{\partial}_J u = 0\},$$

and endow it with the topology inherited from $C^\infty(\mathbb{C}\mathrm{P}^1, X)$. We also define the moduli space of (unparametrized) J-holomorphic spheres with k marked points representing the homology class $A \in H_2(X)$ by

$$\mathcal{M}_{0,k}(A, J) = \{(u : \mathbb{C}\mathrm{P}^1 \to X, p_1, \ldots, p_k)\}/\sim,$$

where $u : \mathbb{C}\mathrm{P}^1 \to X$ is a J-holomorphic map such that $u_*([\mathbb{C}\mathrm{P}^1]) = A$, $p_1, \ldots, p_k$ are k distinct points in $\mathbb{C}\mathrm{P}^1$ and $(u, p_1, \ldots, p_k) \sim (u', p'_1, \ldots, p'_k)$ iff there is $\phi \in \mathrm{Aut}(\mathbb{C}\mathrm{P}^1)$ such that $u \circ \phi = u'$ and $\phi(p'_i) = p_i$ for all i. Note that for two equivalent elements

$$(u, p_1, \ldots, p_k) \sim (u', p'_1, \ldots, p'_k)$$

u is a simple curve if and only if u' is. Therefore, we can speak of simple curves or multiply-covered curves of $\mathcal{M}_{0,k}(A, J)$. We denote by $\mathcal{M}^*(A, J)$ the subspace of $\mathcal{M}(A, J)$ consisting of simple curves and similarly we denote by $\mathcal{M}^*_{0,k}(A, J)$ the subspace of $\mathcal{M}_{0,k}(A, J)$ consisting of simple curves.

Define the evaluation map by

$$\begin{array}{rcl} \mathrm{ev} : \mathcal{M}_{0,k}(A, J) & \longrightarrow & X \times \cdots \times X \\ {[u, p_1, \ldots, p_k]} & \mapsto & (u(p_1), \ldots, u(p_k)). \end{array}$$

It is easy to see that this map is well-defined.

We define the virtual dimension of $\mathcal{M}(A, J)$ by

$$\text{vir-dim}\, \mathcal{M}_{0,k}(A, J) = 2n + 2c_1(A),$$

where $\dim X = 2n$ and we denote $c_1(A) = \langle c_1(TX, J), A\rangle$. Similarly, we define the virtual dimension of $\mathcal{M}_{0,k}(A, J)$ by

$$\text{vir-dim}\, \mathcal{M}_{0,k}(A, J) = 2n + 2c_1(A) - 6 + 2k,$$

where $\dim X = 2n$ and we denote $c_1(A) = \langle c_1(TX, J), A\rangle$.

If we remove multiply-covered curves, for a generic J the parametrized moduli spaces are smooth manifolds of the expected dimension, as the following theorem shows.

Theorem 3.21. *There exists a subspace of $\mathcal{J}_\omega$ of Baire second category (countable intersection of open and dense sets), which we denote by $\mathcal{J}_\omega^{reg}$ and call the space of regular almost complex structures such that the following holds. For every $J \in \mathcal{J}_\omega^{reg}$, the moduli space $\mathcal{M}^*(A, J)$ is a smooth oriented manifold of dimension* vir-dim $\mathcal{M}(A, J)$.

The first step is to describe $\mathcal{M}^*(A, J)$ as the zeros of a section in a certain infinite-dimensional bundle. Let $\mathcal{B}$ be the subspace of all maps in $C^\infty(\mathbb{C}\mathrm{P}^1, X)$ that represent the homology class A. This is an infinite-dimensional manifold whose tangent space at $u \in \mathcal{B}$ is

$$T_u\mathcal{B} = C^\infty(\mathbb{C}\mathrm{P}^1, u^*TX).$$

Denote by $\mathcal{B}^*$ the subset of $\mathcal{B}$ consisiting of maps which are somewhere injective. Let $\mathcal{E}$ be the bundle over $\mathcal{B}$ whose fiber over $u \in \mathcal{B}$ is

$$\mathcal{E}_u = \Omega^{0,1}(\mathbb{C}\mathrm{P}^1, u^*TX),$$

that is, anti-holomorphic 1-forms in $\mathbb{C}\mathrm{P}^1$ with values in u^*TX. There is a natural section of this bundle, given by:

$$\mathcal{S}(u) = (u, \bar{\partial}_J u).$$

Observe that $\mathcal{M}(A, J) = \mathcal{S}^{-1}(0)$. Similarly, $\mathcal{M}^*(A, J) = \mathcal{S}^{-1}(0) \cap \mathcal{B}^*$. Once we have this description, we need to check that, for generic J, the section $\mathcal{S}$ is transversal to the zero section of the bundle $\mathcal{E}|_{\mathcal{B}^*} \to \mathcal{B}^*$ and apply an infinite-dimensional version of the implicit function theorem. There are some technical problems since $\mathcal{B}$ is a not a Banach manifold and the infinite-dimensional implicit function theorem only works for Banach manifolds. As usual, we can solve this problem by considering Sobolev completions of $\mathcal{B}$ and $\mathcal{E}$, which are now Banach manifolds, and applying the implicit function theorem there. The fact that when we consider the set $\mathcal{S}^{-1}(0)$ we are not introducing new solutions follows from elliptic regularity, since the equation $\bar{\partial}_J u = 0$ is elliptic. We will omit details about Sobolev completions, but the reader should bear in mind that in order to obtain a rigorous proof of the theorem, all spaces in what follows should be replaced with suitable completions that make them Banach manifolds. For details, the reader can consult Chapter 3 of [40].

In order to study transversality of the section $\mathcal{S}$ it is useful to introduce the vertical differential of the bundle $\mathcal{E} \to \mathcal{B}$ at a point $u \in \mathcal{B}$, defined as

$$D_u = D_{u,J} = \pi_u \circ D\mathcal{S}(u),$$

where

$$D\mathcal{S}(u) : C^\infty(\mathbb{C}\mathrm{P}^1, u^*TX) \to \Omega^{0,1}(\mathbb{C}\mathrm{P}^1, u^*TX)$$

is the differential of $\mathcal{S}$ at u, and

$$\pi_u : T_{u,0}\mathcal{E} \to \mathcal{E}_u$$

is the vertical projection. Observe that the section $\mathcal{S}$ is transversal to the zero section of the bundle $\mathcal{E} \to \mathcal{B}$ if and only if D_u is surjective for every $u \in \mathcal{M}(A, J)$. The key fact about D_u is that it is a Fredholm operator, with Fredholm index

$$\operatorname{ind} D_u = \operatorname{Ker} D_u - \operatorname{Coker} D_u = 2n + 2c_1(A),$$

where $\dim X = 2n$ and $c_1(A) = \langle c_1(TX, J), A\rangle$. Therefore, if D_u is surjective at every $u \in \mathcal{M}(A, J)$, by the implicit function theorem, $\mathcal{M}(A, J)$ is a smooth manifold of dimension $\operatorname{ind} D_u$. In order to study for which almost complex structures J holds the surjectivity of D_u, we introduce the universal moduli space of simple curves

$$\mathcal{M}^*(A, \mathcal{J}_\omega) = \{(u, J) : J \in \mathcal{J}_\omega, u \in \mathcal{M}^*(A, J)\},$$

and the natural projection

$$\pi : \mathcal{M}(A, \mathcal{J}_\omega) \to \mathcal{J}_\omega.$$

It can be proved that the differential of π at (u, J) is a Fredholm operator of the same index as D_u and that $D_{u,J}$ is surjective iff $d\pi_{(u,J)}$is surjective. By the Sard-Smale theorem, there is a Baire second category subset of $\mathcal{J}_\omega$ of regular values for π. We will call this subset $\mathcal{J}_\omega^{reg}$, and say that J is a regular almost complex structure if $J \in \mathcal{J}_\omega$. By the implicit function theorem, we obtain that for any $J \in \mathcal{J}_\omega$, $\pi^{-1}(J) = \mathcal{M}^*(A, J)$ is a smooth manifold of dimension $\operatorname{ind} D_u$. We skip the proof that $\mathcal{M}^*(A, J)$ can be oriented. For details, see [40, Theorem 3.5]. This proves the theorem.

Using what we have proved, one can prove the following theorem for unparametrized moduli spaces:

Theorem 3.22. *Let $A \in H_2(X)$ be non-zero. There exists a subspace of $\mathcal{J}_\omega$ of Baire second category (countable intersection of open and dense sets), which we denote by $\mathcal{J}_\omega^{reg}$ and call the space of regular almost complex structures such that the following holds. For every $J \in \mathcal{J}_\omega^{reg}$, the moduli space $\mathcal{M}^*_{0,k}(A, J)$ is a smooth manifold of dimension* $\operatorname{vir-dim} \mathcal{M}_{0,k}(A, J)$.

Suppose for simplicity $k = 0$, the case $k > 0$ being similar. The group of Möbius transformations $G = PSL(2, \mathbb{C})$ acts on $\mathcal{M}^*(A, J)$ by reparametrization. This action is free. To see it, note that the maps in $\mathcal{M}^*(A, J)$ are somwhere injective, since they are simple and $A \neq 0$, which implies that if $u \circ \phi = u$, then ϕ is the identity. Moreover, the action of G on $\mathcal{M}^*(A, J)$ is properly discontinuous. Therefore,

$$\mathcal{M}^*_{0,0}(A, J) = \mathcal{M}^*(A, J)/G$$

is a smooth manifold of dimension $\dim \mathcal{M}^*(A, J) - \dim G = 2n + 2c_1(A) - 6$.

We end this section with two criteria for when an almost complex structure is regular in a symplectic 4-manifold. For the proof of the following propositions see section 3.3 of [39]. The next proposition is commonly referred to as automatic transversality.

Proposition 3.23. *Let (X, ω) be a 4-manifold, and let C be an embedded J-holomorphic sphere. Then, J is regular for the class $[C]$ if and only if $C \cdot C \geq -1$.*

Proposition 3.24. *Let $(\tilde{X}, \tilde{\omega})$ be the product of S^2 with a symplectic manifold (X, ω), and let $\tilde{A} \in H_2(\tilde{X})$ be the homology class represented by the spheres $S^2 \times \{pt\}$. Then, for every $J \in \mathcal{J}_\omega$ J, the product almost complex structure $i \times J$ is regular for the class $\tilde{A}$.*

3.5 Gromov compactness

As we have said in the introduction, the moduli spaces $\mathcal{M}(A, J)$ are not compact in general. In this section we describe a compactification of the moduli spaces, due to Kontsevich, and give criteria for when the spaces $\mathcal{M}(A, J)$ are already compact.

Definition 3.25. *A nodal curve is a complex algebraic curve all of whose singularities are nodal (i.e., each singular point is locally of the form $z_1 z_2 = 0$ in $\mathbb{C}^2$). A stable curve is a nodal curve with finite group of automorphisms.*

We can assign to each nodal curve C a graph G_C in the following way. The set of vertices is the set of irreducible components of C, and two vertices $\alpha, \beta \in T$ are connected by an edge if and only if they intersect. We will say that C has genus 0 if each irreducible component of C has genus 0 and G_C is a tree (i.e., has no cycles).

Definition 3.26. *A (genus 0) stable map with k marked points is a tuple $(C, u : C \to X, p_1, \dots, p_k)$ satisfying:*

1. *u is continuous and its pullback to the normalization of C is smooth,*
2. *C is a nodal curve of genus 0,*
3. *$p_1, \dots, p_k$ are k distinct points on C which are not singularities of C,*
4. *each component C_α of C satisfy $\bar{\partial}_J u|_{C_\alpha} = 0$,*
5. *The group*

$$\mathrm{Aut}((u, p_1 \dots, p_k)) = \{\phi \in \mathrm{Aut}(C) : u \circ \phi = u, \phi(p_i) = p_i, i = 1, \dots k\}$$

   is finite.

We can extend definitions from J-holomorphic curves to stable maps. We say that a stable map u represents the homology class $A \in H_2(X)$ if

$$u_*([C]) := \sum_\alpha u_*([C_\alpha]) = A,$$

where α runs over the components of C. We define the energy of a stable map u by

$$E(u) = \sum_\alpha E(u|_{C_\alpha}),$$

where α runs over the components of C.

Definition 3.27. *We define the moduli space of (genus 0) stable maps with k marked points representing the homology class $A \in H_2(X)$, $\overline{\mathcal{M}}_{0,k}(A, J)$, as the set of all stable maps representing the class A, up to isomorphism.*

Observe that a J-holomorphic curve is in particular a stable map. Therefore,

$$\mathcal{M}_{0,k}(A,J) \subseteq \overline{\mathcal{M}}_{0,k}(A,J).$$

In the case where the target manifold X is a point, we obtain the moduli space of (genus 0) stable curves with k marked points. We will denote this moduli space by $\overline{\mathcal{M}}_{0,k}$. Note that here A and J play no role.

In general, if $(u : C \to X, p_1, \ldots, p_k)$ is a stable map, it is not necessarily true that C is a stable curve. For example, let $(u : \mathbb{C}\mathrm{P}^1 \to X, p)$ be a J-holomorphic sphere with one marked point p such that there are two different points $q_1, q_2 \in \mathbb{C}\mathrm{P}^1$ with $u(q_1) = u(q_2)$, $p \neq q_1, q_2$, and u is an embedding outside q_1, q_2. Then, an automorphism of (u, p) must be an automorphism of $\mathbb{C}\mathrm{P}^1$ fixing p and preserving the set $\{q_1, q_2\}$. Since Möbius transformations are simply transitive in triples of points, there are exactly two such automorphisms, so that (u, p) is stable. However, the group of automorphisms of $\mathbb{C}\mathrm{P}^1$ fixing one point p is infinite, hence the curve with one marked point $(\mathbb{C}\mathrm{P}^1, p)$ is not stable. However, the following is true:

Lemma 3.28. *Let $(C, u : C \to X, p_1, \ldots, p_k)$ a stable map with k marked points. There is some $l \geq 0$ and distinct, non-singular points $p_{k+1}, \ldots, p_{k+l} \in C$ such that $(C, p_1 \ldots, p_{k+l})$ is a stable curve.*

We define the universal curve

$$\overline{\mathcal{C}}_{0,k} \to \overline{\mathcal{M}}_{0,k}$$

as the map such that the fiber over $(C, p_1 \ldots, p_k) \in \overline{\mathcal{M}}_{0,k}$ is isomorphic to the curve C. In fact, this is the same as the natural map

$$\overline{\mathcal{M}}_{0,k+1} \to \overline{\mathcal{M}}_{0,k}$$

given by forgetting the last marked point and stabilizing. Stabilization is necessary since a curve with $k+1$ marked points can be stable but become non-stable when forgetting the last marked point. In order to remedy this, we collapse the non-stable components of the curve, thus obtaining a stable curve.

We define a topology on $\overline{\mathcal{M}}_{0,k}(A,J)$ that makes it into a compact topological space. Therefore, we will have a compactification of $\mathcal{M}_{0,k}(A,J)$.

Definition 3.29. *The Gromov topology for $\mathcal{M}_{0,k}(A,J)$ is the topology given by the following basis of neighbourhoods. Let $(u, p_1, \ldots, p_k) \in \mathcal{M}_{0,k}(A,J)$, and let C be the domain of u. Choose l additional marked points in C so that there is no nontrivial automorphism of C fixing these $k+l$ points. Consider the graph $\Gamma_u \subset \overline{\mathcal{C}}_{0,k+l} \times X$ of the map u, where $\overline{\mathcal{C}}_{0,k+l} \to \overline{\mathcal{M}}_{0,k+l}$ is the universal curve. Then, the neighbourhoods of $(u, p_1, \ldots p_k)$ in the Gromov topology are obtained by taking all J-holomorphic curves whose graph is close to Γ_u is the Hausdorff topology, and forgetting the last l marked points.*

It can be seen that the above definition gives rise to a well-defined topology on $\overline{\mathcal{M}}_{0,k}(A,J)$ (in particular, it does not depend on l or the choice of extra marked points). The key result about this topology is the Gromov compactness theorem.

Theorem 3.30. *Let (X,ω) be a closed symplectic manifold, $A \in H_2(X)$ and J an ω-compatible almost complex structure on X. Then, the moduli space of stable maps $\overline{\mathcal{M}}_{0,k}(A,J)$, equipped with the Gromov topology, is compact.*

We will also need the following more general result, concerning sequences of J_n-holomorphic maps for different almost complex structures J_n.

Theorem 3.31. *Let (X,ω) be a closed symplectic manifold. Let J_n be a sequence of ω-compatible almost complex structures on X converging to an ω-compatible almost complex structure J. Let $u_n : \mathbb{C}\mathrm{P}^1 \to X$ a sequence of J_n-holomorphic maps such that $\sup_n E(u_n) < \infty$. Then, there is a subsequence u_{n_k} converging to a J-holomorphic stable map u.*

Proof. See [39, Theorem 5.3.1]. □

To end this section, we prove the following criteria for the compactness of the moduli space of simple J-holomorphic curves.

Proposition 3.32. *Let $A \in H_2(X)$ be such that there are no decomposition $A = B_1 + B_2$ where $\langle [\omega], B_1\rangle > 0$ and $\langle [\omega], B_2\rangle > 0$. Then, $\overline{\mathcal{M}}_{0,k}(A,J) = \mathcal{M}^*_{0,k}(A,J)$. In particular, for a regular J, $\mathcal{M}^*_{0,k}(A,J)$ is a compact smooth manifold of dimension*

$$\dim \mathcal{M}^*_{0,k}(A,J) = 2n + 2c_1(A) + 2k.$$

Proof. If there are multiply covered curves on $\mathcal{M}_{0,k}$, then by the definition of multiply covered curve there exists a homology class B and an integer $k > 1$ such that $A = kB$ and B is represented by a (simple) J-holomorphic curve, so $\langle [\omega], B\rangle > 0$, contradicting the hypothesis. Therefore, $\mathcal{M}_{0,k}(A,J) = \mathcal{M}^*_{0,k}(A,J)$. Consider now the moduli space of stable maps $\overline{\mathcal{M}}_{0,k}(A,J)$. If there is a stable map $(u : C \to X, p_1, \ldots p_k) \in \overline{\mathcal{M}}_{0,k}(A,J)$ with C a curve with more than one component, then, the restriction of u to each component of C gives a J-holomorphic map $u_\alpha : \mathbb{C}\mathrm{P}^1 \to X$. In particular, if B_α is the homology class represented by u_α, we have

$$A = \sum_\alpha B_\alpha,$$

with $\langle [\omega], B_\alpha\rangle > 0$ for every α, contradicting again the hypothesis. Therefore,

$$\overline{\mathcal{M}}_{0,k}(A,J) = \mathcal{M}_{0,k}(A,J) = \mathcal{M}^*_{0,k}(A,J).$$

□

3.6 Gromov-Witten invariants

In this section we will define the (genus 0) Gromov-Witten invariants of a closed symplectic manifold (X, ω).

We start by defining an evaluation map.

Definition 3.33. *The evaluation map of* $\overline{\mathcal{M}}_{0,k}(A, J)$ *is the map*

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,k}(A, J) \to X^n$$

given by $\mathrm{ev}([u, p_1 \ldots, p_k]) = (u(p_1), \ldots, u(p_k))$.

Definition 3.34. *The (genus* 0*) Gromov Witten invariant with* k *marked points and homology class* $A \in H_2(X)$ *is a map*

$$GW^A_{0,k} : H^*(X)^{\otimes k} \to \mathbb{Z}$$

given by

$$GW^A_{0,k}(\beta_1, \ldots, \beta_k) = \int_{\overline{\mathcal{M}}_{0,k}(A,J)} \mathrm{ev}^*(\beta_1 \cup \cdots \cup \beta_k).$$

Some comments about this definition are in order. First, the definition as stated only makes sense for the best possible situation, which will be the only one we will be considering in this thesis. This occurs when $\overline{\mathcal{M}}_{0,k}(A, J)$ is a compact oriented smooth manifold of dimension

$$\text{vir-dim}\, \overline{\mathcal{M}}_{0,k}(A, J) = 2n + 2c_1(A) - 6 + 2k,$$

and the evaluation map ev is transverse to every submanifold of $\overline{\mathcal{M}}_{0,k}(A, J)$. In this case, we can understand the geometric meaning of Gromov-Witten invariants as follows. Let $S_1, \ldots, S_k$ be k submanifolds of X, and let $\beta_i = PD([S_i])$ for all $i = 1, \ldots, k$. Note that $\deg(\beta_i) = 2n - \dim S_i$. Then, $\beta_1 \cup \cdots \cup \beta_k$ is a cohomology class of $H^*(X^n) \simeq H^*(X)^{\otimes k}$ of degree

$$\sum_i \deg(\beta_i) = 2nk - \sum_i \dim S_i,$$

so $\mathrm{ev}^*(\beta_1 \cup \cdots \cup \beta_k)$ is a cohomology class of $H^*(\overline{\mathcal{M}}_{0,k}(A, J))$ of degree $2nk - \sum_i \dim S_i$. From the definition of $GW^A_{0,k}$, we see that it can be non-zero only in case

$$2nk - \sum_i \dim S_i = \text{vir-dim}\, H^*(\overline{\mathcal{M}}_{0,k}(A, J) = 2n + 2c_1(A) - 6 + 2k,$$

and in case this equality holds $GW^A_{0,k}(\beta_1 \cup \cdots \cup \beta_k)$ is exactly the count (with signs, taking into account the orientation) of (unparametrized) J-holomorphic spheres in X such that $u(p_i) \in S_i$ for all i.

In the general situation $\overline{\mathcal{M}}_{0,k}(A, J)$ will not be a manifold. In this case, the integral above does not make sense. Since in the good cases that integral is the same as

$$\langle \mathrm{ev}^*(\beta_1 \cup \cdots \cup \beta_k), [\overline{\mathcal{M}}_{0,k}(A, J)] \rangle$$

where $[\overline{\mathcal{M}}_{0,k}(A,J)]$ is the fundamental class of $\overline{\mathcal{M}}_{0,k}(A,J)$, we can solve the problem by defining a suitable replacement for the fundamental class when $\overline{\mathcal{M}}_{0,k}(A,J)$ is a singular space. This can been done in full generality following several different approaches. However, in this case the Gromov–Witten invariants take in general values in $\mathbb{Q}$, not necessarily in $\mathbb{Z}$. The interested reader can consult [17, 25, 57].

3.7 Seiberg–Witten invariants in symplectic manifolds

In this final section, we discuss the relation of Seiberg–Witten invariants with J-holomorphic curves in 4-manifolds. The main theorems in this direction are due to Taubes, and it is sometimes known as Seiberg–Witten–Taubes theory. A good reference for this section is [28].

Let (X,ω) be a closed symplectic 4-manifold. The symplectic form ω induces a canonical orientation on X, since $\omega\wedge\omega$ is a volume form on X. As we have explained in Section 3.3, there are almost complex structures J on X which are ω-compatible. This means that $g_J(u,v) := \omega(u,Jv)$ is a riemannian metric on X. Note that this implies that u, Ju are g_J-orthogonal. Moreover, the space of ω-compatible almost complex structures on X is contractible. Observe also that with respect to the metric g_J, ω is a self-dual 2-form. Indeed, let $x \in TX$ and fix an oriented orthonormal basis (e_1,e_2,e_3,e_4) at T_xX such that $Je_1 = e_2$ and $Je_3 = e_4$. Noting that $\omega(u,v) = g_J(Ju,v)$, we can write:

$$\omega_x = \sum_{1\le i<j\le 4} g_J(Je_i,e_j)e_i\wedge e_j = e_1\wedge e_2 + e_3\wedge e_4$$

From this, it is immediate that $*\omega = \omega$.

Fix such an almost complex structure J and its associated riemannian metric g_J. Then, one can prove that X carries a canonical $Spin^c$ structure $\mathfrak{s}_{can}$ with determinant bundle $K^{-1} = \Omega^{2,0}X$ (see, for instance, p.44 in [55]). Since $\mathrm{Spin}^c(X)$ is a torsor over $H^2(X)$ we have a canonical identification of $Spin^c$ structures and $H^2(X)$ by sending $\mathfrak{s}_{can}$ to $0 \in H^2(X)$. So, assuming $b_2^+(X) > 1$, we can see the Seiberg-Witten invariants of a symplectic manifold (or more generally, of an almost complex manifold) as a map:

$$SW : H^2(X) \to \mathbb{Z}\,.$$

From now on we will use elements of $H^2(X)$ instead of $Spin^c$ structures, and we will think of the elements of $H^2(X)$ as complex line bundles over X. Let $E \in H^2(X)$ and let $S^+(E)$ be the associated spinor bundle. Then, it can be proved that there are natural identifications

$$S^+(E) = E \oplus (K^{-1}\otimes E)$$
$$S^-(E) = T^{0,1}X \otimes E.$$

Moreover, Clifford multiplication by $v \in T^*X\otimes\mathbb{C}$ acting on $\alpha \in T^{0,*}X\otimes E$ is given by:

$$v\cdot\alpha = \sqrt{2}(v^{0,1}\wedge\alpha - \iota(\overline{v^{1,0}})\alpha),$$

where ι denotes the contraction.

Therefore, we can identify a spinor $\psi \in S^+(E)$ as $\psi = (\alpha, \beta)$, where $\alpha \in C^\infty(E)$ and $\beta \in C^\infty(K^{-1} \otimes E) = \Omega^{0,2}(X, E)$.

It can be shown that there is a canonical choice of connection on K^{-1}, which is induced by the Levi-Civita connection on the underlying riemannian manifold.

Consider now any $Spin^c$ structure on X. Let E be the bundle associated to that $Spin^c$ structure. Then, its determinant bundle is $L = K^{-1} \otimes E^2$. We can write any connection on L as $A = A_0 + 2a$ where a is a connection on E. With this notation, we can write the Dirac operator as follows.

$$D_{A_0+2a}(\alpha, \beta) = \sqrt{2}(\overline{\partial}_a \alpha + \overline{\partial}_a^* \beta),$$

where $\overline{\partial}_a : \Omega^{0,k}(X, E) \to \Omega^{0,k+1}(X, E)$ is the composition of the covariant derivative ∇_a on E with the projection $\Omega^{k+1}(X, E) \to \Omega^{0,k+1}(X, E)$.

We can now write the Seiberg–Witten equations in terms of (α, β) and a.

Proposition 3.35. *With the above identifications, the perturbed Seiberg-Witten equations can be written as:*

$$\overline{\partial}_a \alpha = -\overline{\partial}_a^* \beta$$
$$F_A^+ = i(|\alpha|^2 - |\beta|^2) + 2(\overline{\alpha}\beta + \alpha\overline{\beta}) + ih.$$

In order to analyze these equations and draw conclusions in the symplectic setting, it is useful to consider perturbations of the form

$$h = r\omega - iF_{A_0}^+,$$

with r a large positive real number. We also rescale $\psi = \sqrt{r}(\alpha, \beta)$. Then, the Seiberg-Witten equations become:

$$\overline{\partial}_a \alpha = -\overline{\partial}_a^* \beta \tag{3.2}$$
$$F_a^+ = -\frac{ir}{2}(1 - |\alpha|^2 + |\beta|^2)\omega - r(\alpha\overline{\beta} - \overline{\alpha}\beta) \tag{3.3}$$

Combining the two equations and bounding them (see [28] for details), we arrive to the conclusion that for any $E \in H_2(X)$ such that $SW(E) > 0$ the following inequalities hold:

$$\int \left(\left(1 - \frac{2C}{r}\right) |\nabla_a \alpha|^2 + r(1 - |\alpha|^2)^2 \right) \leq 2\pi[\omega] \cdot e \tag{3.4}$$

$$\int \left(\frac{1}{2}|\nabla_a^* \beta|^2 + \frac{r}{2}|\beta|^4 + \frac{r}{4}|\beta|^2 \right) \leq \frac{C}{r} \int |\nabla_a \alpha|^2, \tag{3.5}$$

where C is a constant independent of r, $[\omega] \in H^2(X; \mathbb{Z})$ is the cohomology class represented by the symplectic form and $e = c_1(E)$.

From (3.4), the next theorem follows.

Theorem 3.36 (Taubes). *Suppose $b_2^+(X) > 1$. Let $e \in H^2(X;\mathbb{Z})$ and let $c = c_1(K)$ be the first Chern class of the canonical bundle of X. Then:*

1. $SW(0) = 1$
2. $SW(c) = \pm 1$
3. $SW(e) = \pm SW(c-e)$
4. *For any $e \in H^2(X;\mathbb{Z})$,*
$$0 \leq \langle [w], e\rangle \leq \langle [w], c\rangle$$
and if some equality holds, then $e = 0$ or $e = c$.

If $b_2^+(X) = 1$, the same Theorem holds, changing SW by SW^+ (see Section 2.6 for the definition of SW^+).

Let now $e \in H_2(X)$ be such that $[\omega]\cdot e > 0$. By (3.4), we see that, for r large enough, $|\alpha|$ must approach 1 almost everywhere. However, $|\alpha|$ cannot be 1 everywhere since it is a section of the bundle E, and the condition $[\omega]\cdot e > 0$ implies that E is not trivial. Moreover, we can also see that there is a bound on $|\nabla_a \alpha|$. Equation (3.5) now gives bounds on $|\beta|$ and $|\nabla_a^* \beta|$. By the Dirac equation (3.2),

$$|\overline{\partial}_a \alpha| = |\overline{\partial}_a \beta| = |\nabla_a^* \beta|,$$

so $|\overline{\partial}_a \alpha|$ must be much smaller than $|\nabla_a \alpha|$. This suggests that the zero set of α must be close to being J-holomorphic. Indeed, Taubes proved that in the limit $r \to \infty$, the zero set of α converges in a suitable sense to a J-holomorphic curve. Moreover, since the zero set of α represents $PD(e)$, this J-holomorphic curve represents the homology class $PD(e)$. More precisely, we have the following statement.

Theorem 3.37 (Taubes). *Let $e \in H^2(X)$. Suppose there exists a sequence r_n of real numbers such that $r_n \to \infty$ and $(a_n, (\alpha_n, \beta_n))$ are solutions of the Seiberg–Witten equations (3.2) with parameter r_n. Then, after taking an appropriate subsequence, there is a closed Riemann surface Σ (not necessarily connected) and a J-holomorphic map $f : \Sigma \to X$ such that:*

1. *$f_*([\Sigma])$ is the Poincaré dual of e,*
2. $\lim_{n\to\infty}\{\sup_{x\in\Sigma} d(x, \alpha_n^{-1}(0)) + \sup_{x\in\alpha_n^{-1}(0)} d(x, f(\Sigma))\} = 0$,
3. *if $G \subset \Sigma$ is a closed set and $G \cap \alpha_n^{-1}(0) \neq \emptyset$ for all n, then $f(\Sigma) \cap G \neq \emptyset$.*

Note that the theorem only requires the existence of a suitable sequence of solutions to the Seiberg–Witten equations. In particular, this holds if $SW(e) \neq 0$ and $b_2^+(X) \geq 2$, or if $SW^+(e) \neq 0$ and $b_2^+(X) = 1$ (see Section 2.6 for conventions about the definition of SW^+).

As a final comment, it is very interesting to note that Taubes in fact constructs a new invariant of symplectic manifolds, called the Gromov invariant, by counting embedded J-holomorphic curves with possibly disconnected domain representing a given homology class for a generic J (note that, though similar, this invariant is not the same as the Gromov–Witten invariants). Moreover, Taubes proves that this Gromov invariant coincides with the Seiberg–Witten invariant of the manifold. Therefore, we obtain a geometric interpretation of the Seiberg–Witten invariant in terms of J-holomorphic curves. We will not say more about this topic, since we will have no need to use the Gromov invariants on this thesis. The interested reader can consult [39, 67, 73] for more details and proofs.

Chapter 4

Which finite groups act on a given 4-manifold?

In this chapter we present the most important contributions of this thesis to the understanding of finite group actions on smooth 4-manifolds. We start by stating the main theorems which we will prove in this chapter, and we then explain the structure of the proof, giving its main ideas, in order to help the reader navigate this chapter. This chapter coincides with the article [53].

