UNIVERSITAT DE BARCELONA

Gorenstein colength of local Artin k-algebras

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Universitat de Barcelona

Facultat de Matemàtiques i Informàtica

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Gorenstein colength of local Artin $\Bbbk$-algebras

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Roser Homs Pons

dirigida per

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Cover design: Joan Gorchs López.

The image of the cover represents the ring $A = \mathbf{k}[\![x, y, z]\!]/(x^2, xy, y^2, z^2)$, which has Gorenstein colength 2 and inverse system $I^\perp = \langle xz, yz\rangle$. $F = xyz$ generates a monomial minimal Gorenstein cover of A, namely $G = R/\operatorname{Ann}_R F$. See case 18 of $\ell(A) = 6$ of Poonen's classification (p.150).

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`https://github.com/zachscrivena/simple-thesis-dissertation`

Gorenstein colength of local Artin $\Bbbk$-algebras

Programa de doctorat de Matemàtiques i Informàtica.
Memòria presentada per aspirar al grau de Doctor en Matemàtiques per la Universitat de Barcelona.

Certifico que la present memòria ha estat realitzada per Roser Homs Pons i dirigida per mi.

Joan Elias Garcia
Maig del 2019

A en Joan

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Contents

Introduction

The purpose of this thesis is to determine how far is an Artin local ring from being Gorenstein and to study those Artin Gorenstein rings that reach this minimal distance.

Huneke claims in [30] that one of the most read articles in commutative algebra is the paper by Hyman Bass named *On the ubiquity of Gorenstein rings*, see [3]. As the title already points out, Gorenstein rings appear in a natural way in many different contexts. Around the decade of 1960, the work of Northcott and Rees on irreducible systems of parameters and Cohen-Macaulay rings, Gorenstein and Rosenlicht on plane curves and complete intersections, Grothendieck and Serre on duality and Bass on rings of finite injective dimension...all of it boiled down to the Gorenstein property.

Living up to their ubiquity expectations, Gorenstein rings appear today far beyond commutative algebra and algebraic geometry. According to Lam in [33], they are widely used in non-commutative algebra, arithmetic geometry, invariant theory, combinatorics and number theory. In fact, a key step of Andrew Wiles's proof of Fermat's Last Theorem involves understanding when certain Gorenstein rings are complete intersections, see [46, p.451].

Quoting Huneke in [30, p.76]: "Regular rings are the most basic rings in the study of commutative rings. However, Gorenstein rings are the next most basic and [...] one can approximate arbitrary local commutative rings quite closely by Gorenstein rings."

In this thesis we address the problem of approximating local rings by Gorenstein rings in the zero-dimensional case. The study of Artin Gorenstein rings is particularly relevant since a local Gorenstein ring of arbitrary dimension can always be turned into zero-dimensional Gorenstein when considering it modulo an ideal generated by a system of parameters.

Let A be an Artin local $\mathbf{k}$-algebra, where $\mathbf{k}$ is an arbitrary field. Hence we may assume that A is a quotient of the ring of formal power series $R = \mathbf{k}[\![x_1, \dots x_n]\!]$, for some integer $n \leq 1$, by an ideal I in R. We denote by $\mathfrak{m} = (x_1, \dots, x_n)$ the maximal ideal of R and by $\mathfrak{n} = \mathfrak{m}/I$ the maximal ideal of $A = R/I$. The length $\ell(A)$ of A stands for the dimension of A as $\mathbf{k}$-vector space.

In [1] Ananthnarayan introduces the notion of Gorenstein colength of an Artin local ring A, denoted by $\operatorname{gcl}(A)$, as the minimum $\ell(G) - \ell(A)$ where G is an Artin Gorenstein ring such that $A \simeq G/H$ for some ideal $H \subset G$, see Definition 1.3.3. We call any such ring G a Gorenstein cover of A. If, in addition, G reaches the $\operatorname{gcl}(A)$, then we say that it is a minimal Gorenstein cover. Notice that A is Gorenstein if and only if $\operatorname{gcl}(A) = 0$.

The next class of rings which are closest to be Gorenstein are Teter rings. In [44], Teter studied the rings that appear when considering an Artin Gorenstein ring G modulo its socle ideal, denoted by $\operatorname{soc}(G)$. Later on, Huneke and Vraciu improved Teter's caracterization in [31] and establish that non-Gorenstein rings $A \simeq G/\operatorname{soc}(G)$ are precisely those that satisfy $\operatorname{gcl}(A) = 1$.

Recall that the lengths of a ring A and its canonical module ω_A always coincide. Moreover, if A is Gorenstein, then $A \simeq \omega_A$. In other words, A is Gorenstein if and only if there is an epimorphism $\varphi : \omega_A \longrightarrow A$. In [31] Huneke and Vraciu proved that A is Teter if and only if there is an epimorphism $\varphi : \omega_A \longrightarrow \mathfrak{n}$. Ananthnarayan extends their result to any ring of low Gorenstein colength in [1, Theorem 5.5]:

THEOREM (Ananthnarayan) Let $A = R/I$ be an Artin ring and let $\mathfrak{m}$ be the maximal ideal of R. Suppose that $I \subseteq \mathfrak{m}^6$ and $\operatorname{char}(\mathbf{k}) \neq 2$. Then the following are equivalent:

(i) $\operatorname{gcl}(A) \leq 2$.
(ii) There exists an ideal $\mathfrak{q} \subseteq A$ such that $\mathfrak{q} \simeq \operatorname{Hom}_A(\mathfrak{q}, \omega_A)$ and $\ell(A/\mathfrak{q}) \leq 2$.
(iii) There exists an epimorphism $f : \omega_A \longrightarrow \mathfrak{q}$, where $\mathfrak{q}$ satisfies the properties in (ii).

Therefore, the approach of Teter, Huneke-Vraciu and Ananthnarayan is based on the existence of these A-module epimorphisms $\varphi : \omega_A \longrightarrow \mathfrak{q}$, where $\mathfrak{q}$ is an ideal of A which is self-dual with respect to the contravariant functor defined by the canonical module and satisfies $\ell(A/\mathfrak{q}) = \operatorname{gcl}(A)$.

An alternative approach is given by Elias and Silva in [20]. Since the canonical module ω_A of an Artin ring $A = R/I$ can be identified with Macaulay's inverse system $I^\perp$ of I, it is natural to apply all the tools available for this device. Considering the R-module structure of $S = \mathbf{k}[y_1, \dots, y_n]$ given by contraction, see Section 1.4.1, in [20, Theorem 3.4] the authors improve the results of Huneke-Vraciu:

THEOREM (Elias-Silva) Let $A = R/I$ be an Artin ring with $n \geq 2$, maximal ideal $\mathfrak{n}$ and socle degree $s \geq 1$. Then the following conditions are equivalent:

(i) $\operatorname{gcl}(A) = 1$.
(ii) There exists a degree $s+1$ polynomial $F \in S$ such that $I^\perp = \langle x_1 \circ F, \dots, x_n \circ F \rangle$.
(iii) There exists an epimorphism of A-modules $I^\perp \twoheadrightarrow \mathfrak{n}$.
(iv) A is a Teter ring.

In particular, if A is a Teter ring, then the Cohen–Macaulay type of A is n and $G = R/\operatorname{Ann_R} F$ is a minimal Gorenstein cover of A.

One of the main results we present in this thesis is the characterization of rings of low colength $A = R/I$ in terms of the relationship between $I^\perp$ and any inverse system $J^\perp$ associated to a minimal Gorenstein cover $G = R/J$ of A. This relation is measured by the colon ideal $K = (I^\perp :_R J^\perp)$, see Definition 2.1.5.

THEOREM (Theorem 2.1.7) Let $A = R/I$ be an Artin ring such that $\operatorname{gcl}(A) \leq 2$. If $G = R/J$ is a minimal Gorenstein cover of A and $K = (I^\perp :_R J^\perp)$, then

(i) $\operatorname{embd}(G) = \operatorname{embd}(A)$,
(ii) $I \subset K$ and $I^2 \subset J \subset I$.

Moreover, after a linear isomorphism of R we may assume:

$$K = \begin{cases} R, & \text{if} \quad \operatorname{gcl}(A) = 0; \\ \mathfrak{m}, & \text{if} \quad \operatorname{gcl}(A) = 1; \\ (x_1, \dots, x_{n-1}, x_n^2), & \text{if} \quad \operatorname{gcl}(A) = 2. \end{cases}$$

In addition, we provide an analogous characterization to Elias-Silva for rings of Gorenstein colength 2, which in turn, improves and extends Ananthnarayan's result:

THEOREM (Theorem 2.2.5) Let $A = R/I$ be an Artin ring with maximal ideal $\mathfrak{n}$ and socle degree $s \geq 1$. We assume that A is neither Gorenstein nor Teter, $I \subset \mathfrak{m}^5$ and $\operatorname{char}(\mathbf{k}) \neq 2$. Then the following conditions are equivalent:

(i) $\operatorname{gcl}(A) = 2$,

(ii) after a linear isomorphism of R there exists a polynomial $F \in S$ of degree $s+1$ or $s+2$ such that $I^\perp = \langle x_1 \circ F, \dots, x_{n-1} \circ F, x_n^2 \circ F \rangle$,

(iii) there exists an epimorphism of A-modules $f : I^\perp \longrightarrow \mathfrak{q}$, where $\mathfrak{q}$ is a self-dual ideal of A such that $\ell(A/\mathfrak{q}) = 2$.

In particular, if any of the previous equivalent conditions hold, $G = R/\operatorname{Ann}_{\mathrm{R}} F$ is a minimal Gorenstein cover of A.

For higher colength, that is, $\operatorname{gcl}(A) \geq 3$, the colon ideal $K = (I^\perp :_R J^\perp)$ has no longer unique analytic type as in Theorem 2.1.7. It may even have infinitely many analytic types when $\operatorname{gcl}(A) \geq 7$, see [40]. Therefore, the previous results cannot be extended to higher Gorenstein colength using analogous arguments, see Section 2.3.

After computing the Gorenstein colength of A and finding a minimal cover G of A, the natural question that arises is whether we can determine all minimal Gorenstein covers of A. In [20], Elias and Silva start addressing this problem for Teter rings. Observe that if $G = R/\operatorname{Ann}_{\mathrm{R}} F$ is a Teter cover of $A = R/I$, then $\langle F \rangle / I^\perp$ is a 1-dimensional sub-$\mathbf{k}$-vector space of $S_{\leq s+1}/I^\perp$, where $S_{\leq s+1}$ is the R-module of all polynomials of degree equal or less than $s+1$ with the contraction structure. Therefore, G defines a point $[\overline{F}]$ in the projective space over $S_{\leq s+1}/I^\perp$.

With this philosophy of identifying Teter covers with certain points of a suitable projective space $\mathbb{P}^N_{\mathbf{k}}$, in [20, Proposition 4.2] the authors introduce the notion of Teter variety $TC(A)$ of A.

THEOREM (Elias-Silva) The Teter variety $TC(A)$ of a Teter ring A is a non-empty Zariski open subset of a linear sub-variety of $\mathbb{P}^N_{\mathbf{k}}$. In particular, $TC(A)$ is an irreducible and non-singular variety of $\mathbb{P}^N_{\mathbf{k}}$.

In order to extend the idea of Teter variety to rings A with arbitrary colength t, we first need to determine where do polynomials F defining minimal Gorenstein covers $G = R/\operatorname{Ann}_{\mathrm{R}} F$ live. If A has socle degree s, then the R-module $S_{\leq s+t}$ would be the natural choice. Nevertheless, it can be refined to the smaller sub-R-module $\int_{\mathfrak{m}^t} I^\perp$ of $S_{\leq s+t}$ formed by polynomials F in S such that $\mathfrak{m}^t \circ F \subseteq I^\perp$.

We introduce this notion of integral of an R-module M with respect to an ideal K, denoted by $\int_K M$, that can be regarded as an inverse operation to contraction, see Definition 3.1.1. We provide a recursive procedure, Algorithm 1 (see p.74), to effectively compute the resulting module based on the integration method for inverse systems proposed by Mourrain in [39].

Our main contribution is the generalization of Teter varieties to varieties of minimal Gorenstein covers $MGC(A)$ via the following existence theorem:

THEOREM (Theorem 3.3.2) Let $A = R/I$ be an Artin ring of Gorenstein colength t. There exists a quasi-projective sub-variety $MGC(A)$ of $\mathbb{P}_{\mathbf{k}}\left(\int_{\mathfrak{m}^t} I^\perp/I^\perp\right)$ whose set of closed points are the points $[\overline{F}]$, $F \in \int_{\mathfrak{m}^t} I^\perp$, such that $G = R/\operatorname{Ann}_R F$ is a minimal Gorenstein cover of A.

We attack the problem of finding an explicit description of $MGC(A)$ from a computational point of view for rings of low Gorenstein colength.

