

Triangle singularities in $B^- \rightarrow K^- \pi^- D_{s0}^+$ and $B^- \rightarrow K^- \pi^- D_{s1}^+$

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We study the appearance of structures in the decay of the B^- into $K^- \pi^- D_{s0}^+$ (2317) and $K^- \pi^- D_{s1}^+$ (2460) final states by forming invariant mass distributions of $\pi^- D_{s0}^+$ and $\pi^- D_{s1}^+$ pairs, respectively. The structure in the distribution is associated to the kinematical triangle singularity that appears when the $B^- \rightarrow K^- K^{*0} D^0$ ($B^- \rightarrow K^- K^{*0} D^{*0}$) decay process is followed by the decay of the K^{*0} into $\pi^- K^+$ and the subsequent rescattering of the $K^+ D^0$ ($K^+ D^{*0}$) pair forming the D_{s0}^+ (2317) (D_{s1}^+ (2460)) resonance. We find this type of non-resonant peaks at 2850 MeV in the invariant mass of $\pi^- D_{s0}^+$ pairs from $B^- \rightarrow K^- \pi^- D_{s0}^+$ (2317) decays and around 3000 MeV in the invariant mass of $\pi^- D_{s1}^+$ pairs from $B^- \rightarrow K^- \pi^- D_{s1}^+$ (2460) decays. By employing the measured branching ratios of the $B^- \rightarrow K^- K^{*0} D^0$ and $B^- \rightarrow K^- K^{*0} D^{*0}$ decays, we predict the branching ratios for the processes $B^- \rightarrow K^- \pi^- D_{s0}^+$ (2317) and $B^- \rightarrow K^- \pi^- D_{s1}^+$ (2460), in the vicinity of the triangle singularity peak, to be about 8×10^{-6} and 1×10^{-6} , respectively. The observation of this reaction would also give extra support to the molecular picture of the D_{s0}^+ (2317) and D_{s1}^+ (2460).

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I. INTRODUCTION

Understanding the observed spectrum of hadron resonances [1] and establishing a connection with the underlying theory of the strong interaction, Quantum Chromodynamics (QCD), is one of the prime goals of the hadron physics community. In conventional quark models, baryons are composed by three quarks, while mesons contain a quark and an antiquark. However, even if these models provide a rather successful description of a wealth of data for meson and baryon resonances [2–4], other more exotic components cannot be ruled out, especially considering that the degrees of freedom of the QCD lagrangian contain not only quarks but also gluons, which interact among each other as a consequence of the non-abelian character of QCD. For many years, the intriguing possibility of finding evidence for these exotic components in the mesonic and baryonic spectrum has motivated a large amount of theoretical and experimental activity, aiming at obtaining a quantitative understanding of the confinement of quarks and gluons in QCD (for recent reviews on this subject, see Refs. [5–9]).

The advent of copious observation of hadron resonances, emerging as peaks in the invariant mass distribution of selected hadrons from the decay of a heavy particle, requires a careful analysis of data, as some of the resonance-like structures could be associated to, or affected by, a triangle singularity of a Feynman diagram contributing to the process. Early introduced by Landau [10], this kinematic singularity occurs if the diagram involves three intermediate-state particles which can be placed simultaneously on-shell with their momenta being collinear in the frame of reference of the decaying particle. These especial conditions permit to fuse two of the loop particles into an external outgoing one, hence allowing for an interpretation of the Feynman diagram as a classical process, which is the essence of the Coleman-Norton theorem [11] and corresponds to placing the singularity on the physical boundary producing an observable effect. It is worth pointing out that a triangle singularity is favored when the two loop hadrons fuse into a hadron molecule mainly composed of these hadrons, as molecules are generally formed close to threshold, hence favoring the fulfillment of the on-shell conditions.

A clear manifestation of a triangle singularity is found in the decays of the $\eta(1405)$. It has been shown that the unusually large ratio of rates between the isospin-violating decay, $\eta(1405) \rightarrow \pi^0 f_0(980) \rightarrow \pi^0, \pi^+ \pi^-$, and the process $\eta(1405) \rightarrow \pi^0 a_0(980) \rightarrow \pi^0, \pi^0 \eta$ found by BESIII [12] can be naturally explained in terms of a triangle singularity [13]. The isospin breaking rate is enhanced by the sequence of processes involved in the triangle diagram, namely the $\eta(1405)$ decaying into a $K^* \bar{K}$ pair, followed by the decay of the K^* into $K \pi$ and the subsequent fusion of the K with the \bar{K} to form the $f_0(980)$. This explanation was corroborated in Ref. [14], where the absolute value for the

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ratio $\Gamma(\eta(1405) \rightarrow \pi^0, \pi^+\pi^-)/\Gamma(\eta(1405) \rightarrow \pi^0, \pi^0\eta)$ was obtained, and it was found, together with the line shapes of the two reactions, to be in good agreement with experiment. One finds a similar example in the signals reported by COMPASS, from the scattering of high energy pions off protons [15]. A peak in the distribution, which could be easily interpreted as the sign of an axial-vector $a_1(1420)$, can naturally receive an explanation in terms of a triangle singularity involving a virtual $a_1(1260)$ which decays into $K^*\bar{K}$, followed by the decay of the K^* into $K\pi$, where the π is emitted and the K merges with the \bar{K} forming the $f_0(980)$. This was noted by the authors of Ref. [16] and explicitly evaluated in [17], finding a good description of the experimental facts observed in [15]. A confirmation of this mechanism was offered by the work of Ref. [18], where the interference between the $K^*K\bar{K}$ and $\rho\pi\pi$ loops was also analyzed.

In the heavy flavor sector, some of the claimed exotic hadrons are also being analyzed on the basis of triangle singularity effects. The possibility that the narrow peak associated to the $P_c(4450)$ pentaquark by the LHCb collaboration [19] could be associated to a triangle singularity was explored in [20–22]. A thorough study of possible charmonium- Λ^* pairs contributing to this signal was carried out in Ref. [23], finding the $\chi_{c1}\Lambda(1890)$ pair as the best candidate if the pentaquark had $J^P = 1/2^+$ or $3/2^+$, but not being able to reproduce the experimental features if the quantum numbers are $J^P = 3/2^-$ or $5/2^+$, as preferred by the experiment for the narrow peak observed. The existence of some exotic meson states, such as the charged charmonium $Z_c(3900)$ [24–27], has also been challenged in favor of a triangle singularity explanation [16, 28, 29], as well as for other quarkonium [30] and bottomonium [31] states. However, in general, no conclusion can be drawn on whether the signal should be associated to a triangular singularity rather than to a resonance-pole, or even on quantifying possible interferences among these two mechanisms, unless the strength of the triangle diagram can be determined in terms of experimentally known vertices. Attempts to parametrize amplitudes in terms of triangular singularity and other elements, as done in Ref. [32], and conducting fits to high quality data, could help in the future.

It is also useful to predict the location of triangular singularity signals in processes that may easily be produced in present experimental facilities, in order to anticipate their nature. A recent example is the decay $B_c \rightarrow B_s\pi\pi$, which has been investigated via the $\bar{K}^*B\bar{K}$ loop in Ref. [33] and has produced a $X(5777)$ structure in the invariant mass distribution of $B_s^0\pi^+$ states. This process perfectly fulfills the triangle singularity condition and it has the advantage that the interaction of the $B\bar{K}$ in the loop, transforming into the final $B_s^0\pi^+$, is too weak to produce a resonance, hence the structure observed would be of pure kinematical origin.

In the present work, we also study the appearance of pure kinematic structures in the decay of the B^- into $K^-\pi^-D_{s0}^+(2317)$ and $K^-\pi^-D_{s1}^+(2460)$ final states by forming invariant mass distributions of $\pi^-D_{s0}^+$ and $\pi^-D_{s1}^+$ pairs, respectively. The $D_{s0}^+(2317)$ was first observed in the BaBar collaboration as a narrow peak in the $D_s\pi$ invariant mass distribution [34]. The state was confirmed by CLEO [35] and Belle [36]. Nowadays is already well established in the PDG [1] with quantum numbers $I(J^P) = 0(0^+)$. The $D_{s1}^+(2460)$ was also observed in the CLEO experiment [35] in the $D_s^*\pi$ channel and BaBar also found a signal in that region. Nowadays it is also well established in the PDG [1] with quantum numbers $0(1^+)$. On the theoretical side, there has been much work claiming that these states are of molecular nature, mostly DK for the $D_{s0}^+(2317)$ and D^*K for the $D_{s1}^+(2460)$ [40–46]. Support for this picture in several reactions has also been discussed in Refs. [47–50]. Lattice QCD calculations have also found indications for this interpretation [51, 52], and a reanalysis of these latter results [53] even provides the amount of DK and D^*K component in the wave function of the $D_{s0}^+(2317)$ and $D_{s1}^+(2460)$, respectively. The $B^- \rightarrow K^-\pi^-D_{s0}^+(2317)$ and $B^- \rightarrow K^-\pi^-D_{s1}^+(2460)$ processes occur via K^*KD or K^*KD^* intermediate loops, which meet the condition of the triangle singularity when the KD or KD^* fuse into the $D_{s0}^+(2317)$ or $D_{s1}^+(2460)$, respectively. We find a peak at 2850 MeV in the invariant mass of $\pi^-D_{s0}^+$ pairs from the $B^- \rightarrow K^-\pi^-D_{s0}^+(2317)$ decay and around 3000 MeV in the invariant mass of $\pi^-D_{s1}^+$ pairs from the $B^- \rightarrow K^-\pi^-D_{s1}^+(2460)$ process. We also quantify the strength under these peaks by explicitly evaluating the triangle singularity diagrams corresponding to these processes and referring them to the decays $B^- \rightarrow K^-K^{*0}D^0$ and $B^- \rightarrow K^-K^{*0}D^{*0}$, whose branching ratios are reported in the PDG [1]. We predict the branching ratios for the processes B^- into $K^-\pi^-D_{s0}^+(2317)$ and $K^-\pi^-D_{s1}^+(2460)$ in the vicinity of the triangle singularity peak to be about 8×10^{-6} and 1×10^{-6} , respectively.

II. FORMALISM

Let us first start with the $B^- \rightarrow K^-K^{*0}D^0$ decay process, which can be visualized in a quark representation in Fig. 1. The process is Cabibbo favored with the topology of external emission [37, 38] and, hence, also favored by the color counting. In the W^- external emission, a $\bar{u}d$ pair is formed, which is hadronized with a $\bar{q}q$ pair with vacuum quantum numbers, and through the $\bar{s}s$ component, gives rise to K^- and a K^{*0} . The $c\bar{u}$ final state in Fig. 1 gives rise to the D^0 , thus completing the final state. The branching ratio for the $B^- \rightarrow K^-K^{*0}D^0$ decay is given in the PDG [1] as a $\text{BR}(B^- \rightarrow K^-K^{*0}D^0) = (7.5 \pm 1.7) \times 10^{-4}$ measured by Belle in Ref. [39]. Similarly, we can also have, with the same topology as in Fig. 1, the $B^- \rightarrow K^-K^{*0}D^{*0}$ decay, which is reported in the PDG with a branching

ratio of $\text{BR}(B^- \rightarrow K^- K^{*0} D^0) = (1.5 \pm 0.4) \times 10^{-3}$, also measured in Ref. [39].