4.1 Main results

The first theorem we prove gives restrictions on the finite groups that can act effectively and smoothly on a closed 4-manifold. To state this theorem we recall some standard terminology. A group G is said to be nilpotent of class at most 2 if $[a,[b,c]]=1$ for every $a,b,c\in G$. Equivalently, $G/Z(G)$ is abelian, where $Z(G)\leq G$ is the center of G. For example, any abelian group is nilpotent of class at most 2. Note that in the literature on nilpotent Lie algebras the analogous property is sometimes called 2-step nilpotency.

Theorem 4.1. *Let X be a closed smooth 4-manifold. There exists a constant C such that every group G acting in a smooth and effective way on X has a subgroup $G_0\leq G$ such that $[G:G_0]\leq C$ and:*

1. *G_0 is nilpotent of class at most 2,*
2. *$[G_0,G_0]$ is a (possibly trivial) cyclic group,*
3. *$X^{[G_0,G_0]}$ is either X or a disjoint union of embedded tori.*

As a qualitative statement this theorem is as good as possible. Namely, if one replaces "nilpotent of class at most 2" by "abelian" then the statement is no longer true. For example, it is false for $T^2\times S^2$, because this manifold has non Jordan diffeomorphism group (see below). In contrast, in dimensions lower than 4 the previous theorem does hold with ”nilpotent of class at most 2” replaced by ”abelian” (the one dimensional case is elementary; see [47] for dimension 2 and [79] for dimension 3).

The following result complements Theorem 4.1 by relating the algebraic structure of nilpotent groups of class at most 2 to the geometry of their potential smooth actions on a given oriented 4-manifold. We are interested on the following invariant of a finite group G:

$$\alpha(G) = \min\{[G : A] \mid A \leq G \text{ abelian}\}.$$

The number $\alpha(G)$ may be understood as a measure of how far G is from being abelian. In the next theorem and in a few other sections of this chapter we are going to use the following standard fact. If X is a closed, connected and oriented 4-manifold and $\Sigma \subset X$ is an embedded closed orientable curve, then picking an orientation of Σ we obtain a homology class $[\Sigma] \in H_2(X)$ whose self intersection can be identified with an integer. This integer is independent of the orientation of Σ and will be denoted by $\Sigma \cdot \Sigma$.

Theorem 4.2. *Let X be a closed, connected and oriented smooth 4-manifold. There exist a constant C and a function $f : \mathbb{N} \to \mathbb{N}$ (both C and f depend on X) satisfying $\lim_{n\to\infty} f(n) = \infty$ and for every finite nilpotent group N of class at most 2 acting in a smooth and effective way on X and satisfying $\alpha(N) \geq C$ there is some $g \in [N, N]$ satisfying:*

1. *the order of g satisfies $\mathrm{ord}(g) \geq f(\alpha(N))$,*
2. *X^g is a nonempty disjoint union of embedded tori $T_1, \dots, T_s \subset X$,*
3. *for every i we have $|T_i \cdot T_i| \geq C\,\alpha(N)$.*

Next, we apply the previous results to the problem of determining which closed 4-manifolds have Jordan diffeomorphism group. The next theorem gives a partial solution to this problem by providing necessary conditions for a 4-manifold to have non Jordan diffeomorphism group. The statement actually applies more generally to subgroups of the group of diffeomorphisms: this will be crucial later when considering automorphisms of geometric structures.

Theorem 4.3. *Let X be a closed connected oriented smooth 4-manifold, and let $\mathcal{G}$ be a subgroup of $\mathrm{Diff}(X)$. If $\mathcal{G}$ is not Jordan then there exists a sequence $(\phi_i)_{i\in\mathbb{N}}$ of elements of $\mathcal{G}$ such that:*

1. *each ϕ_i has finite order $\mathrm{ord}(\phi_i)$,*
2. *$\mathrm{ord}(\phi_i) \to \infty$ as $i \to \infty$,*
3. *all connected components of X^{ϕ_i} are embedded tori,*
4. *for every $C > 0$ there is some i_0 such that if $i \geq i_0$ then any connected component $\Sigma \subseteq X^{\phi_i}$ satisfies $|\Sigma \cdot \Sigma| \geq C$,*

5. *we may pick for each i two connected components $\Sigma_i^-, \Sigma_i^+ \subseteq X^{\phi_i}$ in such a way that the resulting homology classes $[\Sigma_i^\pm] \in H_2(X)$ satisfy $[\Sigma_i^\pm] \cdot [\Sigma_i^\pm] \to \pm\infty$ as $i \to \infty$.*

Proof. This is a consequence of Theorems 4.1 and 4.2, together with Lemma 4.53. □

In the next result we collect a few sufficient conditions for the diffeomorphism group of a closed 4-manifold to be Jordan. We denote by $\chi(X)$, $\sigma(X)$ the Euler characteristic and the signature of a connected, oriented and closed manifold X.

Theorem 4.4. *Let X be a connected, closed, oriented and smooth 4-manifold. If X satisfies any of the following conditions then* $\mathrm{Diff}(X)$ *is Jordan:*

1. $\chi(X) \neq 0$,

2. $\sigma(X) \neq 0$,

3. $b_2(X) = 0$,

4. $b_2^+(X) > 1$ *and X has some nonzero Seiberg–Witten invariant,*

5. $b_2^+(X) > 1$ *and X has some symplectic structure.*

Proof. By the main result in [48], if $\chi(X) \neq 0$ then $\mathrm{Diff}(X)$ is Jordan. By Theorem 4.52 if $\sigma(X) \neq 0$ then $\mathrm{Diff}(X)$ is Jordan. By Theorem 4.36 if $b_2(X) = 0$ then $\mathrm{Diff}(X)$ is Jordan.

Suppose $b_2^+(X) > 1$ and X has some nonzero Seiberg–Witten invariant. If $\mathrm{Diff}(X)$ were not Jordan, then by Theorem 4.3 there would exist in X some embedded torus of positive self-intersection. But this contradicts Corollary 2.51.

If X has $b_2^+(X) > 1$ and admits a symplectic structure, then it has some non-vanishing Seiberg–Witten invariant by Taubes' theorem 3.36. Therefore $\mathrm{Diff}(X)$ must be Jordan by the previous argument. □

Hence if X is a closed 4-manifold such that $\mathrm{Diff}(X)$ is not Jordan, and $H^*(X;\mathbb{Q})$ is not isomorphic to $H^*(T^2 \times S^2;\mathbb{Q})$ as a graded vector space, then all Seiberg–Witten invariants of X are zero.

The following two theorems deal with the question of which automorphisms groups of geometric structures in dimension 4 have the Jordan property for the case of almost complex structures and symplectic structures.

The following theorem extends the main result of [62, 63] from complex structures to almost complex structures.

Theorem 4.5. *Let X be a closed and connected smooth 4-manifold, and let J be an almost complex structure on X. Let* $\mathrm{Aut}(X,J) \subset \mathrm{Diff}(X)$ *be the group of diffeomorphisms preserving J. Then* $\mathrm{Aut}(X,J)$ *is Jordan.*

The proof of Theorem 4.5 is based on Theorem 4.3 (whose proof on its turn is based on [54], which uses the CFSG), and hence is very different from that in [62], which is based on the classification of compact complex surfaces.

The following theorem generalizes the results in [49] to arbitrary symplectic 4-manifolds.

Theorem 4.6. *For any closed symplectic* 4*-manifold* (X, ω) *we have:*

1. $\mathrm{Symp}(X, \omega)$ *is Jordan.*
2. *If* X *is not an* S^2*-bundle over* T^2*, then a Jordan constant for* $\mathrm{Symp}(X, \omega)$ *can be chosen independently of* ω*.*
3. *If* $b_1(X) \neq 2$*, then* $\mathrm{Diff}(X)$ *is Jordan.*

Remark 4.7. *Regarding statement (2) in the previous theorem, note that for a Jordan group* G *depending on a parameter* ω *the following two assertions are in general different:*

(i) the optimal Jordan constant of G *does not depend on* ω*,*

(ii) one can pick a Jordan constant of G *which is independent of* ω*.*

Of course (i) is stronger than (ii), and statement (2) in Theorem 4.6 refers to (ii).

Statement (2) in Theorem 4.6 is sharp in the sense that if X is an S^2-bundle over T^2 then it is impossible to find some number C which is a Jordan constant for $\mathrm{Symp}(X, \omega)$ for all symplectic forms ω on X. More precisely, if $X = T^2 \times S^2$, Theorems 1.1 and 1.2 in [49] imply that the optimal Jordan constant for $\mathrm{Symp}(X, \omega)$ is equal to $\mu(\omega) + C_0(\omega)$, where $\mu(\omega) = |12\langle[\omega], T^2\rangle / \langle[\omega], S^2\rangle|$ and $C_0(\omega)$ is bounded independently of ω, and if X is the twisted S^2-bundle over T^2 then the arguments in the proof of Lemma 4.8 allow to obtain a similar estimate for the optimal Jordan constant for $\mathrm{Symp}(X, \omega)$ for all ω.

Statement (3) in Theorem 4.6 is also sharp: this follows from [13] (see also [50]).

A similar theorem can be proved for isometry groups of closed Lorentz 4-manifolds (see [52], which uses Theorem 4.3 in this chapter). Note that isometry groups of closed Riemannian manifolds (X, g) are always Jordan, because they are compact Lie groups and hence they can be identified, by the Peter–Weyl theorem, with subgroups of some linear group $\mathrm{GL}(n, \mathbb{R})$. This is no longer true for Lorentz metrics: while their isometry group is a finite dimensional Lie group, it may be noncompact and even have infinitely many connected components (see the references in [52]).

4.2 Main ideas of the proofs

The proof of Theorem 4.1 follows different routes depending on whether $b_2(X)$ is zero or not. A common ingredient in both situations is the main result in [54]. This result is concerned with the following analogue of the Jordan property: given positive integers C and d, a collection of finite groups $\mathcal{C}$ satisfies $\mathcal{J}(C, d)$ if each $G \in \mathcal{C}$ has an abelian subgroup A such that $[G : A] \leq C$ and A can be generated by d elements. Let $\mathcal{T}(\mathcal{C})$ be the set of all $G \in \mathcal{C}$ which fit in an exact sequence of groups $1 \to P \to G \to Q \to 1$ such that the orders of P and Q are both prime powers. The main result in [54] (see Theorem 4.48 below) states that if $\mathcal{C}$ is closed under taking subgroups and $\mathcal{T}(\mathcal{C})$ satisfies $\mathcal{J}(C, d)$ for some C and d then $\mathcal{C}$ satisfies $\mathcal{J}(C', d)$ for some C'. We remark that this result uses the classification of finite simple groups.

If M is a closed manifold and $\mathcal{G}$ denotes the collection of all finite subgroups of $\mathrm{Diff}(M)$ then $\mathrm{Diff}(M)$ is Jordan if and only if $\mathcal{G}$ satisfies $\mathcal{J}(C, d)$ for some C and d: this is a nontrivial fact that follows from a theorem of Mann and Su (see Theorem 4.15 below).

In all the results stated in this introduction it suffices to consider connected manifolds, because closed manifolds have finitely many connected components. Let X be a closed connected 4-manifold. By the main result in [49], if the Euler characteristic of X is nonzero then $\mathrm{Diff}(X)$ is Jordan. Consequently, to prove Theorem 4.1 it suffices to consider the case in which X is connected and $\chi(X) = 0$.

Assume first that $b_2(X) = 0$. In this case we directly prove that $\mathrm{Diff}(X)$ is Jordan. This is the main result in Section 4.9 (see Theorem 4.36), and we next briefly explain the structure of the proof. Let $\mathcal{G}$ be the collection of all finite subgroups of $\mathrm{Diff}(X)$, and let $\mathcal{P} \subseteq \mathcal{G}$ be the collection of p-groups (for all primes p). Since $\chi(X) = 0$ we have $b_1(X) = 1$ so $H^1(X) \simeq \mathbb{Z}$. One can prove that if a finite group G acts on X trivially on $H^1(X)$ then there exists a classifying map $c : X \to S^1$ for a generator of $H^1(X)$ which is equivariant with respect to an action of G on S^1 given by a character $\rho : G \to S^1$. The latter is called the rotation morphism, and is defined in Subsection 4.9.1. To study groups G acting on X trivially on $H^1(X)$ we consider separately $\mathrm{Ker}\,\rho$ and $\rho(G)$. In particular we prove that if G is an abelian p-group acting freely on X and $\rho(G)$ is trivial then G must contain a cyclic subgroup of bounded index. This is the main ingredient in the proof that $\mathcal{P}$ satisfies $\mathcal{J}(C, d)$ for some C and d. We deduce from this that $\mathcal{T}(\mathcal{G})$ satisfies $\mathcal{J}(C', d)$ using the main result in Section 4.8. Applying the main result in [54] we conclude that $\mathrm{Diff}(X)$ is Jordan.

Now assume that $b_2(X) \neq 0$. The proof of Theorem 4.1 in this case is contained in Section 4.10. The main step consists in proving that the set $\mathcal{G}_0$ of all finite subgroups of $\mathrm{Diff}(X)$ which are of the form $[G, G]$, where $G < \mathrm{Diff}(X)$ is finite, satisfies the property $\mathcal{J}(C, r)$ for some C and r. To prove this we apply the main result in [54] to $\mathcal{G}_0$, so it suffices to prove $\mathcal{J}(C', r)$ for $\mathcal{T}(\mathcal{G}_0)$. For the purpose of proving Theorem 4.1 we may assume that X is orientable (see Section 4.3). Choose one orientation and let $[X]$ denote the fundamental class. The assumption $b_2(X) \neq 0$, combined with Poincaré duality, implies the existence of line bundles $L_1, L_2 \to X$ such that $\langle c_1(L_1)c_1(L_2), [X]\rangle = 1$. It is proved in [51] that any $\Gamma \in \mathcal{G}_0$ has a central extension $\widehat{\Gamma}$ which acts on L_1, L_2 lifting the action of Γ. If Γ is cyclic a simple trick implies that the action of Γ itself can be lifted to L_1, L_2, and then the equality $\langle c_1(L_1)c_1(L_2), [X]\rangle = 1$ prevents the action of Γ on X from being free. This can be used to prove that any cyclic p-group $\Gamma \in \mathcal{G}_0$ has a subgroup of finite index with nonempty fixed point set (this is Lemma 4.46), and from this one easily deduces that any p-group $\Gamma \in \mathcal{G}_0$ (cyclic or not) has an abelian subgroup of bounded index $B \leq \Gamma$ such that at least one of these properties is true: $X^B \neq \emptyset$, or B fixes a nonorientable embedded surface in X. This then leads to the proof that $\mathcal{G}_0$ satisfies $\mathcal{J}(C', r)$ (Lemma 4.49).

Theorem 4.2 on its turn follows from Theorem 4.1 and from results on actions of finite groups on line bundles over closed surfaces proved in [49] and recalled in Section 4.6.

In the previous sketch we have mentioned some of the sections of the chapter, and we now explain the contents of the other ones. Section 4.3 proves that for our purposes we may assume that all closed manifolds are orientable and that all finite group actions are trivial on cohomology. Section 4.4 contains some auxiliary lemmas on finite abelian p-groups with a bound on the number of generators; these results are used in the present chapter in combination with the theorem of Mann and Su. Section 4.5 gathers some basic consequences of the fact that a group action that preserves a submanifold induces an action on the normal bundle of the submanifold by vector bundle automorphisms. These results are crucial in many of our arguments, and this is the reason why our results can not be automatically transferred from diffeomorphism to to homeomorphism groups. The results in Section 4.6 refer to actions of finite groups on line bundles over closed surfaces. Section 4.7 proves that if a finite abelian p-group acts smoothly on a closed 4-manifold and no element has an isolated fixed point then the homology of the complement of the set where the action is free is bounded independently of p and the group action. This has some important consequences for non-free actions of finite p-groups, which are also proved in Section 4.7. Section 4.8 proves a technical result that in many situations allow to pass from property $\mathcal{J}(C, d)$ for the finite p-subgroups of $\mathrm{Diff}(X)$ to property $\mathcal{J}(C', d)$ for the finite subgroups $G < \mathrm{Diff}(X)$ sitting in an exact sequence of groups $1 \to P \to G \to Q \to 1$ where both P and Q have prime power order. The contents of Sections 4.9 and 4.10 have already been explained. Section 4.11 extracts some consequences of the Atiyah–Singer G-signature theorem. Sections 4.12 and 4.13 contain the proofs of Theorems 4.5 and 4.6 respectively.

4.3 First simplifications

The following is a generalization of the results in [47, Section 2.3].

Lemma 4.8. *Let X be a closed connected manifold and let $X' \to X$ be an unramified covering of finite degree, where X' is connected. If $\mathrm{Diff}(X')$ is Jordan then $\mathrm{Diff}(X)$ is also Jordan. Furthermore, if there exists a constant C such that every finite subgroup $G \leq \mathrm{Diff}(X')$ has a nilpotent subgroup $H \leq G$ of class at most 2 satisfying $[G : H] \leq C$, then the same is true for $\mathrm{Diff}(X)$ for a possibly different value of C.*

Proof. Let $x_0 \in X$ be a base point. Since X is closed, its fundamental group $\pi_1(X, x_0)$ is finitely generated. Let k be the degree of the covering $X' \to X$. Let $\mathrm{Cov}_k(X)$ be the set of isomorphism classes of non-necessarily connected unramified coverings of X of degree k (two coverings $X' \to X$ and $X'' \to X$ are isomorphic if there is a diffeomorphism $X' \to X''$ lifting the identity on X). Let S_k be the permutation group on k letters, and consider the action of S_k on $\mathrm{Hom}(\pi_1(X, x_0), S_k)$ by conjugation. There is a bijection

$$\mathrm{Cov}_k(X) \to \mathrm{Hom}(\pi_1(X, x_0), S_k)/S_k,$$

which sends each element of $\mathrm{Cov}_k(X)$ to its monodromy. Since $\pi_1(X, x_0)$ is finitely generated, $\mathrm{Hom}(\pi_1(X, x_0), S_k)/S_k$ is finite, so $\mathrm{Cov}_k(X)$ is also finite.

Let now $[X'] \in \mathrm{Cov}_k(X)$ be the class of the covering $\pi : X' \to X$. Let $G \leq \mathrm{Diff}(X)$ be a finite subgroup. Then G acts on $\mathrm{Cov}_k(X)$ by pullback. Let $G_0 \leq G$ be the stabilizer of $[X']$. Since the orbit of $[X']$ in $\mathrm{Cov}_k(X)$ can be identified with G/G_0, we have $[G : G_0] \leq \sharp\,\mathrm{Cov}_k(X)$. Define

$$G_1 = \{(g, \phi) \in G_0 \times \mathrm{Diff}(X') \mid \pi \circ \phi = g \circ \pi\}.$$

We have an exact sequence:

$$1 \to \mathrm{Aut}(X') \xrightarrow{\rho} G_1 \xrightarrow{q} G_0 \to 1$$

where $\mathrm{Aut}(X') = \{\phi \in \mathrm{Diff}(X') \mid \pi \circ \phi = \pi\}$ are the automorphisms of the covering, $\rho(\phi) = (1, \phi)$ and $q(g, \phi) = g$. The group $\mathrm{Aut}(X')$ is finite, hence so is G_1. The map $(g, \phi) \mapsto \phi$ defines an inclusion $G_1 \hookrightarrow \mathrm{Diff}(X')$.

If there exists an abelian (resp. nilpotent of class at most 2) subgroup $H \leq G_1$ satisfying $[G_1 : H] \leq C$ then $q(H)$ is also abelian (resp. nilpotent of class at most 2) and satisfies $[G_0 : q(H)] \leq C$, so $[G : q(H)] \leq C \sharp\,\mathrm{Cov}_k(X)$. This proves the Lemma. □

As a corollary, we see that in order to prove that $\mathrm{Diff}(X)$ is Jordan, or to see that there is a constant C such that any finite group $G \leq \mathrm{Diff}(X)$ has a nilpotent subgroup H of class at most 2 with $[G : H] \leq C$, it is enough to show that some finite unramified covering of X has that property. In particular, we may assume without loss of generality that X is orientable.

The following result is a consequence of a classical theorem of Minkowski [45] which states that the size of any finite subgroup of $\mathrm{GL}(k, \mathbb{Z})$ is bounded above by a constant depending only on k. For the proof see [48, Lemma 2.6].

Lemma 4.9. *Let X be a closed manifold. There exists a constant C such that for any continuous action on X of a finite group G there is a subgroup $G_0 \leq G$ satisfying $[G : G_0] \leq C$ and whose action on X is CT.*

This implies that it suffices for our purposes to consider smooth CTE actions of finite groups. In particular these actions are orientation preserving because here we only consider closed manifolds (note that in [48] we consider more generally manifolds with boundary, and for them a cohomologically trivial action need not be orientation preserving).

If a finite group G acts smoothly and preserving the orientation on an oriented 4-manifold X then for every $g \in G$ each connected component of the fixed point set X^g has even codimension. Hence, if $X^g \neq X$ then X^g is the union of finitely many points and finitely many disjoint embedded closed and connected surfaces. We will use this fact repeatedly and without explanation in the arguments that follow.

4.4 Abelian groups with a bound on the number of generators

In this section we collect several lemmas involving finite abelian groups and giving estimates on different quantities as a function of the minimal number of generators of these abelians groups. These results will be used in subsequent sections in combination with the classical theorem of Mann and Su, that we recall in Subsection 4.4.3.

4.4.1 Arbitrary finite groups

Lemma 4.10. *For any natural numbers r, C there exists a number C' with the following property. Let G be a finite group and let $A \leq G$ be an abelian subgroup. Suppose that A can be generated by r elements and that $[G : A] \leq C$. Let $\mathrm{Aut}_A(G) \leq \mathrm{Aut}(G)$ be the group of automorphisms $\phi : G \to G$ satisfying $\phi(A) = A$. We have:*

$$[\mathrm{Aut}(G) : \mathrm{Aut}_A(G)] \leq C'.$$

Proof. Consider the map $\mu : A \to A$ defined as $\mu(a) = a^{C!}$ (we use multiplicative notation on A and later on G), and let $A_0 = \mu(A) \leq A$. Since A can be generated by r elements, we have $[A : A_0] \leq (C!)^r$. Furthermore any subgroup $B \leq A$ satisfying $[A : B] \leq C$ contains A_0. Indeed, for any such B and any $a \in A$ we have $a^{[A:B]} \in B$, so $a^{C!} \in B$. In particular we have $A_0 \leq \phi(A) \cap A$ for every $\phi \in \mathrm{Aut}(G)$, which implies $A_0 \leq \phi(A)$. So $A_1 = \bigcap_{\phi \in \mathrm{Aut}(G)} \phi(A)$ satisfies $A_0 \leq A_1 \leq A$ and consequently

$$[A : A_1] \leq [A : A_0] \leq (C!)^r.$$

By its definition A_1 is clearly a characteristic subgroup of G (i.e., it is invariant under the action of $\mathrm{Aut}(G)$ on G), so in particular it is normal.

Let $\mathcal{S}$ be the collection of all subsets of the quotient group G/A_1. Since

$$\sharp G/A_1 = [G : A_1] = [G : A][A : A_1] \leq C(C!)^r,$$

we can bound $\sharp \mathcal{S} \leq C_1 := 2^{C(C!)^r}$. The action of $\mathrm{Aut}(G)$ on G induces an action on G/A_1 (because $A_1 \leq G$ is characteristic) which on its turn induces an action on $\mathcal{S}$. Denote by $[A] \in \mathcal{S}$ the element corresponding to A/A_1 viewed as a subset of G/A_1. Then $\mathrm{Aut}_A(G)$ is the stabilizer of $[A]$, so we have $[\mathrm{Aut}(G) : \mathrm{Aut}_A(G)] \leq \sharp \mathcal{S} \leq C_1$. □

Lemma 4.11. *For any natural numbers r, C there exists a number C' with the following property. Let G be a finite group, let $G_0 \trianglelefteq G$ be a normal subgroup, and let $A \leq G_0$ be an abelian subgroup. Suppose that $[G_0 : A] \leq C$ and that A can be generated by r elements. Then the normalizer $N_G(A)$ of A in G satisfies*

$$[G : N_G(A)] \leq C'.$$

Proof. Let $c : G \to \mathrm{Aut}(G_0)$ be the morphism defined by the action of G on G_0 given by conjugation. Then $N_G(A) = c^{-1}(\mathrm{Aut}_A(G_0))$, so the lemma follows from Lemma 4.10. □

Lemma 4.12. *Let $1 \to Z \to G \xrightarrow{\pi} A \to 1$ be an exact sequence of finite groups, where $Z \leq G$ is central and A is abelian. Let r be an integer such that every abelian subgroup of G is generated by r elements. Then A is generated by $[r(\log_2(\sharp Z) + 1)]$ elements.*

Proof. For any prime p let $A_p \leq A$ denote the p-part (i.e., the subgroup of elements whose order is a power of p), and let s_p be the minimal number of generators of A_p. Let $A_p[p] \leq A_p$ be the p-torsion. Then $A_p[p] \simeq (\mathbb{Z}/p)^{s_p}$. By the Chinese remainder theorem A can be generated by $\max_p s_p$ elements, where p runs over the set of prime numbers dividing $\sharp A$. Hence it suffices to prove the lemma when $A \simeq (\mathbb{Z}/p)^s$, because the general case can be reduced to it replacing G by $\pi^{-1}(A_p[p])$ for every $p \mid \sharp A$.

Assume for the rest of the proof that $A \simeq (\mathbb{Z}/p)^s$. Then A has a natural structure of s-dimensional vector space over $\mathbb{Z}/p$. Define a map $\Omega : A \times A \to Z$ by $\Omega(a, b) = [\tilde{a}, \tilde{b}]$, where $\tilde{a}, \tilde{b}$ are any lifts of $a, b \in A$ to G. This map is well-defined and it is a skew-symmetric bilinear form on A because Z is central. Hence the image of Ω, which we denote by Z_Ω, is a p-group and all its nontrivial elements have order p. That is, $Z_\Omega \simeq (\mathbb{Z}/p)^r$ for some r, so Z_Ω has a natural structure of vector space over $\mathbb{Z}/p$.

For any vector subspace $I \subseteq A$ we denote $I^\perp = \{a \in A \mid \Omega(a, i) = 1 \text{ for every } i \in I\}$. Alternatively, if we define $\Omega_I : A \to \operatorname{Hom}(I, Z_\Omega)$ by $\Omega_I(a)(i) = \Omega(a, i)$ we can identify

$$I^\perp = \operatorname{Ker} \Omega_I. \tag{4.1}$$

We say that I is isotropic if $I \subseteq I^\perp$. A trivial example of isotropic subspace is $I = 0$. If I is isotropic and there exists some $\gamma \in I^\perp \setminus I$, then $I + \langle \gamma \rangle$ is isotropic (because Ω is skew-symmetric) and strictly bigger than I. Hence, any maximal isotropic subspace I satisfies $I = I^\perp$.

Choose a maximal isotropic subspace $I \subseteq A$. By (4.1) we have $I = I^\perp = \operatorname{Ker} \Omega_I$, so

$$\dim I = \dim \operatorname{Ker} \Omega_I \geq \dim A - \dim \operatorname{Hom}(I, Z_\Omega) = s - \dim I \dim Z_\Omega,$$

and consequently

$$\dim I \geq \frac{s}{1 + \dim Z_\Omega} = \frac{s}{1 + \log_p \sharp Z_\Omega} \geq \frac{s}{1 + \log_2 \sharp Z}.$$

Since I is isotropic, $B := \pi^{-1}(I) \leq G$ is abelian, and it cannot be generated by less than $\dim I$ elements (because B surjects onto I). Consequently $\dim I \leq r$, which, combined with the previous estimates, gives $s \leq \dim I(1 + \log_2 \sharp Z) \leq r(1 + \log_2 \sharp Z)$, so the proof of the lemma is complete. □

4.4.2 Finite p-groups and MNAS's

Lemma 4.13. *Let p be a prime and let $B \leq A$ be finite abelian p-groups. Suppose that A can be generated by r elements. Let $\operatorname{Aut}^0_B(A) \leq \operatorname{Aut}(A)$ denote the automorphisms of A whose restriction to B is the identity. Then*

$$\sharp \operatorname{Aut}^0_B(A) \leq [A : B]^{r^2}.$$

Note that an analogous lemma can be proved for arbitrary finite abelian groups, but for our purposes the case of p-groups will be sufficient.

Proof. Denote $C = [A : B]$. Choose generators $a_1, \ldots, a_r$ of A. An automorphism $\phi \in \mathrm{Aut}(A)$ is determined by the images $\phi(a_1), \ldots, \phi(a_r)$. Suppose that $\phi \in \mathrm{Aut}^0_B(A)$ and write $\phi(a_j) = a_j + d_j$ (additive notation on A) for every j, with $d_j \in A$. For each j we have $Ca_j \in B$, so $Ca_j = \phi(Ca_j) = Ca_j + Cd_j$. It follows that $Cd_j = 0$, so d_j belongs to the C-torsion $A[C] \leq A$. We have $\sharp A[C] \leq C^r$, so the set of all possible choices for $d_1, \ldots, d_r$ has at most $(C^r)^r = C^{r^2}$ elements. Hence, $\sharp\,\mathrm{Aut}^0_B(A) \leq C^{r^2}$. □

Let G be a finite group, and let A be an abelian normal subgroup of G. The action of G on itself by conjugation induces a morphism of groups

$$c : G/A \to \mathrm{Aut}(A).$$

We will write that A is a MNAS (of G) if A is a maximal normal abelian subgroup of G. It is well known that if G is a p-group (for any prime p) and A is a MNAS then c is injective (see e.g. [64, §5.2.3]).

Lemma 4.14. *Let G be a finite p-group and let $A \leq G$ be a MNAS. Suppose that A can be generated by r elements. For every abelian subgroup $B \leq G$ we have*

$$[G : A] \leq [G : B]^{r^2+1}.$$

Proof. Choose an abelian subgroup $B \leq G$. Let $\pi : G \to G/A$ be the quotient map. Then $[G/A : \pi(B)] = [\pi(G) : \pi(B)] \leq [G : B]$. If $b \in B$, then $c(\pi(b)) \in \mathrm{Aut}^0_{A\cap B}(A)$ because B is abelian. Using the injectivity of c and Lemma 4.13 we have

$$\sharp\pi(B) \leq \sharp\,\mathrm{Aut}^0_{A\cap B}(A) \leq [A : A \cap B]^{r^2} \leq [G : B]^{r^2}.$$

Combining the inequalities we have:

$$[G : A] = \sharp G/A = [G/A : \pi(B)] \cdot \sharp\pi(B) \leq [G : B] \cdot [G : B]^{r^2} = [G : B]^{r^2+1}.$$

□

4.4.3 Mann–Su theorem

The following classical result due to Mann and Su will play a prominent role in our arguments.

Theorem 4.15 (Theorem 2.5 in [53])**.** *For any closed manifold Y there exists some integer $r \in \mathbb{Z}$ depending only on $H^*(Y)$ with the property that for any prime p and any elementary p-group $(\mathbb{Z}/p)^s$ admitting an effective action on Y we have $s \leq r$. Equivalently, any finite abelian group acting effectively on Y can be generated by r elements.*

Lemma 4.16. *Given a closed manifold X and natural numbers C_0, C_1 there exists a constant C with the following property. Suppose that G is a finite group sitting in an exact sequence*

$$1 \to G_0 \to G \to G_1 \to 1$$

satisfying $\sharp G_0 < C_0$, and suppose that there exists an abelian subgroup $B \leq G_1$ satisfying $[G_1 : B] \leq C_1$. If there exists a CTE action of G on X then there is an abelian subgroup $A \leq G$ satisfying $[G : A] \leq C$.