THEOREM (Theorem 3.4.6) Let $A = R/I$ be a Teter ring with $n \geq 2$, let h be the dimension of $\int_{\mathfrak{m}} I^\perp/I^\perp$ as $\mathbf{k}$-vector space and let $\mathfrak{a}$ be the homogeneous ideal in a polynomial ring with h variables defined in Section 3.4.2. Then

$$MGC(A) = \mathbb{P}_{\mathbf{k}}^{h-1} \backslash \mathbb{V}_+(\mathfrak{a}).$$

Moreover, for any non-Gorenstein Artin ring A, $\operatorname{gcl}(A) = 1$ if and only if $\mathfrak{a} \neq 0$.

THEOREM (Corollary 3.4.20) Let $A = R/I$ be a ring of Gorenstein colength 2 and let h be the dimension of $\int_{\mathfrak{m}^2} I^\perp/I^\perp$ as $\mathbf{k}$-vector space. Let $\mathfrak{b}$ be a homogeneous ideal in the ring of polynomials with h variables and let $\mathfrak{a}$ and $\mathfrak{c}$ be bihomogeneous ideals in the ring of polynomials with $h+n$ variables as defined in Section 3.4.3. Let π_1 be the projection map from $\mathbb{P}_{\mathbf{k}}^{h-1} \times \mathbb{P}_{\mathbf{k}}^{n-1}$ to $\mathbb{P}_{\mathbf{k}}^{h-1}$. Then

$$MGC(A) = \mathbb{V}_+(\mathfrak{b}) \backslash \pi_1\left(\mathbb{V}_+(\mathfrak{c}) \cap \mathbb{V}_+(\mathfrak{a})\right).$$

In Algorithm 2 (see p.75) and Algorithm 3 (p.87) we provide methods to explicitly calculate the varieties of minimal Gorenstein covers for given rings of Gorenstein colength 1 and 2, respectively.

Next we focus on the study of Gorenstein covers in codimension 2. The approach in this setting is no longer from the inverse system perspective, but instead we use specific tools that only apply to $n = 2$ such as the Hilbert-Burch theorem. Hence we come across with the problem of determining canonical Hilbert-Burch matrices for any $\mathfrak{m}$-primary ideal I of $R = \mathbf{k}[\![x, y]\!]$.

In [8], Conca and Valla parametrize ideals in $\mathbf{k}[x, y]$ with a given leading term ideal E with respect to the lexicographical order. In particular, they parametrize the affine space of all $\mathfrak{m}$-primary ideals K in $\mathbf{k}[x, y]$ such that $\mathrm{Lt}_{\mathrm{lex}}(K) = E$ by defining a canonical Hilbert-Burch matrix of K. A similar result is provided by Constantinescu in [9] for the reverse-degree lexicographical order. See Section 4.1.1 for more details on these parametrizations.

Our main contribution is the extension of Conca-Valla parametrization of ideals in $\mathbf{k}[x, y]$ to the local setting by using a local degree ordering $\overline{\tau}$ induced by the lexicographical order, see Section 1.5. We define a canonical Hilbert-Burch matrix for any ideal with lex-segment leading term ideal L. In other words, we parametrize any $\mathfrak{m}$-primary ideal $K \subset R$ with a given Hilbert function h up to a generic change of coordinates, since $\mathrm{Gin}(K) = \mathrm{Lex}(h)$.

THEOREM (Theorem 4.1.24) Given a lex-segment ideal L in R with canonical Hilbert-Burch matrix H, see Definition 4.1.4, the set $V(L) = \{K \subset R : \mathrm{Lt}_{\overline{\tau}}(K) = L\}$ is an affine space parametrized by the bijection

$$\begin{array}{rccc} \Psi: & \mathcal{M}(L) & \longrightarrow & V(L) \\ & N & \longmapsto & I_t(H+N), \end{array}$$

where $\mathcal{M}(L)$ is the set of matrices with entries $n_{i,j}$ in $\mathbf{k}[y]$ from Definition 4.1.21. Any ideal K in $V(L)$ can be identified with a point p_K in $\mathbb{A}^{\mathbf{N}}_{\mathbf{k}}$ by taking coordinates the coefficients $c^k_{i,j}$ of polynomials $n_{i,j}$.

In particular, this result allows to take $N + H$, with $N = \Psi^{-1}(K)$, as definition of canonical Hilbert-Burch matrix of any ideal K in $V(L)$. Thanks to the connection between the minimal number of generators of an ideal with the rank of a Hilbert-Burch matrix, see Corollary 4.2.3, we can explicitly describe the Gorenstein ideals J in $V(L)$.

COROLLARY (Corollary 4.2.9) Let L be a lex-segment ideal. The set $V_G(L)$ of Gorenstein ideals J such that $\mathrm{Lt}_{\overline{\tau}}(J) = L$ is a quasi-affine variety. In particular,

$$V_G(L) \simeq \mathbb{A}^{\mathbf{N}}_{\mathbf{k}} \backslash \mathbb{V}(c^0_{3,1} \cdots c^0_{t+1,t}).$$

In order to use this approach to find Gorenstein covers $G = R/J$ of a given ring

$A = R/I$, we require the inclusion $J \subset I$, which happens to be a closed condition on variables $c_{i,j}^k$:

COROLLARY (Corollary 4.2.11) Let $A = R/I$ be an Artin ring and let L be a lex-segment ideal. The set $V_{GC(A)}(L)$ of ideals J in $V(L)$ such that $G = R/J$ is a Gorenstein cover of A is a quasi-affine variety. In particular,

$$V_{GC(A)}(L) \simeq \mathbb{V}(p_1, \ldots, p_r) \backslash \mathbb{V}(c_{3,1}^0 c_{4,2}^0 \cdots c_{t+1,t-1}^0),$$

where $c_{i,j}^k$ are the coefficients of the polynomials $n_{i,j}$ in $\mathbf{k}[y]$ of matrices N in $\mathcal{M}(L)$ and p_l are polynomials in variables $c_{i,j}^k$ that occur as coefficients of the reduction of J modulo I.

However, inclusion is not preserved by a generic change of coordinates, hence it is not enough to parametrize ideals with lex-segment leading term ideal when we want to find Gorenstein covers. For a general $\mathfrak{m}$-primary monomial ideal E of R, we give the following result on the set $V(E)$:

PROPOSITION (Proposition 4.1.9) Let E be a monomial $\mathfrak{m}$-primary ideal in R with canonical Hilbert-Burch matrix H, let $V(E)$ be the set of ideals K of R such that $\mathrm{Lt}_{\overline{\tau}}(K) = E$ and let $\mathcal{N}(E)$ be the set of matrices from Definition 4.1.8. Then there is a surjection

$$\begin{array}{rccc} \varphi: & \mathcal{N}(E) & \longrightarrow & V(E) \\ & N & \longmapsto & I_t(H+N). \end{array}$$

Since Proposition 4.1.9 does not provide a notion of canonical Hilbert-Burch matrix for ideals K with monomial leading term ideal $\mathrm{Lt}_{\overline{\tau}}(K) = E$, we cannot replicate the parametrization in Corollary 4.2.11 for $V_{GC(A)}(E)$. Moreover, imposing the Gorenstein property on K requires more effort than in Corollary 4.2.9.

Nevertheless, in Algorithm 4 (see p.125), we propose a routine to compute the affine variety $\mathbb{V}(\mathfrak{a})$ in $\mathbb{A}_{\mathbf{k}}^{\mathbf{N}}$ whose points correspond to non-Gorenstein ideals J in $V(E)$, even though different points might correspond to the same ideal. Since the treatment of the inclusion of ideals $J \subset I$ does not vary, we can ensure that the quasi-affine variety $\mathbb{V}(p_1, \ldots, p_r) \backslash \mathbb{V}(\mathfrak{a})$, where $p_1, \ldots, p_r$ are built as in Corollary 4.2.9, consists of all points p_J that correspond to Gorenstein covers $G = R/J$ of $A = R/I$. Again, this is not a parametrization but it allows us to sweep $V(E)$ for Gorenstein covers.

All the computations in this thesis have been done with the commutative algebra software *Singular*, [11]. We use the *Singular* library **InverseSyst.lib** for inverse system related computations, see [13] for a manual on how to use the library. All the algorithms appearing in this work have been implemented in a new library **GorensteinCovers.lib** created for the purpose of computing Gorenstein covers, see Appendix A.

Let us provide an outline of the contents and structure of this thesis.

In Chapter 1 we provide all the necessary background, adapted to the scope of this work, about Artin and Gorenstein rings, Hilbert functions, Macaulay's inverse systems and how to extend results from the graded setting to the local case.

Chapter 2 is devoted to the study of low Gorenstein colength rings and establishes a connection among Macaulay inverse systems, minimal Gorenstein covers and self-dual ideals.

The first main result of this chapter provides a characterization of rings of low Gorenstein colength in terms of its inverse systems:

THEOREM (See Theorem 2.1.7.) Let A be an Artin ring such that $\operatorname{gcl}(A) \leq 2$. If G is a minimal Gorenstein cover of A, then

(i) $\operatorname{embd}(G) = \operatorname{embd}(A)$,
(ii) if $A = R/I$ with $\dim(R) = \operatorname{embd}(G) = \operatorname{embd}(A)$ and F is a generator of $J^\perp$, $G = R/J$, then $I \subset K_F$ and
$$I^2 \subset J \subset I.$$

Moreover, after a linear isomorphism of R we may assume:

$$K_F = \begin{cases} R, & \text{if} \quad \operatorname{gcl}(A) = 0; \\ \mathfrak{m}, & \text{if} \quad \operatorname{gcl}(A) = 1; \\ (x_1, \dots, x_{n-1}, x_n^2), & \text{if} \quad \operatorname{gcl}(A) = 2. \end{cases}$$

The second essential result is Theorem 2.2.5, which extends and improves the characterization of Artin rings $A = R/I$ of Gorenstein colength two in [1, Theorem 5.5]:

THEOREM (See Theorem 2.2.5.) Let $A = R/I$ be an Artin ring with maximal ideal $\mathfrak{n}$ and socle degree $s \geq 1$. We assume that A is neither Gorenstein nor Teter, $I \subset \mathfrak{m}^5$ and $\operatorname{char}(\mathbf{k}) \neq 2$. Then the following conditions are equivalent:

(i) $\mathrm{gcl}(A) = 2$,

(ii) after a linear isomorphism of R there exists a polynomial $F \in S$ of degree $s+1$ or $s+2$ such that $I^\perp = \langle x_1 \circ F, \dots, x_{n-1} \circ F, x_n^2 \circ F \rangle$,

(iii) there exists an epimorphism of A-modules $f : I^\perp \longrightarrow \mathfrak{q}$, where $\mathfrak{q}$ is a self-dual ideal of A such that $\ell(A/\mathfrak{q}) = 2$.

As a closure of the chapter, we address the complexity of the generalization of these results to rings of higher colength.

In Chapter 3 we study minimal Gorenstein covers of an Artin ring A. We start with the introduction of the notion of integral of a module with respect to an ideal and the extension of Mourrain's integration method to compute it.

The main achievement of this chapter is Theorem 3.3.2, that proves the existence of a quasi-projective sub-variety $MGC^n(A)$ of $\mathbb{P}_{\mathbf{k}}\left(\int_{\mathfrak{m}^t} I^\perp / I^\perp\right)$ whose set of closed points are associated to polynomials F in S such that the ring $G = R/\operatorname{Ann_R} F$ is a minimal Gorenstein cover of A. This result allows us to extend the notion of Teter variety by Elias-Silva to a minimal Gorenstein cover variety $MGC(A)$ for rings A with arbitrary Gorenstein colength and to give a precise description of the $MGC(A)$ variety for $\mathrm{gcl}(A) \leq 2$ as follows:

THEOREM (See Theorem 3.4.6.) Let $A = R/I$ be a Teter ring with $n \geq 2$, let h be the dimension of $\int_{\mathfrak{m}} I^\perp / I^\perp$ as $\mathbf{k}$-vector space and let $\mathfrak{a}$ be the homogeneous ideal defined in Section 3.4.2 in a polynomial ring with h variables. Then

$$MGC(A) = \mathbb{P}_{\mathbf{k}}^{h-1} \backslash \mathbb{V}_+(\mathfrak{a}).$$

Moreover, for any non-Gorenstein Artin ring A, $\mathrm{gcl}(A) = 1$ if and only if $\mathfrak{a} \neq 0$.

THEOREM (See Corollary 3.4.20.) Let $A = R/I$ be a ring of Gorenstein colength 2 and let h be the dimension of $\int_{\mathfrak{m}^2} I^\perp / I^\perp$ as $\mathbf{k}$-vector space. Let $\mathfrak{b}$ be a homogeneous ideal in the ring of polynomials with h variables and let $\mathfrak{a}$ and $\mathfrak{c}$ be bihomogeneous ideals in the ring of polynomials with $h+n$ variables as defined in Section 3.4.3. Let π_1 be the projection map from $\mathbb{P}_{\mathbf{k}}^{h-1} \times \mathbb{P}_{\mathbf{k}}^{n-1}$ to $\mathbb{P}_{\mathbf{k}}^{h-1}$. Then

$$MGC(A) = \mathbb{V}_+(\mathfrak{b}) \backslash \pi_1\left(\mathbb{V}_+(\mathfrak{c}) \cap \mathbb{V}_+(\mathfrak{a})\right).$$

We end the chapter by providing algorithms to explicitly compute $MGC(A)$ for low Gorenstein colength and several computation examples.