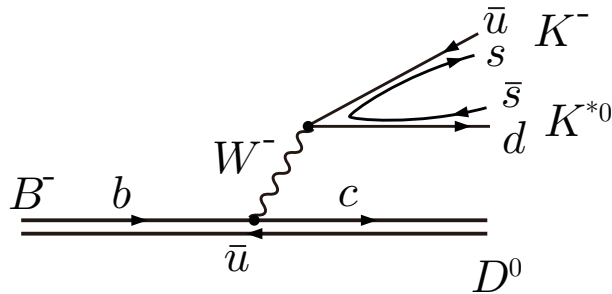


FIG. 1: Quark picture for the $B^- \rightarrow K^- K^{*0} D^0$ decay.

A. $B^- \rightarrow K^- \pi^- D_{s0}^+(2317)$ decay

From the process $B^- \rightarrow K^- K^{*0} D^0$ one can form the triangle mechanism depicted in Fig. 2, which was found in Ref. [16] to develop a singularity. However, the energy at which the singularity appears depends on the invariant mass

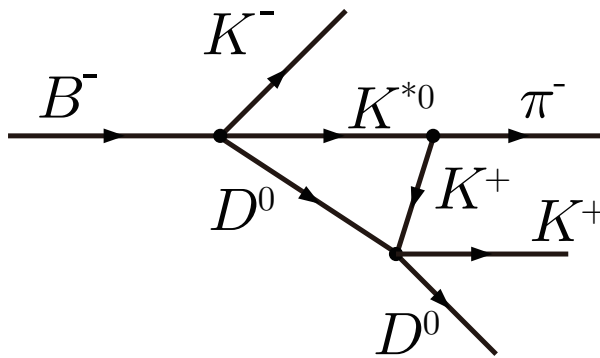


FIG. 2: Triangle diagram stemming from $B^- \rightarrow K^- K^{*0} D^0$, with $K^{*0} \rightarrow \pi^- K^+$ and $D^0 K^+$ rescattering.

of the $D^0 K^+$ pair, and is therefore diluted over the phase space of $D^0 K^+$. Hence, in order to show a neat singular peak, it is preferable to concentrate the $D^0 K^+$ pair strength in one resonance. For this purpose, the $D_{s0}^+(2317)$ is the obvious choice, since this resonance is to a good approximation a DK molecule in $I = 0$ [42–44, 46, 53] and hence it couples strongly to DK states. Thus, we choose to study the process represented in Fig. 3.

In other cases of triangle singularities [16, 20], one starts with the decay of a particle into three final ones, and then one varies the invariant mass of two final particles, as e.g. those emitted in the lower vertex of the triangle, to see for which invariant mass of this pair the singularity appears. However, if this strategy is applied with four particles in the final state, as in Fig. 2, the effect of the triangle singularity would be concentrated in a small region of the whole phase space. Therefore, by producing a resonance from the DK pair, one is effectively rendering the problem into one of three particles in the final state, with a smaller phase space, for which the singularity has a larger weight.

The evaluation of the diagram in Fig. 3 requires first to provide an expression for the $B^- \rightarrow K^- K^{*0} D^0$ vertex. Because a P -wave coupling is needed in this process to conserve angular momentum, we can write the corresponding amplitude as follows:

$$-i t_{B^-, K^- K^{*0} D^0} = -i C \vec{\epsilon}_{K^{*0}} \vec{p}_{K^-}, \quad (1)$$

where \vec{p}_{K^-} is the K^- momentum in a suitable frame and $\vec{\epsilon}_{K^{*0}}$ is the polarization vector of the K^{*0} . The choice of this vertex, involving the \vec{p}_{K^-} momentum, requires a justification. Let us note that the two vertices involved in the exchange of the W^- in Fig. 1 lead to an operator of the type $\gamma^\mu (1 - \gamma_5) \gamma_\mu (1 - \gamma_5)$ at the quark level, neglecting the term inversely proportional to the large mass squared of the W^- propagator. The big contributions of this term will

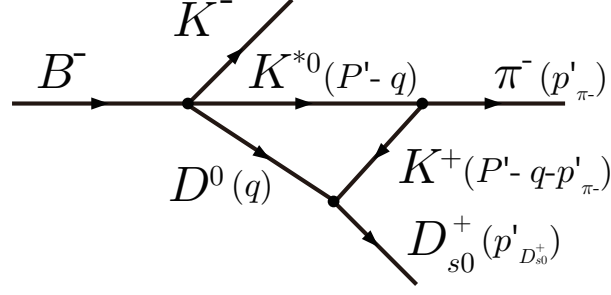


FIG. 3: Triangle diagram for the $B^- \rightarrow K^- \pi^- D_{s0}^+$ process. In parenthesis the momenta of the particles.

be associated to the operators γ^0 and $\gamma^i \gamma_5$, which in the nonrelativistic limit correspond to 1 and σ^i , respectively. The term proportional to $\sigma^i \sigma^i$ will not contribute because, by virtue of the Wigner-Eckart theorem, the expectation value of the σ^i operator between the two 0^- states, corresponding to B^- and D^0 at the lower vertex, vanishes. Moreover, we recall that the $\bar{q}q$ pair at the other vertex, involved in the hadronization process, has spin $S = 1$ and $L = 1$, according to the 3P_0 model [54, 55]. The $S = 1$ carried by the $\bar{q}q$ pair at the quark level allows one to produce the vector meson and, therefore, an operator of the type $\epsilon_{K^* \mu} (p_K - p_{K^*})^\mu$ is expected at the hadronic level in the rest frame of the $K^- K^{*0}$ pair, where $(p_K - p_{K^*})^\mu$ stands for the relative four-momentum, as required by the $L = 1$ of the $\bar{q}q$ pair initiating the hadronization process. The Lorenz condition $\epsilon_{K^* \mu} p_{K^*}^\mu = 0$ allows one to finally write the vertex as $\epsilon_{K^* \mu} p_{K^*}^\mu$, which is the covariant form of Eq. (1) that will also be used later on.

The position of the triangle singularity is of kinematical origin and should not depend strongly on the specific details of the weak decay vertex employed. However, in sect. IV we will also explore how the results are modified if the decay vertex assumes the process $B^- \rightarrow K^- K^{*0} D^{(*)0}$ to be dominated by the subthreshold $a_1(1260)$ resonance coupling strongly to the $K^- K^{*0}$ pair. This will modify the strength of the singularity, as we will see.

We shall see that a singularity appears around 2815 MeV for the $\pi^- D_{s0}^+(2317)$ invariant mass in the $B^- \rightarrow K^- \pi^- D_{s0}^+$ process. This means that, in the $\pi^- D_{s0}^+(2317)$ rest frame where we will evaluate the triangle diagram, the K^{*0} momentum is about 262 MeV/c, which is small compared to the K^{*0} mass. This allows us to neglect the time component ϵ^0 of the K^{*0} polarization vector, and employ the following formula for the polarization sum,

$$\sum_{\text{pol}} \epsilon_{K^{*0}i} \epsilon_{K^{*0}j} = \delta_{ij} , \quad (2)$$

when required. The parameter C in Eq. (1) can be related to the partial decay rate of the $B^- \rightarrow K^- K^{*0} D^0$ process, which is given by

$$\Gamma_{B^- \rightarrow K^- K^{*0} D^0} = \int dM_{\text{inv}}(D^0 K^{*0}) \frac{1}{(2\pi)^3} \frac{|\vec{p}_{K^-}| |\vec{p}'_{K^{*0}}|}{4m_{B^-}^2} \sum_{\text{pol}} |t_{B^-, K^- K^{*0} D^0}|^2 , \quad (3)$$

where $M_{\text{inv}}(D^0 K^{*0})$ is the $D^0 K^{*0}$ invariant mass, \vec{p}_{K^-} and $\vec{p}'_{K^{*0}}$ are the momenta of the K^- in the B^- rest frame and that of the K^{*0} in the $D^0 K^{*0}$ center-of-mass (CM) frame, respectively, given by

$$|\vec{p}_{K^-}| = \frac{1}{2m_{B^-}} \lambda^{1/2}(m_{B^-}^2, m_{K^-}^2, M_{\text{inv}}^2(D^0 K^{*0})) , \quad (4)$$

$$|\vec{p}'_{K^{*0}}| = \frac{1}{2M_{\text{inv}}(D^0 K^{*0})} \lambda^{1/2}(M_{\text{inv}}^2(D^0 K^{*0}), m_{K^{*0}}^2, m_{D^0}^2) , \quad (5)$$

where $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$ is the Källén function. From the vertex of Eq. (1), which assumes the ϵ^0 component of the K^* polarization to give a small contribution, and employing the polarization sum of Eq. (2), we have

$$\sum_{\text{pol}} |t_{B^-, K^- K^{*0} D^0}|^2 = C^2 |\vec{p}'_{K^{*0}}|^2 \quad (6)$$

written in terms of the K^- momentum in the $D^0 K^{*0}$ CM frame. However, the phase space for $B^- \rightarrow K^- D^0 K^{*0}$ decay is such that the invariant mass of $D^0 K^{*0}$ peaks at 3100 MeV, where $|\vec{p}'_{K^{*0}}| = 670$ MeV/c and stretches at even

higher invariant masses. Therefore, $|\vec{p}'_{K^{*0}}|/m_{K^{*0}}$ is not small, and one must employ the covariant form of Eq. (1),

$$-it_{B^-,K^-K^{*0}D^0} = iC\epsilon_{K^{*0}\mu}p_{K^-}^\mu. \quad (7)$$

The sum over polarizations in Eq. (6), evaluated in the D^0K^{*0} rest frame and after performing an angular integration over the K^- and K^{*0} angle in that frame, becomes then

$$\sum_{\text{pol}} |t_{B^-,K^-K^{*0}D^0}|^2 = C^2 \left(-m_{K^-}^2 + \frac{(\tilde{p}'_{K^-} - \tilde{p}'_{K^{*0}})^2 + \frac{1}{3}(|\vec{p}'_{K^-}| |\vec{p}'_{K^{*0}}|)^2}{m_{K^{*0}}^2} \right), \quad (8)$$

where $|\vec{p}'_{K^-}|$ is given by

$$|\vec{p}'_{K^-}| = \frac{1}{2M_{\text{inv}}(D^0K^{*0})} \lambda^{1/2}(m_{B^-}^2, m_{K^-}^2, M_{\text{inv}}^2(D^0K^{*0})), \quad (9)$$

and \tilde{p}'_{K^-} and $\tilde{p}'_{K^{*0}}$ are given by $\sqrt{m_{K^-}^2 + |\vec{p}'_{K^-}|^2}$ and $\sqrt{m_{K^{*0}}^2 + |\vec{p}'_{K^{*0}}|^2}$, respectively. Thus, inserting Eq. (8) into Eq. (3), one obtains C^2 as

$$C^2 = \frac{\Gamma_{B^- \rightarrow K^- K^{*0} D^0}}{\int dM_{\text{inv}}(D^0K^{*0}) \frac{1}{(2\pi)^3} \frac{|\vec{p}_{K^-}| |\vec{p}_{K^{*0}}|}{4m_{B^-}^2} \left(-m_{K^-}^2 + \frac{(\tilde{p}'_{K^-} - \tilde{p}'_{K^{*0}})^2 + \frac{1}{3}(|\vec{p}'_{K^-}| |\vec{p}'_{K^{*0}}|)^2}{m_{K^{*0}}^2} \right)}. \quad (10)$$

Next, we look into the $K^{*0}\pi^-K^+$ vertex in Fig 3. We can obtain this VPP vertex from the chiral invariant lagrangian with local hidden symmetry given in Refs. [56–59],

$$\mathcal{L}_{VPP} = -ig \langle V^\mu [P, \partial_\mu P] \rangle, \quad (11)$$

where the brackets $\langle \dots \rangle$ stand for the trace of the flavor SU(3) matrices and g is the coupling in the local hidden gauge, $g = m_V/2f_\pi$, with $m_V = 800$ MeV and $f_\pi = 93$ MeV. The symbols V_μ and P stand for the octet vector meson field and the octet pseudoscalar meson field SU(3) matrices, respectively (see, *e.g.*, Ref. [59] for their explicit forms). The amplitude of the K^{*0} decay into $K^+\pi^-$ is then

$$-it_{K^{*0},K^+\pi^-} = -ig \vec{\epsilon}_{K^{*0}}(\vec{p}'_{\pi^-} - \vec{p}'_{K^+}), \quad (12)$$

where \vec{p}'_{π^-} and \vec{p}'_{K^+} are the momenta of the π^- and K^+ , respectively, in the chosen reference frame of πD_{s0}^+ at rest, where $\epsilon_{K^{*0}}^0$ can be neglected.