Proof. Denote by $\pi : G \to G_1$ the projection. Substituting G by $\pi^{-1}(B)$ we may assume without loss of generality that G_1 is abelian. Let G' be the centralizer of G_0 inside G. Since $\sharp G_0 \leq C_0$ we have $[G : G'] \leq C_0!$. Define $Z = G_0 \cap G'$ and $G_1' = \pi(G') \leq G_1$. Clearly, $[G_1 : G_1'] \leq C_0!$ and $\sharp Z \leq C_0$, and we have an exact sequence

$$1 \to Z \to G' \to G_1' \to 1,$$

where Z is central in G'. Let r be the number given by Theorem 4.15 applied to X. By Lemma 4.12, G_1' can be generated by $[r(\log_2(\sharp Z) + 1)] \leq [r(\log_2 C_0 + 1)]$ elements. Therefore we can apply [47, Lemma 2.2] to obtain the result. □

4.5 Linearization of finite group actions

The following is a classical theorem of Camille Jordan (see [26], and [14, 47] for modern proofs).

Theorem 4.17 (Jordan)**.** *For any natural number n there exists a constant C_n such that every finite subgroup $G \leq \mathrm{GL}(n, \mathbb{C})$ has an abelian subgroup $A \leq G$ satisfying $[G : A] \leq C_n$.*

The next lemma follows from the results in [8, VI.2].

Lemma 4.18. *Let X be a connected 4-manifold and let G be a finite group acting smoothly and effectively on X. Suppose that $X^G \neq \emptyset$. Then, for every $p \in X^G$ the linearization of the G-action at T_pX defines an embedding $G \hookrightarrow \mathrm{GL}(T_pX)$. In particular, we can identify G with a subgroup of $\mathrm{GL}(4, \mathbb{R})$.*

Combining Theorem 4.17 and Lemma 4.18 and taking $C = C_4$ we obtain:

Lemma 4.19. *There is a constant C with the following property. Let X be a connected 4-manifold and let G be a finite group acting smoothly and effectively on X with $X^G \neq \emptyset$. There exists an abelian subgroup $A \leq G$ such that $[G : A] \leq C$.*

Lemma 4.20. *Let X be a connected 4-manifold and let G be a finite group acting smoothly and effectively on X. Suppose that G preserves a connected embedded surface $\Sigma \subset X$. Let $N = TX|_\Sigma / T\Sigma$ be the normal bundle of Σ.*

1. *Linearizing the action along Σ we obtain an effective action of G on N by bundle automorphisms.*

2. *G sits in an exact sequence $1 \to G_0 \to G \to G_\Sigma \to 1$, where G_0 fixes Σ pointwise and is either cyclic or dyhedral, and G_Σ acts effectively on Σ. If in addition X is oriented and G acts on X preserving the orientation then G_0 is cyclic.*

Proof. Since G acts smoothly on X preserving Σ, there is a naturally induced action of G on N. Let $G_0 \leq G$ be the subgroup of elements of G that fix Σ pointwise. Since G_0 is normal in G we have an exact sequence:

$$1 \to G_0 \to G \to G_\Sigma := G/G_0 \to 1$$

Since G preserves Σ we obtain an effective action of G_Σ on Σ. Take a G-invariant Riemannian metric on X. Let $x \in \Sigma$ be any point, and choose an orthogonal basis $e_1, \dots, e_4$ of T_xX with $e_1, e_2 \in T_x\Sigma$. The pair e_3, e_4 is mapped to a basis of N_x via the projection map $T_xX \to N_x = T_xX/T_x\Sigma$. Expressing the linearization of the action of G_0 in terms of the basis (e_i) we obtain a morphism $G_0 \to \emptyset(4, \mathbb{R})$ which by Lemma 4.18 is injective. The image of any element of G_0 is of the form

$$\begin{pmatrix} \mathrm{Id}_2 & 0 \\ 0 & M \end{pmatrix}$$

where $\mathrm{Id}_2 \in \mathrm{SO}(2, \mathbb{R})$ is the identity and $M \in \emptyset(2, \mathbb{R})$. This proves (1). We thus get a monomorphism $\iota : G_0 \hookrightarrow \emptyset(2, \mathbb{R})$ and hence G_0 being finite is cyclic or dihedral. If X is oriented and the action of G on X is orientation preserving then $\det M = 1$, so $\iota(G_0) \leq \mathrm{SO}(2, \mathbb{R})$ and hence G_0 is cyclic. This finishes the proof of (2). □

Assume for the rest of this section that X is an oriented closed 4-manifold.

Lemma 4.21. *Suppose that $\Sigma \subset X$ is a connected embedded surface and that one of the following two assumptions holds true:*

1. *there exists a finite cyclic group G with more than two elements acting smoothly and effectively on X and fixing Σ pointwise;*

2. *there exists a prime $p > 2$ and a finite p-group G acting smoothly and effectively on X, preserving Σ and inducing a noneffective action on Σ.*

Then Σ is orientable.

Proof. Let N be the normal bundle of Σ. Suppose that assumption (1) holds true, so that G is cyclic. Let γ be a generator of G, and let $d = \sharp G > 2$ be its order. Take any point $x \in \Sigma$. The eigenvalues of the action of γ on the fiber N_x are primitive d-roots of unit, which are not real because $d > 2$. Hence they are of the form $\zeta, \overline{\zeta} = \zeta^{-1}$. These eigenvalues are independent of x because Σ is connected. Let $N_\mathbb{C} = N \otimes \mathbb{C}$ and define

$$N^\pm = \{w \in N_\mathbb{C} \mid \gamma \cdot w = \zeta^{\pm 1} w\}.$$

Then $N_\mathbb{C} = N^+ \oplus N^-$ and N^+ and N^- are complex line bundles preserved by the action of G on $N_\mathbb{C}$. Composing the inclusion $N \hookrightarrow N_\mathbb{C}$, $v \mapsto v \otimes 1$, with the projection

$N_{\mathbb{C}} \to N^+$ we obtain an isomorphism of real vector bundles $N \to N^+$ which can be used to transport the complex structure on N^+ to N. Hence N is orientable and since $TX|_\Sigma$ is also orientable, we conclude that $T\Sigma \simeq TX|_\Sigma/N$ is orientable as well.

Now suppose that assumption (2) holds true. Let G_0 be the normal subgroup of G consisting of the elements of G fixing Σ pointwise. By Lemma 4.20, G_0 acts effectively on N by bundle automorphisms and is cyclic or dyhedral. The action of G on Σ is not effective, so G_0 is a nontrivial p-group with $p > 2$ and hence it has more than two elements and cannot be dihedral, so it is cyclic. Applying the case (1) to the action of G_0 we conclude that Σ is orientable. □

Lemma 4.22. *Let $\Sigma \subset X$ be a connected non-orientable embedded surface. There exists a constant $C > 0$, depending only on X and the genus of Σ, such that any finite group G acting smoothly and in a CTE way on X and preserving Σ has an abelian subgroup $A \leq G$ satisfying $[G : A] < C$.*

Proof. By Lemma 4.20, G sits in an exact sequence $1 \to G_0 \to G \to G_\Sigma \to 1$, where G_0 fixes Σ pointwise and acts effectively on the fibers of the normal bundle N, and G_Σ acts effectively on Σ. Since G acts on X preserving the orientation, G_0 is cyclic. By Lemma 4.21, G_0 has at most 2 elements, for otherwise Σ would be orientable. By Lemma 4.23 below, there is an abelian subgroup $B \leq G_\Sigma$ satisfying $[G_\Sigma : B] \leq C_0$, where C_0 depends only on the genus of Σ. According to Lemma 4.16 this implies the existence of an abelian subgroup $A \leq G$ satisfying $[G : A] \leq C$ for some constant C depending only on C_0 and X. □

4.6 Surfaces and line bundles

Lemma 4.23. *Let Σ be a closed connected surface. There is a constant C, depending only on the genus of Σ, such that any finite subgroup $G < \mathrm{Diff}(\Sigma)$ has an abelian subgroup $A \leq G$ satisfying $[G : A] \leq C$.*

Proof. By Lemma 4.8 it suffices to consider the case of orientable Σ, and this is proved in [47, Theorem 1.3 (1)]. □

Lemma 4.24. *Let Σ be a closed connected surface satisfying $\chi(\Sigma) \neq 0$. There exists a constant C, depending only on the genus of Σ, such that for every finite group G acting smoothly on Σ there exists some point $x \in \Sigma$ such that $[G : G_x] \leq C$.*

Proof. Again by Lemma 4.8 it suffices to consider the case of orientable Σ. Choose an orientation of Σ. If the genus g of Σ is 2 or bigger then any finite group acting effectively on Σ has at most $168(g-1)$ elements (see e.g. [47, Theorem 1.3 (2)]) and this immediately implies the lemma. Now suppose that $\Sigma \cong S^2$ and let G be a finite group acting smoothly on Σ. The subgroup $G' \leq G$ of elements which act preserving the orientation satisfies $[G : G'] \leq 2$. An orientation preserving action on a surface has isolated fixed points, so by [48, Theorem 1.4] there exists some $x \in \Sigma$ such that $[G' : G_x] \leq C$, where C is independent of G. This proves the lemma. □

Lemma 4.25. *Let L be an oriented rank 2 real vector bundle over a manifold Σ, and let G be a finite group acting on L by orientation preserving vector bundle automorphisms, lifting an arbitrary smooth action on Σ. Then L admits a G-invariant complex structure.*

Proof. Choosing an arbitrary euclidean structure on L and averaging over the action of G we obtain a G-invariant euclidean structure on L. There is a unique vector bundle isomorphism $I : L \to L$ lifting the identity on Σ such that for any $\lambda \in L$ the two vectors $\lambda, I\lambda$ are perpendicular and $\lambda \wedge I\lambda$ is compatible with the orientation of L. One checks immediately that I is a G-invariant complex structure on L. □

Lemma 4.26. *Let $E \to \Sigma$ be a rank 2 real vector bundle over a connected surface. Suppose that the total space of E is oriented. Let $\mathrm{Aut}^+(E)$ be the group of vector bundle automorphisms of E, lifting arbitrary diffeomorphisms of Σ, and preserving the orientation of E. Let $G < \mathrm{Aut}^+(E)$ be a finite group and suppose that $\alpha \in G$ lifts the trivial action on Σ.*

1. *If Σ is not orientable then α commutes with all elements of G;*

2. *if Σ is orientable then α commutes with the elements of G that act orientation preservingly on Σ.*

Proof. The case $\alpha = \mathrm{Id}_E$ being trivial, we may assume that $\alpha \neq \mathrm{Id}_E$. Let $G_0 \leq G$ be the subgroup of elements lifting the identity map on Σ. We have $\alpha \in G_0$. Applying Lemma 4.20 to the zero section of E we conclude that G_0 is cyclic.

Suppose that Σ is not orientable. Then by Lemma 4.21 G_0 has at most two elements, so α has order 2. Since $\alpha \in \mathrm{Aut}^+(E)$, the action of α on the fibers of E is multiplication by -1, and this implies that α commutes with all elements of $\mathrm{Aut}^+(E)$.

Now suppose that Σ is orientable. Then, since the total space of E is orientable, E is also orientable. Choose an orientation of E. We may replace for our purposes the group G by its intersection with the elements of $\mathrm{Aut}^+(E)$ that act on Σ orientation preservingly. These elements preserve the orientation of E as a vector bundle. By Lemma 4.25 there is a G-invariant complex structure on E, so we can look at E as a complex line bundle. Since α lifts the identity on Σ, its action is given by multiplication by a smooth map $f : \Sigma \to \mathbb{C}^*$, so $\alpha(\lambda) = f(\pi(\lambda))\lambda$ for every $\lambda \in E$, where $\pi : E \to \Sigma$ is the projection. Since α has finite order, there is some integer k such that $f(x)^k = 1$ for every $x \in \Sigma$. This implies that f is constant because Σ is connected, and this implies that α commutes with all elements of G. □

Lemma 4.27. *Let $L \to T^2$ be a complex line bundle and let $\mathrm{Aut}(L) \subset \mathrm{Diff}(L)$ denote the group of line bundle automorphisms of L, lifting arbitrary diffeomorphisms of T^2. Let $G < \mathrm{Aut}(L)$ be a finite subgroup.*

1. *There is an abelian subgroup $A \leq G$ satisfying $[G : A] \leq 12 \max\{1, |\deg L|\}$.*

2. *There is a nilpotent subgroup $N \leq G$ of class at most 2 such that $[G : N] \leq 12$ and $[N, N]$ is cyclic and acts trivially on T^2.*

Proof. Let $G_0 \leq G$ be the group of elements acting on T^2 preserving the orientation. We have $[G : G_0] \leq 2$. Statement (1) follows from applying [49, Proposition 2.10] to G_0. Let us prove (2). Let $\rho : \mathrm{Aut}(L) \to \mathrm{Diff}(T^2)$ be defined by restricting to the zero section. By [49, Lemma 2.5] there is an abelian subgroup $B \leq \rho(G_0)$ satisfying $[\rho(G_0) : B] \leq 6$. Hence $N = \rho^{-1}(B)$ satisfies $[G_0 : N] \leq 6$. Now $[N, N]$ acts trivially on T^2, so $[[N, N], N] = 1$ follows from (2) in Lemma 4.26. Since the action of N on the total space of L preserves the orientation, Lemma 4.20 implies that $[N, N]$ is cyclic. □

Lemma 4.28. *Let Σ be a closed and connected surface, and let $L \to \Sigma$ be a complex line bundle. Let $\mathrm{Aut}(L) \subset \mathrm{Diff}(L)$ denote the group of line bundle automorphisms of L, lifting arbitrary diffeomorphisms of Σ. Suppose that at least one of the following two conditions holds true.*

1. $\chi(\Sigma) \neq 0$, *or*
2. *L is trivial.*

Then there is a constant C, depending only on the genus of Σ, such that any finite subgroup $G < \mathrm{Aut}(L)$ has an abelian subgroup $A \leq G$ satisfying $[G : A] \leq C$.

Proof. Suppose first that $\chi(\Sigma) \neq 0$ and that G is a finite group acting effectively on L by complex line bundle automorphisms. Then G preserves the zero section of L, which we identify with Σ, and hence by Lemma 4.24 there exists some point $x \in \Sigma$ satisfying $[G : G_x] \leq C_0$ for some C_0 depending only on the genus of Σ. Applying now Lemma 4.19 to the action of G_x on the total space of L we conclude the proof.

It suffices now to consider the case in which L is trivial and $\chi(\Sigma) = 0$. By Lemma 4.8 we need only consider the case $\Sigma \cong T^2$, which follows from (1) in Lemma 4.27. □

4.7 Non-free actions

In this section p denotes a prime number.

The following result is well known, see for example [51, Lemma 5.1].

Lemma 4.29. *Let G be a finite p-group acting on a manifold X. Then*

$$\sum_{j \geq 0} b_j(X^G; \mathbb{Z}/p) \leq \sum_{j \geq 0} b_j(X; \mathbb{Z}/p),$$

where b_j denotes the j-th Betti number.

Lemma 4.30. *Let p be an odd prime, and let $U, V \in \mathrm{SO}(4, \mathbb{R})$ be two commuting matrices of order p. If $\mathrm{Ker}(U - 1) \neq \mathrm{Ker}(V - 1)$ and both kernels are nonzero then $\mathrm{Ker}(UV - 1) = 0$.*

Proof. Let $A = \mathrm{Ker}(U - 1)$ and $B = \mathrm{Ker}(V - 1)$. By assumption $A \neq 0 \neq B$. Since U and V have odd nontrivial order we necessarily have $\dim A = \dim B = 2$, and U (resp. V) acts as a nontrivial rotation on $A^\perp$ (resp. $B^\perp$). One easily checks that the only two dimensional subspaces of $\mathbb{R}^4$ that are preserved by U are A and $A^\perp$. Since U and V commute, U preserves B, so the only possibilities are $B = A$ (which is ruled out by our assumptions) or $B = A^\perp$ (and consequently $B^\perp = A$). Then UV preserves both A and $A^\perp$ and its restriction to each of them is a nontrivial rotation. Hence UV has no nonzero fixed vector. $\square$

4.7.1 The set $W(X, A)$

Let X be a closed 4-manifold and let A be a finite abelian group acting smoothly and effectively on X. Define the following subset of X:

$$W(X, A) = \bigcup_{a \in A \setminus \{1\}} X^a.$$

The set $W(X, A)$ will appear several times in our arguments, especially in situations where no element of A has an isolated fixed point. Assuming this condition, the following lemma gives what for us will be the most important properties of $W(X, A)$.

Lemma 4.31. *Let X be a closed, connected and oriented 4-manifold. There exists a constant C with the following property. Let p be any prime. Suppose that A is a finite abelian p-group acting in a smooth and CTE way on X, and that there exists no $a \in A$ for which the fixed point set X^a has an isolated point. Let $W = W(X, A)$. Then*

1. *$W \subset X$ is a possibly disconnected embedded closed surface, and each connected component of W is a connected component of X^a for some $a \in A \setminus \{1\}$,*
2. *X^a is equal to the union of some connected components of W for each $a \in A \setminus \{1\}$,*
3. *$\sharp\pi_0(W) \leq C$, and each connected component of W has genus at most C.*

Proof. Let $\Gamma \leq A$ be the p-torsion and define:

$$F = \{(a, Z) \mid a \in \Gamma \setminus \{1\},\ Z \text{ connected component of } X^a\}.$$

Since for every $a \in A \setminus \{1\}$ there exists some $r = p^s$ such that a^r has order p, and clearly $X^a \subseteq X^{a^r}$, we have

$$W = \bigcup_{(a,Z) \in F} Z. \tag{4.2}$$

By assumption, for each $(a, Z) \in F$, Z is a connected and embedded surface in X. We claim that for every two elements $(a, Z), (a', Z') \in F$ either $Z = Z'$ or $Z \cap Z' = \emptyset$. Indeed, if $Z \cap Z' \neq \emptyset$ but $Z \neq Z'$, then $Z \cap Z'$ is a proper submanifold of Z. Applying Lemma 4.30 to the linearisation of the action of $\langle a, a' \rangle$ at some point in $Z \cap Z'$ we would then deduce the existence of $a'' \in A$ with an isolated fixed point, contradicting our assumption. This proves the claim, and the claim immediately implies (1).

To prove (2) take an arbitrary $a \in A \setminus \{1\}$ and choose as before some r such that a^r has order p. Since all connected components of X^a have dimension 2 and the same happens with X^{a^r}, we conclude that X^a is the union of some (maybe all) connected components of X^{a^r}. Combining this with formula (4.2) we deduce statement (2).

We next prove (3). Let $U \subset X$ be a Γ-invariant tubular neighborhood of W and let $V = X \setminus W$. Consider the Mayer–Vietoris sequence with $\mathbb{Z}/p$ coefficients applied to the covering $X = U \cup V$:

$$\cdots \to H^n_\Gamma(U \cap V; \mathbb{Z}/p) \to H^{n+1}_\Gamma(X; \mathbb{Z}/p) \to \begin{array}{c} H^{n+1}_\Gamma(U; \mathbb{Z}/p) \\ \oplus \\ H^{n+1}_\Gamma(V; \mathbb{Z}/p) \end{array} \to H^{n+1}_\Gamma(U \cap V; \mathbb{Z}/p) \cdots$$

Since the action of Γ on V (hence also on $U \cap V$) is free, we have

$$H^*_\Gamma(V; \mathbb{Z}/p) \simeq H^*(V/\Gamma; \mathbb{Z}/p)$$

(and similarly for $U \cap V$). Since V/Γ and $(U \cap V)/\Gamma$ are 4-manifolds, for $n > 4$ we have $H^n_\Gamma(V; \mathbb{Z}/p) = H^n_\Gamma(U \cap V; \mathbb{Z}/p) = 0$. Therefore, the above exact sequence gives us isomorphisms

$$H^6_\Gamma(X; \mathbb{Z}/p) \simeq H^6_\Gamma(U; \mathbb{Z}/p) \simeq H^6_\Gamma(W; \mathbb{Z}/p)$$

Considering the Serre spectral sequence for the fibration $X_G \to X$, we obtain

$$\dim H^6_\Gamma(X; \mathbb{Z}/p) \leq \sum_{i+j=6} \dim H^i(X; \mathbb{Z}/p) \otimes H^j(\Gamma; \mathbb{Z}/p). \tag{4.3}$$

By Theorem 4.15 the p-rank of Γ is bounded above by a constant depending only on X, so it follows from (4.3) that

$$\dim H^6_\Gamma(X; \mathbb{Z}/p) \leq C', \tag{4.4}$$

where C' only depends on X.

Let $D = \max_p \sum_{j \geq 0} b_j(X; \mathbb{Z}/p)$, where p runs over the set of all primes. If $a, b \in \Gamma$ and Z is a connected component of X^a then bZ is a connected component of $X^{bab^{-1}} = X^a$. Since, by Lemma 4.29, X^a has at most D connected components, and each connected component of W is a connected component of X^a for some $a \in \Gamma$, the orbits of the action of Γ on $\pi_0(W)$ have at most D elements. Let $Z_1, \dots, Z_l$ be a collection of connected components of W such that W is the disjoint union of the Γ-orbits of $Z_1, \dots, Z_l$. Then

$$l \geq \frac{\sharp \pi_0(W)}{D}. \tag{4.5}$$

Define for each i the following two subgroups of Γ:

$$\Gamma_i = \{a \in \Gamma \mid aZ_i = Z_i\},$$
$$\Gamma_{i0} = \{a \in \Gamma \mid ax = x \text{ for every } x \in Z_i\}.$$

Remark that Γ_{i0} is not the trivial group, since by assumption Z_i is a connected component of X^a for some $a \in \Gamma \setminus \{1\}$. Fix some model $E\Gamma \to B\Gamma$ for the universal principal Γ-bundle. The inclusion

$$E\Gamma \times_{\Gamma_i} Z_i \to E\Gamma \times_{\Gamma_i} (\Gamma Z_i)$$

followed by the projection $E\Gamma \times_{\Gamma_i} (\Gamma Z_i) \to E\Gamma \times_{\Gamma} (\Gamma Z_i)$ gives a homeomorphism

$$E\Gamma \times_{\Gamma_i} Z_i \cong E\Gamma \times_{\Gamma} (\Gamma Z_i),$$

which induces an isomorphism

$$H^*_{\Gamma_i}(Z_i; \mathbb{Z}/p) \simeq H^*_{\Gamma}(\Gamma Z_i; \mathbb{Z}/p). \tag{4.6}$$

Hence

$$H^6_{\Gamma}(X; \mathbb{Z}/p) \simeq \bigoplus_{i=1}^{l} H^6_{\Gamma}(\Gamma Z_i; \mathbb{Z}/p) \simeq \bigoplus_{i=1}^{l} H^6_{\Gamma_i}(Z_i; \mathbb{Z}/p). \tag{4.7}$$

The action of Γ_i on Z_i descends to an action of $J_i := \Gamma_i/\Gamma_{i0}$ on Z_i. We claim that J_i acts freely on Z_i. This is equivalent to the statement that if an element $b \in \Gamma$ preserves Z_i and fixes some point of Z_i then it necessarily fixes all points of Z_i; this is true because X^b is a possibly disconnected embedded surface (without isolated points, as we are assuming at this point) and because $X^b \cap Z_i \neq \emptyset$ implies $Z_i \subseteq X^b$ by Lemma 4.30 (see the argument after formula (4.2)). Now, an argument similar to the one that led to the isomorphism (4.6) combined with Künneth implies

$$H^6_{\Gamma_i}(Z_i; \mathbb{Z}/p) \simeq H^6_{\Gamma_{i0}}(Z_i/J_i; \mathbb{Z}/p) \simeq \bigoplus_{u+v=6} H^u(B\Gamma_{i0}; \mathbb{Z}/p) \otimes H^v(Z_i/J_i; \mathbb{Z}/p),$$

where the second term is the equivariant cohomology of the trivial action of Γ_{i0} on Z_i/J_i. The rightmost term in the previous formula contains the summand

$$H^6(B\Gamma_{i0}; \mathbb{Z}/p) \otimes H^0(Z_i/J_i; \mathbb{Z}/p),$$

which is nonzero because Γ_{i0} is not the trivial group, and hence is of the form $(\mathbb{Z}/p)^s$ for some $s > 0$. It then follows from (4.7) that $\dim H^6_{\Gamma}(X; \mathbb{Z}/p) \geq l$. Using (4.4) we get $C' \geq l$, so using (4.5) we obtain

$$\sharp\pi_0(W) \leq C'D.$$

Finally, if Z is a connected component of W then by (1) there exists some $a \in A\setminus\{1\}$ such that Z is a connected component of X^a. Then Lemma 4.29 implies that $b_0(Z; \mathbb{Z}/p) + b_1(Z; \mathbb{Z}/p) + b_2(Z; \mathbb{Z}/p) \leq D$, which implies that the genus of Z is bounded above by a constant depending on X. □

4.7.2 Normal abelian p-subgroups

Lemma 4.32. *Let X be a closed, connected and oriented 4-manifold. There exists a constant C with the following property. Suppose that G is a finite group acting in a smooth and CTE way on X, let p be any prime and let $A \leq G$ be a normal abelian p-subgroup. If there exists some $a \in A$ and an isolated point in X^a then there is an abelian subgroup $B \leq G$ satisfying $[G : B] \leq C$ and $X^B \neq \emptyset$.*

Proof. Suppose that $a \in A$ and that X^a contains an isolated point. Let $S \subset X$ be the set of isolated points of X^a. Let

$$D = \max_p \sum_{j \geq 0} b_j(X; \mathbb{Z}/p),$$

where p runs over the set of all primes. Applying Lemma 4.29 to the action of $\langle a \rangle$ on X we deduce that $\sharp S \leq D$. Take any point $s \in S$. Since A is abelian, the action of any $a' \in A$ preserves X^a and hence S. Consequently, the stabilizer A_0 of s in A satisfies $[A : A_0] \leq D$. Let $G_0 \leq G$ be the normalizer of A_0. Combining Theorem 4.15 with Lemma 4.11 we conclude that $[G : G_0] \leq C_1$ for some constant C_1 depending only on X. Applying Lemma 4.29 to the action of A_0 on X we deduce that X^{A_0} contains at most D isolated points. Since G_0 normalizes A_0, its action on X preserves the set of isolated fixed points of A_0. Hence there is a subgroup $G_1 \leq G_0$ satisfying $[G_0 : G_1] \leq D$ and preserving (hence, fixing) one of the isolated fixed points of A_0. By Lemma 4.19, G_1 contains an abelian subgroup $B \leq G_1$ satisfying $[G_1 : B] \leq C_2$, where C_2 is a universal constant. It follows that $[G : B] \leq DC_1C_2$, so we are done. □

Lemma 4.33. *Let X be a closed, connected and oriented 4-manifold. There exists a constant C with the following property. Suppose that G is a finite group acting in a smooth and CTE way on X, let p be any prime and let $A \leq G$ be a normal abelian p-subgroup. Suppose that:*

1. *there is no $a \in A$ such that X^a has an isolated fixed point,*
2. *there exists some $a \in A$ such that X^a has a connected component Z which is a nonorientable surface;*

then there is an abelian subgroup $B \leq G$ satisfying $[G : B] \leq C$ and an element $b \in B$ such that X^b has Z as one of its connected components.

Proof. Let C_1 be the constant given by Lemma 4.31 and let $W = W(X, A)$. Since A is normal in G, the action of G on X preserves W. By (2) in Lemma 4.31, Z is a connected component of W. By (3) in Lemma 4.31, W contains at most C_1 connected components and the genus of Z is not bigger than C_1.

Let $G_0 \leq G$ be the subgroup of elements that preserve Z. We have $[G : G_0] \leq C_1$. By Lemma 4.22 there is an abelian subgroup $B_0 \leq G_0$ satisfying $[G_0 : B_0] \leq C_2$, where C_2 depends only on C_1 and X, hence only X. Let $a \in A$ be an element whose fixed point set contains Z as a connected component and let $B = \langle a, B_0 \rangle$. Let $N \to Z$ be the normal bundle. There is a natural morphism $B \to \operatorname{Aut}(N)$ which is injective by (1) in Lemma 4.20. Its image is contained in $\operatorname{Aut}^+(N)$, the automorphisms preserving the orientation of the total space of N (which is orientable because X is). By (1) in Lemma 4.26, it follows that B is abelian. Since $B_0 \leq B$, we have

$$[G : B] \leq [G : B_0] = [G : G_0][G_0 : B_0] \leq C_1C_2,$$

so the proof of the lemma is complete. □

Lemma 4.34. *Let X be a closed, connected and oriented 4-manifold. There exists a constant C with the following property. Suppose that G is a finite group acting in a smooth and CTE way on X, let p be any prime and let $A \leq G$ be a normal abelian p-subgroup. If the action of A on X is not free, then at least one of these statements holds true:*

1. *there exists an abelian subgroup $G_0 \leq G$ such that $[G : G_0] \leq C$,*
2. *there exists an embedded connected orientable surface $Z \subset X$ of genus not bigger than C preserved by a subgroup $G_0 \leq G$ that satisfies $[G : G_0] \leq C$.*

Proof. Let C_1, C_2, C_3 be the constants given by Lemmas 4.31, 4.32 and 4.33 respectively. Let $C = \max\{C_1, C_2, C_3\}$.

If there exists some $a \in A$ such that X^a has an isolated fixed point then we can apply Lemma 4.32 and conclude the existence of an abelian subgroup $G_0 \leq G$ satisfying $[G : G_0] \leq C_2$. If there is no $a \in A$ such that X^a contains an isolated point, and there is some $b \in A$ such that X^b has a connected component which is a nonorientable surface then by Lemma 4.33 there is an abelian subgroup $G_0 \leq G$ satisfying $[G : G_0] \leq C_3$.

Now suppose that for every $a \in A \setminus \{1\}$ the fixed point set X^a is an embedded orientable surface. Then $W := W(X, A)$ is nonempty because by assumption the action of A on X is not free. By our assumptions and Lemma 4.31, W is a possibly disconnected embedded orientable surface, and W has at most C_1 connected components. Let $Z \subseteq W$ be any connected component. Since A is a normal subgroup of G, the action of G on X preserves W. The subgroup $G_0 \leq G$ preserving Z satisfies $[G : G_0] \leq \sharp\pi_0(W) \leq C_1$. By Lemma 4.31 the genus of Z is at most C_1. □

4.8 Diffeomorphisms normalizing an action of $\mathbb{Z}/p$ or $\mathbb{Z}/p^2$

The following basic fact will be used in this section and in several other results on free actions to be proved in Subsection 4.9.2: suppose that a finite group Γ acts freely and orientation preservingly on an oriented, closed and connected 4-manifold X; then the Borel construction X_Γ is homotopy equivalent to X/Γ, which is an orientable, closed and connected 4-manifold; consequently,

$$H^4_\Gamma(X; A) \simeq A, \qquad H^k_\Gamma(X; A) = 0 \text{ if } k > 4, \tag{4.8}$$

for every abelian group A.

Lemma 4.35. *Let X be a closed, connected and oriented 4-manifold. Let $\Gamma = (\mathbb{Z}/p)^r$, where $r = 1$ or 2. Suppose that Γ acts on X in a smooth and CTE way, and that there exists an automorphism $\phi \in \mathrm{Aut}(\Gamma)$ and a diffeomorphism $\psi \in \mathrm{Diff}(X)$ acting trivially on $H^*(X)$ in such a way that the diagram*

$$\begin{array}{ccc} \Gamma \times X & \longrightarrow & X \\ {\scriptstyle \phi\times\psi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \Gamma \times X & \longrightarrow & X, \end{array}$$

in which the horizontal arrows are the maps defining the action of Γ *on* X*, is commutative. If the order of* ϕ *is not divisible by* p *and is bigger than* 4 *then the action of* Γ *on* X *is not free.*

Proof. The commutative diagram in the statement of the lemma gives the following commutative diagram involving the Borel construction of X:

$$\begin{array}{ccc} X_\Gamma & \longrightarrow & B\Gamma \\ \downarrow & & \downarrow \\ X_\Gamma & \longrightarrow & B\Gamma, \end{array}$$

in which the left (resp. right) hand side vertical arrow is induced by (ϕ, ψ) (resp. ϕ). The previous diagram implies the existence of an automorphism of the Serre spectral sequence with coefficients in $\mathbb{Z}/p$ for the fibration $X_\Gamma \to B\Gamma$ which is given, at the level of the second page, by the morphism

$$\phi^* \otimes \psi^* : H^\sigma(B\Gamma; \mathbb{Z}/p) \otimes H^\tau(X; \mathbb{Z}/p) \to H^\sigma(B\Gamma; \mathbb{Z}/p) \otimes H^\tau(X; \mathbb{Z}/p).$$

Crucially, $\phi^* \otimes \psi^*$ commutes with all the differentials of the spectral sequence. Since by assumption the action of Γ on X is CTE, we have $\psi^* = \mathrm{id}$.