Chapter 4 deals with rings of codimension 2. The first part of the chapter is devoted to the extension of Conca-Valla parametrization of ideals in $\mathbf{k}[x,y]$ to the local setting. The main results presented here are the complete parametrization of all the ideals with lex-segment leading term ideal and the partial analogous for general $\mathfrak{m}$-primary monomial ideals.

THEOREM (See Theorem 4.1.24.) Given a lex-segment ideal L in R with canonical Hilbert-Burch matrix H, the set $V(L) = \{K \subset R : \mathrm{Lt}_{\overline{\tau}}(K) = L\}$ is an affine space parametrized by the bijection

$$\begin{array}{rccc} \Psi: & \mathcal{M}(L) & \longrightarrow & V(L) \\ & N & \longmapsto & I_t(H+N), \end{array}$$

where $\mathcal{M}(L)$ is the set of matrices from Definition 4.1.21.

PROPOSITION (See Proposition 4.1.9.) Consider a monomial $\mathfrak{m}$-primary ideal E in R with canonical Hilbert-Burch matrix H, let $V(E)$ be the set of ideals K of R such that $\mathrm{Lt}_{\overline{\tau}}(K) = E$ and let $\mathcal{N}(E)$ be the set of matrices from Definition 4.1.8. The map

$$\begin{array}{rccc} \varphi: & \mathcal{N}(E) & \longrightarrow & V(E) \\ & N & \longmapsto & I_t(H+N), \end{array}$$

is surjective.

In the second part of the chapter we focus on constructing Gorenstein covers from the canonical Hilbert-Burch matrices defined by the previous parametrizations. The main result in this part is the parametrization of all Gorenstein covers $G = R/I$ of $A = R/I$ that occur as a deformation of a lex-segment ideal L:

COROLLARY (See Corollary 4.2.11.) Let $A = R/I$ be an Artin ring and let L be a lex-segment ideal. The set $V_{GC(A)}(L)$ of ideals J in $V_G(L)$ such that $G = R/J$ is a Gorenstein cover of A is a quasi-affine variety. In particular,

$$V_{GC(A)}(L) \simeq \mathbb{V}(p_1,\dots,p_r)\backslash\mathbb{V}(c^0_{3,1}c^0_{4,2}\cdots c^0_{t+1,t-1}),$$

where $c^k_{i,j}$ are the coefficients of the polynomials $n_{i,j}$ in $\mathbf{k}[y]$ of matrices N in $\mathcal{M}(L)$

and p_l are polynomials in variables $c_{i,j}^k$ that occur as coefficients of the reduction of J modulo I.

We also provide a method, Algorithm 4, to compute all Gorenstein covers $G = R/J$ of $A = R/I$ that occur as a deformation of any monomial ideal E, where the parametrization in Corollary 4.2.11 is no longer valid.

Chapter 5 is devoted to the study of certain families of Artin rings such as stretched $\mathbf{k}$-algebras or monomial ideals. On one hand, we study in depth all analytic types of $\mathbf{k}$-algebras A with $\ell(A) \leq 6$ taking as guide Poonen's classification of such algebras in [40]. On the other hand, we put special emphasis on understanding whether the properties of minimal Gorenstein covers from Theorem 2.1.7 hold for higher colength $\operatorname{gcl}(A) > 2$:

PROPOSITION (See Proposition 5.0.3.) Let $A = R/I$ be an Artin ring. In the following cases we have that there exists a minimal Gorenstein cover $G = R/J$ of A such that $\operatorname{embd}(G) = \operatorname{embd}(A)$ and $I^2 \subset J \subset I$:

(i) $\ell(A) \leq 6$,
(ii) A is stretched,
(iii) $I = \mathfrak{m}^t$ for some $t \geq 1$,

Moreover, the preservation of the embedding dimension works for all minimal Gorenstein covers of stretched rings.

Appendix A consists on a manual on how to use the library **GorensteinCovers.lib**. One of its most relevant features is the description and comparison of 3 different methods to compute Macaulay's inverse systems.

In Appendix B, we recall the structure theorem of stretched and Gorenstein almost stretched $\mathbf{k}$-algebras in terms of their analytic types, summarizing the fundamental results of [21] and [15].

In Appendix C, we provide the explicit expression of varieties of minimal Gorenstein covers $MGC(A)$ of all low Gorenstein colength $\mathbf{k}$-algebras A such that $\ell(A) \leq 6$ up to analytic type.

Notation

R	ring of formal power series in variables $x_1, \ldots, x_n$ and coefficients in $\mathbf{k}$
$\mathfrak{m}$	unique maximal ideal $(x_1, \ldots, x_n)$ of R
A	equicharacteristic Artin local ring R/I
$\mathfrak{n}$	maximal ideal $\mathfrak{m}/I$
$\mathbf{k}$	residue field of R, residue field of A
P	polynomial ring in variables $x_1, \ldots, x_n$ and coefficients in $\mathbf{k}$
S	polynomial ring in variables $y_1, \ldots, y_n$ and coefficients in $\mathbf{k}$
$S_{\leq d}$	polynomials in S of degree equal or less than d
$\dim A$	Krull dimension of A
$\dim_{\mathbf{k}} A$	$\mathbf{k}$-vector space dimension of A
$\mathrm{char}(A)$	characteristic of the ring A
$\circ$	contraction operation
$\mathrm{Ann}_{\mathrm{R}}(M)$	annihilator of the R-module M
$\mathrm{Ann}_{\mathrm{A}}(\mathfrak{q})$	annihilator of the ideal $\mathfrak{q}$ of A
$\mathrm{depth}_R(M)$	depth of the R-module M

$\mathrm{embd}(A)$	embedding dimension of A
$GC(A)$	set of Gorenstein covers of A
$\mathrm{gcl}(A)$	Gorenstein colength of an Artin ring
$\mathrm{Gin}(I)$	generic initial ideal of I
$Gr_{\mathfrak{n}}(A)$	associated graded ring of A with respect to the maximal ideal $\mathfrak{n}$
$\mathrm{ht}(I)$	height of the ideal I
HF_A	Hilbert function of the associated graded ring of A
$(-)^{\times}$	continuous dual space
$(-)^{\vee}$	$\mathrm{Hom}_R(-, E)$
$(-)^{*}$	$\mathrm{Hom}_R(-, R)$
$\mathfrak{q}$	ideal of A
f^*	initial form of $f \in R$
I^*	initial ideal of I
$\mathrm{id}_A(M)$	injective dimension of the A-module M
$E_A(M)$	injective hull of the A-module M
E	injective hull of the residue field $\mathbf{k}$ of R
$\int_K M$	integral of the R-module M with respect to the ideal K in R
$\mathcal{L}_{A,t}$	R-module $\int_{\mathfrak{m}^t} I^{\perp}/I^{\perp}$, where $A = R/I$
$I^{\perp}$	inverse system of I
$\langle F_1, \dots, F_r \rangle_{\mathbf{k}}$	$\mathbf{k}$-vector space S generated by polynomials $F_1, \dots, F_r$
$\mathrm{Lt}_{\overline{\tau}}(I)$	leading term ideal of the I with respect to a local ordering $\overline{\tau}$
$\mathrm{LC}_{\overline{\tau}}(f)$	leading coefficient of the series $f \in R$ with respect to $\overline{\tau}$
$\mathrm{Lt}_{\overline{\tau}}(f)$	leading term of the series $f \in R$ with respect to $\overline{\tau}$

$\ell(M)$	length of an A-module, dimension as $\mathbf{k}$-vector space
$\mathrm{Lex}(h)$	lex-segment ideal associated to the Hilbert function h
$MGC(A)$	variety of minimal Gorenstein covers of A
$\mu(I)$	minimal number of generators of the ideal I in R
ω_A	canonical module of A
$\mathrm{ord}(f)$	order of $f \in R$
$\mathrm{pd}_A(M)$	projective dimension of the A-module M
$\mathbb{F}_\bullet$	free resolution
$\mathbb{F}_\bullet^*$	dual free resolution with respect to $(-)^*$
$\langle F_1, \dots, F_r \rangle$	sub-R-module of S generated by polynomials $F_1, \dots, F_r$ with respect to the contraction structure
$\mathrm{soc}\, A$	socle ideal of the ring A
$\mathrm{socdeg}\, A$	socle degree of the ring A
$\mathrm{Supp}(f)$	support of a series $f \in R$
$\mathrm{Syz}(M)$	module of syzygies of an R-module M
$\tau(A)$	Cohen-Macaulay type of A
τ	term ordering in P
$\overline{\tau}$	local ordering in R induced by τ in P
$TC(A)$	Teter variety of A
$V(E)$	set of ideals J in $\mathbf{k}[[x, y]]$ with $\mathrm{Lt}_{\overline{\tau}}(J) = E$
$V_G(E)$	set of Gorenstein ideals J in $V(E)$
$V_{GC(A)}(E)$	set of ideals J in $V(E)$ such that $G = R/J$ is a Gorenstein cover of $A = R/I$.

CHAPTER 1

Preliminaries

In this first chapter, besides fixing the notation, we will provide the necessary background to understand both the object of our study and the different tools we will apply. In order to keep it to a reasonable number of pages, some elementary commutative algebra notions will not be defined or only a partial definition restricted to the zero-dimensional case will be given. For complete proofs and general results, see [5].

1.1 Artin and Gorenstein rings

According to Cohen's structure theorems, any local equicharacteristic Artin ring A is isomorphic to a quotient of the regular local ring $R = \mathbf{k}[\![x_1, \ldots x_n]\!]$, for some $n \geq 1$, by an $\mathfrak{m}$-primary ideal I of R, where $\mathfrak{m} = (x_1, \ldots, x_n)$ is the unique maximal ideal of R. From now on, whenever we consider an Artin ring we refer to

$$A \simeq \mathbf{k}[\![x_1, \ldots x_n]\!]/I,$$

with maximal ideal $\mathfrak{n} = \mathfrak{m}/I$ and residue field $\mathbf{k}$. We denote by $\ell(A)$ the length of A, that is, the dimension of A as $\mathbf{k}$-vector space.

DEFINITION 1.1.1 The **socle degree** of $A = R/I$ is the smallest integer s such that $\mathfrak{m}^{s+1} \subseteq I$ and it is denoted by $\operatorname{socdeg} A$.

Note that we can also characterize the socle degree as the largest integer s such that $\mathfrak{n}^s \neq 0$.

DEFINITION 1.1.2 The **socle** of the Artin ring $A = R/I$, denoted by $\mathrm{soc}(A)$, is the annihilator of the maximal ideal $\mathfrak{n}$ in A, that is, $\mathrm{soc}(A) := \mathrm{Ann}_{\mathrm{A}}(\mathfrak{n})$.

Observe that $\mathrm{soc}(A)$ is the largest ideal of A such that the A-module structure gives at the same time an $A/\mathfrak{n}$-module structure on it. Therefore, the socle ideal is the largest ideal equipped with a natural $\mathbf{k}$-vector space structure.

DEFINITION 1.1.3 The **Cohen-Macaulay type** of $A = R/I$, denoted by $\tau(A)$, is the dimension of the socle ideal $\mathrm{soc}(A)$ as $\mathbf{k}$-vector space.

Note that $0 \neq \mathfrak{n}^{\mathrm{socdeg}\, A} \subseteq \mathrm{soc}(A) \subseteq A$, hence $1 \leq \tau(A) \leq \ell(A)$.

EXAMPLE 1.1.4 Fields have socle degree 0 and Cohen-Macaulay type 1. Indeed, the unique maximal ideal of $\mathbf{k}$ is (0) and $\mathrm{soc}\, \mathbf{k} = \mathbf{k}$.

Let us assume that in the representation R/I of A we are choosing the ring of power series with a minimal number of variables n. In other words, $\mathrm{ht}(I) = \dim R$, where $\mathrm{ht}(I)$ stands for the height of the ideal I. Using the well-known Auslander-Buchsbaum formula and the Cohen-Macaulayness of both R and A, we have

$$\mathrm{pd}_R(A) = \mathrm{depth}_R(R) - \mathrm{depth}_R(A) = \dim R - \dim A = \mathrm{ht}(I) = n,$$

where $\mathrm{pd}_R(A)$ denotes the projective dimension of A as R-module. Therefore, we have a minimal free resolution of A as R-module of length n

$$\mathbb{F}_{\bullet} : 0 \longrightarrow R^{b_n} \xrightarrow{\phi_n} R^{b_{n-1}} \xrightarrow{\phi_{n-1}} \dots \xrightarrow{\phi_2} R^{b_1} \xrightarrow{\phi_1} R \longrightarrow A \longrightarrow 0.$$

The Cohen-Macaulay type can also be retrieved from the last Betti number of A, that is, $\tau(A) = b_n$.

DEFINITION 1.1.5 Consider the left exact contravariant functor $(-)^* = \mathrm{Hom}_R(-, R)$. The **canonical module**, denoted by ω_A, of an Artin ring $A \simeq R/I$ is the cokernel of the dual map $\phi_n^* : \left(R^{b_{n-1}}\right)^* \longrightarrow \left(R^{b_n}\right)^*$. In literature it is often also called dualizing module.