Finally, for the evaluation of the triangle diagram in Fig. 3, we also need to provide the $D_{s0}^+ D^0 K^+$ vertex. We follow the model of Ref. [46], where the D_{s0}^+ is found to be dynamically generated from the DK , $D_s\eta$, and $D_s\eta_c$ interaction in S -wave within a chiral unitary approach. The $D_{s0}^+ D^0 K^+$ vertex is then given by

$$-it_{D_{s0}^+,D^0K^+} = -ig_{D_{s0}^+,D^0K^+}, \quad (13)$$

where the coupling of the D_{s0}^+ to D^0K^+ states, $g_{D_{s0}^+,D^0K^+}$, is obtained from the coupling constant of the D_{s0} to the DK channel in isospin $I = 0$, found to be $g_{D_{s0},DK} = 10.21$ GeV in Ref. [46], multiplied by the appropriate Clebsch-Gordan (CG) coefficient, namely $g_{D_{s0}^+,D^0K^+} = -g_{D_{s0},DK}/\sqrt{2}$.

With the vertices of Eqs. (1), (12), and (13), the amplitude of the triangle diagram of Fig. 3, evaluated in the $\pi^- D_{s0}^+$ CM frame, is given by

$$\begin{aligned} t_{B^-,K^-\pi^-D_{s0}^+} &= C g g_{D_{s0}^+,D^0K^+} \sum_{\text{pol}} i \int \frac{d^4q}{(2\pi)^4} (\vec{\epsilon}_{K^{*0}} \vec{p}'_{K^-}) (\vec{\epsilon}_{K^{*0}} (2\vec{p}'_{\pi^-} + \vec{q})) \\ &\quad \frac{1}{q^2 - m_{D^0}^2 + i\epsilon} \frac{1}{(P' - q)^2 - m_{K^{*0}}^2 + i\epsilon} \frac{1}{(P' - q - p'_{\pi^-})^2 - m_{K^+}^2 + i\epsilon} \\ &\equiv C g g_{D_{s0}^+,D^0K^+} t_T, \end{aligned} \quad (14)$$

where the last identity defines t_T . In the former equation, $P' \equiv p'_{\pi^-} + p'_{D_{s0}^+}$, p'_{π^-} and p'_{K^-} are the momenta in the CM frame of the $\pi^- D_{s0}^+$ pair. Hence, $P'^0 = M_{\text{inv}}(\pi^- D_{s0}^+)$, $\vec{P}' = \vec{0}$,

$$|\vec{p}'_{\pi^-}| = \frac{1}{2M_{\text{inv}}(\pi^- D_{s0}^+)} \lambda^{1/2}(M_{\text{inv}}^2(\pi^- D_{s0}^+), m_{\pi^-}^2, m_{D_{s0}^+}^2), \quad (15)$$

and

$$|\vec{p}'_{K^-}| = \frac{1}{2M_{\text{inv}}(\pi^- D_{s0}^+)} \lambda^{1/2}(m_{B^-}^2, m_{K^-}^2, M_{\text{inv}}(\pi^- D_{s0}^+)^2). \quad (16)$$

Using Eq. (2), the sum over the polarizations of the K^{*0} gives

$$\sum_{\text{pol}} (\vec{\epsilon}_{K^{*0}} \vec{p}'_{K^-}) (\vec{\epsilon}_{K^{*0}} (2\vec{p}'_{\pi^-} + \vec{q})) = \vec{p}'_{K^-} (2\vec{p}'_{\pi^-} + \vec{q}), \quad (17)$$

and, then, the function t_T defined in Eq. (14) can be written as

$$t_T = i \int \frac{d^4 q}{(2\pi)^4} \vec{p}'_{K^-} (2\vec{p}'_{\pi^-} + \vec{q}) \frac{1}{q^2 - m_{D^0}^2 + i\epsilon} \frac{1}{(P' - q)^2 - m_{K^{*0}}^2 + i\epsilon} \frac{1}{(P' - q - p'_{\pi^-})^2 - m_{K^+}^2 + i\epsilon}. \quad (18)$$

We can perform the q^0 integration analytically, in the same way as shown in Refs. [23, 60, 61]. As for the three-momentum integral we note that, since the only three-momentum that is not integrated in Eq. (18) is \vec{p}'_{π^-} , one may employ the following identity

$$\int d^3 q q_i f(\vec{q}, \vec{k}') = k'_i \int d^3 q \frac{(\vec{q} \vec{k}')}{|\vec{k}'|^2} f(\vec{q}, \vec{k}'), \quad (19)$$

with $\vec{k}' = \vec{p}'_{\pi^-}$ and $f(\vec{q}, \vec{k}')$ being the product of the D^0 , K^{*0} , and K^+ propagators. Hence, we obtain:

$$\begin{aligned} t_T &= (\vec{p}'_{K^-} \vec{p}'_{\pi^-}) \int \frac{d^3 q}{(2\pi)^3} \frac{1}{8\omega(\vec{q})\omega'(\vec{q})\omega^*(\vec{q})} \frac{1}{p_{\pi^-}^0 - \omega'(\vec{q}) - \omega^*(\vec{q}) + i\epsilon} \frac{1}{P'^0 - \omega^*(\vec{q}) - \omega(\vec{q}) + i\epsilon} \\ &\quad \frac{2P'^0\omega(\vec{q}) + 2p_{\pi^-}^0\omega'(\vec{q}) - 2[\omega(\vec{q}) + \omega'(\vec{q})][\omega(\vec{q}) + \omega'(\vec{q}) + \omega^*(\vec{q})]}{(P'^0 - \omega(\vec{q}) - \omega'(\vec{q}) - p_{\pi^-}^0 + i\epsilon)(P'^0 + \omega(\vec{q}) + \omega'(\vec{q}) - p_{\pi^-}^0 - i\epsilon)} \left(2 + \frac{\vec{p}'_{\pi^-} \vec{q}}{|\vec{p}'_{\pi^-}|^2}\right) \\ &\equiv (\vec{p}'_{K^-} \vec{p}'_{\pi^-}) \tilde{t}_T, \end{aligned} \quad (20)$$

where the last identity defines the triangle singularity integral \tilde{t}_T out of the amplitude t_T . We recall that, in the integrand of Eq. (20), P'^0 denotes the invariant mass of the $\pi^- D_{s0}^+$ pair, $M_{\text{inv}}(\pi^- D_{s0}^+)$, while $\omega(\vec{q}) = \sqrt{m_{D^0}^2 + |\vec{q}|^2}$ is the energy of the D^0 , $\omega^*(\vec{q}) = \sqrt{m_{K^{*0}}^2 + |\vec{q}|^2}$ that of the K^{*0} , $\omega'(\vec{q}) = \sqrt{m_{K^+}^2 + |\vec{q} + \vec{p}'_{\pi^-}|^2}$ that of the K^+ , and $p_{\pi^-}^0 = \sqrt{m_{\pi^-}^2 + |\vec{p}'_{\pi^-}|^2}$ that of the π^- in the $\pi^- D_{s0}^+$ CM frame. The $|\vec{q}|$ integral in Eq. (20) is regulated by a cutoff $|\vec{q}|_{\text{max}} = 800$ MeV in the D_{s0}^+ rest frame, which reproduces the results obtained in the chiral unitary approach for the D_{s0}^+ meson [46]. We implement the width of the K^{*0} with the replacement of ω^* by $\omega^* - i\Gamma_{K^{*0}}/2$ in the integrand of Eq. (20).

Taking into account the amplitude of Eq. (14), with t_T given in Eq. (18), and performing the integration over the angle formed by the K^- and π^- momenta, the mass distribution of the $B^- \rightarrow K^- \pi^- D_{s0}^+$ decay width is given by

$$\frac{d\Gamma_{B^- \rightarrow K^- \pi^- D_{s0}^+}}{dM_{\text{inv}}(\pi^- D_{s0}^+)} = \frac{1}{(2\pi)^3} \frac{1}{4m_{B^-}^2} |\vec{p}_{K^-}| |\vec{p}'_{\pi^-}| \frac{C^2}{3} |\tilde{t}_{B^-, K^- \pi^- D_{s0}^+}^{\text{eff}}|^2, \quad (21)$$

where \vec{p}_{K^-} is the momentum of the K^- in the B^- rest frame,

$$|\vec{p}_{K^-}| = \frac{1}{2m_{B^-}} \lambda^{1/2}(m_{B^-}^2, m_{K^-}^2, M_{\text{inv}}^2(\pi^- D_{s0}^+)), \quad (22)$$

\vec{p}'_{π^-} is that of the π^- in the CM frame of the $\pi^- D_{s0}^+$ pair [see Eq. (15)], and we have defined $\tilde{t}_{B^-, K^- \pi^- D_{s0}^+}^{\text{eff}} \equiv g g_{D_{s0}^+, D^0 K^+} |\vec{p}'_{K^-}| |\vec{p}'_{\pi^-}| \tilde{t}_T$.

Dividing both sides of Eq. (21) by Γ_B and using Eq. (10), we obtain

$$\frac{1}{\Gamma_B} \frac{d\Gamma_{B^- \rightarrow K^- \pi^- D_{s0}^+}}{dM_{\text{inv}}(\pi^- D_{s0}^+)} = \frac{\text{BR}(B^- \rightarrow K^- K^{*0} D^0) |\vec{p}_{K^-}| |\vec{p}'_{\pi^-}| \frac{1}{3} |\tilde{t}_{B^-, K^- \pi^- D_{s0}^+}^{\text{eff}}|^2}{\int dM_{\text{inv}}(D^0 K^{*0}) |\vec{p}_{K^-}| |\vec{p}_{K^{*0}}| \left(-m_{K^-}^2 + \frac{(\vec{p}_{K^-}^0 - \vec{p}_{K^{*0}}^0)^2 + \frac{1}{3} (|\vec{p}_{K^-}| |\vec{p}_{K^{*0}}|)^2}{m_{K^{*0}}^2} \right)}, \quad (23)$$

where the branching ratio $\text{BR}(B^- \rightarrow K^- K^{*0} D^0) = \Gamma_{B^- \rightarrow K^- K^{*0} D^0} / \Gamma_B$ is $(7.5 \pm 1.7) \times 10^{-4}$ [1]. Integrating Eq. (23) over $M_{\text{inv}}(\pi^- D_{s0}^+)$, we obtain the branching ratio for the $B^- \rightarrow K^- \pi^- D_{s0}^+$ process.