Suppose from now on that Γ acts freely on X. Denote the Serre spectral sequence for the fibration $X_\Gamma \to B\Gamma$ by $\{(E_u^{\sigma,\tau}, d_u^{\sigma,\tau})\}$.

We consider separately the cases $r = 1$ and $r = 2$.

Consider first the case $\Gamma = \mathbb{Z}/p$, and suppose that ϕ acts on $\mathbb{Z}/p$ as multiplication by some $\zeta \in (\mathbb{Z}/p)^*$. Then ϕ^* acts on $H^1(B\Gamma; \mathbb{Z}/p) = \mathrm{Hom}(H_1(B\Gamma), \mathbb{Z}/p)$ as multiplication by ζ. Let $\mathrm{b} : H^*(B\Gamma; \mathbb{Z}/p) \to H^{*+1}(B\Gamma; \mathbb{Z}/p)$ be the Bockstein morphism. To compute the action on higher cohomology groups, note that if $\theta \in H^1(B\Gamma; \mathbb{Z}/p)$ is a generator then $\mathrm{b}(\theta)$ is a generator of $H^2(B\Gamma; \mathbb{Z}/p)$. By the naturality of b, ϕ^* acts on $H^2(B\Gamma; \mathbb{Z}/p)$ as multiplication by ζ. More generally, for any natural number k and any $\epsilon \in \{0, 1\}$, $\theta^\epsilon\, \mathrm{b}(\theta)^k$ is a generator of $H^{2k+\epsilon}(B\Gamma; \mathbb{Z}/p)$, which implies that the action of ϕ^* on $H^n(B\Gamma; \mathbb{Z}/p)$ is given by multiplication by $\zeta^{[(n+1)/2]}$, where $[t]$ denotes the integral part of t.

Now suppose that the order of ζ is bigger than 4. Then in particular the elements $1, \zeta, \zeta^2, \zeta^3 \in (\mathbb{Z}/p)^*$ are pairwise distinct. This implies that the differentials d_2, d_3, d_4, d_5 in the spectral sequence are identically zero, because they commute with $\phi^* \otimes \mathrm{id}$. Since $E_2^{\sigma,\tau} = 0$ for every $\tau > 4$, the vanishing of $d_2, \dots, d_5$ implies that the spectral sequence degenerates. In particular

$$\begin{aligned} \dim H^4_\Gamma(X; \mathbb{Z}/p) &= \dim E_2^{0,4} + \dim E_2^{1,3} + \dim E_2^{2,2} + \dim E_2^{3,1} + \dim E_2^{4,0} \\ &= \sum_{j=0}^{4} b_j(X; \mathbb{Z}/p) \geq 2, \end{aligned}$$

and this contradicts (4.8).

We next consider the case $\Gamma = (\mathbb{Z}/p)^2$. Suppose that α, β are the eigenvalues of ϕ^* acting on $H^1(B\Gamma; \mathbb{Z}/p)$ (in general α, β live in an algebraic extension $\overline{\mathbb{Z}/p}$ of the field

$\mathbb{Z}/p$, which we assume to be fixed for the arguments that follow). We want to compute the action of ϕ^* on $E_4^{4,0} \simeq H^4(B\Gamma;\mathbb{Z}/p)$. Take any basis (θ_1,θ_2) of $H^1(B\Gamma;\mathbb{Z}/p)$. Arguing as in our discussion about $H^*(B\mathbb{Z}/p;\mathbb{Z}/p)$ and using Künneth we deduce that $(\mathrm{b}(\theta_1), b(\theta_2), \theta_1\theta_2)$ is a basis of $H^2(B\Gamma;\mathbb{Z}/p)$. Hence if we denote

$$W = H^1(B\Gamma;\mathbb{Z}/p)$$

then we can identify in a natural way (in particular, *as representations of* $\langle\phi^*\rangle$)

$$H^2(B\Gamma;\mathbb{Z}/p) \simeq W \otimes \Lambda^2 W.$$

Similarly, $(\theta_1\,\mathrm{b}(\theta_1), \theta_2\,\mathrm{b}(\theta_1), \theta_1\,\mathrm{b}(\theta_2), \theta_2\,\mathrm{b}(\theta_2))$ is a basis of $H^3(B\Gamma;\mathbb{Z}/p)$, hence

$$H^3(B\Gamma;\mathbb{Z}/p) \simeq W \otimes W$$

canonically. Similar arguments lead to the following natural isomorphism:

$$H^4(B\Gamma;\mathbb{Z}/p) \simeq S^2W \oplus W \otimes \Lambda^2 W.$$

Accordingly, the eigenvalues of the action of ϕ^* on $H^4(B\Gamma;\mathbb{Z}/p)$ are given by

$$\alpha^2,\ \alpha\beta,\ \beta^2,\ \alpha^2\beta,\ \alpha\beta^2. \tag{4.9}$$

Of course, it may happen that these eigenvalues are not pairwise distinct; in general, the number of times that a given element $\lambda \in \overline{\mathbb{Z}/p}$ appears in the list (4.9) is equal to the dimension of $\mathrm{Ker}(\phi^* - \lambda\,\mathrm{id}_{H^4(B\Gamma;\mathbb{Z}/p)})$.

Since $\dim H^4_\Gamma(X;\mathbb{Z}/p) = 1$ we must have $\dim E_\infty^{4,0} \leq 1$. We have $\dim E_2^{4,0} = 5$, hence the dimensions of the images of the differentials

$$d_2^{2,1} : E_2^{2,1} \to E_2^{4,0}, \quad d_3^{1,2} : E_3^{1,2} \to E_3^{4,0}, \quad d_4^{0,3} : E_4^{0,3} \to E_4^{4,0}$$

have to add up at least 4. The weights of the action of ϕ on $E_2^{2,1}, E_3^{1,2}, E_4^{0,3}$ are the same as the weights of the action on $H^0(B\Gamma;\mathbb{Z}/p) \oplus H^1(B\Gamma;\mathbb{Z}/p) \oplus H^2(B\Gamma;\mathbb{Z}/p)$, namely $1, \alpha, \beta, \alpha\beta$. It follows that at least 4 of the elements in (4.9) must belong to the set $\{1, \alpha, \beta, \alpha\beta\}$. Let us reformulate our last statement in algebraic terms. Define the following subsets of $\mathbb{Z}^2$:

$$S = \{(2,0),(1,1),(0,2),(2,1),(1,2)\}, \qquad T = \{(0,0),(1,0),(0,1),(1,1)\}.$$

We then have:

($\star$) there exists a subset $S' \subseteq S$ such that $S \setminus S'$ contains at most one element, and a map $f = (f_u, f_v) : S' \to T \subset \mathbb{Z}^2$ with the property that for every $(u,v) \in S$ we have $\alpha^u\beta^v = \alpha^{f_u(u,v)}\beta^{f_v(u,v)}$.

Let $R = \{f(w) - w \mid w \in S'\} \subset \mathbb{Z}^2$. We claim that R contains two linearly independent elements of $\mathbb{Z}^2$. First note that $R \neq \{0\}$ for otherwise we would have $f(w) = w$ for all w, which is not compatible with $(\star)$ because $S \cap T$ contains a unique element. We also cannot have $R \subset \mathbb{Z}\, w$ for any $w \in \mathbb{Z}^2$, because for every $w \in \mathbb{Z}^2$ the intersection $S \cap (T + \mathbb{Z}\, w)$ contains at most 3 elements, as one readily checks by plotting the elements of S and T; hence $R \subset \mathbb{Z}\, w$ would again contradict $(\star)$, so the claim is proved.

Suppose $(u, v), (u', v') \in R$ are linearly independent, so that $d := uv' - u'v$ is nonzero. Since $S, T \subset \{0, 1, 2\}^2$, we have $u, v, u', v' \in \{0, 1, 2\}$ and hence $|d| \leq 4$. The equalities $\alpha^u \beta^v = \alpha^{u'} \beta^{v'} = 1$ imply that

$$\alpha^d = \alpha^{uv' - u'v} = (\alpha^u \beta^v)^{v'} (\alpha^{u'} \beta^{v'})^{-v} = 1 = (\alpha^u \beta^v)^{u'} (\alpha^{u'} \beta^{v'})^{-u} = \beta^{vu' - v'u} = \beta^d. \quad (4.10)$$

Consequently, the eigenvalues α^d, β^d of $(\phi^*)^d \in \mathrm{Aut}(H^1(B\Gamma; \mathbb{Z}/p))$ are equal to one, so $(\phi^*)^d$ is a unipotent automorphism. The order of a unipotent automorphism of a vector space over $\mathbb{Z}/p$ is necessarily a power of p. Since the order of $(\phi^*)^d$ is prime to p, it follows that $(\phi^*)^d$ is the identity. Hence ϕ^* is an automorphism of $H^1(B\Gamma; \mathbb{Z}/p)$ of order at most 4. Since there is a natural isomorphism $H^1(B\Gamma; \mathbb{Z}/p) \simeq \mathrm{Hom}(\Gamma, \mathbb{Z}/p)$, the fact that $(\phi^*)^d$ is trivial implies that $\phi^d \in \mathrm{Aut}(\Gamma)$ is trivial, so the order of ϕ is at most 4. $\square$

4.9 Finite groups acting smoothly on 4-manifolds with $b_2 = 0$

The goal of this section is to prove the following:

Theorem 4.36. *Suppose that X is a closed connected 4-manifold satisfying $b_2(X) = 0$. Then $\mathrm{Diff}(X)$ is Jordan.*

Let X be a closed connected 4-manifold satisfying $b_2(X) = 0$. To prove Theorem 4.36 we only need to consider the case $\chi(X) = 0$, for if $\chi(X) \neq 0$ then $\mathrm{Diff}(X)$ is Jordan by the main result in [48]. By Lemma 4.8 we may also assume that X is orientable, so the Betti numbers of X are $b_0(X) = b_4(X) = 1$ and $b_1(X) = b_3(X) = 1$. Let T be the torsion of $H_1(X)$. By the universal coefficient theorem the torsion of $H^2(X)$ is isomorphic to T, and by Poincaré duality we have $H^3(X) \simeq H_1(X)$, so the torsion of $H^3(X)$ is also isomorphic to T. Hence we have

$$H^0(X) \simeq H^1(X) \simeq H^4(X) \simeq \mathbb{Z}, \quad H^2(X) \simeq T, \quad H^3(X) \simeq \mathbb{Z} \oplus T. \quad (4.11)$$

Assuming these conditions, we will prove Theorem 4.36 in Subsection 4.9.4 below, after introducing a number of preliminary results. The manifold X will be fixed in the entire section.

4.9.1 Rotation morphism

The following construction is used in [47]. We explain it here in slightly more intrinsic terms. Let $e : \mathbb{R} \to S^1$ be the map $e(t) = e^{2\pi \mathbf{i} t}$. Fix a generator $\theta \in H^1(X)$.

Suppose that $\phi \in \mathrm{Diff}(X)$ has finite order and acts trivially on $H^1(X)$. By the standard averaging trick, we may then take a ϕ-invariant 1-form $\alpha \in \Omega^1(X)$ representing θ. Take any $x \in X$, choose a path $\gamma : [0,1] \to X$ from x to $\phi(x)$ (which means as usual that $\gamma(0) = x$ and $\gamma(1) = \phi(x)$) and define

$$\rho(\phi) = e\left(\int_\gamma \alpha\right) \in S^1.$$

This is clearly independent of the choice of the path γ. It is also independent of the choice of x. Indeed, if $y \in X$ denotes another point we may take a path η from y to x and take, as a path from y to $\phi(y)$, the concatenation of the paths η, γ, and $\phi \circ \eta_{-1}$, where $\eta_{-1}(t) = \eta(1-t)$. The resulting integral of α is equal to

$$\int_\eta \alpha + \int_\gamma \alpha + \int_{\phi\circ\eta_{-1}} \alpha = \int_\eta \alpha + \int_\gamma \alpha - \int_{\phi\circ\eta} \alpha = \int_\eta \alpha + \int_\gamma \alpha - \int_\eta \alpha = \int_\gamma \alpha,$$

where the second inequality follows from the assumption that α is ϕ-invariant. Finally, we prove that $\rho(\phi)$ is independent of the choice of α. To see this, suppose that β is another ϕ-invariant 1-form representing θ. Then $\beta = \alpha + df$ for some function f. We claim that f is ϕ-invariant. Indeed, the fact that both α and β are ϕ-invariant implies that $\phi^* df = df$, so $d(\phi^* f - f) = 0$ and hence $\phi^* f = f + c$ for some constant c. Writing $c = \phi^* f - f$ and evaluating at a point where f attains its maximum (resp. minimum) we conclude that $c \leq 0$ (resp. $c \geq 0$), so $c = 0$. Now we have, by Stokes's theorem,

$$\int_\gamma \alpha - \int_\gamma \beta = \int_\gamma df = f(\phi(x)) - f(x) = 0.$$

We now prove that if G is a finite group acting smoothly on X and trivially on $H^1(X)$ then the map

$$\rho : G \to S^1$$

is a morphism of groups. Since G is finite we may take a G-invariant 1-form α representing θ. Let $x \in X$ be any point, let $g_1, g_2 \in G$, and let γ_1 (resp. γ_2) be a path from x to $g_1 x$ (resp. from x to $g_2 x$). The concatenation ζ of γ_2 and $g_2\gamma_1$ is a path from x to $g_2 g_1 x$. Hence

$$\int_\zeta \alpha = \int_{\gamma_2} \alpha + \int_{g_2\gamma_1} \alpha = \int_{\gamma_2} \alpha + \int_{\gamma_1} \alpha,$$

where the second equality follows from the fact that α is G-invariant. It now follows that

$$\rho(g_2 g_1) = \rho(g_2)\rho(g_1).$$

Lemma 4.37. *Let a finite group G act smoothly on X and trivially on $H^1(X)$, and assume that $\rho(G) = 1$. Let $\pi : Z \to X$ be the abelian universal cover of X. There exists a smooth action of G on Z lifting the action on X, in the sense that $\pi(g \cdot z) = g \cdot \pi(z)$ for every $g \in G$ and $z \in Z$.*

Proof. Fix some base point $x_0 \in X$. Choose a 1-form α representing θ. We can identify

$$Z = \{(x, \gamma) \mid x \in X,\, \gamma \text{ path from } x_0 \text{ to } x\}/\sim,$$

where the equivalence relation $\sim$ identifies (x, γ) with (x', γ') if and only if $x = x'$ and $\int_\gamma \alpha = \int_{\gamma'} \alpha$. The later equality is independent of the choice of α. Let us assume from now on that α is G-invariant.

Choose, for every $g \in G$, a path η_g from x_0 to $g \cdot x_0$ satisfying

$$\int_{\eta_g} \alpha = 0.$$

This is possible because $\rho(g) = 1$. Define an action of G on Z as follows. If $[(x, \gamma)] \in Z$ and $g \in G$ then set $g \cdot [(x, \gamma)] = [(g \cdot x, g\sharp\gamma)]$, where $g\sharp\gamma = \eta_g * (g \cdot \gamma)$ and the symbol $*$ denotes concatenation of paths. Since α is G-invariant and $\int_{\eta_g} \alpha = 0$, we have

$$\int_{g\sharp\gamma} \alpha = \int_\gamma \alpha.$$

This implies that $g_1 \cdot (g_2 \cdot [(x, \gamma)]) = g_1 g_2 \cdot [(x, \gamma)]$ for every $g_1, g_2 \in G$, which combined with some trivial checks implies that we have defined an action of G on Z lifting the action on X. □

Lemma 4.38. *Suppose that a finite group G acts smoothly and in a CTE way on X, and suppose also that $\rho(G) = 1$. For any abelian group A the differential*

$$d_2^{0,2} : E_2^{0,1} = H^1(X; A) \to E_2^{2,0} = H^2(BG; A)$$

in the second page of the Serre spectral sequence for the fibration $X_G \to BG$ with coefficients in A vanishes identically.

Proof. Let $\pi : Z \to X$ be as in the previous lemma, and take a lift of the action of G on X to an action on Z. We have a Cartesian diagram of fibrations

$$\begin{array}{ccc} Z_G & \xrightarrow{\pi} & X_G \\ \downarrow & & \downarrow \\ BG & = & BG \end{array}.$$

The vanishing of $d_2^{0,2}$ follows from the naturality of the Serre spectral sequence and the fact that $H^1(Z; A) = 0$. □

4.9.2 Fixed points and the rotation morphism

Fix a prime p for the present subsection, and suppose that T (which, recall, is the torsion of $H_1(X)$) has ap^t elements, where $a \geq 1, t \geq 0$ are integers and p does not divide a. Let T_p be the p-part of T, i.e., the subgroup of elements whose order is a power of p. We have

$$\sharp T_p = p^t.$$

The proof of the following theorem was given in Section 1.7.

Theorem 4.39. *Let a, b be natural numbers and let $c = \min\{a, b\}$. For any natural number d, any nonnegative integer k and any prime p we have*

$$H^k((\mathbb{Z}/p^a)^d; \mathbb{Z}/p^b) \simeq (\mathbb{Z}/p^c)^{\binom{k+d-1}{d-1}},$$

where we consider on the coefficient group $\mathbb{Z}/p^b$ the trivial $(\mathbb{Z}/p^a)^d$-module structure.

By Poincaré duality, (4.11) implies

$$H_0(X) \simeq H_3(X) \simeq H_4(X) \simeq \mathbb{Z}$$

and

$$H_1(X) \simeq \mathbb{Z} \oplus T, \qquad H_2(X) \simeq T.$$

Let r be an integer satisfying $r > t$. Then we have (see (1.6) in the Appendix)

$$\operatorname{Hom}(T, \mathbb{Z}/p^r) \simeq T_p, \qquad \operatorname{Ext}(T, \mathbb{Z}/p^r) \simeq T_p.$$

Using the universal coefficient theorem (see (1.4) in the Appendix) we compute

$$H^0(X; \mathbb{Z}/p^r) \simeq H^1(X; \mathbb{Z}/p^r) \simeq H^4(X; \mathbb{Z}/p^r) \simeq \mathbb{Z}/p^r, \tag{4.12}$$

$$H^2(X; \mathbb{Z}/p^r) \simeq \operatorname{Hom}(T, \mathbb{Z}/p^r) \oplus \operatorname{Ext}(\mathbb{Z} \oplus T, \mathbb{Z}/p^r) \simeq T_p \oplus T_p, \tag{4.13}$$

$$H^3(X; \mathbb{Z}/p^r) \simeq \operatorname{Hom}(Z, \mathbb{Z}/p^r) \oplus \operatorname{Ext}(T, \mathbb{Z}/p^r) \simeq \mathbb{Z}/p^r \oplus T_p. \tag{4.14}$$

Lemma 4.40. *Let r be the least integer bigger than $5t/3$. No smooth CTE action of $\Gamma := (\mathbb{Z}/p^r)^2$ on X satisfying $\rho(\Gamma) = 1$ is free.*

Proof. Suppose that Γ acts smoothly, freely, and in a CT way on X. By (4.8) we have

$$H^4(X_\Gamma; \mathbb{Z}/p^r) \simeq \mathbb{Z}/p^r. \tag{4.15}$$

The entries in the second page of the Serre spectral sequence $\{(E_s^{ij}, d_s^{ij})\}$ for the fibration $X_\Gamma \to B\Gamma$ with coefficients in $\mathbb{Z}/p^r$ take the form

$$E_2^{i,j} = H^i(\Gamma; H^j(X; \mathbb{Z}/p^r)),$$

where Γ acts trivially on $H^j(X; \mathbb{Z}/p^r)$. By Theorem 4.39 and (4.12)–(4.14) the matrix $(\log_p \sharp E_2^{ij})_{ij}$ has the following entries (note that $r > t$):

0	0	0	0	0	0	$\dots$
r	$2r$	$3r$	$4r$	$5r$	$6r$	$\dots$
$\mathbf{r+t}$	$2(r+t)$	$3(r+t)$	$4(r+t)$	$5(r+t)$	$6(r+t)$	$\dots$
$2t$	$\mathbf{4t}$	$6t$	$8t$	$10t$	$12t$	$\dots$
r	$2r$	$\mathbf{3r}$	$4r$	$5r$	$6r$	$\dots$
r	$2r$	$3r$	$4r$	$\mathbf{5r}$	$6r$	$\dots$

The isomorphism (4.15) implies that

$$\sharp E_\infty^{4,0} \leq p^r. \tag{4.16}$$

Now assume that $\rho(\Gamma) = 1$. By Lemma 4.38 we have $d_2^{0,1} = 0$, which implies by the multiplicativity of the Serre spectral sequence that $d_2^{2,1} = 0$. Hence, the only way the cardinal of $E_*^{4,0}$ can drop from p^{5r} to an integer not bigger than p^r is by quotienting through the images of the differentials

$$d_2^{1,2} : E_3^{1,2} \to E_3^{4,0} = E_2^{4,0}, \qquad d_4^{0,3} : E_4^{0,3} \to E_4^{4,0}.$$

More precisely, we can estimate

$$\sharp E_5^{4,0} \geq \sharp E_2^{4,0} (\sharp E_3^{1,2})^{-1} (\sharp E_4^{0,3})^{-1} \geq \sharp E_2^{4,0} (\sharp E_2^{1,2})^{-1} (\sharp E_2^{0,3})^{-1} = p^{5r-4t-(r+t)} = p^{4r-5t} > p^r,$$

where the last inequality follows from $r > 5t/3$. This contradicts (4.16), so the proof of the lemma is complete. □

Lemma 4.41. *There exists a constant C, independent of p, such that any finite abelian p-group A acting freely and in a smooth and CT way on X and satisfying $\rho(A) = 1$ has a cyclic subgroup $A_c \leq A$ satisfying $[A : A_c] \leq C$.*

Proof. For every prime q let t_q be defined by $\sharp T_q = q^{t_q}$ and let r_q be the least integer bigger than $5t_q/3$. Let R be the number resulting from applying Theorem 4.15 to X. Define

$$C := \max_q q^{(R-1)(r_q-1)}.$$

This is a finite number, because $\sharp T_q$ is different from 1 only for finitely many primes q.

Let A be an abelian p-group acting freely, smoothly, and in a CT way on X, and satisfying $\rho(A) = 1$. Choose an isomorphism

$$A \simeq \mathbb{Z}/p^{e_1} \oplus \cdots \oplus \mathbb{Z}/p^{e_s},$$

where $e_1 \geq \cdots \geq e_s \geq 1$. By Theorem 4.15 we have $s \leq R$ and by Lemma 4.40 we have $e_i < r_p$ for every $i \geq 2$. Define A_c to be the subgroup of A corresponding to the first summand $\mathbb{Z}/p^{e_1}$. Then we have $[A : A_c] \leq p^{(R-1)(r_p-1)} \leq C$. □

Lemma 4.42. *No smooth CTE action of $\Gamma = (\mathbb{Z}/p^{t+1})^3$ on X is free.*

Proof. Suppose that Γ acts smoothly and in a CTE way on X. Let $r = t+1$. Consider the Serre spectral sequence $\{(E_s^{ij}, d_s^{ij})\}$ for the fibration $X_\Gamma \to B\Gamma$ with coefficients in $\mathbb{Z}/p^r$. The matrix $(\log_p \sharp E_2^{ij})_{ij}$ has entries

0	0	0	0	0	0	...
r	$3r$	$6r$	$10r$	$15r$	$21r$	...
$\mathbf{r+t}$	$3(r+t)$	$6(r+t)$	$10(r+t)$	$15(r+t)$	$21(r+t)$	...
$2t$	$\mathbf{6t}$	$12t$	$20t$	$30t$	$42t$	...
r	$3r$	$\mathbf{6r}$	$10r$	$15r$	$21r$	...
r	$3r$	$6r$	$10r$	$\mathbf{15r}$	$21r$	...

Arguing as in the previous lemma we estimate

$$\begin{aligned}\sharp E_\infty^{4,0} &= \sharp E_4^{4,0} \geq \sharp E_2^{4,0}(\sharp E_2^{2,1})^{-1}(\sharp E_3^{1,2})^{-1}(\sharp E_4^{0,3})^{-1}\\ &\geq \sharp E_2^{4,0}(\sharp E_2^{2,1})^{-1}(\sharp E_2^{1,2})^{-1}(\sharp E_2^{0,3})^{-1}\\ &= p^{15r-6r-6t-(r+t)} = p^{8r-7t} > p^r,\end{aligned}$$

so we have $\sharp H^4_\Gamma(X;\mathbb{Z}/p^r) > p^r$, which is not compatible with the action of Γ being free. $\square$

Lemma 4.43. *Let Γ be a finite p-group sitting in a short exact sequence*

$$1 \to K \to \Gamma \xrightarrow{\pi} Q \to 1$$

with Q cyclic. Suppose that $A \leq K$ is a cyclic subgroup, and that A is normal in Γ. Assume that Γ acts smoothly and in a CT way on X and that $\rho(A) = 1$. If the action of A on X is free, then there is an abelian subgroup $A' \leq \Gamma$ containing A and satisfying

$$[\Gamma : A'] \leq 2p^t[K : A].$$

Proof. Take an element $\gamma \in \Gamma$ such that $\pi(\gamma)$ generates Q. Since A is normal in Γ, conjugation defines a morphism $\Gamma \to \mathrm{Aut}(A)$. Applying this to γ we obtain, as in the proof of Lemma 4.35, a commutative diagram

$$\begin{array}{ccc} A \times X & \longrightarrow & X \\ \downarrow & & \downarrow \\ A \times X & \longrightarrow & X, \end{array}$$

in which the horizontal arrows are the maps defining the action of A on X, the left hand vertical arrow sends (g, x) to $(\gamma g \gamma^{-1}, \gamma x)$, and the right hand side vertical arrow sends x to γx. At the level of Borel constructions we obtain a commutative diagram

$$\begin{array}{ccc} X_A & \longrightarrow & BA \\ \downarrow & & \downarrow \\ X_A & \longrightarrow & BA, \end{array}$$

in which the right hand side vertical arrow is induced by the map

$$c(\gamma) : A \to A, \qquad c(\gamma)(g) = \gamma g \gamma^{-1}.$$

Consider the Serre spectral sequence for $X_A \to BA$ with integer coefficients. The previous diagram implies the existence of an automorphism of the Serre spectral sequence which is given, at the level of the second page, by the morphism

$$\phi : H^\sigma(BA; H^\tau(X)) \to H^\sigma(BA; H^\tau(X)).$$

Crucially, ϕ commutes with all the differentials of the spectral sequence.

Since A acts trivially on $H^*(X)$, in order to understand ϕ it will suffice for our purposes to compute $c(\gamma)^* : H^\sigma(BA) \to H^\sigma(BA)$. Suppose that

$$A \simeq \mathbb{Z}/p^a.$$

Then, thinking of the group structure on A in additive terms, the action of $c(\gamma)$ on A is given by multiplication by some

$$\zeta \in (\mathbb{Z}/p^a)^*.$$

We claim that $H^\sigma(BA) = 0$ if σ is odd and $H^\sigma(BA) \simeq \mathbb{Z}/p^a$ if $\sigma > 0$ is even. Furthermore if λ is a generator of $H^2(BA)$ then λ^k is a generator of $H^{2k}(BA)$. To prove these claims, identify A with the group μ_{p^a} of p^a-th roots of units in S^1. Taking as a model for the classifying space of A the quotient ES^1/μ_{p^a}, we identify BA with the total space of a circle bundle over BS^1 whose first Chern class is p^a times a generator of $H^2(BS^1) \simeq \mathbb{Z}$. Then the claim follows from applying the Gysin exact sequence to this bundle. As a consequence, it suffices to understand $c(\gamma)^*$ acting on $H^2(BA)$. By the universal coefficient theorem we have

$$H^2(BA) = \mathrm{Ext}^1(H_1(BA), \mathbb{Z}).$$

We have a natural identification $H_1(BA) \simeq A$, so $c(\gamma)^*$ acts on $H_1(BA)$ as multiplication by ζ. Fix a surjection $\mathbb{Z} \to H_1(BA)$ and consider the resulting commutative diagram with exact rows, and whose vertical arrows are multiplication by some integer $z \in \mathbb{Z}$ representing $\zeta \in (\mathbb{Z}/p^a)^*$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & H_1(BA) & \longrightarrow & 0\,. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & H_1(BA) & \longrightarrow & 0 \end{array}$$

Applying $\mathrm{Hom}_{\mathbb{Z}}(\cdot, \mathbb{Z})$ and its derived functors we get a commutative diagram with exact rows from which it is easy to conclude that $c(\gamma)^*$ acts on $\mathrm{Ext}^1(H_1(BA), \mathbb{Z}) = H^2(BA)$ as multiplication by ζ. Consequently, $c(\gamma)^*$ acts on $H^4(BA)$ as multiplication by ζ^2. In other words, we may identify $H^4(BA)$ with A in such a way that the action of $c(\gamma)$ on $H^4(BA)$ corresponds in A with conjugation by γ^2.

Suppose from now on that A acts freely on X, which implies $H^4_A(X) \simeq \mathbb{Z}$. From (4.11) and the previous description of $H^*(BA)$ we deduce that the left bottom corner of the second page of the spectral sequence with integer coefficients $\{(E_u^{\sigma,\tau}, d_u)\}$ for the fibration $X_A \to BA$ is isomorphic to:

0	0	0	0	0	0
$H^0(BA;\mathbb{Z})$	0	$H^2(BA;\mathbb{Z})$	0	$H^4(BA;\mathbb{Z})$	0
$H^0(BA;\mathbb{Z}\oplus T_p)$	$H^1(BA;T_p)$	$H^2(BA;\mathbb{Z}\oplus T_p)$	$H^3(BA;T_p)$	$H^4(BA;\mathbb{Z}\oplus T_p)$	0
$H^0(BA;T_p)$	$H^1(BA;T_p)$	$H^2(BA;T_p)$	$H^3(BA;T_p)$	$H^4(BA;T_p)$	0
$H^0(BA;\mathbb{Z})$	0	$H^2(BA;\mathbb{Z})$	0	$H^4(BA;\mathbb{Z})$	0
$H^0(BA;\mathbb{Z})$	0	$H^2(BA;\mathbb{Z})$	0	$H^4(BA;\mathbb{Z})$	0

Furthermore, the action of ϕ is induced in each term by the action of $c(\gamma)$ on BA and the trivial action on coefficients.

The convergence of the spectral sequence to the equivariant cohomology implies that $E_\infty^{4,0}$ can be naturally identified with a subgroup of $H_A^4(X)$. Since $E_\infty^{4,0}$ is a quotient of $E_2^{4,0} \simeq H^4(BA) \simeq \mathbb{Z}/p^a$ and $H_A^4(X)$ is torsion free, we necessarily have $E_\infty^{4,0} = 0$. There are only three differentials that can contribute to kill $E_2^{4,0}$:

$$d_2^{2,1} : E_2^{2,1} \to E_2^{4,0}, \qquad d_3^{1,2} : E_3^{1,2} \to E_3^{4,0} \qquad \text{and} \qquad d_4^{0,3} : E_4^{0,3} \to E_4^{4,0}.$$

We have $d_2^{2,1} = 0$ by Lemma 4.38 and the multiplicativity of the spectral sequence. So we can naturally identify $E_3^{4,0} \simeq E_2^{4,0} \simeq H^4(BA)$. We also have $E_3^{1,2} \simeq E_2^{1,2} \simeq H^1(BA; T_p)$. Hence we can identify the source and target of $d_3^{1,2}$ with

$$d_3^{1,2} : H^1(BA; T_p) \to H^4(BA).$$

Denote by

$$M \subseteq H^4(BA)$$

the image of $d_3^{1,2}$. We have

$$\sharp M \leq \sharp H^1(BA; T_p) \leq p^t,$$

where the second inequality follows from the universal coefficient theorem

$$H^1(BA; T_p) \simeq \mathrm{Hom}(H_1(BA), T_p) \oplus \mathrm{Ext}(H_0(BA), T_p) = \mathrm{Hom}(H_1(BA), T_p),$$

the equality $\sharp T_p = p^t$, and the fact that A is cyclic.