It can be proved that $\mathrm{Ann}_{\mathrm{R}}(\omega_A) = I$, hence the canonical module ω_A is also an A-module.

REMARK 1.1.6 The cohomology of $\mathbb{F}_{\bullet}^*$ are precisely the $\mathrm{Ext}_R(A, R)$ modules. Observe that $\operatorname{Ext}_R^i(A,R) = 0$ for any $i \neq n$ and $\operatorname{Ext}_R^n(A,R) \simeq \omega_A$. In fact, $\mathbb{F}_\bullet^*$ is a free resolution of ω_A as R-module:

$$\mathbb{F}_\bullet^* : 0 \longrightarrow R^\times \xrightarrow{\phi_1^*} \dots \xrightarrow{\phi_{n-1}^*} \left(R^{b_{n-1}}\right)^* \xrightarrow{\phi_n^*} \left(R^{b_n}\right)^* \longrightarrow \omega_A \longrightarrow 0.$$

We denote by $E_A(\mathbf{k})$ the injective hull of the residue field, that is, the minimal injective A-module containing $\mathbf{k}$. Since A is Artin local, thanks to Matlis theorem, any injective module is isomorphic to a power of the unique indecomposable injective A-module $E_A(\mathbf{k})$. It can be proved that the canonical module ω_A is isomorphic to the injective hull $E_A(\mathbf{k})$.

DEFINITION 1.1.7 We denote by $\operatorname{id}_A(A)$ the **injective dimension** of A as A-module, that is, the length of the minimal exact sequence of injective A-modules

$$0 \longrightarrow A \longrightarrow E_A(\mathbf{k})^{c_0} \xrightarrow{d^0} E_A(\mathbf{k})^{c_1} \xrightarrow{d^1} \dots \xrightarrow{d^{i-1}} E_A(\mathbf{k})^{c_i} \xrightarrow{d^i} \dots .$$

In the zero-dimensional case, all rings are Cohen-Macaulay. Therefore, any Artin ring $A = R/I$ can be placed in one of the layers of the following hierarchy:

$$\begin{array}{c} \text{regular local ring} \\ \Downarrow \\ \text{complete intersection ring} \\ \Downarrow \\ \text{Gorenstein ring} \\ \Downarrow \\ \text{Cohen-Macaulay ring} \end{array}$$

Artin regular local rings are fields. Zero-dimensional complete intersections are quotients of $R = \mathbf{k}[\![x_1, \dots x_n]\!]$ by an ideal I generated by a regular sequence of n elements.

We now want to focus on Artin Gorenstein rings. To finish this section, we will provide several equivalent characterizations of such rings in terms of the socle ideal, injective dimension or canonical modules, to mention a few of them. For a complete review on Gorenstein rings both in arbitrary and zero dimension, see [30].

DEFINITION 1.1.8 A **zero-dimensional Gorenstein** ring is an Artin local ring with minimal socle ideal, that is, 1-dimensional socle.

In other words, A is Gorenstein if and only if $\tau(A) = 1$. Therefore, in terms of minimal resolutions of A, this can be translated into $b_n = 1$. In fact, in this situation, the Betti numbers $b_n, \dots, b_0$ are symmetric around the middle of the resolution.

We can also approach the characterization of Gorenstein rings in terms of its canonical module: A is Gorenstein if and only if its canonical module ω_A is a free A-module of rank 1, that is, $\omega_A \simeq A$. In this case, $\mathbb{F}_\bullet^*$ can also be regarded as a free resolution of a A. Following this philosophy, we can say that free resolutions of a Gorenstein ring are self-dual.

From the point of view of injective modules, Gorenstein rings are precisely rings of finite injective dimension. In dimension zero, this can be translated into self-injective rings $A \simeq E_A(\mathbf{k})$.

Now let us now summarize the previous equivalent characterizations of Gorenstein zero-dimensional rings:

THEOREM 1.1.9 Let A be an Artinian local ring. The following are equivalent:

(i) $\mathrm{id}_A(A) < \infty$.
(ii) $\mathrm{id}_A(A) = 0$.
(iii) $A \cong \omega_A$.
(iv) A is injective as a module over itself.
(v) $A \cong E_A(\mathbf{k})$.
(vi) $\mathrm{soc}(A)$ is a 1-dimensional k-vector space, i.e. $\tau(A) = 1$.
(vii) The ideal (0) in A is irreducible.
(viii) For every ideal $\mathfrak{q}$ in A, $(0 :_A (0 :_A \mathfrak{q})) = \mathfrak{q}$.

1.2 Hilbert functions

The Hilbert function of a local ring A with maximal ideal $\mathfrak{n}$ is defined as the Hilbert function of the associated graded ring $Gr_\mathfrak{n}(A) = \sum_{i\geq 0} \mathfrak{n}^i/\mathfrak{n}^{i+1}$, hence $\mathrm{HF}_A : \mathbb{N} \longrightarrow \mathbb{N}$ with

$$\mathrm{HF}_A(i) = \dim_\mathbf{k} \mathfrak{n}^i/\mathfrak{n}^{i+1}.$$

By definition, $\mathrm{HF}_A(0) = 1$.

DEFINITION 1.2.1 We call the **embedding dimension** of A, denoted by $\operatorname{embd}(A)$, the value of the Hilbert function of A evaluated at 1, that is, $\operatorname{embd}(A) := \operatorname{HF}_A(1)$.

Observe that, if $I \subset \mathfrak{m}^2$, then $\operatorname{embd}(A) = \dim R$. Hence, in the representation $A = R/I$, we can always choose R to have $\operatorname{HF}_A(1)$ variables.

Again from the definition, $\operatorname{HF}_A(i) = 0$ for all $i > \operatorname{socdeg} A$. Therefore, the Hilbert function produces a finite succession of integers $\{1, n, \operatorname{HF}_A(2), \dots, \operatorname{HF}_A(s)\}$, where s is the socle degree of A.

Besides having finitely many non-zero values, a lot more is known about the shape of Hilbert functions of Artin rings.

DEFINITION 1.2.2 The expansion

$$c = \binom{c_n}{n} + \binom{c_{n-1}}{n-1} + \cdots + \binom{c_j}{j},$$

such that $c_n > c_{n-1} > c_j \geq j \geq 1$, is called the **Macaulay's n-th representation of** c. The values $c_n, \dots, c_j$ are called **Macaulay's n-th coefficients of** c.

Such a decomposition of c exists and it is unique. The algorithm of this construction is simple: take the greatest c_n satisfying $c \geq \binom{c_n}{n}$. Repeat the step changing c for $c - \binom{c_n}{n}$ and n for $n-1$. Proceed in a similar way until the difference is zero or we reach c_1.

DEFINITION 1.2.3 For any $n \geq 1$, we define $0^{\langle n \rangle} = 0$, and for $c \geq 1$,

$$c^{\langle n \rangle} = \binom{c_n + 1}{n+1} + \binom{c_{n-1}+1}{n} + \cdots + \binom{c_j + 1}{j+1}.$$

The following result describes exactly how Hilbert functions of Artin rings look like. Even more, it says that given any such numerical function, there exist an Artin ring realizing it.

THEOREM 1.2.4 Let $F : \mathbb{N} \longrightarrow \mathbb{N}$ be a numerical function. The following are equivalent:

(i) Exists an Artin local ring $A = R/I$ such that $\operatorname{HF}_A(i) = F(i)$, for any $i \geq 0$.

(ii) $F(0) = 1$, $F(i+1) \leq F(i)^{\langle i \rangle}$, for all $i \geq 1$, and $F(i) = 0$ for i large enough.

1.2.1 Hilbert functions of Gorenstein rings

What do Hilbert functions of Artin Gorenstein rings $G = R/I$ look like? A lot of literature exists on this subject, see [32]. In the general case, the known results only provide necessary conditions on numerical functions in order to correspond to Hilbert functions of Gorenstein rings. However, in codimension 2, they are explicitly characterized.

Consider a Gorenstein ring G of socle degree s. Note that $\mathfrak{n}^s = \operatorname{soc} G$ is a one dimensional $\mathbf{k}$-vector space and $\mathfrak{n}^{s+1} = 0$, hence $\mathrm{HF}_G(s) = 1$. Therefore, any Hilbert function associated to a zero-dimensional Gorenstein ring must be of the form $\{1, n, \mathrm{HF}_G(2), \ldots, \mathrm{HF}_G(s-1), 1\}$.

Another useful tool is the so-called shell formula provided by Iarrobino in [32]. Again, it only enables us to discard some particular numerical functions from being the Hilbert function of a Gorenstein ring whenever it fails the test, but it never ensures this Gorenstein ring exists whenever it passes the test.

The idea of the shell formula is based on the fact that in the graded situation we know that Gorenstein rings have symmetric Hilbert functions. We will briefly introduce the Q-decomposition of a Gorenstein algebra G, that allows us to link the Hilbert function of G with the Hilbert function of a suitable Gorenstein graded algebra.

Consider the Artin Gorenstein ring $G = R/I$ of socle degree s and its associated graded ring

$$Gr_{\mathfrak{n}}(G) = \bigoplus_{i=0}^{s+1} \mathfrak{n}^i/\mathfrak{n}^{i+1}.$$

Several different filtrations can be considered in G. Combining the standard $\mathfrak{n}$-adic filtration $\{\mathfrak{n}^i\}_{i\geq 0}$ and the Löwy filtration $\{(0 :_G \mathfrak{n}^i)\}_{i\geq 0}$, we can define for any a in $\{0, 1, \ldots, s+1\}$ the $Gr_{\mathfrak{n}}(G)$-module

$$C(a)_i = \frac{(0 : \mathfrak{n}^{s+1-a-i}) \cap \mathfrak{n}^i}{(0 : \mathfrak{n}^{s+1-a-i}) \cap \mathfrak{n}^{i+1}} \subseteq G_i,$$

where G_i denotes the piece of degree i of the graded ring $Gr_{\mathfrak{n}}(G)$. Then

$$C(a) = \bigoplus_{i\geq 0} C(a)_i$$

is a graded $Gr_{\mathfrak{n}}(G)$-module and, in particular, $G_i C(a)_j \subseteq C(a)_{i+j}$.

PROPOSITION 1.2.5 [45, Proposition 7.1.1] With the previous notations one has:

(i) $C(0)_i = 0$ for all $i \geq s$.
(ii) If $a \geq 1$ then $C(a)_i = 0$ for all $i \geq s - a$.
(iii) $Gr_{\mathfrak{n}}(G) = C(0) \supset C(1) \supset \cdots \supset C(s) = 0$.
(iv) $C(a)$ is a $\mathbf{k}$-vector space of finite dimension.

DEFINITION 1.2.6 For any $a \in \{0, 1, \ldots, s-1\}$ we define the graded $Gr_{\mathfrak{n}}(G)$-module

$$Q(a) = C(a)/C(a+1).$$

The set $\{Q(0), Q(1), \ldots, Q(s-1)\}$ is called the **Q-decomposition** of G.

PROPOSITION 1.2.7 [45, Proposition 7.1.4] Let G be an Artin Gorenstein ring of socle degree s. Then $Q(0) = Gr_{\mathfrak{n}}(G)/C(1)$ is, up to isomorphism, the only Artinian graded Gorenstein quotient of $Gr_{\mathfrak{n}}(G)$ of socle degree s.

PROPOSITION 1.2.8 [45, Proposition 7.1.5] Let G be an Artinian Gorenstein ring of socle degree s. The following facts are equivalent:

(i) $Gr_{\mathfrak{n}}(G)$ is Gorenstein of socle degree s;
(ii) $C(1) = 0$;
(iii) $C(a) = 0$, for all $a \geq 1$;
(iv) $Q(a) = 0$, for all $a \geq 1$;
(v) $Gr_{\mathfrak{n}}(G) \simeq Q(0)$.

In general, the associated graded algebra $Gr_{\mathfrak{n}}(G)$ of a Gorenstein ring G is not Gorenstein. In fact, $Gr_{\mathfrak{n}}(G)$ is Gorenstein if and only if HF_G is symmetric, see [45, Theorem 7.2.6].

THEOREM 1.2.9 (Shell formula) Let G be an Artinian Gorenstein ring of socle degree s. Then, for all $i \geq 0$,

$$\mathrm{HF}_G(i) = \sum_{a=0}^{s-1} H_{Q(a)}(i),$$

where $H_{Q(a)}$ are symmetric functions satisfying $H_{Q(a)}(i) = H_{Q(a)}(s-a-i)$ and $\mathrm{HF}_{Q(0)}$ satisfies Macaulay's conditions.

EXAMPLE 1.2.10 Consider an Artin ring A with Hilbert function $\{1,3,4,1\}$. If A is Gorenstein, then

$$\mathrm{HF}_A(i) = \sum_{a=0}^{2} \mathrm{HF}_{Q(a)}(i),$$

where $\mathrm{HF}_{Q(a)}$ are symmetric functions. In particular, $\mathrm{HF}_{Q(1)}(0) = \mathrm{HF}_{Q(1)}(2)$ and $\mathrm{HF}_{Q(2)}(0) = \mathrm{HF}_{Q(2)}(1)$. Hence any Q-decomposition of A has the following possible associated Hilbert function decomposition:

i	0	1	2	3
$H_A(i)$	1	3	4	1
$Q(0)$	1	b	b	1
$Q(1)$	0	c	0	0
$Q(2)$	0	0	0	0

Note that | is the symmetry axis of $\mathrm{HF}_{Q(0)}$ and | is the symmetry axis of $\mathrm{HF}_{Q(2)}$.