B. $B^- \rightarrow K^- \pi^- D_{s1}^+$ (2460) decay

The triangle diagram for the $B^- \rightarrow K^- \pi^- D_{s1}^+$ decay mode is shown in Fig. 4. We can evaluate the branching ratio of this process in a similar way to that in Sec. II A for the $B^- \rightarrow K^- \pi^- D_{s0}^+$ process. The differences from the previous case are the amplitudes $B^- \rightarrow K^- K^{*0} D^{*0}$ and $D^{*0} K^+ \rightarrow D_{s1}^+$.

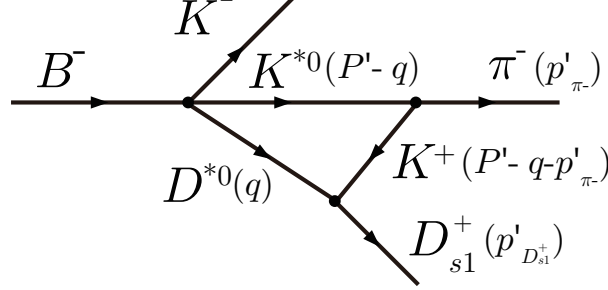


FIG. 4: Triangle diagram for $B^- \rightarrow K^- \pi^- D_{s1}^+$.

The first vertex is related to the decay $B^- \rightarrow K^- K^{*0} D^{*0}$, which can proceed in S -wave. The corresponding amplitude, $t_{B^-, K^- K^{*0} D^{*0}}$, in a suitable reference frame is then given by

$$-i t_{B^-, K^- K^{*0} D^{*0}} = -i C' \vec{\epsilon}_{K^{*0}} \vec{\epsilon}_{D^{*0}}, \quad (24)$$

where $\vec{\epsilon}_{D^{*0}}$ is the polarization vector of the D^{*0} . We fix the coupling constant C' in Eq. (24) from the branching ratio of the $B^- \rightarrow K^- K^{*0} D^{*0}$ process. The partial width for this decay is written as

$$\Gamma_{B^- \rightarrow K^- K^{*0} D^{*0}} = \int dM_{\text{inv}}(D^{*0} K^{*0}) \frac{1}{(2\pi)^3} \frac{|\vec{p}_{K^-}| |\vec{p}'_{K^{*0}}|}{4m_{B^-}^2} \sum_{\text{pol}} |t_{B^-, K^- K^{*0} D^{*0}}|^2 \quad (25)$$

with $M_{\text{inv}}(D^{*0} K^{*0})$ being the invariant mass of the $D^{*0} K^{*0}$ pair, while \vec{p}_{K^-} and $\vec{p}'_{K^{*0}}$ are the momenta of the K^- in the B^- rest frame and that of the K^{*0} in the $D^{*0} K^{*0}$ CM frame, given by

$$|\vec{p}_{K^-}| = \frac{1}{2m_{B^-}} \lambda^{1/2}(m_{B^-}^2, m_{K^-}^2, M_{\text{inv}}^2(D^{*0} K^{*0})), \quad (26)$$

$$|\vec{p}'_{K^{*0}}| = \frac{1}{2M_{\text{inv}}(D^{*0} K^{*0})} \lambda^{1/2}(M_{\text{inv}}^2(D^{*0} K^{*0}), m_{K^{*0}}^2, m_{D^{*0}}^2). \quad (27)$$

Using Eq. (2) for the D^{*0} and K^{*0} polarization sums of Eq. (25), we find

$$\sum_{\text{pol}} |t_{B^-, K^- K^{*0} D^{*0}}|^2 = C'^2 \sum_{\text{pol}} |\vec{\epsilon}_{K^{*0}} \vec{\epsilon}_{D^{*0}}|^2 = 3C'^2. \quad (28)$$

However, once again, the momentum of the K^{*0} is not small compared to the K^{*0} mass in the region of invariant $D^{*0} K^{*0}$ masses covered by the phase space of the process. One must then use the covariant form of Eq. (24),

$$-i t_{B^-, K^- K^{*0} D^{*0}} = i C' \epsilon_{K^{*0} \mu} \epsilon_{D^{*0}}^\mu, \quad (29)$$

finding

$$\sum_{\text{pol}} |t_{B^-, K^- K^{*0} D^{*0}}|^2 = C'^2 \left(2 + \frac{(M_{\text{inv}}^2(D^{*0} K^{*0}) - m_{D^{*0}}^2 - m_{K^{*0}}^2)^2}{4m_{K^{*0}}^2 m_{D^{*0}}^2} \right). \quad (30)$$

Then, the constant C'^2 can be determined from

$$C'^2 = \frac{\Gamma_{B^- \rightarrow K^- K^{*0} D^{*0}}}{\int dM_{\text{inv}}(D^{*0} K^{*0}) \frac{1}{(2\pi)^3} \frac{|\vec{p}_{K^-}| |\vec{p}'_{K^{*0}}|}{4m_{B^-}^2} \left(2 + \frac{(M_{\text{inv}}^2(D^{*0} K^{*0}) - m_{D^{*0}}^2 - m_{K^{*0}}^2)^2}{4m_{K^{*0}}^2 m_{D^{*0}}^2} \right)}. \quad (31)$$

Similarly to the previous case, we rely on results from the chiral unitary approach [62] to obtain the coupling constant of the D_{s1}^+ state to a $D^{*0}K^+$ pair. This coupling is determined from the one of D_{s1}^+ to D^*K in isospin $I = 0$ quoted in [62], $g_{D_{s1}, D^*K} = 9.82$ GeV, multiplied by the appropriate CG coefficient. Hence, $g_{D_{s1}^+, D^{*0}K^+} = -g_{D_{s1}, D^*K}/\sqrt{2}$. The corresponding S -wave amplitude is given by

$$-i t_{D_{s1}^+, D^{*0}K^+} = -i g_{D_{s1}^+, D^{*0}K^+} \vec{\epsilon}_{D_{s1}^+} \vec{\epsilon}_{D^{*0}}, \quad (32)$$

where $\vec{\epsilon}_{D_{s1}^+}$ and $\vec{\epsilon}_{D^{*0}}$ are the polarization vectors of the D_{s1}^+ and D^{*0} , respectively. In the region of the invariant masses where the singularity appears, the momentum of the D_{s1}^+ is of the order of 10% of its mass and, again, it is proper to neglect the $\epsilon_{D_{s1}^+}^0$ component.

Using the expressions for the vertices given by Eqs. (12), (24), and (32), the amplitude of the triangle diagram of Fig. 4 for the decay $B^- \rightarrow K^- \pi^- D_{s1}^+$ in the $\pi^- D_{s1}^+$ CM frame is given by

$$\begin{aligned} t_{B^-, K^- \pi^- D_{s1}^+} &= C' g g_{D_{s1}^+, D^{*0}K^+} \sum_{\text{pol}(K^{*0}, D^{*0})} i \int \frac{d^4 q}{(2\pi)^4} (\vec{\epsilon}_{K^{*0}} \vec{\epsilon}_{D^{*0}}) (\vec{\epsilon}_{K^{*0}} (2\vec{p}'_{\pi^-} + \vec{q})) (\vec{\epsilon}_{D^{*0}} \vec{\epsilon}_{D_{s1}^+}) \\ &\quad \frac{1}{q^2 - m_{D^{*0}}^2 + i\epsilon} \frac{1}{(P' - q)^2 - m_{K^{*0}}^2 + i\epsilon} \frac{1}{(P' - q - p'_{\pi^-})^2 - m_{K^+}^2 + i\epsilon} \\ &\equiv C' g g_{D_{s1}^+, D^{*0}K^+} (\vec{p}'_{\pi^-} \vec{\epsilon}_{D_{s1}^+}) \tilde{t}'_T, \end{aligned} \quad (33)$$

where Eq. (2) has been employed to sum over the polarizations of the K^{*0} and the D^{*0} . The last identity of the previous equation defines the triangle singularity integral \tilde{t}'_T , which is formally the same as \tilde{t}_T in Eq. (20) with the replacement of m_{D^0} by $m_{D^{*0}}$, $P^{0'}$ by $M_{\text{inv}}(\pi^- D_{s1}^+)$ and

$$|\vec{p}'_{\pi^-}| = \frac{1}{2M_{\text{inv}}(\pi^- D_{s1}^+)} \lambda^{1/2}(M_{\text{inv}}^2(\pi^- D_{s1}^+), m_{\pi^-}^2, m_{D_{s1}^+}^2). \quad (34)$$

Following similar steps as in the previous case, the mass distribution of the branching ratio for $B^- \rightarrow K^- \pi^- D_{s1}^+$ decay is given by

$$\frac{1}{\Gamma_B} \frac{d\Gamma_{B^- \rightarrow K^- \pi^- D_{s1}^+}}{dM_{\text{inv}}(\pi^- D_{s1}^+)} = \frac{\text{BR}(B^- \rightarrow K^- K^{*0} D^{*0}) \cdot |\vec{p}_{K^-}| |\vec{p}'_{\pi^-}| |\tilde{t}_{B^-, K^- \pi^- D_{s1}^+}^{\text{eff}}|^2}{\int dM_{\text{inv}}(D^{*0} K^{*0}) |\vec{p}_{K^-}| |\vec{p}'_{K^{*0}}| \left(2 + \frac{(M_{\text{inv}}^2(D^{*0} K^{*0}) - m_{D^{*0}}^2 - m_{K^{*0}}^2)^2}{4m_{K^{*0}}^2 m_{D^{*0}}^2} \right)}, \quad (35)$$

where \vec{p}_{K^-} is the K^- momentum in the B^- rest frame, given by

$$|\vec{p}_{K^-}| = \frac{1}{2m_{B^-}} \lambda^{1/2}(m_{B^-}^2, m_{K^-}^2, M_{\text{inv}}^2(\pi^- D_{s1}^+)), \quad (36)$$

and we have used

$$\sum_{\text{pol}} (\vec{p}'_{\pi^-} \vec{\epsilon}_{D_{s1}^+})^2 = |\vec{p}'_{\pi^-}|^2, \quad (37)$$

which is readily obtained employing again Eq. (2). In Eq. (35) we have defined $\tilde{t}_{B^-, K^- \pi^- D_{s1}^+}^{\text{eff}} = g g_{D_{s1}^+, D^{*0}K^+} |\vec{p}'_{\pi^-}| \tilde{t}'_T$, and the branching ratio $\text{BR}(B^- \rightarrow K^- K^{*0} D^{*0}) = \Gamma_{B^- \rightarrow K^- K^{*0} D^{*0}}/\Gamma_B$ is $(1.5 \pm 0.4) \times 10^{-3}$ [1]. The integral of Eq. (35) over $M_{\text{inv}}(\pi^- D_{s1}^+)$ gives the branching ratio of the $B^- \rightarrow K^- \pi^- D_{s1}^+$ process.