Next we can identify the source and target of $d_4^{0,3}$ with

$$d_4^{0,3} : \mathrm{Ker}\, d_3^{0,3} \to H^4(BA)/M.$$

This map has to be surjective in order for $E_4^{4,0}$ to vanish. At this point we are going to use the fact that $d_4^{0,3}$ commutes with the action induced by ϕ. We can identify $\mathrm{Ker}\, d_3^{0,3}$ with a subgroup of $H^0(BA; \mathbb{Z} \oplus T_p)$, on which the action of ϕ is trivial. So in order for $d_4^{0,3}$ to be surjective the action on $H^4(BA)/M$ induced by ϕ has to be trivial.

Denote for convenience $N = H^4(BA)$. Since the action induced by $\phi \in \mathrm{Aut}(N)$ on N/M is trivial, one can define a morphism (using additive notation)

$$\delta : N \to M, \qquad \delta(n) = n - \phi(n).$$

We have $\mathrm{Ker}\, \delta = N^\phi = \{n \in N \mid \phi(n) = n\}$ so

$$[N : N^\phi] = [N : \mathrm{Ker}\, \delta] \leq \sharp M \leq p^t.$$

We have seen above that we can identify $N \simeq A$ in such a way that ϕ corresponds to the map $A \to A$ given by $a \mapsto \gamma^2 a \gamma^{-2}$. It thus follows that

$$A_\gamma = \{a \in A \mid a = \gamma^2 a \gamma^{-2}\}$$

satisfies $[A : A_\gamma] \leq p^t$. Let $A' \leq \Gamma$ be the subgroup generated by A_γ and γ^2. It follows from the definition of A_γ that A' is abelian. Since $\pi(\gamma)$ is a generator of Q we may bound

$$[\Gamma : A'] \leq 2[K : A_\gamma] \leq 2[K : A][A : A_\gamma] \leq 2p^t[K : A],$$

so the proof of the lemma is complete. □

4.9.3 CTE actions of finite p-groups

Lemma 4.44. *There exists a constant C, independent of p, such that for any finite p-group G and any smooth CTE action of G on X the following holds. Let $A \leq G$ be a MNAS. If the action of A on X is not free, then $[G : A] \leq C$.*

Proof. Let C_0 be the constant given by Lemma 4.34. Suppose that G and A satisfy the hypothesis of the statement. By Lemma 4.34 there exists a subgroup $G_0 \leq G$ satisfying $[G : G_0] \leq C_0$ and such that G_0 is abelian or there exists an embedded connected oriented surface $Z \subset X$ preserved by G_0 and of genus $\leq C_0$. If G_0 is abelian then by Lemma 4.14 we have

$$[G : A] \leq C_0^{r^2+1},$$

where r is the constant given by Mann–Su's Theorem 4.15 applied to X.

Now assume that G_0 is not abelian, so that we have the surface Z at our disposal. Since Z is orientable, so is its normal bundle $N \to Z$. We can identify the degree of N with the self-intersection $Z \cdot Z$, which is equal to 0 because $b_2(X) = 0$. Hence Z is an oriented embedded surface with trivial normal bundle $N \to Z$. By Lemma 4.25, N admits a G_0-invariant complex structure. By (2) in Lemma 4.28 there is an abelian subgroup $B \leq G_0$ satisfying $[G_0 : B] \leq C_1$, where the constant C_1 depends only on the genus of Z and hence can be bounded above by a constant depending only on X. Applying again Lemma 4.14 we conclude that

$$[G : A] \leq (C_0 C_1)^{r^2+1},$$

where r is as above, so the proof of the lemma is now complete. □

Lemma 4.45. *There exists a constant C such that: for any prime p, any finite p-group G and any smooth CTE action of G on X there is an abelian subgroup $A \leq G$ such that $[G : A] \leq C$.*

Proof. Recall that for every prime p we denote by T_p the p-torsion of $H_1(X)$. Since $H_1(X)$ is finitely generated, we have $\sharp T_p = 1$ except for finitely many p's, so

$$C_T := \max_p \sharp T_p$$

is finite.

Let C_R be the number resulting from applying Lemma 4.41 to X.

Let p be a prime. Suppose given a smooth CTE action of a finite p-group G on X and let $\rho : G \to S^1$ be the rotation morphism. Let $G_0 = \operatorname{Ker} \rho$. Let $K \leq G_0$ be a MNAS.

We distinguish two cases, depending on whether the action of K on X is free or not.

Suppose first of all that the action of K on X is not free. By Lemma 4.44 we have $[G_0 : K] \leq C$. Let $G' \leq G$ be the normalizer of K. Since G_0 is a normal subgroup of G, by Theorem 4.15 and Lemma 4.11 we have $[G : G'] \leq C'$. Let $A \leq G'$ be a MNAS containing K. Then the action of A on X is not free, so by Lemma 4.44 we have $[G' : A] \leq C''$. It follows that $[G : A]$ is bounded above by a constant which depends neither on p nor on G.

Assume, for the rest of the proof, that the action of K on X is free. By Lemma 4.41, there is a cyclic subgroup $A \leq K$ satisfying $[K : A] \leq C_R$.

Let $Q = G_0/K$. Since K is a MNAS of G_0, the action of G_0 on K given by conjugation induces an effective action of Q on K, which allows us to identify Q with a subgroup of $\mathrm{Aut}(K)$. Let

$$f : \mathrm{Aut}_A(K) \to \mathrm{Aut}(A)$$

be the restriction map (we use here and below the notation introduced in Lemma 4.10). Since A is cyclic, $\mathrm{Aut}(A)$ is also cyclic. Hence $S := f(Q \cap \mathrm{Aut}_A(K))$ is cyclic. Let $q \in Q \cap \mathrm{Aut}_A(K)$ be an element such that $f(q)$ generates S. Let $Q' = \langle q \rangle \leq Q$. We have

$$[Q : Q'] \leq [Q : Q \cap \mathrm{Aut}_A(K)] \cdot \sharp \operatorname{Ker} f \leq [\mathrm{Aut}(K) : \mathrm{Aut}_A(K)] \cdot \sharp \operatorname{Ker} f.$$

Applying Lemmas 4.10 and 4.13 (and noting that $\operatorname{Ker} f = \mathrm{Aut}^0_A(K)$) we conclude that there is a constant C_L, depending only on X, such that

$$[Q : Q'] \leq C_L. \tag{4.17}$$

Let G_0' be the preimage of Q' via the projection $G_0 \to Q$. We have a short exact sequence

$$1 \to K \to G_0' \to Q' \to 1,$$

and A is normal in G_0' because the elements of Q' belong to $\mathrm{Aut}_A(K)$. We may thus apply Lemma 4.43 and conclude the existence of an abelian subgroup $A' \leq G_0'$ containing A and satisfying

$$[G_0' : A'] \leq 2C_T[K : A] \leq 2C_T C_R.$$

Hence

$$[G_0 : A'] \leq [G_0 : G_0'][G_0' : A'] \leq 2C_L C_T C_R.$$

By Lemma 4.41 there is a cyclic subgroup

$$A_c' \leq A'$$

satisfying $[A' : A_c'] \leq C_R$. Since

$$[G_0 : A_c'] \leq 2C_L C_T C_R^2$$

gives an upper bound that depends only on X, applying Lemma 4.11 and Theorem 4.15 we conclude that the normalizer

$$G' = N_G(A_c')$$

satisfies $[G : G'] \leq C_N$ for a constant C_N depending only on X. We have a short exact sequence

$$1 \to G' \cap G_0 \to G' \to G'/(G' \cap G_0) \to 1.$$

The group $G'/(G' \cap G_0)$ can be identified with a subgroup of $G/G_0 \simeq \rho(G) < S^1$, so $G'/(G' \cap G_0)$ is cyclic, and clearly $A_c' \leq G' \cap G_0$. So we can apply Lemma 4.43 to the inclusion $A'_c \leq G' \cap G_0$ and conclude the existence of an abelian subgroup $A'' \leq G'$ satisfying

$$[G' : A''] \leq 2C_T[G' \cap G_0 : A'_c] \leq 2C_T[G_0 : A'_c] \leq 4C_L C_T^2 C_R^2$$

and hence

$$[G : A''] \leq 4C_L C_T^2 C_R^2 C_N.$$

This finishes the proof of the lemma. □

4.9.4 Proof of Theorem 4.36

Let $\mathcal{P}$ be the collection of all finite p-subgroups (for all primes p) of $\mathrm{Diff}(X)$ which act in a CT way on X. Let $\mathcal{T}$ be the collection of all finite subgroups $G < \mathrm{Diff}(X)$ which act in a CT way on X and such that there exist different primes p, q, a normal Sylow p-subgroup $P \leq G$ and a Sylow q-subgroup $Q \leq G$ such that $G = PQ$ and both P and Q are nontrivial. By the main theorem in [54] it suffices to prove the existence of a constant C such that any $G \in \mathcal{P} \cup \mathcal{T}$ has an abelian subgroup $A \leq G$ satisfying $[G : A] \leq C$.

The existence of C for elements of $\mathcal{P}$ is a consequence of Lemma 4.45.

Let R be the number given by Theorem 4.15 applied to X, let

$$C_A = \max\{4, \max\{\sharp\, \mathrm{GL}(R, \mathbb{Z}/p) \mid \sharp T_p \neq 1\}\}.$$

Suppose that $G = PQ \in \mathcal{T}$, with P a normal p-subgroup of G and Q a q-subgroup of G, $p \neq q$. By Lemma 4.45 there are abelian subgroups $P_0 \leq P$ and $Q_0 \leq Q$ satisfying $[P : P_0] \leq C$ and $[Q : Q_0] \leq C$. Let Q'_0 be the normalizer of P_0 in Q_0. Since $Q'_0 = Q_0 \cap N_G(P_0)$, by Theorem 4.15 and Lemma 4.11, there exists a constant C' such that

$$[Q_0 : Q'_0] \leq [G : N_G(P_0)] \leq C'.$$

Then $G_0 = P_0 Q'_0$ satisfies $[G : G_0] \leq CC'$.

Conjugation gives a morphism $c : Q'_0 \to \mathrm{Aut}(P_0)$. Let $P_0[p]$ be the p-torsion of P_0. This is a characteristic subgroup of P_0, so restriction gives a natural morphism $r : \mathrm{Aut}(P_0) \to \mathrm{Aut}(P_0[p])$. Since Q'_0 is a q-group and $q \neq p$, $\mathrm{Ker}\, c = \mathrm{Ker}\, r \circ c$. (This is a standard fact in finite group theory, but we sketch an argument for the reader's convenience: if $\phi \in \mathrm{Aut}(P_0)$ belongs to $\mathrm{Ker}\, r$ then we may write $\phi = \mathrm{Id} + \psi$ using additive notation on P_0, where $\psi \in \mathrm{Hom}(P_0, P_0)$ satisfies $\psi(x) \in pP_0$ for every x, and hence $\psi(p^k P_0) \leq p^{k+1} P_0$ for every k; using the binomial's formula and induction on r we prove that $\phi^{p^r} - \mathrm{Id}$ sends P_0 to $p^r P_0$, so if r is big enough then $\phi^{p^r} = \mathrm{Id}$; this proves that ϕ is a p-element in $\mathrm{Aut}(P_0)$ and justifies the equality $\mathrm{Ker}\, c = \mathrm{Ker}\, r \circ c$.)

To finish the proof we distinguish two cases.

Suppose first of all that $[Q'_0 : \mathrm{Ker}\, r \circ c] > C_A$. We claim that in this case $\sharp T_p = 1$. Indeed, otherwise we would have $\sharp\, \mathrm{Aut}\, P_0[p] \leq \sharp\, \mathrm{GL}(R, \mathbb{Z}/p) \leq C_A$, which combined with $[Q'_0 : \mathrm{Ker}\, r \circ c] \leq \sharp\, \mathrm{Aut}\, P_0[p]$ would lead to a contradiction. Next we claim that the action of P_0 on X is not free. If the rank of $P_0[p]$ is 1 or 2 this follows from Lemma 4.35, and if it is ≥ 3 then it follows from Lemma 4.42. Once we know that the action of P_0 is not free, applying Lemma 4.44 we conclude that G_0 has an abelian subgroup of bounded index.

Next suppose that $[Q_0' : \operatorname{Ker} r \circ c] \leq C_A$. Then the group $Q_1 = \operatorname{Ker} c = \operatorname{Ker} r \circ c$ commutes with P_0, so $A = P_0 Q_1 \leq G$ is an abelian subgroup satisfying $[G : A] \leq C_A C C'$. This concludes the proof of the theorem.

4.10 Proofs of Theorems 4.1 and 4.2

Assume for the entire present section that X is a closed, connected and oriented 4-manifold. This is more restrictive than the situation considered in Theorem 4.1, but Lemma 4.8 allows us to reduce the general case to this setting. If $b_2(X) = 0$ then both Theorems 4.1 and 4.2 follow from Theorem 4.36. Hence, we also assume in this section that $b_2(X) \neq 0$.

By Lemma 4.9, both in Theorems 4.1 and 4.2 it suffices to consider CTE actions. Indeed, for Theorem 4.2 note that if G is any finite group and $G' \leq G$ is a subgroup we have

$$\alpha(G') \geq \frac{\alpha(G)}{[G : G']},$$

because $[G : A] = [G : G'][G' : A]$ for any subgroup $A \leq G'$ (in particular, for any abelian subgroup).

Let $D = \max_p \sum_{j \geq 0} b_j(X; \mathbb{Z}/p)$.

4.10.1 Commutator subgroups

Let us denote by $\mathcal{G}_0$ the collection of all finite groups Γ such that there exists a finite group G acting smoothly and in a CTE way on X and a monomorphism $\Gamma \hookrightarrow [G, G]$.

Lemma 4.46. *There exists a constant C with the following property. Suppose that $\Gamma \in \mathcal{G}_0$ is a cyclic group of prime power order and that there exists no $g \in \Gamma$ such that X^g contains a connected component which is a nonorientable surface. Then there exists a subgroup $\Gamma_0 \leq \Gamma$ satisfying $[\Gamma : \Gamma_0] \leq C$ and $X^{\Gamma_0} \neq \emptyset$.*

Proof. We are going to prove that $C = 2^D$ does the job.

Let G be a finite group acting smoothly in a CTE way on X and let $\Gamma \leq [G, G]$ be a cyclic subgroup of prime power order. Since $b_2(X) \neq 0$, Poincaré duality implies the existence of classes $\alpha, \beta \in H^2(X)$ such that $\alpha\beta$ is a generator of $H^4(X)$. Let L_α, L_β be complex line bundles on X with first Chern classes α, β respectively. By [51, Theorem 6.5] there exists a short exact sequence

$$1 \to S \to \widehat{\Gamma} \xrightarrow{\pi} \Gamma \to 1,$$

where S is a finite cyclic group, and an action of $\widehat{\Gamma}$ on L_α lifting the action of Γ on X. Denote by

$$\mu : \widehat{\Gamma} \times L_\alpha \to L_\alpha, \qquad (h, \lambda) \mapsto h \cdot \lambda$$

the map given by this action.

Let $g \in \Gamma$ be a generator, let $\gamma \in \widehat{\Gamma}$ be a lift of g, and let $\Gamma' \leq \widehat{\Gamma}$ be the subgroup generated by γ. Let $S' = S \cap \Gamma'$, so that we have an exact sequence

$$1 \to S' \to \Gamma' \xrightarrow{\pi} \Gamma \to 1.$$

Since the action of the elements in S' on L_α lift the trivial action on X, it is given by a morphism of groups $\xi : S' \to S^1$. Since Γ' is cyclic, we may choose an extension of ξ to Γ', which we denote by the same symbol $\xi : \Gamma' \to S^1$. Now the map

$$\nu : \Gamma' \times L_\alpha \to L_\alpha, \qquad \nu(h, \lambda) = \xi(h)^{-1}\mu(h, \lambda)$$

defines an action of Γ' on L_α lifting the action of Γ on X, whose restriction to S' is trivial. Consequently, this action descends to an action of Γ on L_α lifting the action of Γ on X. Replacing L_α by L_β we similarly obtain a lift of the action of Γ to L_β.

Let $E = L_\alpha \oplus L_\beta$. This is a rank 2 complex vector bundle with $c_2(E) = \alpha\beta$, and the lifts of the action of Γ to L_α and L_β combine to give an action on E.

The argument that follows can be seen as a toy model of the proof of [51, Theorem 1.11]. The setting is more restricted in that it applies only to dimension 4, but more general in that no almost complex structure on X is assumed to exist.

We first prove that the action of Γ on X is not free. Arguing by contradiction, let us assume that it is free. Then X/Γ is a closed, oriented and connected 4-manifold. Let $q : X \to X/\Gamma$ be the quotient map and let $p^r = \sharp\Gamma$, where p is prime and $r \geq 1$. Then the image of the map $q^* : H^4(X/\Gamma) \to H^4(X)$ is equal to the set of integral multiples of $p^r\,\alpha\beta$. Using the action of Γ on E we obtain a rank 2 complex vector bundle $E_0 \to X/\Gamma$ together with an isomorphism $E \simeq q^*E_0$. By the naturality of Chern classes this implies that $\alpha\beta = c_2(E) = q^*c_2(E_0)$, which contradicts the previous claim on the image of $q^* : H^4(X/\Gamma) \to H^4(X)$. Hence $X^g \neq \emptyset$ for every $g \in \Gamma$ of order p.

Suppose that there exists some $g \in \Gamma$ such that X^g contains an isolated point. Let $S \subseteq X^g$ be the set of isolated fixed points. By Lemma 4.29 we have $\sharp S \leq D$. Since Γ is abelian, its action on X preserves S. Choose some $s \in S$ and let $\Gamma_0 \leq \Gamma$ be the stabilizer of S. Then $[\Gamma : \Gamma_0] \leq D$ and $s \in X^{\Gamma_0}$, so $X^{\Gamma_0} \neq \emptyset$. Hence we are done in this case.

Assume for the remainder of the proof that there exists no $g \in \Gamma$ such that X^g has an isolated fixed point.

Let $g \in \Gamma$ be an element of order p and let $Y = X^g$. Then Y is a nonempty embedded, possibly disconnected surface. Let $\Theta \leq \Gamma$ be the subgroup generated by g. Recall that $H^r(B\Theta; \mathbb{Z}/p) \simeq \mathbb{Z}/p$ for every $r \geq 0$.

In the arguments that follow we will somehow abusively denote by $c_k^\Theta(V)$ the image of the k-th Chern class of an equivariant vector bundle V under the map $H^{2k}_\Theta(\cdot) \to H^{2k}_\Theta(\cdot; \mathbb{Z}/p)$ induced by the projection $\mathbb{Z} \to \mathbb{Z}/p$. Let $\pi : X \to \{*\}$ denote the projection to a point. Since $c_2(E)$ is a generator of $H^4(X)$, we have

$$\pi_* c_2^\Theta(E) \neq 0,$$

where $\pi_* : H^*_\Theta(X; \mathbb{Z}/p) \to H^{*-4}_\Theta(\{*\}; \mathbb{Z}/p) = H^{*-4}(B\Theta; \mathbb{Z}/p)$ is the pushforward map (see e.g. [51, §2.1], but take into account that here we use coefficients in $\mathbb{Z}/p$ while the discussion in [op.cit.] uses integer coefficients). Our aim is to apply localization to relate the nonvanishing of $\pi_* c_2^\Theta(E)$ to the existence of points with big stabilizer, and for that we need to have an invariant orientation of Y.

By assumption all connected components of Y are orientable. Let ν be the number of connected components of Y. By Lemma 4.29 we have $\nu \le D$. Let $o(Y)$ be the set of orientations of Y. We have $\sharp o(Y) = 2^\nu$, and there is a natural action of Γ on $o(Y)$. Choose some element $o \in o(Y)$ and let $\Gamma_0 \le \Gamma$ be the stabilizer of o. We have $[\Gamma : \Gamma_0] \le 2^\nu \le 2^D$. If $\Gamma_0 = \{1\}$ then we are done, since clearly $X^{\Gamma_0} \neq \emptyset$.

Suppose from now on that $\Gamma_0 \neq \{1\}$. Then $\Theta \le \Gamma_0$. Let $N \to Y$ be the normal bundle of the inclusion $Y \hookrightarrow X$, and orient it in a way compatible with the orientation of X and with $o \in o(Y)$. The bundle N carries a natural action of Γ_0 which preserves the orientation. Hence we may consider the equivariant Euler class $e^{\Gamma_0}(N)$, which we assume, abusively as before, to lie in $H^2_{\Gamma_0}(Y;\mathbb{Z}/p)$. By Lemma 4.25 we may endow N with a Γ_0-invariant complex structure compatible with the orientation. Then we have $e^{\Gamma_0}(N) = c_1^{\Gamma_0}(N)$, and the same formula holds replacing Γ_0 by any of its subgroups.

Let $\rho : Y \to \{*\}$ be the projection to a point. Fix some monomorphism $\zeta : \Theta \to S^1$ and let $t = c_1(E\Theta \times_\zeta S^1) \in H^2(B\Theta;\mathbb{Z}/p)$. By the localization formula we have

$$\pi_* c_2^\Theta(E) = \rho_* \left(\frac{c_2^\Theta(E|_Y)}{c_1^\Theta(N)} \right). \tag{4.18}$$

This follows from the properties of the pushforward map listed in [51, §2.1], together with the fact that $H^4_\Theta(X \setminus Y;\mathbb{Z}/p) = 0$, so that $c_2^\Theta(E)$ can be lifted to $H^4_\Theta(X, X \setminus Y;\mathbb{Z}/p)$ and hence belongs to the image of $i_* : H^*_\Theta(Y;\mathbb{Z}/p) \to H^{*+2}_\Theta(X;\mathbb{Z}/p)$ (here $i : Y \hookrightarrow X$ is the inclusion). The term inside $\rho_*(\cdot)$ in the RHS of (4.18) should be understood as an element of the localized ring $H^*_\Theta(Y;\mathbb{Z}/p)[t^{-1}]$. The invertibility of $c_1^\Theta(N)$ inside this ring is a standard fact, but the computations below give a proof of it.

The RHS of (4.18) can be written as a sum of contributions from each connected component of Y. We next compute in concrete (nonequivariant) terms these contributions.

Fix some connected component $Z \subseteq Y$. Suppose the action of Θ on $L_\alpha|_Z$ (resp. $L_\beta|_Z$, $N|_Z$) is given by a character $\zeta^{a_Z} : \Theta \to S^1$ (resp. $\zeta^{b_Z} : \Theta \to S^1$, $\zeta^{n_Z} : \Theta \to S^1$). The integers a_Z, b_Z, n_Z are of course well defined only up to multiples of p. With respect to the Künneth isomorphism

$$H^*_\Theta(Z;\mathbb{Z}/p) \simeq H^*(Z;\mathbb{Z}/p) \otimes H^*(B\Theta;\mathbb{Z}/p)$$

we have $c_1^\Theta(L_\alpha|_Z) = c_1(L_\alpha|_Z) + a_Z t = \alpha|_Z + a_Z t$, $c_1^\Theta(L_\beta|_Z) = c_1(L_\beta|_Z) + b_Z t = \beta|_Z + b_Z t$, and $c_1^\Theta(N|_Z) = c_1(N|_Z) + n_Z t$.

The fact that Θ acts effectively on N (which follows from (1) in Lemma 4.20) implies that n_Z is not divisible by p, so we may choose an integer m_Z such that $m_Z n_Z \equiv 1 \mod p$. Then we compute in $H^*_\Theta(Z;\mathbb{Z}/p)[t^{-1}]$:

$$(c_1(N|_Z) + n_Z t)^{-1} = m_Z t^{-1}(1 - t^{-1} m_Z c_1(N|_Z)).$$

Hence we have:

$$\begin{aligned}\rho_* \frac{c_2^{\Theta}(E|_Z)}{c_1^{\Theta}(N)} &= \rho_*((\alpha|_Z + a_Z t)(\beta|_Z + b_Z t) m_Z t^{-1}(1 - t^{-1} m_Z c_1(N|_Z))) \\ &= \rho_*(m_Z b_Z \alpha|_Z + m_Z a_Z \beta|_Z - m_Z^2 a_Z b_Z c_1(N|_Z)) \\ &= m_Z b_Z \langle \alpha, [Z]\rangle + m_Z a_Z \langle \beta, [Z]\rangle - m_Z^2 a_Z b_Z \langle c_1(N|_Z), [Z]\rangle,\end{aligned}$$

where $[Z] \in H_2(Z)$ denotes the fundamental class. Let use denote for convenience

$$f(Z) := m_Z b_Z \langle \alpha, [Z]\rangle + m_Z a_Z \langle \beta, [Z]\rangle - m_Z^2 a_Z b_Z \langle c_1(N|_Z), [Z]\rangle.$$

We can now translate the fact that $\pi_* c_2^{\Theta}(E)$ is nonzero into the following statement:

$$\sum_Z f(Z) \qquad \text{is not divisible by } p,$$

where Z runs over the set of connected components of Y.

Let us decompose $Y = Y_1 \sqcup \cdots \sqcup Y_s$, where for each j there is a connected component Z of Y such that $Y_j = \bigcup_{g \in \Gamma_0} gZ$. We claim that at least for one j we have $Y_j \subseteq X^{\Gamma_0}$. With this claim the proof of the lemma will be complete. The claim is an immediate consequence of the following two observations.

If Y_j contains more than one connected component then $\sum_{Z \subset Y_j} f(Z)$ is divisible by p. Indeed, on the one hand for every connected component Z of Y and any $g \in \Gamma_0$ we have $f(Z) = f(gZ)$, because Γ_0 is abelian and $\Theta \leq \Gamma_0$, and on the other hand the cardinality of $\pi_0(Y_j)$ divides $\sharp\Gamma_0$, which is a power of p.

If Y_j is connected but $Y_j \not\subset X^{\Gamma_0}$ then $f(Y_j)$ is divisible by p. To see this, let us denote $Z = Y_j$ and let $\Gamma_1 \leq \Gamma_0$ be the subgroup of elements acting trivially on Z. Then $\Xi := \Gamma_0/\Gamma_1$ acts on Z preserving the orientation and without isolated fixed points (this follows easily from the assumption that there is no $g \in \Gamma$ such that X^g has an isolated fixed point). Hence Ξ acts freely on Z. If we now prove that the action of Ξ on Z lifts to actions on $L_\alpha|_Z$, $L_\beta|_Z$ and $N|_Z$ then we will deduce that $f(Z)$ is divisible by p, by the same arguments that we used at the beginning of the proof to justify that the action of Γ on X is not free. Since Γ_1 acts trivially on Z, its action on $L_\alpha|_Z$, $L_\beta|_Z$ and $N|_Z$ will be given by characters $\xi_\alpha, \xi_\beta, \nu : \Gamma_1 \to S^1$ respectively. Using the fact that Γ_0 is abelian we deduce that these characters can be extended to characters of Γ_0. Denote the extensions by the same symbols. Then we may twist the action of Γ_0 on $L_\alpha|_Z$, $L_\beta|_Z$ and $N|_Z$ by ξ_α^{-1}, ξ_β^{-1}, ν^{-1} respectively. The resulting new action will be trivial on Γ_1, and hence will define a lift of the action of Ξ on Z to the bundles $L_\alpha|_Z$, $L_\beta|_Z$ and $N|_Z$. □

Lemma 4.47. *There exists a constant C such that for every prime p and any p-group $\Gamma \in \mathcal{G}_0$ there exists an abelian subgroup $B \leq \Gamma$ satisfying $[\Gamma : B] \leq C$. Furthermore, at least one of the following statements is true.*

1. *for every $b \in B$ we have $X^b \neq \emptyset$;*
2. *there exists some $b \in B$ such that X^b has a connected component which is a nonorientable surface of genus not bigger than C.*

Proof. Let p be any prime and let $\Gamma \in \mathcal{G}_0$ be a p-group. Choose a MNAS $A \leq \Gamma$. Recall that since $A \leq \Gamma$ is a MNAS, conjugation gives a monomorphism $c : \Gamma/A \hookrightarrow \mathrm{Aut}(A)$ (see [64, §5.2.3]).

Suppose that there exists some $a \in A$ such that X^a contains an isolated fixed point. (resp. a connected component Z which is a nonorientable surface). Then we may apply Lemma 4.32 (resp. Lemma 4.33) and conclude the existence of an abelian subgroup $B \leq \Gamma$ satisfying $[\Gamma : B] \leq C$ (where C depends only on X) and furthermore one of the following statements are true:

1. $X^B \neq \emptyset$ (this happens if we are applying Lemma 4.32), or
2. there is some $b \in B$ such that X^b has Z as a connected component (this happens if we are applying Lemma 4.33).

So we are done in this case.

Suppose from now on that the fixed point set of every $a \in A \setminus \{1\}$ is a possibly disconnected orientable embedded surface. Let C_1 be the constant given by Lemma 4.31 and let $W = W(X, A)$. Since A is normal in Γ, the action of Γ on X preserves W. By (1) in Lemma 4.31, $W \subset X$ is a possibly disconnected closed embedded surface, and each connected component of W is a connected component of X^a for some $a \in A$. So, by our assumption, W is orientable. By (3) in Lemma 4.31, W contains at most C_1 connected components (but beware that we have not proved that W is nonempty).

Let r be the number given by Theorem 4.15 applied to X, so that every finite abelian group acting effectively on X can be generated by r elements.

Let C_2 be the constant given by Lemma 4.46. Let p^k be the biggest power of p not bigger than C_2. Let $A_0 \leq A$ be the image of the multiplication map $A \to A$, $a \mapsto p^k a$ (we use additive notation on A). Since A can be generated by r or fewer elements, we have

$$[A : A_0] \leq p^{kr} \leq C_2^r. \tag{4.19}$$

Hence if $A_0 = \{1\}$ then $\sharp A \leq C_2^r$, so $\sharp\,\mathrm{Aut}(A) \leq (C_2^r)!$. Since there is a monomorphism $\Gamma/A \to \mathrm{Aut}(A)$, we have $\sharp\Gamma \leq C_2^r(C_2^r)!$. Setting $B = \{1\}$ we are done in this case.

Suppose from now on that $A_0 \neq \{1\}$. By Lemma 4.46 we have $X^a \neq \emptyset$ for every $a \in A_0$. Indeed, any $a \in A_0$ can be written as $a = p^k b$ for some $b \in A$, so a is contained in any subgroup $F \leq \langle b \rangle$ satisfying $[\langle b \rangle : F] \leq C_2$. It follows that $W \neq \emptyset$.

Since $[A : A_0]$ is bounded above by a constant depending only on X, by Lemma 4.11 and Theorem 4.15 the normalizer $\Gamma_0 \leq \Gamma$ of $A_0 \leq A$ satisfies

$$[\Gamma : \Gamma_0] \leq C_3$$

for some constant C_3 that depends only on X.