But this decomposition is not possible because $b = 4$ and $b + c = 3$. Therefore, there exists no Gorenstein ring with Hilbert function $\{1,3,4,1\}$.

In codimension 2, there is a numerical characterization of the Hilbert function of the ring $A = R/I$ in terms of the minimal number of generators $\mu(I)$ of I, see [4] for more details:

THEOREM 1.2.11 Let $F = \{1, 2, \dots, d, h_d, \dots, h_s\}$ be a numerical function satisfying $d = h_{d-1} \geq h_d \geq h_{d+1} \geq \cdots \geq h_s \geq 1$, let $p = \max\{h_{j-1} - h_j : j \geq d\}$ and let m be a positive integer. The following facts are equivalent:

(i) There exists an ideal $I \subseteq R = \mathbf{k}[\![x,y]\!]$ such that $\mathrm{HF}_{R/I} = F$ and $\mu(I) = m$.

(ii) $p + 1 \leq m \leq d + 1$.

Gorenstein rings of codimension 2 are complete intersections, hence they are of the form $A = \mathbf{k}[\![x,y]\!]/I$, where I is minimally generated by two elements. From Theorem 1.2.11, it follows that the jump between two consecutive elements of its Hilbert function cannot be bigger than 1.

EXAMPLE 1.2.12 Any ring with Hilbert function $\{1,2,3,4,3,3,1\}$ will not be Gorenstein, whereas $\{1,2,3,4,3,3,2,1,1\}$ does admit a Gorenstein ring.

1.3 Gorenstein covers and Gorenstein colength

The following fact is a well-known commutative algebra result:

LEMMA 1.3.1 Let $A = R/I$ be a local Artin ring. Then A is a quotient of an Artin Gorenstein ring $G = R/J$.

In fact, G can be taken as Nagata's idealization $G = A \ltimes \omega_A$, see [5, Theorem 3.3.6].

DEFINITION 1.3.2 We say that an Artin Gorenstein $\mathbf{k}$-algebra G is a **Gorenstein cover** of A if there is a power series ring $R = \mathbf{k}[\![x_1, \dots, x_n]\!]$ such that $A \cong R/I$, $G \cong R/J$ and $J \subset I$. We denote by $GC(A)$ the set of Gorenstein covers of A.

Then we can define the Gorenstein colength of A as follows:

DEFINITION 1.3.3 The **Gorenstein colength** of A is

$$\operatorname{gcl}(A) = \min\{\ell(G) - \ell(A) \mid G \text{ is a Gorenstein cover of } A\}.$$

A Gorenstein cover G of an Artin ring A is **minimal** if $\ell(G) = \ell(A) + \operatorname{gcl}(A)$.

1.4 Inverse systems

Inverse systems are a useful tool to deal with local Artin $\mathbf{k}$-algebras and, in a more general setting, to study isolated points in a variety. Some properties of ideals in R that have a difficult computational approach have a particularly nice translation into inverse systems: quotient ideals, elimination of variables or even differential equations. See [23, Sections 7.1.5-7.1.8] for more details.

Macaulay's inverse systems are the main tool we use along this thesis to study the Gorenstein colength of an Artin ring A and to find minimal Gorenstein covers G of A. We devote this section to the introduction of the basic notions surrounding inverse systems. In Section 1.4.1, inverse systems are introduced as Matlis duals, see [14] for more details. On the other hand, in Section 1.4.2 inverse systems are presented as orthogonal $\mathbf{k}$-vector spaces, see [39] and [23].

1.4.1 Matlis and Macaulay dualities

Let E be the injective hull $E_R(\mathbf{k})$ of the residue field $\mathbf{k}$ of R. Recall that the contravariant functor $(-)^\vee = \operatorname{Hom}_R(-, E)$ is exact.

DEFINITION 1.4.1 Given an R-module M, we call $\operatorname{Hom}_R(M, E_R(\mathbf{k}))$, denoted by $M^\vee$, the **Matlis dual** of M.

Matlis duals satisfy the following properties, see [14]:

THEOREM 1.4.2 Let M be a finitely generated R-module. Then

(i) $R^\vee \simeq E$ and $E^\vee \simeq R$.
(ii) $\ell(M) = \ell(M^\vee)$.
(iii) $\mathbf{k} \simeq \mathbf{k}^\vee$, $R \simeq R^{\vee\vee}$ and $E \simeq E^{\vee\vee}$.
(iv) E is Artin.

The previous result applies to a more general setting of noetherian complete local rings, but we restrict to the ring of formal power series R for the sake of simplicity, given that it is enough for the scope of this work. Now we state the well-known Matlis duality in our setting:

THEOREM 1.4.3 **(Matlis duality)** The functor $(-)^\vee$ defines an anti-equivalence between finitely generated R-modules and Artin R-modules. $(-)^\vee$ is the identity functor in both the category of R-modules and the category of Artin R-modules. In particular, if M is either a finitely generated R-module or an Artin R-module, $M^{\vee\vee} \simeq M$.

Let $S = \mathbf{k}[y_1, \ldots, y_n]$ be the polynomial ring with n variables and let us we denote by $\mathfrak{m}$ the homogeneous maximal ideal $(y_1, \ldots, y_n)$ of S. The ring S can be considered as an R-module by contraction:

$$\begin{array}{ccc} R \times S & \longrightarrow & S \\ (x^\alpha, y^\beta) & \mapsto & x^\alpha \circ y^\beta = \begin{cases} y^{\beta-\alpha}, & \beta \geq \alpha; \\ 0, & otherwise. \end{cases} \end{array}$$

Note that we are using multi-index notation: $\alpha = (\alpha_1, \ldots, \alpha_n)$ in $\mathbb{N}^n$, $|\alpha| = \sum_{i=1}^n \alpha_i$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We say that $\beta \geq \alpha$ if and only if $\beta_i \geq \alpha_i$ for all $1 \leq i \leq n$.

If $\mathrm{char}(\mathbf{k}) = 0$ then S is also an R-module with the module structure induced by the usual derivation.

THEOREM 1.4.4 **(Gabriel)** [26] If $\mathbf{k}$ is of characteristic zero then

$$E_R(\mathbf{k}) \cong (S, derivation) \cong (S, contraction).$$

If $\mathbf{k}$ is of positive characteristic then $E_R(\mathbf{k}) \cong (S, contraction)$.

Since the characteristic of the ground field $\mathbf{k}$ is arbitrary, from now on we will use the structure of S as R-module defined by contraction.

We denote by $\langle F_1, \dots, F_r\rangle$ the sub-R-module of S generated by polynomials F_1, $\dots$, F_r of S. Note that $\langle F_1, \dots, F_r\rangle$ can also be regarded as a $\mathbf{k}$-vector space generated by all the contractions of $F_1, \dots, F_r$.

DEFINITION 1.4.5 Given an $\mathfrak{m}$-primary ideal $I \subset R$, we call the **Macaulay inverse system** of I, denoted by $I^\perp$, the sub-R-module $\{g \in S \mid I \circ g = 0\}$ of S. Given a sub-R-module M of S, we denote by $M^\perp$ the ideal $\{f \in R \mid f \circ g = 0 \text{ for all } g \in M\}$ of R.

Observe that the inverse system of I is precisely the Matlis dual of the Artin R-module R/I:

$$\begin{array}{rcl} \text{Artin } R\text{-modules} & \longleftrightarrow & \text{finitely generated } R\text{-modules} \\ R/I & \longmapsto & (R/I)^\vee = I^\perp \\ M^\vee = M^\perp & \longleftarrow\!\!\shortmid & M \end{array}$$

Now, from Theorem 1.4.3, we can recover the classical result of Macaulay, see [36], [24] and [32].

PROPOSITION 1.4.6 **(Macaulay's duality)** There is an order-reversing bijection $\perp$ between the set of finitely generated sub-R-submodules of S and the set of $\mathfrak{m}$-primary ideals of R given by: if M is a submodule of S, then $M^\perp = (0 :_R M)$ and $I^\perp = (0 :_S I)$ for an ideal $I \subset R$. Moreover, $A = R/I$ is Gorenstein of socle degree s if and only if $I^\perp$ is a cyclic R-module generated by a polynomial of degree s.

Observe that we can identify $I^\perp \simeq (0 :_S I) \simeq (0 :_{E_R(\mathbf{k})}) \simeq E_A(\mathbf{k}) \simeq \omega_A$. For the sake of simplicity, we will only use the inverse systems notation, that is, $I^\perp$.

1.4.2 The orthogonal of an ideal

The goal of this section is to introduce inverse systems in an analogous way as Mourrain and Elkadi did in their book [23]. Thus we can use the tools presented in [39] to deal with inverse systems, that is, the integration method (see Chapter 3). Their framework is more general but we will focus on the local zero-dimensional case. See [24] for more details.

The ring $R = \mathbf{k}[\![x_1, \dots, x_n]\!]$ is a topological $\mathbf{k}$-vector space with the $\mathfrak{m}$-adic topology. The field $\mathbf{k}$ can also be endowed with a topological structure by considering the discrete topology.

DEFINITION 1.4.7 We denote by $R^\times$ the **continuous dual space** of R, that is, the $\mathbf{k}$-vector space of continuous $\mathbf{k}$-linear maps $\varphi : R \longrightarrow \mathbf{k}$.

REMARK 1.4.8 Observe that $R^\times$ is a sub-$\mathbf{k}$-vector space of the dual space of R, that is, $\mathrm{Hom}_{\mathbf{k}}(R, \mathbf{k})$.

LEMMA 1.4.9 $\varphi : R \longrightarrow \mathbf{k}$ is continuous with respect to the $\mathfrak{m}$-adic topology in R and the discrete topology in $\mathbf{k}$ if and only if $\varphi(\mathfrak{m}^t) = 0$ for some $t \geq 0$.

Proof: Recall that 0 is an open set in $\mathbf{k}$ with respect to the discrete topology, then it is enough to check that $\ker \varphi$ is an open set with respect to the $\mathfrak{m}$-adic topology. □

Therefore, any continuous $\mathbf{k}$-linear map $\varphi \in R^\times$ is completely determined by its image at finitely many monomials x^α in R. Set

$$\begin{array}{cccc} \varphi : & R & \longrightarrow & \mathbf{k} \\ & x^\alpha & \longmapsto & d_\alpha \end{array}$$

such that $d_\alpha = 0$ for any $|\alpha| \geq t$, where t is some positive integer. The image of any series $f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \in R$ can be defined by $\mathbf{k}$-linearity as

$$\varphi(f) = \varphi\left(\sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha\right) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \varphi(x^\alpha).$$

Note that this is well-defined because only finitely many terms in the formal sum are non-zero. We can think of $(d_\alpha)_{\alpha \in \mathbb{N}^n}$ as a sequence in $\bigoplus_{\alpha \in \mathbb{N}^n} \mathbf{k}$, hence $\Lambda = \sum_{\alpha \in \mathbb{N}^n} d_\alpha y^\alpha$

is a polynomial in $S = \mathbf{k}[y_1, \ldots, y_n]$.

Recalling the contraction structure we defined in the previous section, we have

$$\left(x^\alpha \circ y^\beta\right)(0) = \begin{cases} 1, & \text{if } \alpha = \beta; \\ 0, & \text{otherwise.} \end{cases}$$

Given the polynomial $\Lambda = \sum_{\alpha \in \mathbb{N}^n} d_\alpha y^\alpha$ in S, note that $(x^\alpha \circ \Lambda)(0) = d_\alpha$ for any $\alpha \in \mathbb{N}^n$, hence we get the following maps:

$$\begin{array}{ccccc} R^\times & \longleftrightarrow & \bigoplus_{\mathbb{N}^n} \mathbf{k} & \longleftrightarrow & S \\ & & & & \\ \varphi & \longmapsto & (\varphi(x^\alpha))_{\alpha \in \mathbb{N}^n} & \longmapsto & \sum_{\alpha \in \mathbb{N}^n} \varphi(x^\alpha) y^\alpha \\ \begin{array}{ccc} R & \to & \mathbf{k} \\ x^\alpha & \mapsto & (x^\alpha \circ \Lambda)(0) \end{array} & \longleftarrow & ((x^\alpha \circ \Lambda)(0))_{\alpha \in \mathbb{N}^n} & \longleftarrow & \Lambda \end{array}$$

PROPOSITION 1.4.10 (See [24, Proposition 1].) The map $\lambda : R^\times \longrightarrow S$ defined by $\lambda(\varphi) = \sum_{\alpha \in \mathbb{N}^n} \varphi(x^\alpha) y^\alpha$ is an isomorphism of topological $\mathbf{k}$-vector spaces.