III. RESULTS

In Fig. 5, we show the absolute value, square of the absolute value, real part, and imaginary part of the triangle amplitude \tilde{t}_T in Eq. (20) for the $B^- \rightarrow K^- \pi^- D_{s0}^+$ decay process, as functions of the $\pi^- D_{s0}^+$ invariant mass. The peak in $|\tilde{t}_T|^2$ is located at 2800 MeV and it has a width of about 200 MeV. This width comes from the K^* decay width and the fact that the condition for the triangle singularity of Ref. [23] cannot be strictly fulfilled for the actual mass of the D_{s0}^+ (2317), because this state is bound with respect to the DK threshold at 2359 MeV and the DK pair cannot be placed on-shell. Yet, one can see that the peak is related to a “nearly missed” triangle singularity by taking

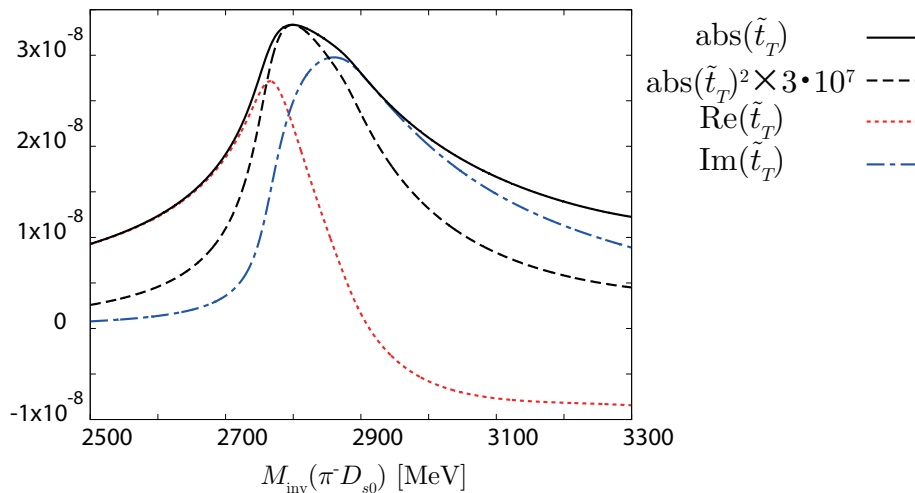


FIG. 5: Triangle amplitude for the $B^- \rightarrow K^- \pi^- D_{s0}^+$ process, $|\tilde{t}_T|$, $|\tilde{t}_T|^2$, $\text{Re}(\tilde{t}_T)$, and $\text{Im}(\tilde{t}_T)$ as functions of $M_{\text{inv}}(\pi^- D_{s0}^+)$. The unit of \tilde{t}_T is MeV^{-2} .

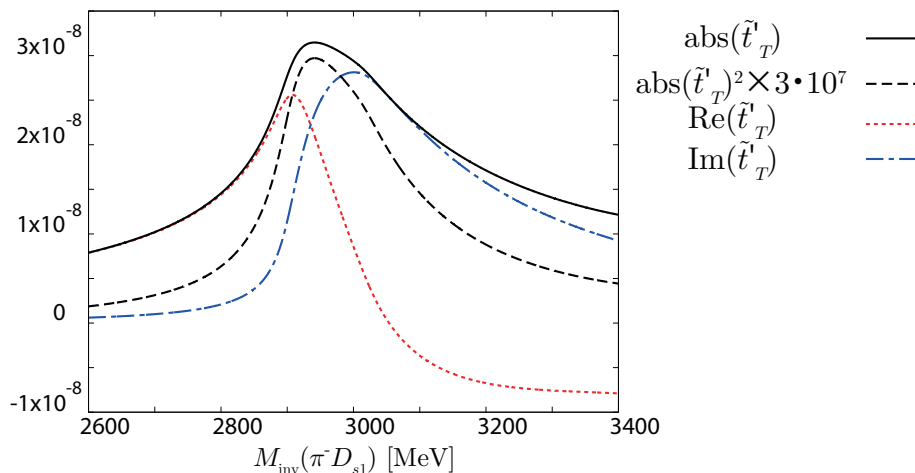


FIG. 6: Triangle amplitude for the $B^- \rightarrow K^- \pi^- D_{s1}^+$ process, $|\tilde{t}'_T|$, $|\tilde{t}'_T|^2$, $\text{Re}(\tilde{t}'_T)$, and $\text{Im}(\tilde{t}'_T)$ as functions of $M_{\text{inv}}(\pi^- D_{s1}^+)$. \tilde{t}'_T is given in the unit of MeV^{-2} .

values of the $D_{s0}^+(2317)$ mass just above the DK threshold. If we take a mass of the $D_{s0}^+(2317)$ close to the DK threshold, from 2360 MeV to 2365 MeV, the condition of Eq. (18) of Ref. [23] gives the singularity between 2827 MeV and 2801 MeV, close to the position of the peak in Fig. 5. The results of Fig. 6, shown as functions of $M_{\text{inv}}(\pi^- D_{s1}^+)$, are the equivalent ones for the \tilde{t}'_T amplitude in Eq. (33), corresponding to the $B^- \rightarrow K^- \pi^- D_{s1}^+$ decay process. One can see a peak around 2950 MeV with a 150 MeV width. The peak position is also similar to the value expected from the formula in Ref. [23], which ranges between 2971 MeV and 2943 MeV when we take a D_{s1}^+ mass from 2502 MeV to 2507 MeV just above the $D^{*0}K^+$ threshold (2501 MeV). Note that, around the peaks, these amplitudes could be described by an expression of the type $iBW(M_{\text{inv}}) + iB$, with $BW(M_{\text{inv}}) = 1/(M_{\text{inv}}^2 - m_R^2 + iM_{\text{inv}}\Gamma)$ and B as a real background, therefore resembling the behavior expected for a hadron resonance.

Before proceeding further, we would like to perform a few tests that allow us to interpret properly the peaks that appear both in the real and imaginary parts of \tilde{t}_T and \tilde{t}'_T displayed in Figs. 5 and 6, respectively. Let us discuss the case of Fig. 5 since the one of Fig. 6 would be identical. A triangle singularity appears in the diagram of Fig 3 when the K^{*0} , D^0 , and K^+ particles are placed on-shell and \vec{p}'_π and $-\vec{q}$ are parallel (K^{*0} and π^- go in the same direction), namely when q_{on} and q_{a-} in the nomenclature of Ref. [23] are equal (see Eq. (18) of this reference). As mentioned before, since the $D_{s0}^+(2317)$ is bound with respect to the D^0K^+ component, this meson pair cannot be placed on-shell. Hence, the condition $q_{\text{on}} = q_{a-}$ cannot be strictly fulfilled. However, since the binding is moderate, instead of the singularity that one would obtain (if Γ_{K^*} was zero) we find a finite enhancement. As discussed before, the condition $q_{\text{on}} = q_{a-}$ of Ref. [23] is fulfilled with masses of $D_{s0}^+(2317)$ slightly bigger than the DK threshold. We have seen that

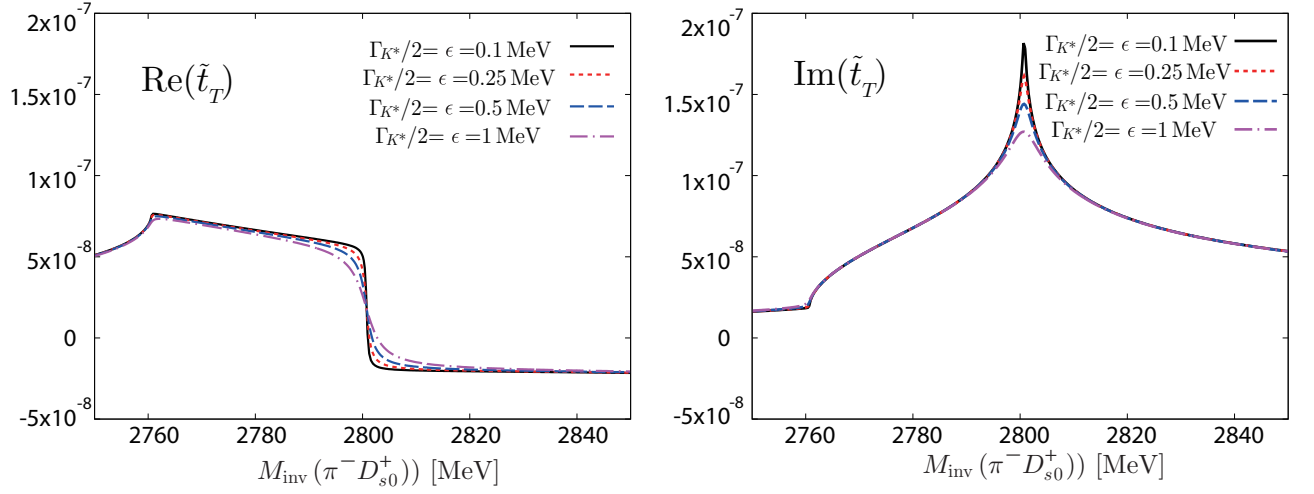


FIG. 7: Real (left) and imaginary (right) parts of \tilde{t}_T in Eq. (20), with m_K replaced by $m_K - 60$ MeV and several values of $\Gamma_{K^*}/2 = \epsilon = 1, 0.5, 0.25$ and 0.1 MeV.

the singularity moves towards higher energies as the $D_{s_0}^+$ mass decreases and that at a mass of 2360 MeV, just 1 MeV above the DK threshold, it appears at 2827 MeV. It is then logical to associate the peak of $\text{Im}(\tilde{t}_T)$ at 2850 MeV to this “nearly missed” triangle singularity. Note, however, that there is also a peak in the real part around 2758 MeV that instead corresponds to the threshold of the $K^{*0}D^0$ system. If the K^{*0} had no width, this would have appeared as a cusp tied to having a K^{*0} and a D^0 on-shell at threshold, while the K^+ is off-shell. In order to corroborate this interpretation, we perform calculations where $\Gamma_{K^*}/2$ is artificially reduced while ϵ is taken finite but close to zero, which will help visualizing the building up of the singularity. The results are shown in Fig. 7. There we have also artificially reduced the mass of the kaon by 60 MeV so that the D^0K^+ mass is below the $D_{s_0}^+$ (2317) mass by about 17 MeV, and hence the K^{*0} , D^0 , and K^+ mesons can be placed simultaneously on-shell, producing a triangle singularity. One can now see two structures in both $\text{Re}(\tilde{t}_T)$ and $\text{Im}(\tilde{t}_T)$: a cusp at the threshold of the $K^{*0}D^0$ state and the triangle singularity around 2800 MeV, seen as a narrow peak in $\text{Im}(\tilde{t}_T)$ and as a sharp downfall in $\text{Re}(\tilde{t}_T)$. In order to further stress this point, we take decreasing values of $\Gamma_{K^*}/2$ and ϵ , namely 1, 0.5, 0.25 and 0.1 MeV, and we then see that the cusp associated to the threshold in the real part converges to a finite value, while the peak of the singularity in the imaginary part keeps growing and becomes sharp as corresponds to a singularity.

Coming back to Fig. 5 for the realistic case, we should note that as we increase $M_{\text{inv}}(\pi^- D_{s_0})$, so will the on-shell momentum of D^0K^{*0} , and since we apply a momentum cutoff $|\vec{q}|_{\text{max}}$, this could impose constraints in the phase space and modify the shape of the calculated spectrum. For this reason, we also investigate whether the falldown of $\text{Im}(\tilde{t}_T)$ is due to the cutoff value employed or to the triangle singularity behavior. We increase $|\vec{q}|_{\text{max}}$ from the value 800 MeV, constrained by the unitary approach to obtain the $D_{s_0}^+$ from the DK component, to 1000 MeV and 1200 MeV. The corresponding results are shown in Fig. 8. We see that, as the cutoff value is increased, the falldown of $\text{Im}(\tilde{t}_T)$ at higher $M_{\text{inv}}(\pi^- D_{s_0})$ becomes softer, but in all cases the peak that we have associated to the triangle singularity remains.