Since the action of Γ on X preserves W, so does the action of Γ_0. Let $\Gamma_1 \leq \Gamma_0$ be the subgroup of elements preserving each connected component of W. Then

$$[\Gamma_0 : \Gamma_1] \leq C_1!.$$

Choose some orientation of W. The set of possible orientations of W contains $2^{\sharp\pi_0(W)} \leq 2^{C_1}$ elements, so the subgroup $\Gamma_2 \leq \Gamma_1$ preserving the orientation of W satisfies

$$[\Gamma_1 : \Gamma_2] \leq 2^{C_1}.$$

We claim that the elements of Γ_2 centralize A_0. Let $g \in \Gamma_2$ and $a \in A_0$. Let $Z \subseteq X^a$ be a connected component. Then Z is a connected component of W as well and thus g preserves Z and acts on Z preserving the orientation, while a acts trivially on Z. This implies that g and a commute, by (1) in Lemma 4.20 and Lemma 4.26.

Let $\mathrm{Aut}^0_{A_0}(A) \leq \mathrm{Aut}_{A_0}(A)$ denote the automorphisms of A which fix each element of A_0. From the bound (4.19), Lemma 4.13, and Theorem 4.15, we conclude that

$$\sharp\,\mathrm{Aut}^0_{A_0}(A) \leq C_4$$

for some constant C_4 depending only on X. Using once again the fact that $A \leq \Gamma$ is a MNAS, we deduce that conjugation gives a monomorphism $\Gamma_2/\Gamma_2 \cap A \hookrightarrow \mathrm{Aut}(A)$. Since Γ_2 centralizes A_0, the image of this monomorphism lies in $\mathrm{Aut}^0_{A_0}(A)$. Hence

$$[\Gamma_2 : \Gamma_2 \cap A] = \sharp(\Gamma_2/\Gamma_2 \cap A) \leq C_4.$$

Define $B := \Gamma_2 \cap A_0$. Then we have $X^b \neq \emptyset$ for every $b \in B$ and

$$\begin{aligned}[\Gamma : B] &= [\Gamma : \Gamma_0][\Gamma_0 : \Gamma_1][\Gamma_1 : \Gamma_2][\Gamma_2 : \Gamma_2 \cap A][\Gamma_2 \cap A : \Gamma_2 \cap A_0] \\ &\leq [\Gamma : \Gamma_0][\Gamma_0 : \Gamma_1][\Gamma_1 : \Gamma_2][\Gamma_2 : \Gamma_2 \cap A][A : A_0] \\ &\leq C_3 C_1! 2^{C_1} C_4 C_2^r.\end{aligned}$$

The proof of the lemma is now complete. □

Let C and d be positive integers. Recall that a collection of finite groups $\mathcal{C}$ satisfies $\mathcal{J}(C,d)$ if each $G \in \mathcal{C}$ has an abelian subgroup A such that $[G : A] \leq C$ and A can be generated by d elements. Denote by $\mathcal{T}(\mathcal{C})$ the set of all $T \in \mathcal{C}$ such that there exist primes p and q, a normal Sylow p-subgroup P of T, and a Sylow q-subgroup Q of T, such that $T = PQ$. Note that here Q might be trivial. The following is the main result in [54]:

Theorem 4.48. *Let d and C_0 be positive integers. Let $\mathcal{C}$ be a collection of finite groups which is closed under taking subgroups and such that $\mathcal{T}(\mathcal{C})$ satisfies $\mathcal{J}(C_0,d)$. Then there exists a positive integer C such that $\mathcal{C}$ satisfies $\mathcal{J}(C,d)$.*

Lemma 4.49. *$\mathcal{G}_0$ satisfies the property $\mathcal{J}(C,r)$ for some constant C.*

Proof. By Theorem 4.48 it suffices to prove the existence of a constant C_0 such that $\mathcal{T}(\mathcal{G}_0)$ satisfies $\mathcal{J}(C_0,r)$.

Let $\Gamma \in \mathcal{T}(\mathcal{G}_0)$ and write $\Gamma = PQ$, where $P \leq \Gamma$ (resp. $Q \leq \Gamma$) is a Sylow p-subgroup (resp. q-subgroup), p, q are different primes, and P is a normal subgroup of Γ. By Lemma 4.47 there is an abelian subgroup $P_0 \leq P$ satisfying $[P : P_0] \leq C_1$, where C_1 depends only on X, and, furthermore, at least one of these statements is true:

1. for any $g \in P_0$ we have $X^g \neq \emptyset$,
2. there is some $g \in P_0$ such that X^g has a connected component which is a nonorientable surface.

Using again Lemma 4.47 we may pick an abelian subgroup $Q' \leq Q$ satisfying $[Q : Q'] \leq C_1$. Let $Q_0 \leq Q'$ be the normalizer of P_0 in Q'. Since $Q_0 = Q' \cap N_\Gamma(P_0)$, by Theorem 4.15 and Lemma 4.11, there exists a constant C_2 depending only on X such that

$$[Q' : Q_0] \leq [\Gamma : N_\Gamma(P_0)] \leq C_2.$$

By Lemmas 4.32 and 4.33, if there exists some $g \in P_0$ such that X^g has a connected component which is an isolated point or a nonorientable surface then there exists an abelian subgroup $B \leq P_0Q_0$ satisfying $[P_0Q_0 : B] \leq C_3$, where C_3 only depends on X. Since $[PQ : P_0Q_0] \leq C_1^2 C_2$, it follows that

$$[\Gamma : B] = [PQ : B] \leq C_1^2 C_2 C_3$$

and we are done in this case.

Let us assume for the remainder of the proof that the fixed point set of every $g \in P_0 \setminus \{1\}$ is a possibly disconnected orientable embedded surface. Define $W = W(X, P_0)$. By Lemma 4.31, W is a possibly disconnected embedded closed surface (orientable, by our previous assumption), for each $g \in P_0 \setminus \{1\}$ the fixed point set X^g is equal to the union of some connected components of W, and W has at most C_4 connected components, where C_4 only depends on X. Furthermore, the genus of each connected component of W is at most C_4. Since Q_0 normalizes P_0, the action of Q_0 on X preserves W.

Our hypothesis implies that statement (1) above holds true. Let $Q_1 \leq Q_0$ be the subgroup of those elements preserving each connected component of W, and acting orientation preservingly on each connected component of W. We have $[Q_0 : Q_1] \leq 2^{C_4} C_4!$. We claim that if $p \in P_0$ and $q \in Q_1$ then p and q commute. To see this, take a connected component Z of X^p. By (2) in Lemma 4.31, Z is a connected component of W, so Q_1 preserves Z and acts on Z orientation preservingly. Then the commutativity of p and q follows from (1) in Lemma 4.20 and from Lemma 4.26. Hence P_0Q_1 is abelian, and combining our previous bounds we obtain

$$[\Gamma : P_0Q_1] \leq C_1^2 C_2 2^{C_4} C_4!,$$

so the proof of the lemma is now complete. □

4.10.2 Proof of Theorem 4.1

Let X be an oriented and connected closed 4-manifold. Let r be the number resulting from applying Theorem 4.15 to X, so that every finite abelian group A acting effectively on X can be generated by r elements.

Let G be a finite group acting in a smooth and CTE way on X. Let $\Gamma = [G, G]$. By Lemma 4.49 there is an abelian subgroup $A \leq \Gamma$ satisfying

$$[\Gamma : A] \leq C_1,$$

where C_1 depends only on X. We distinguish two cases, according to whether the action of A on X is free or not.

Suppose that A acts freely on X. Let r, C_F be the constants given by Theorem 4.15 and Lemma 4.46 applied to X. If p is a prime bigger than C_F and the p-part $A_p \leq A$ is nontrivial, then by Lemma 4.46 the action of A_p on X has nontrivial fixed points, which contradicts the assumption that A acts freely. Hence we may write

$$A \simeq A_{p_1} \times \cdots \times A_{p_s},$$

where $p_1, \ldots, p_s$ are the prime numbers in $\{1, \ldots, C_F\}$. By Lemma 4.46 the exponent of A_{p_i} cannot be bigger than C_F, for otherwise the action of A_{p_i} would not be free. This implies that $\sharp A_{p_i} \leq C_F^r$, and consequently

$$\sharp A \leq C_F^{rs},$$

so $\sharp\Gamma \leq C_1 C_F^{rs}$. Applying Lemma 4.16 to the exact sequence $1 \to \Gamma \to G \to G/\Gamma \to 1$ we conclude the existence of an abelian subgroup $B \leq G$ such that $[G : B]$ is bounded above by a constant depending only on X. In this case we set $G_0 := B$ and we are done.

Assume, for the remainder of the proof, that A does not act freely on X. Let

$$G' = N_G(A) \leq G$$

be the normalizer of A in G. By Lemma 4.11 we have

$$[G : G'] \leq C_2,$$

where C_2 depends only on X. Let p be a prime such that $A_p \neq 1$ and the action of A_p on X is not free. Since A_p is a characteristic subgroup of A, A_p is normal in G'.

If there is some $a \in A_p$ such that X^a has an isolated fixed point then by Lemma 4.32 there is an abelian subgroup $B \leq G'$ such that $[G' : B]$ is bounded above by a constant depending only on X, so setting $G_0 := B$ we are done in this case.

Now assume that there is no $a \in A_p$ such that X^a has an isolated fixed point. If there is some $b \in A_p$ such that X^b has a connected component which is a nonorientable surface then by Lemma 4.33 there is an abelian subgroup $B \leq G'$ such that $[G' : B]$ is bounded above by a constant depending only on X, and hence setting $G_0 := B$ we are also done in this case.

At this point we may assume that for every $a \in A_p \setminus \{1\}$ the fixed point set $X^a \subset X$ is a possibly empty embedded orientable surface and that the set $W = W(X, A_p)$ defined in Subsection 4.7.1 is nonempty. Let C_3 be the constant given by applying Lemma 4.31 to X (so C_3 only depends on X). Then W has at most C_3 connected components and the absolute value of the genus of each of its connected components is not bigger than

C_3. Furthermore, since A_p is a normal subgroup of G' the action of G' on X preserves W.

We distinguish two cases according to whether $\chi(Z)$ vanishes for all connected components $Z \subseteq W$ or not.

Suppose first that there is a connected component $Z \subseteq W$ such that $\chi(Z) \neq 0$. Let $G'' \leq G'$ be the subgroup of elements preserving Z. We have $[G' : G''] \leq C_3$. From Lemmas 4.20 and 4.28 we deduce the existence of an abelian subgroup $B \leq G''$ such that $[G'' : B]$ is bounded above by a constant depending only on X. Hence, setting $G_0 := B$ we are done.

Finally, suppose that $\chi(Z) = 0$ for all connected components $Z \subseteq W$. Choose any connected component $Z \subset X$ and let $G'' \leq G'$ be the subgroup of elements preserving Z. We have $[G' : G''] \leq C_3$ and by Lemmas 4.20 and 4.27 there exists a nilpotent subgroup $G_0 \leq G''$ of class at most 2 satisfying $[G'' : G_0] \leq 12$ and, furthermore, $[G_0, G_0]$ is cyclic and acts trivially on Z. We thus have

$$Z \subseteq X^{[G_0,G_0]} \subseteq W,$$

so $X^{[G_0,G_0]}$ is a nonempty union of embedded tori because all connected components of W are orientable and have zero Euler characteristic. Combining the previous estimates we have

$$[G : G_0] \leq [G : G'][G' : G''][G'' : G_0] \leq 12\, C_2 C_3,$$

so the proof of the theorem is now complete.

4.10.3 Proof of Theorem 4.2

Suppose that N is a finite nilpotent group of class at most 2 acting in a smooth and CTE way on X. Then $[N, N]$ is abelian and central in N. The arguments in Subsection 4.10.2 imply the existence of a constant C_1, depending only on X, such that if $\alpha(N) \geq C_1$ then the group $[N, N]$ does not act freely on X, and any nontrivial $g \in [N, N]$ whose action on X has fixed points satisfies (2) in the statement of Theorem 4.2. Furthermore, (3) holds for any such g (with a suitable choice of C depending only on X) thanks to (1) in Lemma 4.27.

To conclude the proof of Theorem 4.2 assume that $\alpha(N) \geq C_1$ and let us prove that there exists a nontrivial $g \in [N, N]$ which does not act freely on X and whose order satisfies $\operatorname{ord}(g) \geq f(\alpha(N))$ for some function f depending on X and satisfying $\lim_{n\to\infty} f(n) = \infty$.

We may write

$$[N, N] \simeq \Gamma_1 \times \cdots \times \Gamma_s,$$

where each Γ_i is cyclic of prime power order. By Theorem 4.15, $s \leq C_2$, where C_2 depends only on X. For any $g \in [N, N]$ which does not act freely on X the fixed point set X^g is the disjoint union of some tori (because we are assuming $\alpha(N) \geq C_1$), so in particular X^g has no connected component which is a nonorientable surface. Consequently, Lemma 4.46

implies that for every i there exists some $\Gamma_i' \leq \Gamma_i$ such that $X^{\Gamma_i'} \neq \emptyset$ and $[\Gamma_i : \Gamma_i'] \leq C_3$ for some constant C_3 depending only on X. Then we have

$$\max_i \sharp\Gamma_i' \geq \frac{\max_i \sharp\Gamma_i}{C_3} \geq \frac{[N,N]^{1/C_2}}{C_3}.$$

By Lemma 4.16 there exists a function $h : \mathbb{N} \to \mathbb{N}$ depending only on X and satisfying $\lim_{n\to\infty} h(n) = \infty$ and $\sharp[N,N] \geq h(\alpha(N))$ (just take $G = N$ and $G_0 = [N,N]$, so that $G_1 = N/[N,N]$ is abelian). The function $f : \mathbb{N} \to \mathbb{N}$ defined as

$$f(n) := \frac{h(n)^{1/C_2}}{C_3}$$

depends only on X, it satisfies $\lim_{n\to\infty} f(n) = \infty$, and by the previous estimate we have $\max_i \sharp\Gamma_i' \geq f(\alpha(N))$, so picking some i realizing the previous maximum, any generator g of Γ_i' satisfies $\mathrm{ord}(g) \geq f(\alpha(N))$.

4.11 Using the Atiyah–Singer G-signature theorem

Theorem 4.50. *Let X be a closed connected and oriented 4-manifold satisfying $\sigma(X) \neq 0$. If $\phi \in \mathrm{Diff}(X)$ has finite order and acts trivially on cohomology then $X^\phi \neq \emptyset$.*

Proof. This is an immediate consequence of the G-signature theorem [3, Theorem 6.12] and the fact that $\sigma(\phi, X) = \sigma(X) \neq 0$ if ϕ acts trivially on cohomology. □

Theorem 4.51. *Let X be a closed connected and oriented 4-manifold. Suppose that $\phi \in \mathrm{Diff}(X)$ has finite order bigger than 2 and acts trivially on cohomology, and that the fixed point set X^ϕ has no isolated fixed points (so all the connected components of X^ϕ are embedded surfaces). Suppose that $X^\phi = S_1 \sqcup \cdots \sqcup S_n$ with each S_i connected, and that the action of ϕ on the normal bundle of S_k is by rotation of angle $\theta_k \in S^1$. Then all connected components of X^ϕ are orientable and*

$$\sigma(X) = \sum_{k=1}^{n} \sin^{-2}(\theta_k/2)\, S_k \cdot S_k.$$

Proof. The orientability of the connected components of X^ϕ is guaranteed by (1) in Lemma 4.21. If the order of ϕ is odd then the formula for $\sigma(X)$ follows from [3, Proposition 6.18]. For the general case note that the proof of [3, Proposition 6.18] works equally well if the order of ϕ is even and bigger than 2. Indeed, in this case the normal bundle N of every connected component $Y \subseteq X^\phi$ supports an invariant almost complex structure (by Lemma 4.25, because Y is orientable and hence so is N) and ϕ acts on N through multiplication by a complex number different from ± 1 (so in the notation of [3, §6] we have $N^\phi(-1) = 0$). □

Theorem 4.52. *Let X be a closed, connected and oriented 4-manifold. If $\sigma(X) \neq 0$ then $\mathrm{Diff}(X)$ is Jordan.*

Proof. The same argument that we used in Lemma 4.49 to prove that the family of finite groups $\mathcal{G}_0$ is Jordan works in our case if we replace Lemma 4.46 by Theorem 4.50. □

The following lemma is used in the proof of Theorem 4.3.

Lemma 4.53. *Let X be a closed connected and oriented 4-manifold satisfying $\sigma(X) = 0$. There exists a real number $\lambda > 0$ with the following property. Suppose that $\phi \in \mathrm{Diff}(X)$ has finite order and acts trivially on cohomology, that the fixed point set X^ϕ has no isolated fixed points, and that all connected components of X^ϕ (which, by assumption, are embedded surfaces) are orientable. Write $X^\phi = S_1 \sqcup \cdots \sqcup S_n$ and define*

$$\mu_M = \max_i S_i \cdot S_i, \qquad \mu_m = \min_i S_i \cdot S_i.$$

Then $\mu_M \geq -\lambda\mu_m \geq 0$ and $\mu_m \leq -\lambda\mu_M \leq 0$.

Proof. We first prove that the number n of connected components of X^ϕ is bounded above by a constant depending only on X: more precisely, if we define $D = \max_p \sum_{j\geq 0} b_j(X; \mathbb{Z}/p)$ then $n \leq D/2$. Indeed, if $\phi \in \mathrm{Diff}(X)$ satisfies the hypothesis of the lemma and its order is equal to ps, where p is a prime and s an integer, then applying Lemma 4.29 to the fixed point set of ϕ^s and noting that each connected component of X^ϕ is a connected component of X^{ϕ^s} we conclude that

$$\sum_i \sum_k b_k(S_i; \mathbb{Z}/p) \leq D.$$

Since each S_i contributes at least two units to the left hand side, the bound $n \leq D/2$ follows.

Once we have an upper bound on the number of connected components of X^ϕ, the proof is concluded combining Theorem 4.51 and the following lemma. □

Lemma 4.54. *Given an integer $n > 0$ there exists a real number $\delta > 0$ and an integer $k_0 > 0$ such that for every integer $k \geq k_0$ and any choice of primitive k-th roots of unity*

$$\theta_1, \ldots, \theta_n \in S^1$$

there is an integer a such that $|\sin \theta_j^a| \geq \delta$ for every j.

Proof. We consider the standard measure on S^1 of total volume 2π. For every integer $k \neq 0$ we denote by μ_k the set of all k-th roots of unity, and for every $\epsilon > 0$ we denote $A_\epsilon = \{e^{2\pi \mathbf{i}\theta} \mid |\theta| < \epsilon\} \subseteq S^1$ and $S_\epsilon = A_\epsilon \cup (-A_\epsilon)$.

Define $\epsilon = 1/(4(n+1))$ and $k_0 = 4(n+1)$. Suppose that $k \geq k_0$. For every $\theta, \theta' \in \mu_k$ the sets $\theta S_{1/(2k)}$ and $\theta' S_{1/(2k)}$ are disjoint. Since $1/2k \leq 1/2k_0 = \epsilon/2$, we have

$$\bigcup_{\theta \in \mu_k \cap A_{\epsilon/2}} \theta S_{1/(2k)} \subseteq A_\epsilon.$$

Combining this inclusion with $\mathrm{Vol}(A_\epsilon) = 8\pi\epsilon = 2\pi/(n+1)$ and $\mathrm{Vol}(\theta S_{1/(2k)}) = 2\pi/k$, it follows that

$$\sharp\mu_k \cap A_{\epsilon/2} \leq \frac{2\pi/(n+1)}{2\pi/k} = \frac{k}{n+1}.$$

Let $[k] = \{1, 2, \dots, k\}$. Suppose that $\theta_1, \dots, \theta_n$ are k-th primitive roots of unity. Then for every j the map $e_j : [k] \to \mu_k$ defined as $e_j(a) = \theta_j^a$ is a bijection. Define $C_j = \{a \in [k] \mid \theta_j^a \in A_{\epsilon/2}\}$. The previous estimate implies that $\sharp C_j \leq k/(n+1)$, and hence the set $C = C_1 \cup C_2 \cup \dots \cup C_n$ satisfies $\sharp C < k$. Therefore $[k] \setminus C$ is nonempty. Take any $a \in [k] \setminus C$. For every j we have $\theta_j^a \notin A_{\epsilon/2}$, so

$$|\sin \theta_j^a| \geq \delta := \sin 2\pi\epsilon/2 = \sin \pi/4(n+1),$$

so the proof of the lemma is complete. □

4.12 Automorphisms of almost complex manifolds: proof of Theorem 4.5

Let us prove Theorem 4.5. Let (X, J) be a closed almost complex 4-manifold, and let $\mathcal{G} = \mathrm{Aut}(X, J)$ be its group of automorphisms. Assume that $\mathcal{G}$ is not Jordan. Then, by Theorem 4.3 we can find some $\phi \in \mathcal{G}$ of finite order such that X^ϕ has a connected component T which is an embedded torus of negative self-intersection. Since ϕ preserves J and has finite order, its fixed point locus is a (possibly disconnected) almost complex submanifold. In particular T is an almost complex submanifold of (X, J) and hence can be identified with the image of a holomorphic embedding $\psi : \Sigma \to (X, J)$ where Σ is a closed connected Riemann surface of genus 1.

Let $\mathcal{G}_0 \leq \mathcal{G}$ denote the subgroup of automorphisms acting trivially on $H^*(X)$. We claim that the elements of $\mathcal{G}_0$ preserve T. Indeed, if $\zeta \in \mathcal{G}_0$ then applying Proposition 3.15 to ψ and $\zeta \circ \psi$ we conclude that $\zeta(T) = T$ because $T \cdot T < 0$.

Let $G \leq \mathcal{G}$ be a finite subgroup. By Lemma 4.9 the intersection $G_0 = G \cap \mathcal{G}_0$ satisfies $[G : G_0] \leq C$ for some constant C depending only on X. By our previous observation, every element of G_0 preserves T. So, if we denote by $N \to T$ the normal bundle of the inclusion in X then by Lemma 4.20 the action of G_0 on X induces a monomorphism $G_0 \hookrightarrow \mathrm{Aut}(N)$. By Lemma 6.5 there is an abelian subgroup $A \leq G_0$ satisfying

$$[G_0 : A] \leq 12\,|T \cdot T|,$$

so we have

$$[G : A] \leq 12\,C|T \cdot T|.$$

We have thus proved that $\mathcal{G}$ is Jordan, contradicting our initial assumption that it was not.

4.13 Symplectomorphisms: proof of Theorem 4.6

By (1), (2) and (4) in Theorem 4.4, in order to prove Theorem 4.6 it suffices to consider the case where

$$\chi(X) = \sigma(X) = 0, \qquad b_2^+(X) = 1. \tag{4.20}$$

For the latter condition note that $b_2(X) > 0$ on any symplectic manifold, and the vanishing of the signature implies that $b_2(X) = 2b_2^+(X)$. Under these conditions we have $b_2(X) = 2$ and consequently (using the vanishing of χ and Poincaré duality) $b_1(X) = b_3(X) = 2$. In particular, statement (3) in Theorem 4.6 follows from Theorem 4.4.

So throughout this section (X, ω) will denote a fixed closed symplectic 4-manifold satisfying the previous conditions (4.20).

Let J be any ω-compatible almost complex structure on X. We can define the canonical bundle K_X of X as the complex line bundle $K_X = \bigwedge^2 T^*X$, where the complex structure is induced by J. We denote by

$$K \in H_2(X)$$

the Poincaré dual of $c_1(K_X)$. Since the space of ω-compatible almost complex structures on X is contractible, K is independent of the chosen J.

We say that a class $A \in H_2(X)$ is representable by J-holomorphic curves if there is a possibly disconnected closed Riemann surface Σ and a J-holomorphic map $\psi : \Sigma \to X$ such that $\psi_*[\Sigma] = A$.

Lemma 4.55. *Suppose that X is not an S^2-bundle over T^2. Then, for every ω-compatible almost complex structure J on X, K or $2K$ are representable by J-holomorphic curves.*

Proof. Before we prove the lemma, let us recall some facts about Seiberg–Witten invariants of symplectic manifolds with $b_2^+(X) = 1$.

For any closed connected 4-manifold X the set of Spin^c structures on X has a natural structure of torsor over $H^2(X)$ (see e.g. [46, §3.1]). If $\mathfrak{s}$ is a Spin^c structure and $\beta \in H^2(X)$ then we denote by $\beta \cdot \mathfrak{s}$ the Spin^c structure given by the action of β on $\mathfrak{s}$. If (X, ω) is a symplectic manifold (which we assume in all the following discussion) then there is a canonical Spin^c structure on X, denoted by $\mathfrak{s}_{can}$, with determinant line bundle K_X^{-1} (actually to define this structure we need to choose an almost complex structure compatible with ω, but the outcome only depends on ω, see e.g. [46, §3.4]). This Spin^c structure allows us to identify $H^2(X)$ with the set of Spin^c structures on X, by assigning to $\beta \in H^2(X)$ the Spin^c structure $\beta \cdot \mathfrak{s}_{can}$. In terms of this identification we can regard the Seiberg–Witten invariant as a map

$$SW : H^2(X) \to \mathbb{Z}.$$

For closed 4-manifolds X with $b_2^+(X) > 1$ the moduli spaces of Seiberg–Witten solutions for two generic pairs of metric and perturbation $(g_1, \eta_1), (g_2, \eta_2)$ can be connected by a smooth cobordism[1]. This implies that the invariant SW is independent of the generic metric and perturbation chosen to define it.

When $b_2^+(X) = 1$ this is not true anymore, as there might exist generic pairs (g_1, η_1), (g_2, η_2) whose moduli spaces cannot be connected by any smooth cobordism. More precisely, for any $\beta \in H^2(X)$ the space $\mathcal{S}_\beta$ of all pairs (metric, perturbation) whose moduli space of Seiberg–Witten solutions contain no reducible solution (that is, solutions (A, ψ) with $\psi = 0$) has two connected components. The Seiberg–Witten moduli spaces associated to two generic elements of $\mathcal{S}_\beta$ can be connected by a smooth cobordism if the two elements belong to the same connected component of $\mathcal{S}_\beta$, but there is no reason to expect the existence of such a cobordism if they belong to different connected components. Hence, we should consider two possibly different Seiberg–Witten invariants, one for each connected component of $\mathcal{S}_\beta$.

One can prove that it is possible to label the connected components of $\mathcal{S}_\beta$ as $\mathcal{S}_\beta^+$ and $\mathcal{S}_\beta^-$ in such a way that the following holds. For any metric g on X let us denote by ω_g the unique self-dual g-harmonic 2-form of L^2-norm 1 whose cohomology class belongs to the same connected component of $H^2_+(X; \mathbb{R}) \setminus \{0\}$ as $[\omega]$. Then $(g, \pm \mathbf{i}\lambda\omega_g) \in \mathcal{S}_\beta^\pm$ for $\lambda > 0$ sufficiently big. Hence we may encode the Seiberg–Witten invariants of X through two maps

$$SW^\pm : H^2(X) \to \mathbb{Z},$$

where $SW^\pm(\beta)$ is the invariant obtained from a generic pair belonging to $\mathcal{S}_\beta^\pm$. For further details, see Section 7.4 of [67].

Define

$$w(\beta) = SW^+(\beta) - SW^-(\beta).$$

This difference $w(\beta)$ can be computed by means of a wall-crossing formula. We will just decribe the relevant formula needed for our purposes. For the general form of the wall-crossing formulas we refer the reader to [67, Theorem 9.4]. By [67, Proposition 12.5] (see [67, Remark 13.7]) we have

$$SW^-(\beta) = SW^+(c_1(K_X) - \beta)$$

for every β. Therefore,

$$w(\beta) = SW^+(\beta) - SW^+(c_1(K_X) - \beta). \tag{4.21}$$

A theorem of Taubes implies that $SW^+(0) = 1$ (see [72] and [67, Theorem 13.8]).

For a manifold with $b_1(X) = 2$, we can compute $w(\beta)$ as follows (see [38, Definition 2.2]). Let α_1, α_2 be a basis of $\in H^1(X)$, and define $a = \alpha_1 \cup \alpha_2$. Let $\beta \in H^2(X)$. Let

$$d(\beta) = -\langle \beta, K \rangle + \int_X \beta^2.$$

[1]Here and below generic means as usual that the Seiberg–Witten equations define a section of a Banach vector bundle over the parameter space (connections) $\times$ (sections of the spinor bundle) which is transverse to the zero section, so in particular the moduli space is a smooth manifold of the expected dimension.

Then

$$w(\beta) = \int_X a \cup (\beta - c_1(K_X)/2) \qquad \text{if } d(\beta) \geq 0, \tag{4.22}$$

and $w(\beta) = 0$ if $d(\beta) < 0$.

We are now ready to prove the lemma.

We claim that if $SW^+(\beta) \neq 0$, then for any ω-compatible almost complex structure J, $PD(\beta)$ is representable by J-holomorphic curves. Indeed, let $g_J = \omega(\cdot, J\cdot)$ be the metric associated with ω and J. With respect to this metric, ω is self-dual and of positive norm. The fact that $SW^+(\beta) \neq 0$ means that for any perturbation η satisfying $(g_J, \eta) \in \mathcal{S}^+_\beta$ there exists some solution to the Seiberg–Witten equations with metric g_J and perturbation η (this follows by definition for generic η and by a compactness argument for general perturbations). Then, the existence of the J-holomorphic curve representing $PD(\beta)$ follows from [73, Theorem 1.3].

Therefore, we only need to show that $SW^+(c_1(K_X))$ and $SW^+(2c_1(K_X))$ cannot be both zero.

If $w(0) \neq 1$, from (4.21) and $SW^+(0) = 1$ we obtain $SW^+(c_1(K_X)) \neq 0$, so K is representable by J-holomorphic curves, and we are done in this case.

Suppose for the remainder of the proof that $w(0) = 1$. We have $d(0) = 0$. By the Hirzebruch signature theorem we have $K \cdot K = 2\chi(X) + 3\sigma(X) = 0$, and this implies $d(2c_1(K_X)) = 0$. We then compute, using (4.22),

$$\begin{aligned} w(2c_1(K_X)) &= \int_X a \cup 3c_1(K_X)/2 && \text{because } d(2c_1(K_X)) = 0 \\ &= 3\int_X a \cup c_1(K_X)/2 \\ &= -3w(0) && \text{because } d(0) = 0 \\ &= -3. \end{aligned}$$

Hence,

$$SW^+(2c_1(K_X)) - SW^+(-c_1(K_X)) = -3.$$

We claim that $-K$ is not representable by J-holomorphic curves and therefore

$$SW^+(-c_1(K_X)) = 0.$$

Indeed, if $-K$ were representable by J-holomorphic curves, then by the positive energy condition of J-holomorphic curves we would have $\langle [\omega], -K \rangle > 0$. However, Theorem B in [34] implies that in this case (X, ω) is a ruled or rational surface or a blow up of a ruled or rational surface. Since $b_1(X) = b_2(X) = 2$, (X, ω) must be a ruled surface over T^2. Hence, X is an S^2-bundle over T^2, contradicting the assumption of the lemma.

Therefore, $SW^+(2c_1(K_X)) = -3$ and consequently $2K$ is representable by J-holomorphic curves, thus finishing the proof of the lemma. □

Lemma 4.56. *Suppose that X is not an S^2-bundle over T^2. Let $\phi \in \mathrm{Symp}(X, \omega)$ be an element of finite order acting on X in a CT way, and suppose that X^ϕ is a disjoint union of embedded tori. Then, every connected component $T \subseteq X^\phi$ satisfies $T \cdot T = 0$.*

Proof. Choose some almost complex structure J on X which is ω-compatible and ϕ-invariant (see e.g. [39, Lemma 5.5.6]). If there is some connected component $T' \subseteq X^\phi$ with negative self-intersection then from Theorem 4.51 and the assumption $\sigma(X) = 0$ we conclude that there is some $T \subseteq X^\phi$ with $T \cdot T > 0$.