Moreover, we can define an R-module structure on $R^\times$ via the multiplicative action

$$\begin{array}{cccc} g \cdot \varphi : & R & \longrightarrow & \mathbf{k} \\ & f & \longmapsto & \varphi(gf) \end{array}$$

for any $g \in R$ and $\varphi \in R^\times$. Note that $g \cdot \Lambda$ is indeed $\mathbf{k}$-linear and continuous.

PROPOSITION 1.4.11 Consider the R-module structure in S given by contraction. Then λ is an isomorphism of R-modules.

Proof: For any $\beta \in \mathbb{N}^n$, define φ_β such that $\varphi_\beta(x^\alpha) = \delta_{\alpha,\beta}$, where $\delta_{\alpha,\beta}$ is the Kronecker delta. Note that this is the dual $\mathbf{k}$-basis in $R^\times$. Consider $x^\gamma \in R$, then

$$\lambda(x^\gamma \cdot \varphi_\beta) = \sum_{\alpha \in \mathbb{N}^n} (x^\gamma \cdot \varphi_\beta)(x^\alpha) y^\alpha = \sum_{\alpha \in \mathbb{N}^n} \varphi_\beta(x^{\gamma+\alpha}) y^\alpha = y^{\beta-\gamma}.$$

On the other hand,

$$x^\gamma \circ \lambda(\varphi_\beta) = x^\gamma \circ \sum_{\alpha \in \mathbb{N}^n} \varphi_\beta(x^\alpha) y^\alpha = x^\gamma \circ y^\beta = y^{\beta-\gamma}.$$

Therefore, the R-module structures on $R^\times$ and S given by $\cdot$ and $\circ$, respectively, are compatible. $\square$

From now on we will identify maps $\varphi : R \longrightarrow \mathbf{k}$ in $R^\times$ with polynomials Λ in S. Observe that the multiplication by x_i acts on the elements Λ of S as the product by the inverse of the variable y_i. Indeed, taking $\Lambda = y^\beta$, we get

$$(x_i \cdot y^\beta)(f) = y^\beta(x_i f) = \left((x_i f) \circ y^\beta\right)(0) = \left(f \circ (x_i \circ y^\beta)\right)(0) =$$
$$\left(f \circ y_1^{\beta_1} \cdots y_i^{\beta_i - 1} \cdots y_n^{\beta_n}\right)(0) = y_1^{\beta_1} \cdots y_i^{\beta_i - 1} \cdots y_n^{\beta_n}(f),$$

for any $f \in R$. That is, x_i can be identified with y_i^{-1}, and this justifies the terminology of inverse systems as claimed in [23].

DEFINITION 1.4.12 Given an $\mathfrak{m}$-primary ideal $I \subseteq R$, we define its **orthogonal** as the sub-$\mathbf{k}$-vector space of $R^\times$ given by

$$I^\perp = \{\Lambda \in R^\times : \Lambda(f) = 0 \text{ for any } f \in I\}.$$

REMARK 1.4.13 The $\mathfrak{m}$-primality condition on I arises naturally when we require I to be contained in $\ker \varphi$ for a continuous map φ.

Note that the definition of orthogonal ideal is consistent with the notion of inverse system in Definition 1.4.5:

PROPOSITION 1.4.14 Let I be an ideal in R. Then

$$\{\Lambda \in S \mid f \circ \Lambda = 0 \text{ for any } f \in I\} = \{\Lambda \in S \mid (f \circ \Lambda)(0) = 0 \text{ for any } f \in I\}.$$

Proof: The right inclusion is direct. If $(f \circ \Lambda)(0) = 0$ for any $f \in I$, in particular it holds for a system of generators $f_1, \ldots, f_m$ of I and hence, for any $1 \leq i \leq m$,

$$f_i \circ \Lambda = \sum_{1 \leq |L| \leq N} a_L y^L \in S, \quad a_L \in \mathbf{k}.$$

Consider the highest non-zero term $a_L y^L$ of $f_i \circ \Lambda$, then

$$x^L \circ (f_i \circ \Lambda) = a_L \in \mathbf{k}.$$

But $x^L f_i \in I$, hence $\left(x^L f_i\right) \circ \Lambda = 0$. Therefore, $a_L = 0$ and hence $f_i \circ \Lambda = 0$. $\square$

Observe that the elements of $I^\perp$ can be regarded as continuous $\mathbf{k}$-linear maps on R/I. Consider the projection map $\pi : R \longrightarrow R/I$. For any $\Lambda' : R/I \longrightarrow \mathbf{k}$ we obtain a linear map $\Lambda' \circ \pi$ on R. For any $f \in I$, $(\Lambda' \circ \pi)(f) = 0$ and hence $\Lambda' \circ \pi \in I^\perp$. On the other hand, consider a linear map Λ on R such that it vanishes on all polynomials in I, that is, $I \subseteq \ker(\Lambda)$. Then it factors through π in the sense that there exists $\Lambda' \in (R/I)^*$ such that $\Lambda = \Lambda' \circ \pi$. Therefore, π induces an isomorphism

$$\begin{array}{rccc} \pi_* : & (R/I)^* & \longrightarrow & I^\perp \\ & \Lambda' & \longmapsto & \Lambda' \circ \pi \end{array}$$

REMARK 1.4.15 Note that the continuous dual space $(R/I)^*$ is, in fact, the dual space of R/I. Indeed, the $\mathfrak{m}$-primality of I ensures continuity of any $\mathbf{k}$-linear map $\varphi : R \longrightarrow \mathbf{k}$ that vanishes on I, since $\mathfrak{m}^t \subseteq I \subseteq \ker \varphi$ for some $t \geq 0$, see Lemma 1.4.9.

Hence Macaulay's duality can then be reprashed as:

THEOREM 1.4.16 The $\mathfrak{m}$-primary ideals in R are in bijection with sub-$\mathbf{k}$-vector spaces of S stable by contraction.

See [24], [36], [28] and [37] for more details about these bijections:

$$\left\{ \begin{array}{c} \mathfrak{m}\text{-primary} \\ \text{ideals of } R \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{finitely generated} \\ \text{sub-}R\text{-modules of } S \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{finitely generated} \\ \mathbf{k}\text{-vector spaces of } S \\ \text{stable by contraction} \end{array} \right\}$$

REMARK 1.4.17 Observe that in [23], I is considered as an ideal of the ring of polynomials S and $I^\perp$ is defined in the dual of S, which is isomorphic to the ring of power series R. Consider the maximal ideal $\mathfrak{m}_S$ in S corresponding to the point at the origin in $\mathbf{k}^n$ and assume that I is an $\mathfrak{m}_S$-primary ideal. By [23, Proposition 7.30], it can be proved that $I^\perp$ is actually formed only by polynomials. On the other hand, since $I \subset \mathfrak{m}_S$ contains no polynomials with non-zero constant terms, the extension IR of I in R is $\mathfrak{m}$-primary and $S/I \simeq R/IR$. Therefore, I can be regarded as an $\mathfrak{m}$-primary ideal of R and $I^\perp$ as a sub-R-module of S.

1.4.3 Dictionary

Well-known results and properties can be reproved using inverse systems. A paradigmatic example is Lemma 1.3.1, where adding this tool simplifies the proof considerately:

LEMMA 1.4.18 (See Lemma 1.3.1.) Let $A = R/I$ be a local Artin ring. Then A is a quotient of an Artin Gorenstein ring $G = R/J$.

Proof: If s is the socle degree of A then $I^\perp$ is generated by polynomials of degree at most s, by [20, Proposition 2.5], so $I^\perp \subset S_{\leq s}$. Since $S_{\leq s} \subset \langle y_1^s \cdots y_n^s \rangle$, the ideal $J = \operatorname{Ann_R}(y_1^s \cdots y_n^s)$ satisfies the claim. □

Moreover, several invariants of Artin rings can be translated easily in terms of inverse systems. Next we will provide some examples. For extended details, see [14].

PROPOSITION 1.4.19 Let $A = R/I$ be an Artin local ring. For any $i \geq 0$,

$$(I^\perp)_i := \frac{I^\perp \cap S_{\leq i} + S_{<i}}{S_{<i}}$$

is an R-module with the contraction structure and

- for all $i \leq 0$, $\operatorname{HF}_A(i) = \dim_{\mathbf{k}} (I^\perp)_i$;
- $(\operatorname{soc} A)^\vee = I^\perp / \left(\mathfrak{m} \circ I^\perp\right)$;
- $\tau(A) = \dim_{\mathbf{k}} I^\perp / \left(\mathfrak{m} \circ I^\perp\right) = \mu(I^\perp)$.

In particular, if $n = 2$, then $\mu(I) = \tau(A) + 1$.

1.5 From graded rings to local rings

In this section, we will recall some facts around the idea of extending results from graded rings to local rings. In particular, we introduce local orderings, standard basis and Grauert's division theorem.

REMARK 1.5.1 Let $P = \mathbf{k}[x_1, \dots, x_n]$. Given a zero-dimensional ring $A = R/I$, we can find polynomial generators $f_1, \dots, f_m$ of I. Hence

$$\frac{R}{I} \simeq \frac{P}{(f_1, \dots, f_m)P}.$$

We will abuse notation and denote by I the ideals generated by $f_1, \dots, f_m$ in both R and P, which will always be clear by context.

If I is a homogeneous ideal, then $A = R/I$ can be identified with the graded ring P/I and everything we know about graded rings applies. However, if I is not homogeneous we can still find a homogeneous ideal J and a monomial ideal E in P such that

$$\mathrm{HF}_{R/I} = \mathrm{HF}_{P/J} = \mathrm{HF}_{P/E}\,.$$

From both the computational and theoretical point of view, to deal with ideals, it is extremely practical to extend the notion of Gröbner basis in a polynomial ring P to the local case. To that end, we need to define a total ordering compatible with the local structure.

Any element $f \in R$ can be written as $\sum_{|\alpha| \geq 0} c_\alpha x^\alpha$.

DEFINITION 1.5.2 We define the **order** of $f \in R$ as $\mathrm{ord}(f) = \min\{|\alpha| : c_\alpha \neq 0\}$ and the **initial form** of $f \in R$ as the homogeneous polynomial $f^* = \sum_{|\alpha| = \mathrm{ord}(f)} c_\alpha x^\alpha$.

DEFINITION 1.5.3 Given an ideal $I \subset R$, we define the **initial ideal** of I as the homogeneous ideal of P generated by the initial forms, i.e.

$$I^* = \langle f^* : f \in I \rangle_P \subset P = \mathbf{k}[x_1, \dots, x_n].$$

DEFINITION 1.5.4 We call $f_1, \dots, f_m$ a **standard basis of** I if $I = (f_1, \dots, f_m)$ and $I^* = (f_1^*, \dots, f_m^*)$. That is, $f_1, \dots, f_m$ is a system of generators of I, not necessarily minimal, that also generates its initial ideal.

Consider a term ordering τ in P. This induces a reverse-degree ordering $\overline{\tau}$ in R such that for any monomials m, m' in R, $m >_{\overline{\tau}} m'$ if and only if

$$\deg(m) < \deg(m')$$

or

$$\deg(m) = \deg(m') \text{ and } m >_\tau m'.$$

We will call $\overline{\tau}$ a **local degree ordering** induced by the global ordering τ.

DEFINITION 1.5.5 We call the **support** of $f = \sum_{|\alpha|\geq 0} c_\alpha x^\alpha \in R$, the set

$$\operatorname{Supp}(f) := \{x^\alpha : c_\alpha \neq 0\}.$$

We define the **leading term** of $f \in R$ with respect to $\overline{\tau}$, denoted by $\operatorname{Lt}_{\overline{\tau}}(f)$, as the monomial $x^\alpha \in \operatorname{Supp}(f)$ such that $x^\alpha \geq_{\overline{\tau}} m$ for any $m \in \operatorname{Supp}(f)$. $\operatorname{LC}_{\overline{\tau}}(f) = c_\alpha$ is its **leading coefficient** and $\operatorname{tail}(f) = f - \operatorname{LC}_{\overline{\tau}}(C)\operatorname{Lt}_{\overline{\tau}}(f)$ is its **tail**.

DEFINITION 1.5.6 Given an ideal $I \subset \mathbf{k}[\![x_1, \ldots, x_n]\!]$, we define the **leading term ideal** of I as the monomial ideal in P generated by the leading terms with respect to the local degree ordering $\overline{\tau}$, i.e.

$$\operatorname{Lt}_{\overline{\tau}}(I) = \langle \operatorname{Lt}_{\overline{\tau}}(f) : f \in I \rangle_P \subset P = \mathbf{k}[x_1, \ldots, x_n].$$

DEFINITION 1.5.7 We call $f_1, \ldots, f_m$ a $\overline{\tau}$**-enhanced standard basis of** I if $I = (f_1, \ldots, f_m)$ and $\operatorname{Lt}_{\overline{\tau}}(I) = (\operatorname{Lt}_{\overline{\tau}}(f_1), \ldots, \operatorname{Lt}_{\overline{\tau}}(f_m))$. That is, $f_1, \ldots, f_m$ is a system of generators of I, not necessarily minimal, that also generates its leading term ideal with respect to $\overline{\tau}$.