In Fig. 9, we show the mass distribution of the $B^- \rightarrow K^- \pi^- D_{s_0}^+$ decay as a function of $M_{\text{inv}}(\pi^- D_{s_0}^+)$ given in Eq. (23). A peak appears around 2850 MeV, reflecting the behavior observed in Fig. 5 for \tilde{t}_T but with the additional weight of some kinematical factors. Analogously, Fig. 10 shows the mass distribution of the $B^- \rightarrow K^- \pi^- D_{s_1}^+$ decay, given in Eq. (35), as a function of $M_{\text{inv}}(\pi^- D_{s_1}^+)$. The peak of the mass distribution in this case is located at 3000 MeV, which lies somewhat above the peak of the corresponding \tilde{t}_T amplitude shown in Fig. 6. Upon inspecting the final results in Fig. 9 for $d\Gamma_{B^- \rightarrow K^- \pi^- D_{s_0}^+}/dM_{\text{inv}}(\pi^- D_{s_0}^+)$, we note that the peak around 2850 MeV corresponds to where $|\tilde{t}_T|^2$ gets most of its strength from $\text{Im}(\tilde{t}_T)$, the triangle singularity amplitude displayed in Fig. 5, and hence we can associate the structure seen in Fig. 9 (and Fig. 10 for the $K^- \pi^- D_{s_1}^+$ decay) mostly to the effect of the triangle singularity.

We also integrate the mass distributions of Figs. 9 and 10 up to 400 MeV above the peak, and we obtain a branching ratio of 7.8×10^{-6} for the $B^- \rightarrow K^- \pi^- D_{s_0}^+$ decay process and of 4.2×10^{-6} for the $B^- \rightarrow K^- \pi^- D_{s_1}^+$ one. These branching ratios have not yet been reported in the PDG and, since they are likely to be measured in a near future, our prediction and the anticipation on establishing the nature of these unavoidable peaks, as coming from triangle singularities, is most opportune. Note, however, that the strength under the peak of the $B^- \rightarrow K^- \pi^- D_{s_0}^+$ and $B^- \rightarrow K^- \pi^- D_{s_1}^+$ distributions depends on the constants C^2 [Eq. (10)] and C'^2 [Eq. (31)], respectively, which in turn

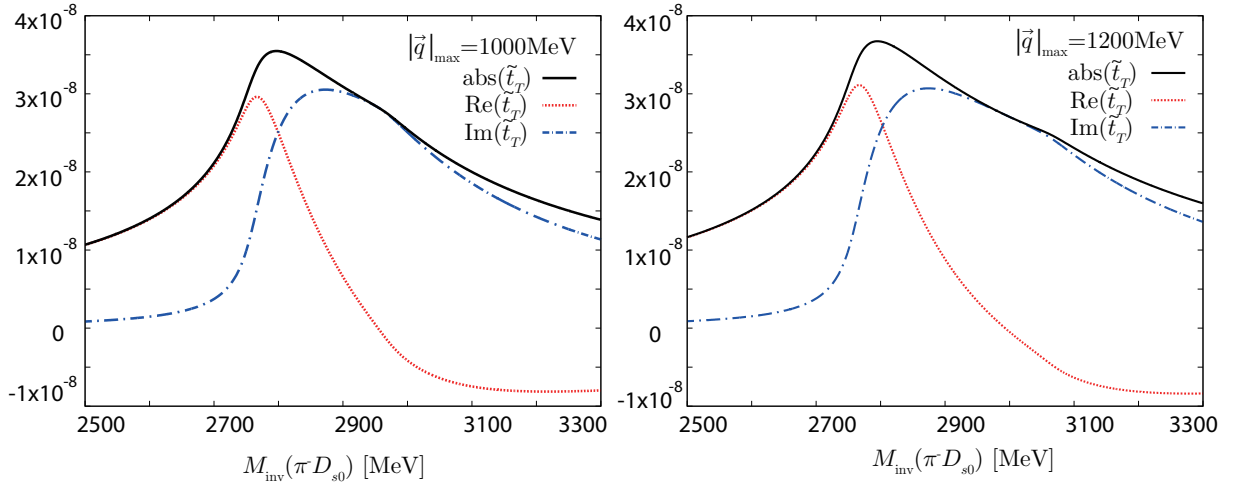


FIG. 8: $|\tilde{t}_T|$, $\text{Re}(\tilde{t}_T)$ and $\text{Im}(\tilde{t}_T)$ with $|\vec{q}|_{\text{max}} = 1000$ and 1200 MeV.

depend on the particular shape adopted for the respective vertices, Eqs. (7) and (29). Therefore, our final results for the branching ratios of the $B^- \rightarrow K^- \pi^- D_{s0}^+$ and $B^- \rightarrow K^- \pi^- D_{s1}^+$ processes under their respective peaks will be given in the next section, after implementing a more realistic form of the decay vertex, which accounts for the dominance of the $a_1(1260)$ resonance coupling strongly to a $K^- K^{*0}$ pair.

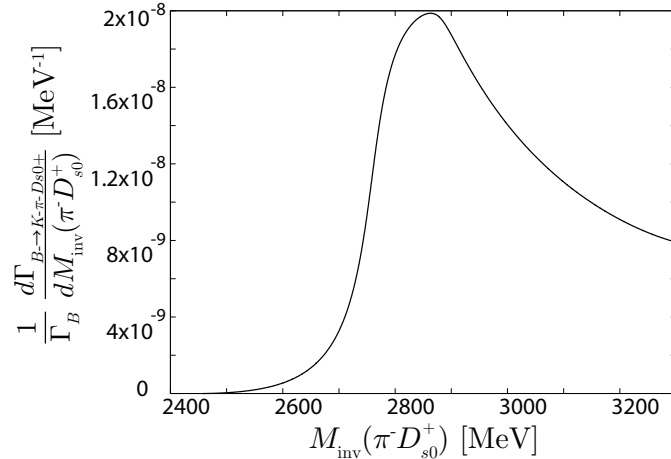


FIG. 9: Mass distribution of the $B^- \rightarrow K^- \pi^- D_{s0}^+$ decay as a function of $M_{\text{inv}}(\pi^- D_{s0}^+)$.

One may wonder about possible sources of background to the $K^- \pi^- D_{s0}^+(2317)$ or $K^- \pi^- D_{s1}^+(2460)$ processes. In this sense, these reactions are special. Indeed, one notes that there is no resonance with strangeness and charge -2 that could decay to $K^- \pi^-$. On the other hand, one might also wonder what would happen if a $K^- DK'^*$ intermediate state was produced, with K'^* being any resonance which can decay to πK , and then we added the corresponding triangle diagram contribution of Fig. 3, upon substituting K^* by K'^* . First of all, we note that there is no evidence in the PDG [1] for the processes $B^- \rightarrow K^- K'^* D$, with K'^* being any of the states which decay into πK . Hence, an evaluation of the strength of the corresponding triangle amplitudes would not be possible. In any case, this would not be necessary here since, by testing Eq. (18) of Ref. [23] with $m_{D_{s0}^+}$ slightly larger than the DK pair mass (3 MeV above the threshold), the singularity appears at the values shown in Table I for different K'^* states up to 2000 MeV, and all of them are much higher than the value around 2815 MeV that we have obtained with our mechanism.

We would also like to estimate possible sources of errors. In the first place, for the evaluation of the triangle diagram we made the approximation of neglecting the zeroth component of the K^* polarization vector $\epsilon^\mu(K^*)$ in Eq. (12). This was justified because the K^* momentum for the peak of the singularity is 262 MeV/c, which is smaller than the K^* mass. For parity reasons, the omission of the ϵ^0 component goes in corrections of the type $(p_{K^*}/m_{K^*})^2$, which are of the order of 8%. Actually, there are factors that make this error even smaller and, as shown in the appendix,

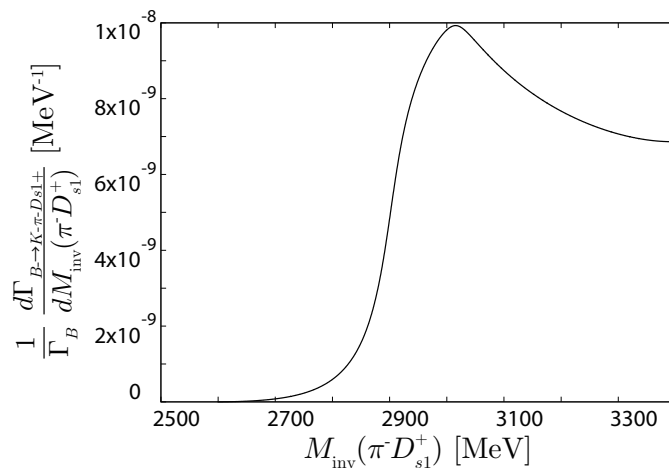


FIG. 10: Mass distribution of the $B^- \rightarrow K^- \pi^- D_{s1}^+$ decay as a function of $M_{\text{inv}}(\pi^- D_{s1}^+)$.

TABLE I: The energy value $\sqrt{s_{TS}}$ at which a triangle singularity appears in the process of Fig. 3, for different K'^* resonances instead of $K^*(892)$. The values of $\sqrt{s_{TS}}$ are evaluated taking $m_{D_{s0}^+} = 2362$ MeV.

$\sqrt{s_{TS}}$ [MeV]	$K^*(1410)(1^-)$	$K_0^*(1430)(0^+)$	$K^*(1680)(1^-)$	$K(1780)(3^-)$	$K_0^*(1950)(0^+)$
	3617	3650	4085	4264	4575

the final corrections turn to be of the order of 1.5%. Another source of uncertainty in our results is the coupling of the $D_{s0}^+(2317)$ to DK ($D_{s1}^+(2460)$ to D^*K). From the analyses of Refs. [46, 53], we estimate this uncertainty below 10%, which would induce uncertainties of 20% in our results. However, the biggest uncertainty comes from the experimental error in the $B^- \rightarrow K^- K^{*0} D^{*0}$ branching ratio $(1.5 \pm 0.4) \times 10^{-3}$, which is 27%. Summing all the uncertainties in quadrature we get 34%. It is thus realistic to attribute some 30 – 40% uncertainties to the absolute values of the obtained rates. Yet the shape of the distribution as a function of energy, which is the distinctive feature of the triangle singularity, should be much more accurate.