We prove the lemma by contradiction. By the previous comment, it suffices to assume that there is a connected component $T \subseteq X^\phi$ satisfying $T \cdot T > 0$. Since $d\phi$ and J commute, J preserves the tangent bundle of X^ϕ, and hence the tangent bundle of T. In particular, T is J-holomorphic. Denote by $[T] \in H_2(X)$ the fundamental class of T corresponding to the standard orientation as a closed Riemann surface. We have

$$0 < T \cdot T = [T] \cdot [T] = -K \cdot [T],$$

where the second equality is given by the adjunction formula. By Lemma 4.55, K or $2K$ are representable by J-holomorphic curves. Let $n = 1$ if K is representable, and let $n = 2$ if $2K$ is representable and K is not.

By definition there is a possibly disconnected closed Riemann surface Σ and a J-holomorphic map $\psi : \Sigma \to X$ such that $nK = \psi_*[\Sigma]$. Let $\{\Sigma_i\}$ be the connected components of Σ and let $A_i = \psi_*[\Sigma_i]$, so that $nK = \sum_i A_i$.

We have $A_i \cdot [T] \geq 0$ for all i. This follows from Proposition 3.15 if $\psi(\Sigma_i) \neq T$, and from the assumption $T \cdot T > 0$ if $\psi(\Sigma_i) = T$ because in this case A_i is a positive multiple of $[T]$. Therefore we have

$$0 < n[T] \cdot [T] = -nK \cdot [T] = -\sum_i A_i \cdot [T] \leq 0.$$

We have thus reached a contradiction, finishing the proof of the lemma. □

Lemma 4.57. *Let X be an S^2-bundle over T^2. For any symplectic form ω on X the symplectomorphism group* $\mathrm{Symp}(X, \omega)$ *is Jordan.*

Proof. This is [49, Corollary 1.5]. However, we provide here a shorter version of the proof. We prove the lemma for the case where X is diffeomorphic to $T^2 \times S^2$. The general case then follows from the observation that $T^2 \times S^2$ is a double cover of the twisted S^2-bundle over T^2 by applying Lemma 4.8.

Suppose $\mathrm{Symp}(X, \omega)$ is not Jordan. Then, by Theorem 4.3 there is, for every n, an element $\phi_n \in \mathrm{Symp}(X, \omega)$ of finite order such that X^{ϕ_n} has a connected component which is a torus T_n of self-intersection number $T_n \cdot T_n < -n$. We will prove that this is impossible.

Fix some n and choose an ω-compatible almost complex structure J on X which is invariant by ϕ_n. Then, T_n is an embedded J-holomorphic curve. By the adjunction formula, we have

$$K \cdot [T_n] = -[T_n] \cdot [T_n].$$

Taking into account that X is diffeomorphic to $T^2 \times S^2$, it is easy to compute its canonical class K as

$$K = -2[T^2 \times pt].$$

Hence, if we write

$$[T_n] = a[T^2 \times pt] + b[pt \times S^2]$$

from the adjunction formula we obtain $-2b = -2ab$, so that either $b = 0$ (and then T_n has self-intersection number 0) or $a = 1$. Taking $n > 0$, we must have $a = 1$. If ω_{T^2} and ω_{S^2} are pullbacks to $T^2 \times S^2$ of area forms of total area 1 in T^2 and S^2, we have that $[\omega_{T^2}], [\omega_{S^2}]$ form a basis of $H^2(X)$, so we can write

$$[\omega] = \alpha[\omega_{T^2}] + \beta[\omega_{S^2}],$$

for some $\alpha, \beta > 0$. Since T_n is J-holomorphic, it must satisfy the condition of positive energy, which is given by

$$0 < \langle [\omega], [T_n] \rangle = \alpha + b\beta.$$

Hence,

$$b > -\frac{\alpha}{\beta}.$$

On the other hand, $T_n \cdot T_n = 2b < -n$. Taking $n > 2\alpha/\beta$ we obtain a contradiction.

This finishes the proof of the lemma. □

4.13.1 Proof of statements (1) and (2) in Theorem 4.6

If X is an S^2-bundle over T^2, this follows from Lemma 4.57. Assume that (X, ω) is not an S^2-bundle over T^2 and that $\mathrm{Symp}(X, \omega)$ is not Jordan. Then, by Theorem 4.3, there is some $\phi \in \mathrm{Symp}(X, \omega)$ of finite order and acting in a CT way on X with the property that some connected component T of X^ϕ is diffeomorphic to a torus and has positive self-intersection. This contradicts Lemma 4.56, so the proof of the first statement of Theorem 4.6 is complete.

Statement (2) in Theorem 4.6 follows from combining Theorem 4.1, (4) in Theorem 4.2, and Lemma 4.56.

Chapter 5

Symplectic actions on S^2-bundles over S^2

In this chapter we provide another contribution of this thesis: the classification of finite groups that act effectively and symplectically on symplectic 4-manifolds which are S^2-bundles over S^2. Up to diffeomorphism, there are only two such manifolds: $S^2 \times S^2$ and a twisted bundle $X_S \to S^2$ with fiber S^2. However, both manifolds admit infinitely many, non symplectomorphic, symplectic structures. A remarkable fact about these manifolds is that they are among the few 4-manifolds where a complete classification of their symplectic structures is known (this is due to a deep theorem of Lalonde and McDuff), and this theorem will be the key that will allow us to prove classification theorems for any symplectic structure. Some ideas used in this chapter are an adaptation and generalization of those in [49].

In the first section, we will recall some facts about S^2-bundles over S^2, together with some remarks about the cohomology of these manifolds. In the next section, we prove some lemmas about complex line bundles over S^2, and in the final two sections we present our classification theorem, first for $S^2 \times S^2$ and then for X_S.

5.1 S^2-bundles over S^2

In this section we will describe some basic facts about symplectic S^2-bundles over S^2

Let $B = S^2$ and let $\pi : E \to B$ be a smooth and orientable fiber bundle with fiber $F = S^2$.

The first observation is that there are only two such bundles up to diffeomorphism. For the proof of the following theorem we refer the reader to [40, Lemma 6.2.3].

Proposition 5.1. *There are two orientable S^2-bundles with base S^2. These bundles are the trivial bundle $S^2 \times S^2$ and a non-trivial bundle which we denote by X_S.*

We now give a description of these bundles that will be useful for our purposes. Let $\pi_d : L \to S^2$ be a complex line bundle over S^2 of degree d. Then, we can consider the bundles $\pi_d : \mathbb{P}(L \oplus \underline{\mathbb{C}}) \to S^2$ (where $\underline{\mathbb{C}} \to S^2$ is the trivial complex line bundle over

S^2). Since $\pi_d^{-1}(p) \simeq \mathbb{P}(\mathbb{C}^2) \simeq \mathbb{CP}^1$, they are S^2-bundles over S^2. Let $X_d = \mathbb{P}(L \oplus \underline{\mathbb{C}})$. Observe also that $X_d \simeq P_d \times_{S^1} S^2$, where P_d is the principal S^1-bundle over S^2 with Euler number d, and the action of S^1 on S^2 is the usual action by rotations around a fixed axis. We have defined a bundle $X_d \to S^2$ for each integer d. It turns out that the isomorphism class of these bundles only depends on the parity of d.

In the next proposition and elsewhere in this chapter, we say that two bundles are diffeomorphic if their total spaces are diffeomorphic via a diffeomorphism commuting with the projections.

Proposition 5.2. *The bundles $X_d \to S^2$ and $X_{d'} \to S^2$ are diffeomorphic if and only if $d \equiv d' \mod 2$.*

Proof. See Example 4.4.2 and Exercise 6.2.4 in [40]. □

By the previous description, we know that for every integer d the smooth section $s_d \in C^\infty(S^2, X_d)$ defined by $s_d(p) = [0 : 1]_p$ has self-intersection number d. Since $X_0 \cong X_{2k}$ for each k and $X_1 \cong X_{2k+1}$ for each k, we have in $S^2 \times S^2 \cong X_0 \to S^2$ smooth sections s_{2k} of self-intersection number $2k$, and we have in $X_S \cong X_1 \to S^2$ smooth sections s_{2k+1} of self-intersection $2k+1$. Since the image of s_{2k} is a smooth submanifold of dimension 2 of the 4-dimensional manifold X_{2k} with self-intersection $2k$, the normal bundle of s_{2k} is isomorphic to the complex line bundle $L \to S^2$ of degree $2k$. We denote by S_+ (resp. S_-) the submanifold of X_S which is the image of s_1 (resp. s_{-1}).

We now describe the homology and cohomology of $S^2 \times S^2$ and X_S. First we treat the case of the trivial bundle. In this case, $H_2(S^2 \times S^2)$ is generated by the classes $[S^2 \times pt] = [s_0]$ and the class of the fiber $[pt \times S^2]$. The cohomology $H^2(S^2 \times S^2)$ can be described as follows. Let $\omega_{S^2} \in \Omega^2(S^2)$ be the standard area form on S^2 with total area 1. Define $\omega_{S^2 \times \mathrm{pt}} = \pi_1^*(\omega_{S^2})$ and $\omega_{\mathrm{pt} \times S^2} = \pi_2^*(\omega_{S^2})$, where $\pi_i : S^2 \times S^2 \to S^2$, $i = 1, 2$, are the projections onto each factor. Then, $[\omega_{S^2 \times \mathrm{pt}}]$ and $[\omega_{\mathrm{pt} \times S^2}]$ form a basis for $H^2(S^2 \times S^2)$. Observe that in fact $[\omega_{S^2 \times \mathrm{pt}}] = PD([S^2 \times \mathrm{pt}])$ and $[\omega_{\mathrm{pt} \times S^2}] = PD([\mathrm{pt} \times S^2])$ The intersection product on $H_2(S^2 \times S^2)$ is easily computed:

$$[S^2 \times \mathrm{pt}] \cdot [\mathrm{pt} \times S^2] = 1$$
$$[S^2 \times \mathrm{pt}] \cdot [S^2 \times \mathrm{pt}] = [\mathrm{pt} \times S^2] \cdot [\mathrm{pt} \times S^2] = 0$$

For X_S we can take as a basis of $H_2(X_S)$ the classes $[S_+], [S_-]$. Note that with this basis, the fiber of the bundle $X_S \to S^2$ represents the class $[S_+] - [S_-]$. For the cohomology, one can pick $\omega_+, \omega_- \in \Omega^2(X_S)$ such that $[\omega_\pm] = PD(S_\pm)$. Then, a basis for $H^2(X_S)$ is formed by $[\omega_+]$ and $[\omega_-]$.

The intersection product on $H_2(X_S)$ is:

$$[S_+] \cdot [S_-] = 0$$
$$[S_+] \cdot [S_+] = 1$$
$$[S_-] \cdot [S_-] = -1$$

A useful fact about the homology of $S^2 \times S^2$ and X_S is that they admit a uniform description. Consider cohomology with real coefficients, and define $B = [S^2 \times \mathrm{pt}]$ in

$S^2 \times S^2$ and $B = 1/2([S_+] + [S_-]$ in X_S. Note that B is not an integral homology class in X_S. Similarly, define $F = [\mathrm{pt} \times S^2]$ in $S^2 \times S^2$ and $F = [S_+] - [S_-]$ in X_S. Note that in both cases F is the homology class represented by the fiber of the bundle. With these, definitions, note that we obtain the following intersection product:

$$B \cdot B = F \cdot F = 0$$
$$B \cdot F = 1,$$

regardless of whether we work in $S^2 \times S^2$ or X_S. Any integral homology class in $H_2(S^2 \times S^2)$ can be written as

$$aB + bF,$$

for some a, b integers. Any integral homology class in $H_2(X_S)$ can be written as

$$cB + dF,$$

where $c \in \mathbb{Z}$ and, if c is even then d is integer, and if c is odd then $d = k/2$ for some odd k.

To end this section we will discuss symplectic forms on S^2-bundles over S^2. It is easy to see that every

$$\omega_{\alpha,\beta} = \alpha\omega_{S^2 \times \mathrm{pt}} + \beta\omega_{\mathrm{pt} \times S^2}$$

for α, β positive real numbers is a symplectic form on $S^2 \times S^2$. Similarly, one can construct symplectic forms $\omega_{\lambda_+,\lambda_-}$ on X_S representing cohomology classes

$$\lambda_+[\omega_+] + \lambda_-[\omega_-]$$

for every $\lambda_+ > \lambda_- > 0$. (See for instance [40, Theorem 6.2.5].)

The following theorem states that in fact these are essentially the only examples.

Theorem 5.3 (Lalonde-McDuff). *Let ω be a symplectic form on $S^2 \times S^2$. Then, there is a diffeomorphism $\phi : S^2 \times S^2 \to S^2 \times S^2$ such that $\phi^*(\omega) = \omega_{\alpha,\beta}$.*

Similarly, if ω is a symplectic form on X_S, there is a diffeomorphism $\phi : S^2 \times S^2 \to S^2 \times S^2$ such that $\phi^*(\omega) = \omega_{\lambda_+,\lambda_-}$.

It is worth mentioning that the S^2-bundles over S^2 (and, more generally, S^2-bundles over a closed surface Σ) are one of the few cases where a complete classification of symplectic forms on a manifold of dimension greater than 2 has been obtained.

We will need the following lemma in the sequel, which computes the action on homology of a symplectomorphism of $S^2 \times S^2$ or X_S.

Lemma 5.4. *Let $\phi \in Symp(S^2 \times S^2)$. If $\alpha \neq \beta$, then ϕ induces the identity on $H_2(S^2 \times S^2)$. If $\alpha = \beta$, then either ϕ induces the identity on $H_2(X)$ or ϕ interchanges $[S^2 \times \mathrm{pt}]$ and $[\mathrm{pt} \times S^2]$.*

Let $\phi \in \mathrm{Symp}(X_S)$. Then, ϕ induces the identity on $H_2(X_S)$.

Proof. Since ϕ is an orientation-preserving diffeomorphism, it must preserve the intersection product. Hence,

$$\phi_*([S^2 \times \mathrm{pt}]) \cdot \phi_*([\mathrm{pt} \times S^2]) = 1,$$

$$\phi_*([S^2 \times \mathrm{pt}]) \cdot \phi_*([S^2 \times \mathrm{pt}]) = \phi_*([\mathrm{pt} \times S^2]) \cdot \phi_*([\mathrm{pt} \times S^2]) = 0.$$

Let

$$\phi_*([S^2 \times \mathrm{pt}]) = a[S^2 \times \mathrm{pt}] + b[\mathrm{pt} \times S^2]$$

and

$$\phi_*([\mathrm{pt} \times S^2]) = c[S^2 \times \mathrm{pt}] + d[\mathrm{pt} \times S^2],$$

with a, b, c, d integers. Then, by the above conditions we get $ad + bc = 1$, $ab = 0$ and $cd = 0$.

Moreover, since ϕ is a symplectomorphism,

$$\alpha = \langle [\omega], [S^2 \times \mathrm{pt}] \rangle = \langle \phi^* \omega, [S^2 \times \mathrm{pt}] \rangle = \langle \omega, \phi_*[S^2 \times \mathrm{pt}] \rangle = a\alpha + b\beta,$$

and similarly, $\beta = a\alpha + b\beta$.

Since $ab = 0$, either $a = 0$ or $b = 0$. If $a = 0$ we have $\alpha = b\beta$ and $b = c = \pm 1$. Since $\alpha, \beta > 0$, we get $\alpha = \beta$ and $b = c = 1$, so ϕ_* interchanges $[S^2 \times \mathrm{pt}]$ and $[\mathrm{pt} \times S^2]$. If $b = 0$, then $c = 0$, $a = d = \pm 1$ and $\alpha = a\alpha$, so $a = d = 1$ and ϕ_* is the identity.

The second part of the statement follows by an analogous computation. □

5.2 Finite groups acting on S^2

In this section we discuss the classification of finite groups acting smoothly and effectively on S^2. This will play a key role in the subsequent sections of the chapter.

The first step is to reduce the problem to that of classifying finite subgroups of $SO(3)$.

Lemma 5.5. *Let G be a finite group acting smoothly and effectively on S^2. Then, G is isomorphic to a subgroup of* $\mathrm{SO}(3)$.

Proof. Pick any G-invariant riemannian metric g on S^2, and let j be the almost complex structure on S^2 given by the conformal class of g, which is clearly also G-invariant. Since any almost complex structure on a surface is integrable, (S^2, j) is a Riemann surface. By the Riemann uniformization theorem, we have an isomorphism $(S^2, j) \simeq \mathbb{C}\mathrm{P}^1$. Therefore, G is isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{C}\mathrm{P}^1) \simeq \mathrm{PSL}(2, \mathbb{C})$. Since $\mathrm{SO}(3)$ is the maximal compact Lie subgroup of $\mathrm{PSL}(2, \mathbb{C})$, G is indeed isomorphic to a finite subgroup of $\mathrm{SO}(3)$. □

We now give a description of the finite subgroups of $\mathrm{SO}(3)$. We refer the reader to [77, Section 2.6] for further discussion and proofs.

The list of finite subgroups of $\mathrm{SO}(3)$ is the following:

- The cyclic groups C_n. C_n acts on S^2 by rotations around a fixed axis.

- The dihedral groups D_m. Recall that D_m is the group of symmetries of a polygon with m sides, and it can be presented as

$$D_m = \langle A, B \mid A^m = B^2 = 1, BAB^{-1} = A^{-1} \rangle.$$

  An action of D_m on S^2 is given as follows. A acts by a rotation of angle $2\pi/m$ around a fixed axis l of S^2, and B acts as a rotation of angle π around an axis of S^2 perpendicular to l.

- The group A_4. This is isomorphic to the group of orientation-preserving symmetries of a regular tetrahedron. A_4 acts on S^2 as the group of symmetries of a regular tetrahedron inscribed in S^2.

- The group S_4. This is isomorphic to the group of orientation-preserving symmetries of a regular octahedron, and also to the group of orientation-preserving symmetries of a regular hexahedron. S_4 acts on S^2 as the group of symmetries of a regular octahedron (or a regular hexahedron) inscribed in S^2.

- The group A_5. This is isomorphic to the group of orientation-preserving symmetries of a regular dodecahedron, and also to the group of orientation-preserving symmetries of a regular icosahedron. A_5 acts on S^2 as the group of symmetries of a regular dodecahedron (or a regular icosahedron) inscribed in S^2.

The next theorem shows that every finite subgroup of SO(3) is isomorphic to one of the above list.

Theorem 5.6. *Every finite subgroup of* SO(3) *is isomorphic to a cyclic group, a dihedral group, A_4, S_4 or A_5. Moreover, any two isomorphic subgroups of* SO(3) *are conjugated.*

For the proof, we refer the reader to [77, Theorem 2.6.5].

5.3 Actions on line bundles over S^2

In this section we prove some results about finite group actions by vector bundle automorphisms on line bundles over S^2 that we will use in later sections.

Recall that any finite subgroup of SO(3) is isomorphic to a cyclic group, a dihedral groups, or to one of the three polyhedral groups A_4, A_5 and S_4.

Lemma 5.7. *Let $L \to S^2$ be a complex line bundle of degree* 0. *A finite group G acts effectively on L by bundle automorphisms if and only if $G \simeq Z \times H$, where Z is finite cyclic and H is a finite subgroup of* SO(3).

Proof. Let us see first that there are effective actions on L of any group of the form $Z \times H$. Choose a trivialization of L. We only have to prove that G admits an effective action on $S^2 \times \mathbb{C} \to S^2$ by bundle automorphisms. Since H is a finite subgroup of SO(3) it admits a linear action on S^2, and identifying Z with a group of roots of the unity, Z

admits an effective action on $\mathbb{C}$ by multiplication. Therefore, G has an effective action on $S^2 \times \mathbb{C}$ by bundle automorphisms, given by

$$(\gamma, h) \cdot (p, z) = (h \cdot p, \gamma z),$$

for $(\gamma, h) \in Z \times H$ and $(p, z) \in S^2 \times \mathbb{C}$.

Conversely, assume that G is a finite group acting effectively on $L \to S^2$ by bundle automorphisms. Observe that G acts (maybe non-effectively) on S^2. By choosing a G-invariant riemannian metric on S^2 and considering its conformal class, we can assume that G acts on S^2 preserving a complex structure on it. Since by the uniformization theorem $(S^2, j) \simeq \mathbb{CP}^1$, we may assume that G acts on $L \to \mathbb{CP}^1$ preserving the complex structure on $\mathbb{CP}^1$. Moreover, we may pick a G-invariant connection on the bundle $L \to \mathbb{CP}^1$ (by the usual averaging trick, since the space of all connections is an affine space). By considering the $\bar{\partial}$ operator associated to this connection, we obtain a holomorphic structure on the bundle $L \to \mathbb{CP}^1$ such that the action of G is holomorphic.

By the Riemann–Roch theorem, we have

$$\dim H^0(\mathbb{CP}^1, L) - \dim H^1(\mathbb{CP}^1, L) = 1.$$

By Serre duality,

$$\dim H^1(\mathbb{CP}^1, L) = \dim H^0(\mathbb{CP}^1, L^{-1} \otimes K).$$

However, since the canonical bundle $K \to \mathbb{CP}^1$ is a line bundle of degree -2, $L^{-1} \otimes K$ is a line bundle of degree -2. Using that holomorphic bundles of negative degree have no non-zero holomorphic sections, we conclude that $\dim H^1(\mathbb{CP}^1, L) = 0$. Therefore, $\dim H^0(\mathbb{CP}^1, L) = 1$. Pick any holomorphic section s of $L \to \mathbb{CP}^1$. Since $\deg L = 0$, s does not vanish at any point. Since G acts holomorphically on $L \to \mathbb{CP}^1$, there is an action of G on the space of holomorphic sections $H^0(\mathbb{CP}^1, L) \simeq \mathbb{C}$. This action is given by a character $k : G \to \mathbb{C}$, where G acts on the space of sections $H^0(\mathbb{CP}^1, L)$ by $g \cdot s = k(g)s$. Consider the decomposition

$$0 \to G_0 \to G \to G_S \to 1,$$

where $G_0 \leq G$ is the central subgroup of the elements fixing S^2, and $G_S := G/G_0$ acts effectively on $\mathbb{CP}^1$, so it is a finite subgroup of $\mathrm{SO}(3)$. We may identify G_0 with a subgroup of $S^1 \subseteq \mathbb{C}$. Then, in fact $k : G \to G_0$, and it gives a retraction showing that the exact sequence above splits. Therefore, $G \simeq G_0 \times G_S$. □

Lemma 5.8. *Let $L \to S^2$ be a complex line bundle of degree 1. A finite group G acts effectively on L by bundle automorphisms if and only if G is isomorphic to a finite subgroup of $U(2)$.*

Proof. Suppose first that G is a finite group acting effectively on $L \to S^2$ by bundle automorphisms. We are going to show that G must be a finite subgroup of $U(2)$. Proceeding exactly like in the previous lemma, we may suppose that the bundle $L \to \mathbb{CP}^1$ is holomorphic, and that G acts preserving this holomorphic structure. By an argument analogous to that of the previous proof using the Riemann–Roch theorem and Serre duality, we obtain that $\dim H^0(\mathbb{C}\mathrm{P}^1, L) = 2$. Therefore, we have an induced action of G on $\mathbb{C}^2$. We now show that this action is effective. Let $g \in G$ be such that $g \cdot s = s$ for all sections $s \in H^0(\mathbb{C}\mathrm{P}^1, L)$. Then, g fixes pointwise the zero section of L . Indeed, each point $p \in \pi^{-1}(0)$ is the intersection of the zero section s_0 with some section s_p, and $g \cdot p$ is the intersection of $g \cdot s_0$ and $g \cdot s_p$. Therefore g acts on L by a homothety of the fibers. But since g preserves every section, this implies that g acts trivially on L. Since the action of G on L is effective, this implies $g = 1$. This shows that the action of G on $\mathbb{C}^2$ is effective, and hence G can be identified with a subgroup of $GL(2, \mathbb{C})$. Since G is finite, G is conjugated to a subgroup of $U(2)$, which is the maximal compact subgroup of $GL(2, \mathbb{C})$.

Now we prove that any finite subgroup of $U(2)$ acts effectively on $L \to S^2$ by bundle automorphisms. It is clearly enough to prove that there is an action of G on the associated principal S^1-bundle $P \to S^2$. This bundle is just the Hopf bundle $S^3 \to \mathbb{C}\mathrm{P}^1$ given by

$$(z_1, z_2) \mapsto [z_1 : z_2],$$

where $(z_1, z_2) \in S^3 \subseteq \mathbb{C}^2$. Since $U(2)$ acts by matrix multiplication on $\mathbb{C}^2$ preserving the Hopf fibration, any finite subgroup of $U(2)$ admits an effective action on $P \to S^2$ by bundle automorphisms. □

Lemma 5.9. *Let $L \to S^2$ be a complex line bundle of degree d. Let G be a finite group acting effectively on L by bundle automorphisms. Then, G admits an effective action by bundle automorphisms on a bundle $L' \to S^2$, where $\deg(L') = 0$ if d is even, and $\deg(L') = 1$ if d is odd.*

Proof. By Lemma 4.20, G sits on an exact sequence

$$1 \to Z \to G \xrightarrow{\pi} H \to 1,$$

where Z is a central cyclic subgroup of G and H is a finite subgroup of $\mathrm{SO}(3)$. There is a (possibly non-effective) action of G on TS^2, where H acts on TS^2 via the action induced by that of H on S^2, and Z acts trivially. Let $k \in \mathbb{Z}$ be such that $d = 2k$ (if d is even) or $d = 2k + 1$ (if d is odd). There is an induced action of G on the vector bundle $L \otimes (TS^2)^{-k} \to S^2$. We claim that this action is effective. Let $g \in G$. If $\pi(g) \neq 1$, then g does not act trivially on the base S^2, and so it does not act trivially on L. If $\pi(g) = 1$, then $g \in Z$ and its action is by a homothety. Since Z acts trivially on TS^2, if g acts trivially on $L \otimes (TS^2)^{-k}$, then it already acts trivially on L. Since this last action is effective, we must have $g = 1$. Therefore, the action is effective. Since $TS^2 \to S^2$ is a vector bundle of degree 2, the bundle $L \otimes (TS^2)^{-k} \to S^2$ is of degree 0 if d is even and of degree 1 if d is odd. □

5.4 The trivial bundle

In this section we give a complete classification of the finite groups that act effectively and symplectically on $(S^2 \times S^2, \omega)$, for any symplectic form ω. We define, for every pair of real numbers $\alpha, \beta > 0$,

$$\omega_{\alpha,\beta} := \alpha\omega_{S^2\times \mathrm{pt}} + \beta\omega_{\mathrm{pt}\times S^2}.$$

Each of these forms is a symplectic form on $S^2 \times S^2$. By Theorem 5.3, there is a unique symplectic form on $S^2 \times S^2$ up to diffeomorphism whose cohomology class is $\alpha[\omega_{S^2\times \mathrm{pt}}] + \beta[\omega_{\mathrm{pt}\times S^2}]$. Therefore, it is enough to consider the case where $\omega = \omega_{\alpha,\beta}$.

We start by constructing some simple examples of effective and symplectic actions of finite groups on $(S^2 \times S^2, \omega)$.

Lemma 5.10. *Let ω be any symplectic form on S^2. The finite subgroups of $Symp(S^2, \omega)$ coincide with the finite subgroups of $SO(3)$.*

Proof. Since any symplectic action is an orientation preserving smooth action, any group acting symplectically on (S^2, ω) must be isomorphic to a subgroup of $SO(3)$. We prove the converse. Let G be a group acting smoothly on S^2 by orientation preserving diffeomorphisms. We can find an area form dA on S^2 such that

$$\int_{S^2} dA = \int_{S^2} \omega.$$

By averaging it over the action of the G, we obtain another area form with the same total area which is G-invariant. Since any area form on a 2-dimensional orientable manifold is a symplectic form, G acts symplectically on (S^2, dA). Finally, Moser's trick in symplectic geometry implies that there is a symplectomorphism $(S^2, dA) \simeq (S^2, \omega)$ (see for instance Section 3.2 in [39]). Hence, G acts symplectically on (S^2, ω). □

Lemma 5.11. *Let H_1, H_2 be two finite subgroups of* $\mathrm{SO}(3)$. *Then, for any symplectic form $\omega_{\alpha,\beta}$, there exists an effective and symplectic action of $H_1 \times H_2$ on $(S^2 \times S^2, \omega_{\alpha,\beta})$. Moreover, in the case $\alpha = \beta$, there is an effective and symplectic action on $(S^2\times S^2, \omega_{\alpha,\alpha})$ of any group G lying on an exact sequence*

$$1 \to H \times H \to G \to \mathbb{Z}/2 \to 1,$$

for some finite subgroup H of $\mathrm{SO}(3)$, *and where the action by conjugation of a lift $g \in G$ of the non-trival element of $\mathbb{Z}/2$ on $H \times H$ is given by*

$$g(h_1, h_2)g^{-1} = (\phi_1 h_2 \phi_1^{-1}, \phi_2 h_1 \phi_2^{-1}),$$

for some $\phi_1, \phi_2 \in SO(3)$ such that $\phi_1\phi_2 \in H$ and $\phi_2\phi_1 \in H$.

Proof. By the previous lemma, H_1 (resp. H_2) acts effectively and symplectically on $(S^2, \alpha\omega_{S^2})$ (resp. on $(S^2, \beta\omega_{S^2})$). Therefore, there is an effective and symplectic action of $H_1 \times H_2$ on $(S^2 \times S^2, \alpha\omega_{S^2\times \mathrm{pt}} + \beta\omega_{\mathrm{pt}\times S^2})$, where H_1 acts on the first factor while fixing the second, and H_2 acts on the second factor while fixing the first.

We now consider the case $\alpha = \beta$. Then, as before, we have an action of $H \times H$ on $(S^2 \times S^2, \omega_{\alpha,\alpha})$. Moreover, in this case the interchanging involution

$$\tau : S^2 \times S^2 \to S^2 \times S^2$$

defined by $\tau(x,y)=(y,x)$, is a symplectomorphism. Let $\phi_1,\phi_2 \in \mathrm{SO}(3)$ be such that $\phi_1\phi_2, \phi_2\phi_1 \in H$. Let $g=(\phi_1,\phi_2)\circ\tau$, and let G be the subgroup of $\mathrm{Symp}(S^2\times S^2,\omega_{\alpha,\alpha})$ generated by $H\times H$ and g. Then, noting that $g^2=(\phi_1\phi_2,\phi_2\phi_1)\in H\times H$, we get the desired exact sequence

$$1\to H\times H\to G\to \mathbb{Z}/2\to 1.$$

The action of g on $H\times H$ by conjugation is easily computed to be the one in the statement. This finishes the proof of the Lemma. □

We devote the rest of this section to prove that the groups of the previous lemma (and its subgroups) are the only ones that act effectively and symplectically on $(S^2\times S^2,\omega_{\alpha,\beta})$. We break the proof into a series of lemmas.