REMARK 1.5.8 The terminology of standard basis is not consistent in literature. The notation used here is the same as in [4], [19]. However, in another reference we often cite, [27], a $\overline{\tau}$-enhanced standard basis is called standard basis.

The analogous notion to the enhanced standard basis in the polynomial case corresponds, as expected, to Gröbner basis. Let us rephrase the definition as follows:

DEFINITION 1.5.9 Consider an ideal $J \subset P$ and a term ordering τ in P. We call $g_1, \ldots, g_m$ a **τ-Gröbner basis** of J if it is a system of generators of J and generates the leading term ideal of J with respect to τ, i.e.

$$\mathrm{Lt}_\tau(J) = (\mathrm{Lt}_\tau(g_1), \ldots, \mathrm{Lt}_\tau(g_m)).$$

Some basic properties are proved in [4, Proposition 1.5, Corollary 1.6] and [19]:

PROPOSITION 1.5.10 Given an $\mathfrak{m}$-primary ideal I of R, the following properties hold:

(i) $\mathrm{Lt}_{\overline{\tau}}(f) = \mathrm{Lt}_\tau(f^*)$.
(ii) $\mathrm{Lt}_{\overline{\tau}}(I) = \mathrm{Lt}_\tau(I^*)$.
(iii) $f_1, \ldots, f_m$ is a $\overline{\tau}$-enhanced standard basis of I if and only if $f_1^*, \ldots, f_m^*$ is a τ-Gröbner basis of I^*.
(iv) Any $\overline{\tau}$-enhanced standard basis is also a standard basis.
(v) $\mathrm{HF}_{R/I} = \mathrm{HF}_{P/I^*} = \mathrm{HF}_{P/\,\mathrm{Lt}_{\overline{\tau}}(I)}$.

The following example from [19] shows that a standard basis is not always a $\overline{\tau}$-enhanced standard basis:

EXAMPLE 1.5.11 Consider the ideal $I = (g_1, g_2)$ in $\mathbf{k}[\![x, y]\!]$, where $g_1 = x^2 + y^2$ and $g_2 = xy + y^3$. Then $g_1^* = x^2 + y^2$, $g_2^* = xy$, $\mathrm{Lt}_{\overline{\tau}}(g_1) = x^2$, $\mathrm{Lt}_{\overline{\tau}}(g_2) = xy$. Observe that $(g_1, g_2) = (g_1^*, g_2^*)$, hence I is homogeneous and g_1,g_2 is a standard basis. In particular, $y^3 = yg_1^* - xg_2^*$ is in I. However, y^3 does not belong to $(\mathrm{Lt}_{\overline{\tau}}(g_1), \mathrm{Lt}_{\overline{\tau}}(g_2))$, hence g_1,g_2 is not a $\overline{\tau}$-enhanced standard basis of I.

THEOREM 1.5.12 (Grauert's Division Theorem) Let $f, f_1, \ldots, f_m$ be elements in R. Then, there are $q_1, \ldots, q_m, r$ in R such that

$$f = \sum_{i=1}^{m} q_i f_i + r$$

satisfying the following properties:

(i) No monomial of r is divisible by any $\mathrm{Lt}_{\overline{\tau}}(f_i)$, for $1 \leq i \leq m$.
(ii) If $q_i \neq 0$, $\mathrm{Lt}_{\overline{\tau}}(q_i f_i) \leq_{\overline{\tau}} \mathrm{Lt}_{\overline{\tau}}(f)$.

DEFINITION 1.5.13 We say that a finite subset G of R is **reduced** with respect to $\overline{\tau}$ if the following conditions hold:

(i) $0 \in G$.
(ii) $\mathrm{Lt}_{\overline{\tau}}(f) \nmid \mathrm{Lt}_{\overline{\tau}}(g)$ for any two different elements $f, g \in G$.
(iii) $\mathrm{LC}_{\overline{\tau}}(f) = 1$ for any $f \in G$.
(iv) For any $f \in G$, if $M \in \mathrm{Supp}(\mathrm{tail}(f))$, then $M \notin \mathrm{Lt}_{\overline{\tau}}(G)$.

DEFINITION 1.5.14 The residue r of this division is called the **normal form** of f with respect to a finite subset G of R and it is denoted by

$$\mathrm{NF}(f \mid G) = r.$$

This normal form is, in fact, a reduced normal form. The existence of a reduced normal form is fundamental in order to inherit in R all properties from Gröbner basis in P:

- If S_1, S_2 are $\overline{\tau}$-enhanced standard basis of I and $f \in R$, then $\mathrm{NF}(f \mid S_1) = \mathrm{NF}(f \mid S_2)$. In other words, the normal form of an element in R is unique when computed with respect to any $\overline{\tau}$-enhanced standard basis.
- Buchberger's criterion. See Theorem 1.5.16 below, for complete details see [27, Theorem 1.7.3].
- A reduced $\overline{\tau}$-enhanced standard basis is uniquely determined.

DEFINITION 1.5.15 Consider f, g in R with leading terms $\mathrm{Lt}_{\overline{\tau}}(f) = x^\alpha$ and $\mathrm{Lt}_{\overline{\tau}}(g) = x^\beta$. Set $\gamma := (\max(\alpha_1, \beta_1), \ldots, \max(\alpha_n, \beta_n))$. Then we define the **S-polynomial** of f and g as

$$S(f, g) = x^{\gamma-\alpha} f - \frac{\mathrm{LC}_{\overline{\tau}}(f)}{\mathrm{LC}_{\overline{\tau}}(g)} x^{\gamma-\beta} g.$$

THEOREM 1.5.16 (Buchberger's Criterion) Consider a finite subset G of elements $f_1, \ldots,$ f_m in R. Let $\mathrm{NF}(- \mid G)$ be the reduced normal form provided by Grauert's division theorem. The following are equivalent:

(i) G is a $\overline{\tau}$-enhanced standard basis of I.
(ii) $\mathrm{NF}(f, G) = 0$ for all $f \in I$.
(iii) Each $f \in I$ has a standard representation with respect to $\mathrm{NF}(- \mid G)$.

(iv) G generates I and $\mathrm{NF}(S(f_i, f_j) \mid G) = 0$, for $1 \leq i < j \leq m$.
(v) G generates I and $\mathrm{NF}(S(f_i, f_j) \mid G_{ij}) = 0$, for a suitable subset $G_{ij} \subset G$ and $1 \leq i < j \leq m$.

THEOREM 1.5.17 (See [27, Theorem 6.4.3].) Let $P \subset R$ be equipped with compatible local degree orderings $\overline{\tau}', \overline{\tau}$ respectively. Let $f_1, \ldots, f_m$ be the generators of an ideal J in P such that $\mathrm{Lt}_{\overline{\tau}'}(J) = (\mathrm{Lt}_{\overline{\tau}'}(f_1), \ldots, \mathrm{Lt}_{\overline{\tau}'}(f_m))$, then $f_1, \ldots, f_m$ is a $\overline{\tau}$-enhanced standard basis of $I = JR$.

The previous theorem implies that, whenever an ideal I of R is generated by polynomials $f_1, \ldots, f_r$ in R, we can look for its standard basis in $J = I \cap P$. We must be careful here, since in general, $f_1, \ldots, f_r$ is not a system of generators of J in P, even when it is a reduced $\overline{\tau}$-enhanced standard basis.

PROPOSITION 1.5.18 Given a $\overline{\tau}$-enhanced standard basis $f_1, \ldots, f_m$ of a zero-dimensional ideal $I \subset R$, we can always find polynomials $f'_1, \ldots, f'_m$ that form a $\overline{\tau}$-enhanced standard basis of I such that $\mathrm{Lt}_{\overline{\tau}}(f_i) = \mathrm{Lt}_{\overline{\tau}}(f'_i)$ and $\deg f_i \leq s + 1$, where s is the socle degree of R/I.

Proof: Consider a $\overline{\tau}$-enhanced standard basis $f_1, \ldots, f_m \in R$ of I, where

$$f_i = \sum_{|\alpha| \geq 0} c^i_\alpha x^\alpha.$$

Now set new elements f'_i by removing from f_i the terms of degree higher than $s+1$ and define

$$f'_i = f_i - \sum_{|\alpha| > s+1} c^i_\alpha x^\alpha = \sum_{|\alpha| \leq s+1} c^i_\alpha x^\alpha.$$

Clearly $f'_i \in P$ and $\mathrm{Lt}_{\overline{\tau}}(f'_i) = \mathrm{Lt}_{\overline{\tau}}(f_i)$. Set $I' = (f'_1, \ldots, f'_m) \subset R$, note that $\mathrm{Lt}_{\overline{\tau}}(f_i) \in \mathrm{Lt}_{\overline{\tau}}(I')$, hence $\mathrm{Lt}_{\overline{\tau}}(I) \subset \mathrm{Lt}_{\overline{\tau}}(I')$.

Since $a_i = \sum_{|\alpha| > s+1} c^i_\alpha x^\alpha \in \mathfrak{m}^{s+1} \subset I$, then $f'_i = f_i - a_i \in I$. Hence $I' \subset I$. But then $\mathrm{Lt}_{\overline{\tau}}(I) \subset \mathrm{Lt}_{\overline{\tau}}(I') \subset \mathrm{Lt}_{\overline{\tau}}(I)$ and hence $\mathrm{Lt}_{\overline{\tau}}(I') = \mathrm{Lt}_{\overline{\tau}}(I)$.

Therefore, by [4, Corollary 1.6], $\mathrm{HF}_{R/I} = \mathrm{HF}_{P/\,\mathrm{Lt}_{\overline{\tau}}(I)} = \mathrm{HF}_{P/\,\mathrm{Lt}_{\overline{\tau}}(I')} = \mathrm{HF}_{R/I'}$. In particular, $\ell(R/I) = \ell(R/I)$, and the equality $I = I'$ follows. $\square$

Another result that can be translated into the local setting is Schreyer's theorem. In [4, Theorem 1.10], Bertella proves that a system of generators of the module of syzygies of $\mathrm{Lt}_{\overline{\tau}}(I)$ can be lifted to a system of generators of the module of syzygies of I:

THEOREM 1.5.19 Let I be an ideal of R and let $f_1, \dots, f_m$ be a $\overline{\tau}$-enhanced standard basis of I. Let $\Sigma = \{\sigma_1, \dots, \sigma_t\}$ be a homogeneous system of generators of $\mathrm{Syz}(\mathrm{Lt}_{\overline{\tau}}(I))$. Then $\mathrm{Syz}(I)$ is generated by $m_1, \dots, m_t$, where m_i is a lifting of σ_i.

CHAPTER 2

Low Gorenstein colength

The Gorenstein colength of an Artin local ring $A = R/I$, denoted by $\operatorname{gcl}(A)$, is an invariant introduced by Ananthnarayan in [1] that tells us how far from A is the closest Artin Gorenstein ring $G = R/J$, see Definition 1.3.3. Although the computation of this invariant is still an open problem except for very special families of rings, see [2], some characterizations have been provided for $\operatorname{gcl}(A) \leq 2$. We say that these rings have low Gorenstein colength.

In [44, Theorem 2.3], Teter characterized what later on Huneke and Vraciu, see [31], would call Teter rings or almost Gorenstein rings. They are local Artin rings A for which there exists an Artin Gorenstein ring G such that $A \simeq G/\operatorname{soc}(G)$. Then

$$\phi : G \twoheadrightarrow G/\operatorname{soc}(G) \simeq A \quad \text{and hence } \operatorname{gcl}(A) \leq \ell(G) - \ell(A) = 1.$$

For $\operatorname{embd}(A) \geq 2$, Teter rings correspond exactly to rings of Gorenstein colength 1, see Proposition 2.1.3.

The approach of Teter, Huneke-Vraciu and Ananthnarayan is based on the existence of self-dual ideals of A with respect to the functor $(-)^{+} = \operatorname{Hom}_A(-, \omega_A)$, where ω_A is the canonical module of A.

THEOREM 2.0.1 (Teter) Let $A = R/I$ be an Artin ring. Then the following are equivalent:

(i) $\operatorname{gcl}(A) \leq 1$.
(ii) Either A is Gorenstein or there is an isomorphism $\varphi : \mathfrak{n} \to \mathfrak{n}^{+}$ such that $\varphi(x)(y) = \varphi(y)(x)$, for every x, y in $\mathfrak{n}$.

We name the symmetric property of φ as Teter's condition, which is precisely what Huneke and Vraciu overcome in [31, Theorem 2.5]:

THEOREM 2.0.2 (Huneke-Vraciu) Let $A = R/I$ be an Artin ring such that $\mathrm{char}(\mathbf{k}) \neq 2$ and $\mathrm{soc}(A) \subseteq \mathfrak{n}^2$. Then the following are equivalent:

(i) $\mathrm{gcl}(A) \leq 1$.
(ii) Either A is Gorenstein or $\mathfrak{n}$ is a self-dual ideal of A.
(iii) There exists an epimorphism $f : \omega_A \longrightarrow \mathfrak{n}$.

Ananthnarayan extends in [1, Theorem 5.5] the previous result to any ring of low Gorenstein colength.