IV. FURTHER CONSIDERATIONS

The position of a triangle singularity is of kinematical origin, tied to the possibility to have $K^{*0} D^0 K^+$ ($K^{*0} D^{*0} K^+$) on-shell with collinear momenta, and hence not strongly dependent on the $B^- \rightarrow K^- D^0 K^{*0}$ ($K^- D^{*0} K^{*0}$) vertex. We have given a suitable structure for these vertices based on angular momentum conservation by means of Eqs. (7) and (29). In practice, it could be more complex. In this section, we examine what happens if we consider that this primary decay is dominated by the $a_1(1260)$ resonance coupled to $K^- K^{*0}$, as implied by the $K^- K^{*0}$ invariant mass distribution of Ref. [39] (see Fig. 3 (a) of this reference). Actually, in that work the invariant mass distribution is given for the combined $B^- \rightarrow K^- D^0 K^{*0}$ and $K^- D^{*0} K^{*0}$ decays. We discuss here what happens if we introduce also a dominance of the $a_1(1260)$ resonance for the $K^- K^{*0}$ system. We make it by multiplying the amplitudes that we have in Eqs. (7) and (29) by an extra factor accounting for the possible $a_1(1260)$ resonant shape in the $K^- K^{*0}$ invariant mass distribution,

$$B_W(M_{\text{inv}}(K^- K^{*0})) = \frac{m_{a_1} \Gamma_{a_1}}{M_{\text{inv}}^2(K^- K^{*0}) - m_{a_1}^2 + im_{a_1} \Gamma_{a_1}}, \quad (38)$$

and, as in Ref. [39], we take $m_{a_1} = 1230$ MeV and $\Gamma_{a_1} = 460$ MeV. This requires now some changes in the formalism. Equation (3) is now changed to

$$\Gamma_{B^- \rightarrow K^- K^{*0} D^0} = \int dM_{\text{inv}}(K^- K^{*0}) \frac{1}{(2\pi)^3} \frac{|\vec{p}_{D^0}||\vec{p}_{K^{*0}}|}{4m_{B^-}^2} \sum_{\text{pol}} |t_{B^-, K^- K^{*0} D^0}|^2 |B_W(M_{\text{inv}}(K^- K^{*0}))|^2, \quad (39)$$

where the integration runs over the variable on which both the amplitude and B_W depend, and \vec{p}_{D^0} and \vec{p}''_{K^-} are the momenta of D^0 in the B^- rest frame and K^- in the K^-K^{*0} CM frame given by

$$|\vec{p}_{D^0}| = \frac{\lambda^{1/2}(m_{B^-}^2, m_{D^0}^2, M_{\text{inv}}^2(K^-K^{*0}))}{2m_{B^-}}, \quad (40)$$

$$|\vec{p}''_{K^-}| = \frac{\lambda^{1/2}(M_{\text{inv}}^2(K^-K^{*0}), m_{K^-}^2, m_{K^{*0}}^2)}{2M_{\text{inv}}(K^-K^{*0})}, \quad (41)$$

respectively. The polarization sum of the square of the matrix element is given by

$$\sum_{\text{pol}} |t_{B^-, K^-K^{*0}D^0}|^2 = C'^2 \left(-m_{K^-}^2 + \frac{(p_{K^-} \cdot p_{K^{*0}})^2}{m_{K^{*0}}^2} \right) \quad (42)$$

with $p_{K^-} \cdot p_{K^{*0}} = [M_{\text{inv}}^2(K^-K^{*0}) - m_{K^-}^2 - m_{K^{*0}}^2]/2$. Similarly, Eq. (25) is changed to

$$\Gamma_{B^- \rightarrow K^-K^{*0}D^0} = \int dM_{\text{inv}}(K^-K^{*0}) \frac{1}{(2\pi)^3} \frac{|\vec{p}_{D^0}| |\vec{p}''_{K^-}|}{4m_{B^-}^2} \sum_{\text{pol}} |t_{B^-, K^-K^{*0}D^0}|^2 |B_W(M_{\text{inv}}(K^-K^{*0}))|^2 \quad (43)$$

with the D^{*0} momentum in the B^- rest frame,

$$|\vec{p}_{D^{*0}}| = \frac{\lambda^{1/2}(m_{B^-}^2, m_{D^{*0}}^2, M_{\text{inv}}^2(K^-K^{*0}))}{2m_{B^-}}. \quad (44)$$

The square of the $B^- \rightarrow K^-K^{*0}D^{*0}$ matrix element with the polarization sum is given by

$$\sum_{\text{pol}} |t_{B^-, K^-K^{*0}D^{*0}}|^2 = C''^2 \left(2 + \frac{(\tilde{p}_{D^{*0}}'' \tilde{p}_{K^{*0}}'')^2 + \frac{1}{3}(|\vec{p}_{D^{*0}}''| |\vec{p}_{K^{*0}}''|)^2}{m_{D^{*0}}^2 m_{K^{*0}}^2} \right) \quad (45)$$

where $\vec{p}_{D^{*0}}''$ is the momentum of the D^{*0} in the K^-K^{*0} CM frame,

$$|\vec{p}_{D^{*0}}''| = \frac{\lambda^{1/2}(m_{B^-}^2, m_{D^{*0}}^2, M_{\text{inv}}^2(K^-K^{*0}))}{2M_{\text{inv}}(K^-K^{*0})}; \quad \tilde{p}_{D^{*0}}'' = \sqrt{m_{D^{*0}}^2 + |\vec{p}_{D^{*0}}''|^2}, \quad (46)$$

with $|\vec{p}_{K^{*0}}''|$ given by Eq. (41) and $\tilde{p}_{K^{*0}}'' = \sqrt{m_{K^{*0}}^2 + |\vec{p}_{K^{*0}}''|^2}$. In the triangle diagram, which we evaluate in the CM frame of $\pi^- D_{s0}^+$ ($\pi^- D_{s1}^+$), we have to multiply the integrand of t_T by $B_W(M_{\text{inv}}(K^-K^{*0}))$ which is given by Eq. (38) with

$$M_{\text{inv}}^2(K^-K^{*0}) = (p_{K^-}^0 + P^0 - q^0)^2 - |\vec{p}'_{K^-} - \vec{q}|^2 \quad (47)$$

In the loop integration of Eq. (18), q^0 becomes $\omega_D(\vec{q})$ ($\omega_{D^*}(\vec{q})$) by taking the relevant positive energy parts of the intermediate propagators. Then, we have for the $K^- \pi^- D_{s0}^+$ production (the same for the $K^- \pi^- D_{s1}^+$ production by changing $D_{s0}^+ \rightarrow D_{s1}^+$ and $\omega_D(\vec{q}) \rightarrow \omega_{D^*}(\vec{q})$)

$$M_{\text{inv}}^2(K^-K^{*0}) = (p_{K^-}^0 + M_{\text{inv}}(\pi^- D_{s0}^+) - \omega_D(\vec{q}))^2 - (|\vec{p}'_{K^-}|^2 + |\vec{q}|^2 - 2\vec{p}'_{K^-} \cdot \vec{q}), \quad (48)$$

where \vec{p}'_{K^-} is the momentum of the K^- in the CM frame of $\pi^- D_{s0}^+$. We can see now that we have a dependence on a new angle, the one between \vec{p}'_{K^-} and \vec{q} which we define as $\theta_{K^-, q}$. Rather than redoing the formulation including three new angular integrals, we make an approximation of omitting the linear term in $|\vec{q}|$ because $|\vec{q}|$ is much smaller than $|\vec{p}'_{K^-}|$. Indeed, for the relevant $\pi^- D_{s0}^+$ invariant mass where the singularity appears we have $|\vec{p}'_{K^-}| = 3386$ MeV and $|\vec{q}| = 335$ MeV. This, plus the $\cos \theta_{K^-, q}$ angle dependence, justifies such an approximation. It essentially corresponds to taking $\cos \theta_{K^-, q} = 0$, but we check also the results for other angles, and make an average and evaluate the errors.

In Fig. 11, we show the mass distributions for the $B^- \rightarrow K^-K^{*0}D^0$ and $B^- \rightarrow K^-K^{*0}D^{*0}$ reactions as functions of $M_{\text{inv}}(K^-K^{*0})$. Since in the experiment of Ref. [39] the K^-K^{*0} mass distribution for the sum of the two decays is shown, we also sum the two distributions, but weighted by their respective branching ratios, 7.5×10^{-4} for $B^- \rightarrow K^-K^{*0}D^0$ and 1.5×10^{-3} for $B^- \rightarrow K^-K^{*0}D^{*0}$. It is curious that the second reaction shows a neat signal for the tail of the $a_1(1260)$, but this is not the case for the $B^- \rightarrow K^-K^{*0}D^0$. This is due to the \vec{p}_{K^-} factor in Eq. (1) that peaks where the K^-K^{*0} invariant mass is large. In the absence of the B_W factor, the mass distribution has a dip in

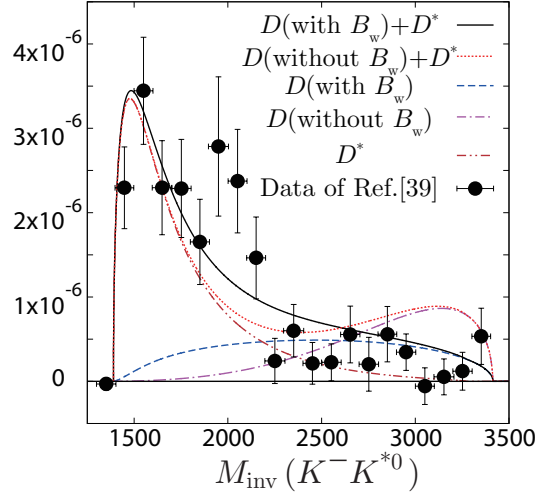


FIG. 11: The mass distributions $d\Gamma/dM_{\text{inv}}(K^- K^{*0})$ for the $B^- \rightarrow K^- K^{*0} D^0$ and $K^- K^{*0} D^{*0}$ processes, and their sum as functions of $M_{\text{inv}}(K^- K^{*0})$. The normalization of the data $dN/dM_{\text{inv}}(K^- K^{*0})$ from Ref. [39] has been fixed at the peak to agree with the calculated $d\Gamma/dM_{\text{inv}}(K^- K^{*0})$.

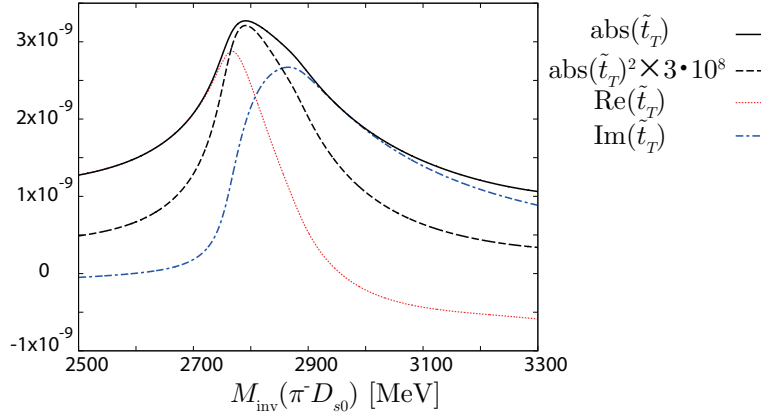


FIG. 12: The triangle amplitude for the $B^- \rightarrow K^- \pi^- D_{s0}^+$ process including $B_W(M_{\text{inv}}(K^- K^{*0}))$ of Eq. (38).

the low $K^- K^{*0}$ invariant mass region. It is noteworthy that the weighted sum of the two distributions agrees well with the experimental distribution of Ref. [39]. It would be very interesting to disentangle these two distributions to see if what we obtain agrees with the data, or one should rather construct more elaborated vertices. Actually, with the \vec{p}_K - momentum dependence in Eq. (1) which we have justified from the microscopic picture for the decay, it is not easy to generate the $K^* \bar{K}$ s -wave $a_1(1260)$, so we also show the results for the $B^- \rightarrow K^- K^{*0} D^0$ decay in the absence of the resonance. We see that the sum of the two distributions is not much different and still compatible with the experiment except at very large invariant mass.