Lemma 5.12. *Let $A_{a,b}=a[S^2\times \mathrm{pt}]+b[\mathrm{pt}\times S^2]\in H_2(S^2\times S^2)$, and let J be any $\omega_{\alpha,\beta}$-compatible almost complex structure. The set $\mathbb{Z}_{>0}\times\mathbb{Z}_{<0}$ contains at most one element (a,b) such that $\mathcal{M}(A_{a,b},J)$ is non-empty. Moreover, if there is one such non-empty moduli space, then there is only one (unparametrized) J-holomorphic curve representing the class $A_{a,b}$, and the moduli spaces $\mathcal{M}(A_{a,0},J)$ are empty for all $a>0$. The same statement works interchanging a and b.*

Proof. The lemma follows easily from positivity of intersections. Indeed, suppose that there are two different (unparametrized) J-holomorphic curves, C and C', representing the classes $A_{a,b}$ and $A_{c,d}$ respectively, where $a,c>0$ and $b,d<0$. Then, $[C]\cdot[C']=ad+bc<0$, a contradiction with positivity of intersections. To prove the second part of the statement, suppose that there is one J-holomorphic curve C representing the class $A_{a,b}$ with $a>0, b<0$. If there exists some curve C' representing the class $A_{a,0}$ with $a>0$, we get $[C]\cdot[C']<0$, a contradiction. The last statement of the lemma is clearly analogous to what we have proved. □

Lemma 5.13. *Let J be an $\omega_{\alpha,\beta}$-compatible almost complex structure on $S^2\times S^2$. Let $A=[S^2\times\mathrm{pt}]$ and $B=[\mathrm{pt}\times S^2]$. Then, through each point in X there is at least one stable curve representing the class A and at least one stable curve representing the class B.*

Proof. It is enough to prove that the genus 0 Gromov-Witten invariants with 1 marked point of $(S^2\times S^2,\omega_{\alpha,\beta})$ are nonzero. In fact, we will show

$$GW^X_{0,1}(A,J)=GW^X_{0,1}(B,J)=1.$$

The Gromov-Witten invariants can be computed by using any regular ω-compatible almost complex structure J. Consider the product almost complex structure $J_0=j\oplus j$ on

$$(S^2\times S^2,\omega_{\alpha,\beta}),$$

where j is the usual complex structure on $\mathbb{C}\mathrm{P}^1\simeq S^2$. J_0 is shown to be regular in [40, Example 3.3.6]. However, it is clear that there are exactly two J_0-holomorphic spheres through each point $(p,q)\in S^2\times S^2$, namely $S^2\times q$ and $p\times S^2$. Hence, for all regular J', there are J'-holomorphic stable curves C^A and C^B representing A and B respectively. Picking a sequence J_n converging to J, and J_n-holomorphic stable curves C_n^A and C_n^B, by the Gromov compactness theorem these curves converge to J-holomorphic stable curves representing A and B, thus proving the lemma. □

In all the following, by a fibration of manifolds we mean a locally trivial fibration in the category of smooth manifolds.

In order to state the next proposition, we introduce some terminology. Let $\pi : E \to X$ be a fibration, and let $\phi : E \to E$ be a diffeomorphism. We say that ϕ preserves the fibration π if ϕ sends fibers of π to fibers of π. Suppose now there are two different fibrations $\pi_i : E \to X$, for $i = 1, 2$. We say that a diffeomorphism $\phi : E \to E$ interchanges the two fibrations π_1 and π_2 if ϕ sends each fiber of π_1 to a fiber of π_2 and viceversa. Let J be an almost complex structure on E. We say that a fibration $\pi : E \to X$ is by J-holomorphic spheres if every fiber $\pi^{-1}(p)$ is a J-holomorphic sphere (in particular, π is a fibration with fiber S^2).

The following proposition is essentially an adaptation of [49, Proposition 2.2] to the case of fibrations over $S^2 \times S^2$. However, here there are additional difficulties due to the extra symmetry of $S^2 \times S^2$ given by swapping the two factors, which is not present in $T^2 \times S^2$.

Proposition 5.14. *Let G be a finite group acting effectively and symplectically on $(S^2 \times S^2, \omega_{\alpha,\beta})$. Let J be a G-invariant and $\omega_{\alpha,\beta}$-compatible almost complex structure on $S^2 \times S^2$.*

Then, there exists a fibration $\pi : S^2 \times S^2 \to S^2$ by J-holomorphic spheres, where the homology class represented by the fiber is either $[S^2 \times \mathrm{pt}]$ or $[\mathrm{pt} \times S^2]$.

Proof. Let $A = [S^2 \times \mathrm{pt}]$ and $B = [\mathrm{pt} \times S^2]$, and take any ω-compatible almost complex structure J on X. We claim that at least one of the moduli spaces $\mathcal{M}(A, J)$ and $\mathcal{M}(B, J)$ is non-empty. We also claim that if $\mathcal{M}(A, J)$ (resp. $\mathcal{M}(B, J)$) is non-empty, then $\mathcal{M}(A, J)/G$ (resp. $\mathcal{M}(B, J)/G$) is a closed smooth oriented manifold of dimension 2, where $G = PSL(2, \mathbb{C})$ is the group of Möbius transformations.

By Lemma 5.13, there is at least one stable curve representing the homology class A. Let C be any such stable curve. We can decompose A as a sum of non-trivial homology classes is by writing $A = A_1 + \cdots + A_n$, such that there are J-holomorphic curves C_i with $[C_i] = A_i$ for all $i = 1, \ldots, n$. Let

$$A_i = a_i[S^2 \times \mathrm{pt}] + b_i[\mathrm{pt} \times S^2]$$

for some integers a_i, b_i.

Observe that if there is some index i for which $a_i \geq 0$ and $b_i \geq 0$, then necessarily $A_i = A$, and in particular $n = 1$ in the above decomposition. Indeed, if for some i we have $a_i, b_i \geq 0$, then the class

$$\sum_{j \neq i} A_j = (1 - a_i)[S^2 \times \mathrm{pt}] - b_i[\mathrm{pt} \times S^2]$$

can be represented by a stable curve. Then,

$$\langle [\omega_{\alpha,\beta}], \sum_{j\neq i} A_j \rangle = \alpha(1-a_i) - \beta b_i.$$

This quantity is negative unless $a_i = 1$ and $b_i = 0$. By the positivity of energy of J-holomorphic curves, we must then have $A_i = A$. Hence, either $\mathcal{M}(A,J)$ is non-empty, or for all i we have $a_i < 0$ or $b_i < 0$.

Suppose that $\mathcal{M}(A,J)$ is non-empty. In this case, by positivity of interections, there cannot be any A_i with $b_i < 0$. This implies that any stable curve representing A is in fact a simple J-holomorphic curve, since all its components must represent a homology class $a_i[S^2 \times \mathrm{pt}] + b_i[\mathrm{pt} \times S^2]$ with $a_i > 0$, so the stable curve can have only one component. In particular, we obtain that $\mathcal{M}(A,J)/G = \overline{\mathcal{M}}_{0,0}(A,J)$ is compact, by the Gromov compactness theorem. Each J-holomorphic sphere in class A is embedded, by the adjunction formula (see [40, Theorem 2.6.4]), and

$$\langle c_1(A), TX \rangle = 2 \geq 0.$$

The automatic transversality theorem in dimension 4 (see [40, Corollary 3.3.4]), implies then that each J-holomorphic curve in the class A is regular. Hence, $\mathcal{M}(A,J)$ is a smooth manifold of dimension 6. Moreover, the group $G := PSL(2,\mathbb{C})$ acts freely on $\mathcal{M}(A,J)$, since all curves are embedded. Then, $\mathcal{M}(A,J)/G$ is a closed smooth oriented manifold of dimension 2.

Suppose now that $\mathcal{M}(A,J) = \emptyset$, so that we have a collection of J-holomorphic curves representing homology classes $a_i[S^2 \times \mathrm{pt}] + b_i[\mathrm{pt} \times S^2]$ such that

$$A = \sum_{i=1}^{n} (a_i[S^2 \times \mathrm{pt}] + b_i[\mathrm{pt} \times S^2])$$

with $a_i < 0$ or $b_i < 0$ for all i, and $n \geq 2$. Clearly, there must be at least one summand with $a_i > 0$, and hence also $b_i < 0$. By Lemma 5.12, there is exactly one (unparametrized) J-holomorphic curve representing a homology class $a[S^2 \times \mathrm{pt}] + b[\mathrm{pt} \times S^2]$ with $a > 0, b < 0$.

Repeating the same argument with B in the place of A, we get that either $\mathcal{M}(B,J)$ is non-empty and $\mathcal{M}(B,J)/G$ is a closed smooth oriented manifold of dimension 2, or there is exactly one (unparametrized) J-holomorphic curve representing a homology class $a[S^2 \times \mathrm{pt}] + b[\mathrm{pt} \times S^2]$ with $a < 0, b > 0$.

Hence, it only remains to discard the case where there is exactly one J-holomorphic curve C_1 representing a homology class $a[S^2 \times \mathrm{pt}] + b[\mathrm{pt} \times S^2]$ with $a < 0, b > 0$, and exactly one J-holomorphic curve C_2 representing a homology class $c[S^2 \times \mathrm{pt}] + d[\mathrm{pt} \times S^2]$ with $c > 0, d < 0$.

But if this were the case, then we should have that any stable curve representing the homology class A has as components the curves C_1 and C_2 (maybe with multiplicity), contradicting Lemma 5.13. This finishes the proof of the claim.

We assume that $\mathcal{M}(A,J)$ is non-empty. Otherwise, we change A by B in all the arguments that follow. The evaluation map $ev : \mathcal{M}(A,J) \times_G S^2 \to X$ is an orientation-preserving diffeomorphism, since it is a map degree 1. The projection

$$f : \mathcal{M}(A,J) \times_G S^2 \to \mathcal{M}(A,J)/G$$

is a fibration. Using multiplicativity of Euler characteristic we see that $\chi(\mathcal{M}(A,J)/G) = 2$, so we have a fibration over S^2 with fibers diffeomorphic to S^2. Since there are only two orientable fibrations over S^2 with non-diffeomorphic total spaces, f must be the trivial fibration.

Choosing a G-invariant almost complex structure J, we obtain our desired fibration by G-invariant J-holomorphic spheres. □

Lemma 5.15. *Let G be a finite group acting effectively and symplectically on $(S^2 \times S^2, \omega_{\alpha,\beta})$, and let J be an $\omega_{\alpha,\beta}$-compatible almost complex structure on $S^2 \times S^2$ which is invariant by G. Then, exactly one of the following holds true.*

1. *There is an embedded J-holomorphic curve of negative self-intersection,*
2. *There exist two fibrations $\pi_i : S^2 \times S^2 \to S^2$, $i = 1, 2$, by J-holomorphic spheres, with fibers representing $[S^2 \times \mathrm{pt}]$ and $[\mathrm{pt} \times S^2]$ respectively, and such that every $g \in G$ either preserves both fibrations or interchanges them. Moreover, if $\alpha \neq \beta$, every element of G preserves both fibrations.*

Proof. By Proposition 5.14, there is a fibration $\pi_1 : S^2 \times S^2 \to S^2$ by J-holomorphic spheres, such that the fiber represents either $A = [S^2 \times \mathrm{pt}]$ or $B = [\mathrm{pt} \times S^2]$. Suppose that the class of the fiber is B. This will not affect the arguments that follow. Then, $\mathcal{M}(B,J) \neq \emptyset$. For every integer k let

$$A_k = [S^2 \times \mathrm{pt}] - k[\mathrm{pt} \times S^2] \in H_2(S^2 \times S^2).$$

We claim that there is exactly one integer $k \geq 0$ such that the moduli space $\mathcal{M}(A_k, J)$ is not empty. By Lemma 5.13, there exists a stable curve C representing $[S^2 \times \mathrm{pt}]$. Let $C_1, \ldots, C_n$ be the components of C, so that each C_i is a J-holomorphic curve. Then,

$$[C] = \sum_{i=1}^{n} [C_i].$$

Since there are no J-holomorphic curves representing a homology class of the form $-mB$ for any $m > 0$ (by the condition of positive energy of J-holomorphic curves), we conclude from Lemma 5.12 (after some rearranging) that

$$[C_1] = [S^2 \times \mathrm{pt}] - k[\mathrm{pt} \times S^2],$$

for some $k \geq 0$, while $[C_i] = k_i[\mathrm{pt} \times S^2]$ for some $k_i \geq 0$, for $i > 1$. The existence of C_1 shows that $\mathcal{M}(A_k, J) \neq \emptyset$. Moreover, if $k' \geq 0$ is such that $k' \neq k$, Lemma 5.12 implies that $\mathcal{M}_{0,0}(A_{k'}, J) = \emptyset$. This proves the claim.

We now prove the proposition. Suppose first that $\mathcal{M}(A_k, J) \neq \emptyset$ for some $k > 0$, and let S be a J-holomorphic sphere representing A_k. The adjunction formula [40, Theorem 2.6.4] implies that S is embedded. Moreover, $S \cdot S = -2k < 0$. Therefore, we are in the first case of the statement.

Assume now that all $\mathcal{M}(A_k, J)$ are empty for $k > 0$. Then, $\mathcal{M}(A, J) \neq \emptyset$. The proof of Proposition 5.14 then implies that there is a fibration $\pi_2 : S^2 \times S^2 \to S^2$ by J-holomorphic spheres whose fibers represent A. If $\alpha \neq \beta$, by Lemma 5.4 every element of G induces the identity on the homology. Therefore the image by G of a J-holomorphic curve representing A is another J-holomorphic curve representing A, and the fibrations π_1 and π_2 are preserved by the action of G. If $\alpha = \beta$, just as above any element of G inducing the identity on homology preserves the fibration. However, it may happen that G has elements whose induced map in homology interchange the classes A and B. In this situation, the action of an element of G either preserves the two fibrations (if it induces the identity on homology) or interchanges them (if not).

This finishes the proof of the lemma. □

Lemma 5.16. *Let G be a finite group acting effectively and symplectically on $(S^2 \times S^2, \omega_{\alpha,\beta})$, and let J be an $\omega_{\alpha,\beta}$-compatible almost complex structure on $S^2 \times S^2$ which is invariant by G. If there is an embedded J-holomorphic sphere S with negative self-intersection, then G is isomorphic to $Z \times H$, where Z is a finite cyclic group and H is a finite subgroup of* $\mathrm{SO}(3)$*.*

Proof. We claim that S is preserved by G. Let $g \in G$. Since g is a symplectomorphism of $S^2 \times S^2$, Lemma 5.4 implies that $[g(S)] = [S]$, so if $g(S) \neq S$, $S \cdot g(S) = S \cdot S < 0$, which is impossible by positivity of intersections.

Therefore, by Proposition 1.7, we obtain an effective action of G on the complex line bundle $N \to S$ by line bundle automorphisms. By Lemma 5.9, G acts effectively on a line bundle of degree 0 over S^2 by bundle automorphisms. Now the result follows from Lemma 5.7. □

We are now ready to prove the main result of this section.

Theorem 5.17. *The finite groups that act effectively and symplectically on $(S^2 \times S^2, \omega_{\alpha,\beta})$ are:*

1. *If $\alpha \neq \beta$, the groups that are isomorphic to a subgroup of $H_1 \times H_2$, for some finite subgroups H_1, H_2 of* $\mathrm{SO}(3)$*.*

2. *If $\alpha = \beta$, the groups that are isomorphic to a subgroup of $G_1 \times G_2$, for some finite groups G_1, G_2 of* $\mathrm{SO}(3)$*, or groups G lying on an exact sequence*

$$1 \to H \times H \to G \to \mathbb{Z}/2 \to 1,$$

for some finite subgroup H of $\mathrm{SO}(3)$*, and where the action by conjugation of a lift $g \in G$ of the non-trival element of $\mathbb{Z}/2$ on $H \times H$ is given by*

$$g(h_1, h_2)g^{-1} = (\phi_1 h_2 \phi_1^{-1}, \phi_2 h_1 \phi_2^{-1}),$$

for some $\phi_1, \phi_2 \in SO(3)$ *such that* $\phi_1\phi_2 \in H$ *and* $\phi_2\phi_1 \in H$*.*

Proof. That the claimed groups act effectively and symplectically on $(S^2 \times S^2, \omega_{\alpha,\beta})$ was proved in Lemma 5.11. We prove that these are the only groups acting effectively and symplectically on $(S^2 \times S^2, \omega_{\alpha,\beta})$. Let G be a finite group acting symplectically and effectively on $(S^2 \times S^2, \omega_{\alpha,\beta})$. Let J be an ω-compatible almost complex structure which is invariant by G. The existence of such a J is proved, for example, in [39, Lemma 5.5.6].

If there is a G-invariant embedded J-holomorphic sphere with negative self-intersection, we obtain the result by Lemma 5.16.

Therefore, it sufficies now to consider the case where there are no G-invariant embedded J-holomorphic spheres with negative self-intersection. In this case, Proposition 5.14 implies that there are two fibrations $\pi_i : S^2 \times S^2 \to S^2$, $i = 1, 2$, such that every element of G either preserves or interchanges them. The fibers of these two fibrations represent the homology classes $[S^2 \times \mathrm{pt}]$ and $[\mathrm{pt} \times S^2]$ respectively, and each fiber of each of the two fibrations is a J-holomorphic sphere. Positivity of intersections then implies that the fibers $\pi_1^{-1}(p)$ and $\pi_2^{-1}(q)$ for any two points $p, q \in S^2$ intersect transversely in exactly one point of $S^2 \times S^2$, while $\pi_i^{-1}(p)$ and $\pi_i^{-1}(q)$ do not intersect for $i = 1, 2$ and $p \neq q \in S^2$. In this way, we obtain a smooth bijection

$$\psi : S^2 \times S^2 \to S^2 \times S^2,$$

such that $\psi(p, q) = (\pi_1(p, q), \pi_2(p, q))$. In fact, ψ is a diffeomorphism. To see that the inverse is smooth, use the local triviality of the fibrations π_i and the fact that the fibers of the two fibrations intersect transversely to see that for all $(p, q) \in S^2 \times S^2$, $d_{(p,q)}\psi$ is an isomorphism, and then use the inverse function theorem. Conjugating the action of G by ψ, we may assume that the two fibrations $\pi_i : S^2 \times S^2 \to S^2$ are the projections onto each factor.

Let $G_0 \leq G$ be the subgroup of G consisting of those elements that preserve both projections. It is easy to see that G_0 is normal in G, so we get an exact sequence:

$$1 \to G_0 \to G \to G/G_0 \to 1.$$

Note that G_0 has index at most 2 in G. Therefore, either $G = G_0$ or $G/G_0 \simeq \mathbb{Z}/2$. The action of every element $g \in G_0$ on $S^2 \times S^2$ is given by:

$$g \cdot (p, q) = (g \cdot_1 p, g \cdot_2 q),$$

where in the right hand side we consider the two (possibly non-effective) induced actions of G_0 on $\pi_i(S^2 \times S^2) = S^2$, $i = 1, 2$. Let $H_i = G_0/\operatorname{Ker}\Phi_i$, where $\Phi_i : G_0 \to \operatorname{Diff}(S^2)$ are the induced actions of G_0 on $\pi_i(S^2 \times S^2)$, $i = 1, 2$. We claim that $G_0 \leq H_1 \times H_2$. Indeed, define a map

$$f : G_0 \to H_1 \times H_2,$$

given by $f(g) = (p_1(g), p_2(g))$, where $p_i : G_0 \to H_i$ are the quotient maps. f is clearly a morphism of groups, and since the action of G_0 on $S^2 \times S^2$ is effective, f is injective.

This proves the theorem in the case $\alpha = \beta$, since Lemma 5.15 implies that in this case we must have $G = G_0$.

Assume now that $G/G_0 \simeq \mathbb{Z}/2$, so G lies in an exact sequence

$$1 \to H_1 \times H_2 \to G \to \mathbb{Z}/2 \to 1,$$

for some finite subgroups H_1, H_2 of SO(3). We claim that in fact $H_1 \simeq H_2$. Pick some $g \in G$ lifting the non-trivial element in $\mathbb{Z}/2$. Since g interchanges the fibrations $S^2 \times S^2 \to S^2$, we can write $g \cdot (p, q) = (\phi_1 q, \phi_2 p)$, for some elements $\phi_1, \phi_2 \in SO(3)$. For any $h_1 \in H_1, h_2 \in H_2$ and any $(p, q) \in S^2$, we have

$$g(h_1, h_2)g^{-1} \cdot (p, q) = (\phi_1 h_2 \phi_1^{-1}, \phi_2 h_1 \phi_2^{-1}) \cdot (p, q).$$

In particular, conjugation with ϕ_2 gives an isomorphism $H_1 \simeq H_2$. By conjugating all elements of G by (ϕ_2, ϕ_2), we may assume $H_1 = H_2$. Noting that $(\phi_1\phi_2, \phi_2\phi_1) = g^2 \in H \times H$, we must have $\phi_1\phi_2 \in H$ and $\phi_2\phi_1 \in H$. Therefore, the proof of the theorem is complete. □

5.5 The non-trivial bundle

Denote by X_S be the total space of the non-trivial orientable S^2-bundle over S^2. In this section, we provide a classification of all finite groups acting symplectically on (X_S, ω), for any symplectic form ω on X_S.

One way to do that is to proceed like in the previous section, proving the existence of a fibration preserved by the group and studying the moduli spaces of J-holomorphic curves in order to find an embedded J-holomorphic sphere. However, we give here an alternative proof that exploits the fact that $X_S \cong (\mathbb{C}\mathrm{P}^2)\#\overline{\mathbb{C}\mathrm{P}^2}$, that is, X_S is a (smooth) blow-up of $\mathbb{C}\mathrm{P}^2$.

We start by recalling a few facts about exceptional spheres in symplectic 4-manifolds. A good reference for this is Section 13.3 of [40].

Definition 5.18. *Let (X, ω) be a closed symplectic 4-manifold.*

1. *An exceptional sphere is a smoothly embedded sphere $S \subset X$ such that $S \cdot S = -1$.*
2. *A symplectic exceptional sphere is a symplectically embedded sphere $S \subset X$ such that $S \cdot S = -1$.*

The relation between these two notions and J-holomorphic curves is given by the following two theorems.

Theorem 5.19. *Let (X, ω) be a closed symplectic 4-manifold. Let E be the homology class of a symplectic exceptional sphere in X. For generic ω-compatible almost complex structure J, there is exactly one J-holomorphic sphere (up to reparametrization) representing the class E.*

Proof. See Section 1.3.1 in [76]. □

Theorem 5.20. *Let (X, ω) be a closed symplectic 4-manifold, and let E be the homology class of an exceptional sphere in X.*

1. *If $\langle c_1(TX), E\rangle = 1$, then there is a symplectic exceptional sphere representing E.*
2. *If $b_2^+(X) > 1$, then $\langle c_1(TX), E\rangle = \pm 1$.*

In particular, if $b_2^+(X) > 1$, either E or $-E$ can be represented by a symplectic exceptional sphere.

Proof. See [40, Corollary 13.3.27]. □

Therefore, in the case $b_2^+(X) > 1$, every exceptional sphere in a closed symplectic manifold is in the same homology class of a J-holomorphic sphere for generic J. However, we are interested in the case $b_2^+(X) = 1$. In this case, we must check the additional condition $\langle c_1(TX), E\rangle = 1$.

Lemma 5.21. *Let ω be any symplectic form on X_S. Then the homology class $[S_-]$ satisfies $\langle c_1(TX_S), [S_-]\rangle = 1$.*

Proof. Note that $\langle c_1(TX_S), [S_-]\rangle = -K \cdot [S_-]$. Therefore it is enough to compute this intersection product. Since K is independent of the ω-compatible almost complex structure J, we can compute it using a suitable J. Choosing a generic J there are embedded J-holomorphic spheres representing the classes $[S_+] - [S_-]$ and $[S_+]$. Using the adjunction formula, we then obtain:

$$-K \cdot ([S_+] - [S_-]) = 2 + ([S_+] - [S_-]) \cdot ([S_+] - [S_-]),$$

$$-K \cdot [S_+] = 2 + [S_+] \cdot [S_+].$$

From here, we conclude that $-K \cdot [S_-] = 1$, finishing the proof of the lemma. □

Lemma 5.22. *Let G be a finite group acting symplectically on (X_S, ω). For any G-invariant and ω-compatible almost complex structure J, there exists a G-invariant embedded J-holomorphic sphere S such that $S \cdot S$ is an odd negative number.*

Proof. By Lalonde-McDuff theorem, conjugating the action of the group by a suitable diffeomorphism, we may assume without loss of generality that $\omega = \omega_{\lambda_+,\lambda_-}$ for some $\lambda_+ > \lambda_- > 0$. Since X_S is the blow-up of $\mathbb{CP}^2$ at one point, by Theorems 5.19 and 5.20, and Lemma 5.21, for a generic ω-compatible almost complex structure J' there exists a J'-holomorphic sphere S representing the homology class of the exceptional divisor, that is, $[S_-]$.

Let J_n be a sequence of generic ω-compatible almost complex structures converging to J, and S_n a sequence of J_n-holomorphic spheres representing $[S_-]$. By Gromov compactness theorem, we obtain the existence of a J-holomorphic stable curve C representing $[S_-]$. Let $C_1, \dots, C_s$ be the components of C. Then, there are positive integers $n_1, \dots n_s$ such that

$$[S_-] = n_1 A_1 + \cdots + n_s A_s,$$

and each $A_i \in H_2(X_S)$ is represented by a simple J-holomorphic curve.

We claim that there is some i such that $A_i \cdot A_i < 0$. Indeed, if $A_i \cdot A_i \geq 0$ for all i, by positivity of intersections we would have

$$-1 = [S_-] \cdot [S_-] = \sum_{i,j} [C_i] \cdot [C_j] \geq 0,$$

a contradiction. Therefore there is some J-holomorphic sphere S with negative self-intersection. An argument analogous to that of the proof of Lemma 5.15 then allows us to conclude that in fact

$$[S] = -k[S_+] + (k+1)[S_-]$$

for some $k > 0$. In particular, $S \cdot S = -2k - 1 < 0$. Finally, the adjunction formula implies that S is embedded. This finishes the proof of the lemma. □

The following lemma is an adaptation of [49, Lemma 3.4].

Lemma 5.23. *Let $\pi : L \to S^2$ be a complex line bundle of degree 1. If G acts effectively on L by line bundle automorphisms, then G acts effectively and symplectically on the twisted bundle X_S for any symplectic form ω.*

Proof. We will use the fact that $X_S \simeq \mathbb{P}(L \oplus \underline{\mathbb{C}})$ (here, $\underline{\mathbb{C}}$ denotes the trivial line bundle over S^2), and also the fact that $X_S \simeq P \times_{S^1} S^2$, where P is the principal S^1-bundle over S^2 with first Chern number 1.

It is easy to show that G acts effectively and smoothly on X_S. Indeed, if we define the action of G on the trivial bundle $\underline{\mathbb{C}}$ by $g \cdot (p, z) = (\pi(g) \cdot p, z)$, we have a canonically defined linear action on the bundle $L \oplus \underline{\mathbb{C}}$, which descends to the projectivization X_S. Observe that this action preserves the fibration $X_S \to S^2$. Moreover, this is equivalent to an effective action of G on the principal S^1-bundle P.

Let us analyze now what symplectic structures on X are compatible with this action. Let A be a G-invariant connection on P. We can obtain such a connection by averaging an arbitrary connection, since the space of all connections on P_d is an affine space. By Chern-Weil theory, its curvature F_A must satisfy $\langle [F_A/2\pi i], [S^2] \rangle = 1$. The 2-form $F_A/2\pi i \in \Omega^2(S^2)$ is closed and G-invariant. Define

$$\omega_S := \frac{F_A}{2\pi i}.$$

Then, ω_S is a symplectic form on S^2 of area 1 which is G-invariant with respect to the action of G induced on the base S^2 of the bundle $P \to S^2$.

Let ω_{FS} be the area form on S^2 associated to the restriction of the euclidean metric of $\mathbb{R}^3$. Observe that ω_{FS} has area 4π. Its associated moment map is the map:

$$\begin{aligned} \mu_{FS} : S^2 &\to i\,\mathbb{R} \\ (x, y, z) &\mapsto iz \end{aligned}$$

We define a 2-form $\widetilde{\omega}_0 \in \Lambda^2(X_S)$ as follows. The connection A induces for every $p \in X_S$ an splitting $T_pX_S = V_p \oplus H_p$, where V_p is the vertical space of the fibration, $V_p = \operatorname{Ker} \pi$. Let $\Pi : T_pX_S \to V_p$ be the linear projection with kernel H_p. Then, we define

$$\widetilde{\omega}_0(v, w) = \omega_{FS}(\Pi(v), \Pi(w)),$$

for every v, w vector fields on X_S. Observe that $\widetilde{\omega}_0$ is degenerate and, in general, it is not closed. Define

$$\omega_0 = \widetilde{\omega}_0 + \mu_{FS}\pi^* F_A.$$

This form is G-invariant, because A is, and it is closed, but may be degenerate. In order to obtain a symplectic form, pick some real number δ and define

$$\omega_\delta = \omega_0 + \delta\pi^*\omega_S.$$

We claim that ω_δ is a G-invariant symplectic form on P for every $\delta > 2\pi$. In particular, G acts symplectically on (X_S, ω_δ) for every $\delta > 2\pi$. ω_δ is clearly closed and its G-invariance follows from the invariance of ω_0 and of ω_S. Therefore, it only remains to see that it is also non-degenerate. Since the vertical and horizontal distributions on TX_S are ω_δ-orthogonal, it suffices to prove that it is non-degenerate restricted to these distributions. For the vertical distributions this is clear since its restriction coincides with ω_{FS}. For the horizontal distribution, the restriction of ω_δ is the 2-form $(\mu_{FS}2\pi i + \delta)\pi^*\omega_S$, so it will be non-degenerate if $\mu_{FS}2\pi i + \delta > 0$. Since $\mu_{FS}(S^2) = i[-1, 1]$, this will be true as long as $\delta > 2\pi$, as claimed.

Let us compute the cohomology class defined by ω_δ. Recall that S_+, S_- are the submanifolds of X_S defined by

$$S_\pm = P \times_{S^1} \{(0, 0, \pm 1)\},$$

and that the homology of $P \times_{S^1} S^2$ is generated by $[S_+]$ and $[S_-]$. Since $\mu_{FS}(0, 0, 1) + \mu_{FS}(0, 0, -1) = 0$,

$$\langle [\omega_0], [S_+] + [S_-] \rangle = 0.$$

This implies that $[\omega_0]$ is a multiple of $[\omega_+] - [\omega_-]$. Since the area of a fiber of $P \times_{S^1} S^2 \to S^2$ is 4π, and the fiber represents the homology class $[S_+] - [S_-]$ we must have

$$[\omega_0] = 2\pi([\omega_+] - [\omega_-]).$$

On the other hand, since

$$\langle [\pi^*\omega_S], [S_\pm] \rangle = \langle [\omega_S], [S^2] \rangle = 1,$$

we have $[\pi^*\omega_S] = [\omega_+] + [\omega_-]$. Putting all together, we obtain

$$[\omega_\delta] = (\delta + 2\pi)[\omega_+] + (\delta - 2\pi)[\omega_-].$$

By Lalonde-McDuff's theorem, for any pair of symplectic forms ω_1, ω_2 on X_S representing the same cohomology class there is a diffeomorphism

$$\psi : X_S \to X_S$$

such that $\psi^*(\omega_1) = \omega_2$. Since clearly $\text{Symp}(X_S, \omega_\delta) \simeq \text{Symp}(X_S, \lambda\omega_\delta)$ for $\lambda > 0$, it is enough to show that any symplectic form represents the same cohomology class than a multiple of ω_δ, for some $\delta > 2\pi$. Since any symplectic form on X_S represents a cohomology class of the form $\lambda_+[\omega_+] + \lambda_-[\omega_-]$ for some $\lambda_+ > \lambda_- > 0$, this follows from our computation of $[\omega_\delta]$. This finishes the proof of the lemma. □

Theorem 5.24. *Let ω be any symplectic form on X_S. Then, G acts effectively and symplectically on (X_S, ω) if and only if G is isomorphic to a finite subgroup of $U(2)$.*

Proof. Observe that by Lemma 5.8, it suffices to prove that G acts effectively and symplectically on (X_S, ω) if and only if G acts effectively on a line bundle $L \to S^2$ of degree 1 by vector bundle automorphisms.

If G acts effectively on $L \to S^2$ by bundle automorphisms, we can apply Lemma 5.23 and obtain a symplectic and effective action of G on (X_S, ω).

Conversely, assume that there is an effective and symplectic action of G on (X_S, ω). Let J be a G-invariant ω-compatible almost complex structure on X_S. By Lemma 5.22, there is a J-holomorphic sphere S with negative and odd self-intersection number that is invariant by the action of G. By Proposition 1.7, we obtain an effective action of G on the normal bundle $N \to S^2$ by bundle automorphisms. Since S is J-holomorphic, J induces a complex structure on the normal bundle $N \to S$. Since $\deg N = [S] \cdot [S]$ is odd, applying Lemma 5.9, we obtain an effective action of G on a bundle $L \to S^2$ of degree 1 by bundle automorphisms. This completes the proof. □

The classification of finite subgroups of $U(2)$ is known. See page 57 of [16].

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