In Fig. 12, we show now the triangle amplitude for the $B^- \rightarrow K^- \pi^- D_{s0}^+$ process corresponding to Fig. 5. The units are arbitrary as in Fig. 5. We can see that there are some minor modifications in the shape, but the structure of the triangle singularity remains. In Fig. 13, we show the triangle amplitude for the $B^- \rightarrow K^- \pi^- D_{s1}^+$ corresponding to Fig. 6. Once again, we can see the peak structure from the triangle singularity with some minor modifications from $B_W(K^- K^*)$ compared with Fig. 6.

Finally, in Figs. 14 and 15 we show the mass distributions for the $B^- \rightarrow K^- \pi^- D_{s0}^+$ and $B^- \rightarrow K^- \pi^- D_{s1}^+$ processes including the $B_W(M_{\text{inv}}(K^- K^{*0}))$ factor corresponding to Figs. 9 and 10, respectively. We show the results for the average of $\cos \theta_{K^-, q}$. Within the normalization obtained using the vertices of Eq. (7) and (29), the normalization in the figures is absolute. In Fig. 14, we find a small variation in the size, but the shape is similar to that obtained before. In Fig. 15, we find a similar behavior, but now the size has been reduced by about a factor of five. This reduces the integrated branching ratios to about 8×10^{-6} and 1×10^{-6} with errors of about 60%. Yet, the message is clear. The triangle singularities are very solid, and the absolute rates are still within the measurable range.

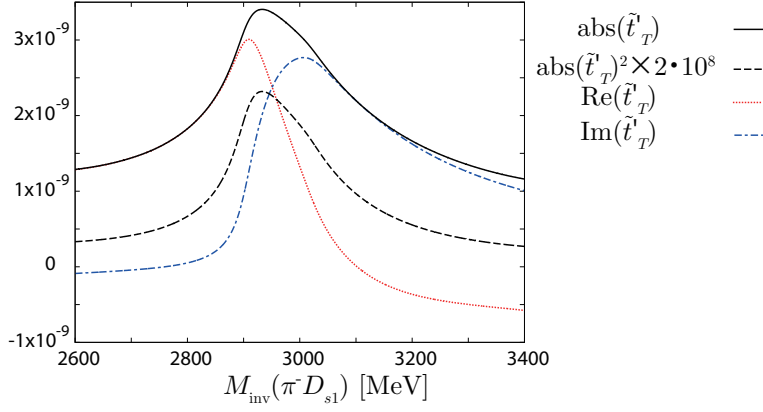


FIG. 13: The triangle amplitude for the $B^- \rightarrow K^- \pi^- D_{s1}^+$ process including $B_W(M_{\text{inv}}(K^- K^{*0}))$ of Eq. (38).

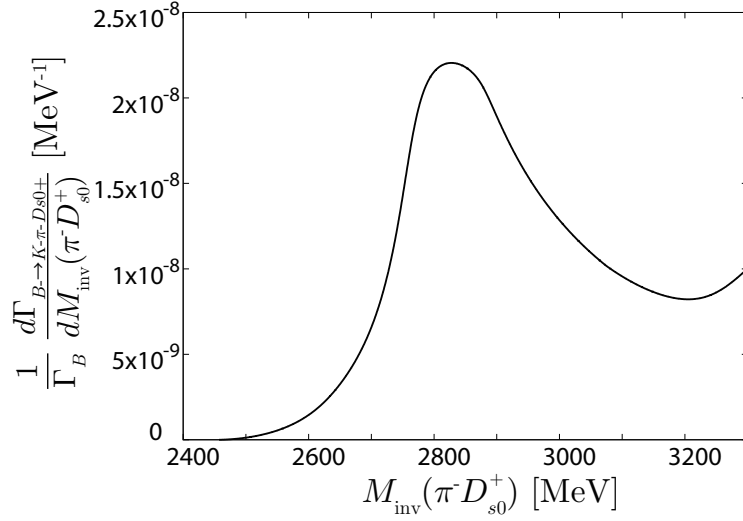


FIG. 14: The mass distribution $d\Gamma_{B^- \rightarrow K^- \pi^- D_{s0}^+} / dM_{\text{inv}}(\pi^- D_{s0}^+)$ as a function of $M_{\text{inv}}(\pi^- D_{s0}^+)$ including $B_W(M_{\text{inv}}(K^- K^{*0}))$ in Eq. (38).

It is also important to stress that the strength of the $B^- \rightarrow K^- \pi^- D_{s0}^+$ reaction is tied to the coupling of the $D_{s0}^+(2317)$ to DK , one important output of the picture in which the $D_{s0}^+(2317)$ would be basically a molecular state of DK . The same is applied to the reaction $B^- \rightarrow K^- \pi^- D_{s1}^+$ and the interpretation of the $D_{s1}^+(2460)$ as a D^*K molecule. The measurement of these reactions and comparison of their strength to the predictions done here would come to further support the molecular hypothesis for these states.

V. CONCLUSION

We have performed a study of the $B^- \rightarrow K^- \pi^- D_{s0}^+(2317)$ and $B^- \rightarrow K^- \pi^- D_{s1}^+(2460)$ reactions and shown that they develop a triangle singularity for an invariant mass of 2850 MeV in $\pi^- D_{s0}^+(2317)$ and 3000 MeV in $\pi^- D_{s1}^+(2460)$, respectively. This triangle singularity shows up as a peak in the invariant mass distribution of these pairs with an apparent width of about 200 MeV. The integrated strength in a region of about 400 MeV around the peaks gives branching ratios of about 8×10^{-6} and 1×10^{-6} , respectively, which are within present measurable range [1].

The singularity in the $B^- \rightarrow K^- \pi^- D_{s0}^+(2317)$ reaction is initiated by the transition $B^- \rightarrow K^- K^{*0} D^0$, followed by the $K^{*0} \rightarrow \pi^- K^+$ decay and the fusion of $K^+ D^0$ to give the $D_{s0}^+(2317)$ state. The choice of the $D_{s0}^+(2317)$ in the final state is motivated by the large coupling of DK to this resonance, which in several theoretical works, as well as from lattice QCD simulations, qualifies mostly as a DK molecule. The second reaction, $B^- \rightarrow K^- \pi^- D_{s1}^+(2460)$, proceeds in an analogous way, first the $B^- \rightarrow K^- K^{*0} D^0$ transition occurs, the K^{*0} decays into $\pi^- K^+$ and the $K^+ D^{*0}$ fuse

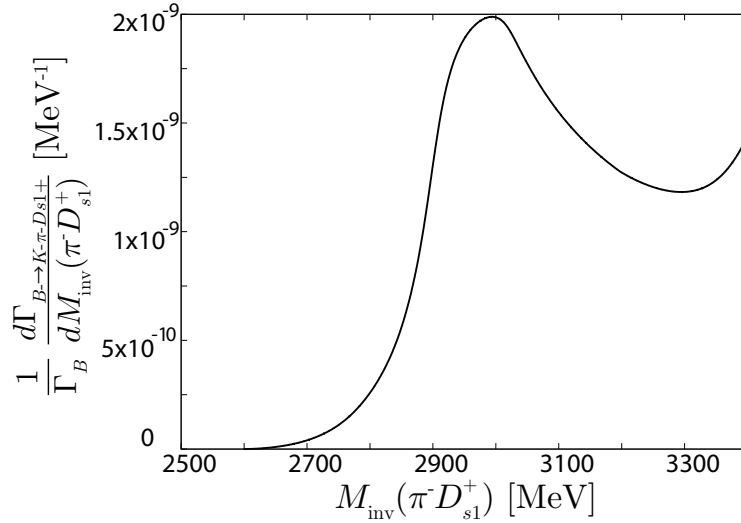


FIG. 15: The mass distribution $d\Gamma_{B^- \rightarrow K^- \pi^- D_{s1}^+} / dM_{\text{inv}}(\pi^- D_{s1}^+)$ as a function of $M_{\text{inv}}(\pi^- D_{s1}^+)$ including $B_W(M_{\text{inv}}(K^- K^{*0}))$ in Eq. (38).

to produce the $D_{s1}^+(2460)$, which also, according to theoretical calculations and lattice QCD simulations, corresponds to a molecular state mostly composed of D^*K .

The predictions found here constitute a clear case of a peak produced by a triangle singularity, which could be misidentified with a resonance when the experiment is done. This work has then the value of targeting a suitable reaction to identify a triangle singularity, and then having the results and the study ready to correctly interpret the peaks when they are observed. We also stressed that the observation of those reactions would provide further support for the molecular picture of the $D_{s0}^+(2317)$ and $D_{s1}^+(2460)$ states. With the steady advances in some of the experimental facilities, particularly, with the LHCb and Belle experiments, we hope that these measurements can be done in the near future, helping us get a better understating of hadronic physics.

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Appendix: The approximation of neglecting the ϵ^0 component in the K^* polarization

We take Eq. (12) and we write the covariant form of it

$$-it'_{K^{*0}, K^+ \pi^-} = + ig \epsilon_{K^{*0} \mu} (p_\pi - p_K)^\mu, \quad (\text{A.1})$$

then we calculate

$$\sum_{\text{pol}} |t'|^2 = g^2 \left(-g_{\mu\nu} + \frac{P_\mu^* P_\nu^*}{m_{K^*}^2} \right) (p_\pi - p_K)^\mu (p_\pi - p_K)^\nu = g^2 (\vec{p}_\pi - \vec{p}_K)^2 \quad (\text{A.2})$$

where P_μ^* , $p_{\pi\mu}$, and $p_{K\mu}$ are the momenta of K^* , π , and K in the rest frame of the K^* . Now, if we take Eq. (12),

$$-it_{K^{*0}, K^+ \pi^-} = - ig \epsilon_{K^{*0} j} (p'_\pi - p'_K)_j, \quad (\text{A.3})$$

we find

$$\sum_{\text{pol}} |t|^2 = g^2 \delta_{ij} (p'_\pi - p'_K)_i (p'_\pi - p'_K)_j = g^2 (\vec{p}'_\pi - \vec{p}'_K)^2 \quad (\text{A.4})$$

with \vec{p}'_π, \vec{p}'_K evaluated in the K^*D rest frame, where K^{*0} has 262 MeV/c. In order to calculate \vec{p}'_π and \vec{p}'_K we make a boost from the rest frame of K^{*0} to the one where it has a momentum $|\vec{p}^*| = 262$ MeV/c. This boost is readily done and we get the result

$$\vec{p}'_\pi = \left[\left(\frac{E^*}{m^*} - 1 \right) \frac{\vec{p}_\pi \cdot \vec{p}^*}{|\vec{p}^*|^2} + \frac{E_\pi}{m^*} \right] \vec{p}^* + \vec{p}_\pi \quad (\text{A.5})$$

and similar for the K , where $m^* = m_{K^*}$, $E^* = \sqrt{m^{*2} + |\vec{p}^*|^2}$, and \vec{p}_π, E_π are the momentum of the pion and its energy in the K^* rest frame. Upon integrating $\sum |t|^2$ over the solid angle, $\int d\Omega$, to get the decay width of the K^* , we readily obtain

$$\frac{\int d\Omega (|t|^2 - |t'|^2)}{\int d\Omega |t|^2} = \frac{(E_\pi - E_K)^2 |\vec{p}^*|^2}{4|\vec{p}_\pi|^2 m^{*2}} + O\left(\frac{|\vec{p}^*|^4}{m^{*4}}\right), \quad (\text{A.6})$$

which gives an effect of the order of 1.5%.

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