THEOREM 2.0.3 (Ananthnarayan) Let $A = R/I$ be an Artin ring and let $\mathfrak{m}$ be the maximal ideal of R. Suppose that $I \subseteq \mathfrak{m}^6$ and $\mathrm{char}(\mathbf{k}) \neq 2$. Then the following are equivalent:

(i) $\mathrm{gcl}(A) \leq 2$.
(ii) There exists a self-dual ideal $\mathfrak{q} \subseteq A$ such that $\ell(A/\mathfrak{q}) \leq 2$.
(iii) There exists an epimorphism $f : \omega_A \longrightarrow \mathfrak{q}$, where $\mathfrak{q}$ is a self-dual ideal of A such that $\ell(A/\mathfrak{q}) \leq 2$.

Since the canonical module ω_A can be identified with the inverse system $I^{\perp}$, see Section 1.4.1, another natural approach to the problem is considering it from the inverse system perspective. This was first done by Elias and Silva to study Teter rings. In [20, Theorem 3.4], the restrictions on the characteristic and the socle of Huneke-Vraciu are dumped and a new characterization of Teter rings in terms of their Macaulay inverse system is provided:

THEOREM 2.0.4 (Elias-Silva) Let $A = R/I$ be a non-Gorenstein Artin ring with maximal ideal $\mathfrak{n}$ and socle degree $s \geq 1$. Then the following conditions are equivalent:

(i) $\mathrm{gcl}(A) = 1$.
(ii) There exists a degree $s+1$ polynomial $F \in S$ such that $I^{\perp} = \langle x_1 \circ F, \dots, x_n \circ F \rangle$.
(iii) There exists an epimorphism of A-modules $I^{\perp} \twoheadrightarrow \mathfrak{n}$.
(iv) A is a Teter ring.

In particular, if A is a Teter ring, then the Cohen–Macaulay type of A is n.

This chapter is devoted to the study of low Gorenstein colength rings and establishes a connection among Macaulay inverse systems, minimal Gorenstein covers and self-dual ideals.

In Section 2.1, we present the first main result of this chapter. Theorem 2.1.7 shows the exact relationship between the inverse system $I^\perp$ of $A = R/I$ and the inverse system $J^\perp$ of a minimal Gorenstein cover $G = R/J$ by considering the colon ideal K_F defined as $(J^\perp :_R \langle F \rangle)$:

THEOREM 2.0.5 (See Theorem 2.1.7.) Let A be an Artin ring such that $\mathrm{gcl}(A) \leq 2$. If G is a minimal Gorenstein cover of A, then

(i) $\mathrm{embd}(G) = \mathrm{embd}(A)$,

(ii) if $A = R/I$ with $\dim(R) = \mathrm{embd}(G) = \mathrm{embd}(A)$ and F is a generator of $J^\perp$, $G = R/J$, then $I \subset K_F$ and

$$I^2 \subset J \subset I.$$

Moreover, after a linear isomorphism of R we may assume:

$$K_F = \begin{cases} R, & \text{if} \quad \mathrm{gcl}(A) = 0; \\ \mathfrak{m}, & \text{if} \quad \mathrm{gcl}(A) = 1; \\ (x_1, \dots, x_{n-1}, x_n^2), & \text{if} \quad \mathrm{gcl}(A) = 2. \end{cases}$$

Several examples are given to answer natural questions regarding the uniqueness of such covers. Section 2.2 contains the second main result, Theorem 2.2.5, which extends and improves the characterization of Artin rings $A = R/I$ of Gorenstein colength two in [1, Theorem 5.5]:

THEOREM 2.0.6 (See Theorem 2.2.5 for more details.) Let $A = R/I$ be an Artin ring with maximal ideal $\mathfrak{n}$ and socle degree $s \geq 1$. We assume that A is neither Gorenstein nor Teter, $I \subset \mathfrak{m}^5$ and $\mathrm{char}(\mathbf{k}) \neq 2$. Then the following conditions are equivalent:

(i) $\mathrm{gcl}(A) = 2$,

(ii) after a linear isomorphism of R there exists a polynomial $F \in S$ of degree $s+1$ or $s+2$ such that $I^\perp = \langle x_1 \circ F, \dots, x_{n-1} \circ F, x_n^2 \circ F \rangle$,

(iii) there exists an epimorphism of A-modules $f : I^\perp \longrightarrow \mathfrak{q}$, where $\mathfrak{q}$ is a self-dual ideal of A such that $\ell(A/\mathfrak{q}) = 2$.

Examples and families of Gorenstein colength two rings are provided with explicit descriptions of both the self-dual ideals and the epimorphisms that appear in the previous characterization. In Section 2.3, we finish the chapter giving some hints on what occurs in higher colength and a detailed example for $\mathrm{gcl}(A) = 3$.

We perform all the computations in *Singular*, [11], using the Singular library [13] for inverse system related computations.

Part of the results of this chapter are published in [16].

2.1 Gorenstein covers

Let us start by redefining the notion of minimal Gorenstein cover in terms of the Macaulay's inverse system.

DEFINITION 2.1.1 We say that $G = R/J$, with $J = \operatorname{Ann}_{\mathrm{R}} F$, is a **minimal Gorenstein cover** of $A = R/I$ if and only if $I^{\perp} \subset \langle F \rangle$ and $\ell(G) = \ell(A) + \operatorname{gcl}(A)$.

REMARK 2.1.2 Note that, a priori, $\operatorname{embd}(A)$ and $\operatorname{embd}(G)$ are not necessarily the same. Nevertheless, according to the main result in this section, Theorem 2.1.7, if $\operatorname{gcl}(A) \leq 2$ then the embedding dimensions of A and any minimal Gorenstein cover G of A coincide and the number of variables n of R can always be taken as $\operatorname{HF}_A(1) = \operatorname{HF}_G(1)$ in this case.

In the next proposition we recall some basic results on Gorenstein colength.

PROPOSITION 2.1.3 Let A be a local Artin $\mathbf{k}$-algebra.

(i) $0 \leq \operatorname{gcl}(A) \leq \ell(A)$,
(ii) $\operatorname{gcl}(A) = 0$ if and only if A is Gorenstein,
(iii) if $\operatorname{embd}(A) \geq 2$, then $\operatorname{gcl}(A) = 1$ if and only if A is Teter.

Proof: (i) We know that any Artin ring A is a quotient of the Artin Gorenstein ring $G = A \ltimes \omega_A$, where $\ltimes$ stands for Nagata's idealization [5, Theorem 3.3.6]. On the other hand, since A is a $\mathbf{k}$-algebra, G is a $\mathbf{k}$-algebra as well. Notice that if the embedding dimension of A is b, then the embedding dimension of G is $n = b + \tau(A)$, so A and G are quotients of R and G is a Gorenstein cover of A. Since the length of G is $2\ell(A)$, we get the claim.
(ii) is trivial. (iii) Assume that $\operatorname{gcl}(A) = 1$. Then there exists a Gorenstein cover $G = R/J$ of $A = R/I$ such that $\ell(G) = \ell(A) + 1$. In particular $\ell(I/J) = 1$. Hence $H = I/J \subset G$ is the socle of G. From this is easy to deduce that $A = G/\operatorname{soc}(G)$.

Conversely, if $A = G/\operatorname{soc}(G)$, where G is a Gorenstein cover of A, then $\operatorname{gcl}(A) \leq 1$. From [20, Proposition 3.7] we get the claim. $\square$

From now on we will assume that the embedding dimension of $A = R/I$ is the dimension of R. Note that, since we are interested in non-Gorenstein rings A, we can also assume that $\operatorname{embd}(A) \geq 2$. Otherwise, if $R = \mathbf{k}[\![x]\!]$, any ideal of R is of the form $I = (x^n)$, $n \geq 1$. Then $I^\perp = \langle y^{n-1} \rangle$ is cyclic and hence $A = \mathbf{k}[\![x]\!]/(x^n)$ is Gorenstein (in fact, it is much more: a complete intersection ring).

We can present $G = R'/J$, with R' a power series ring such that the embedding dimension of G coincides with the dimension of R', see Remark 2.1.2.

Let $A = R/I$ be an Artin ring with R a power series ring over $\mathbf{k}$ such that $n = \dim R = \operatorname{embd}(A)$. Then there is an R-module monomorphism

$$\xi_A : I^\perp \longrightarrow S = \mathbf{k}[y_1, \dots, y_n].$$

Let $G = R'/J$ be a Gorenstein cover of an Artin ring $A = R/I$ with R' a power series ring over $\mathbf{k}$ such that $n+t = \dim R' = \operatorname{embd}(G)$. We assume that R is a quotient of R' by a linear regular sequence, i.e. we may assume that $R' = \mathbf{k}[\![x_1, \dots, x_n, x_{n+1}, \dots, x_{n+t}]\!]$. Then we have a commutative diagram:

$$\begin{array}{ccc} S & \hookrightarrow & S' \\ \xi_A \uparrow & & \uparrow \xi_G \\ I^\perp & \longrightarrow & J^\perp \end{array}$$

where S' is a polynomial ring of dimension $n+t$ over $\mathbf{k}$ on the variables $y_1, \dots, y_n, y_{n+1}, \dots, y_{n+t}$. We denote by $S'_{\leq i}$ the sub-R'-module of S' consisting of all polynomials of degree equal or less that i. Note that $S'_{\leq 0} \simeq \mathbf{k}$ and it is contained in any non-zero sub-R'-module of S'. In other words, if $F = \sum_{i=0}^{s} F_i$ is a polynomial in S', where F_i are homogeneous polynomials of degree i, then $\langle F \rangle = \langle F - F_0 \rangle$.

PROPOSITION 2.1.4 Let $G = R'/J$ be a Gorenstein cover of an Artin ring $A = R/I$ with $\dim R \geq 2$. Then there exists a generator $F \in J^\perp$ such that $\operatorname{ord}(F) \geq 2$.

Proof: Let $F = \sum_{i=0}^{s} F_i$ be a generator of $J^\perp$, where $s = \operatorname{socdeg} G$ and F_i are forms of degree i. Since $J \subset (\mathfrak{m}')^2$, where $\mathfrak{m}'$ is the maximal ideal of R', we have $S'_{\leq 1} \subset J^\perp$. From $F_0 + F_1 \in S'_{\leq 1}$, we deduce that $\langle F - (F_0 + F_1) \rangle \subseteq \langle F \rangle$ is an inclusion of R'-modules.

Since $\operatorname{embd}(G) \geq \operatorname{embd}(A) \geq 2$, then $\mathrm{HF}_G(1) = n + t \geq 2$ cannot be the last non-zero value of the Hilbert function because G is Gorenstein, i.e. $\operatorname{socdeg}(G) \geq 2$. Hence $S'_{\leq 1} \subset \mathfrak{m}' \circ F$. Therefore, the equality

$$\langle F \rangle = \langle F - (F_0 + F_1) \rangle + \mathfrak{m}' \circ F$$

holds and we can apply Nakayama's Lemma. We get $\langle F \rangle = \langle F - (F_0 + F_1) \rangle$ or, in other words, $J^\perp = \langle F_s + \cdots + F_2 \rangle$. $\square$

We are now interested in knowing what is the exact relationship between the inverse system of a Gorenstein cover and the inverse system of the base ring. To that aim, we consider the following colon ideal:

DEFINITION 2.1.5 Let $A = R/I$ be an Artin ring. For all $F \in S'$ such that $I^\perp \subset \langle F \rangle$, we consider the ideal K_F of R' defined by

$$K_F = (I^\perp :_{R'} \langle F \rangle).$$

If $J = \operatorname{Ann}_{\mathrm{R}}(F)$, then $G = R'/J$ is a Gorenstein cover of A and $J \subset K_F$. Indeed, $J \circ J^\perp = 0$ by definition and hence, $J \circ J^\perp \subset I^\perp$.

The following proposition enables us to establish a connection between the colon ideal K_F of any cover $G = R/\operatorname{Ann}_{\mathrm{R}} F$ (not necessarily minimal) of A and the Cohen-Macaulay type of A. Even more, it provides an upper bound on $\operatorname{gcl}(A)$ given by the length of R'/K_F.

PROPOSITION 2.1.6 Let $A = R/I$ be an Artin ring. For all $F \in S'$ such that $I^\perp \subset \langle F \rangle$ we write $J = \operatorname{Ann}_{\mathrm{R}}(F)$. It holds:

(i) $I^\perp = K_F \circ F$, $I = (J :_R K_F)$,

(ii) there is an R'-module isomorphism

$$\begin{array}{ccc} \frac{R'}{K_F} & \longrightarrow & \frac{\langle F \rangle}{I^\perp} \\ a & \mapsto & a \circ F \end{array}$$

(iii) if $G = R'/J$, then $\ell(G) - \ell(A) = \ell(R'/K_F)$,

(iv) $\tau(A) = \dim_{\mathbf{k}}(K_F/\mathfrak{m}K_F + J)$.

Proof: (i) $K_F \circ F \subseteq I^\perp$ directly from the definition of K_F. To prove the reverse inclusion, it is enough to observe that if $g \in I^\perp$, then $g \in \langle F \rangle$, i.e. $g = a \circ F$ for some $a \in R'$. But then $a \in K_F$ again by definition of K_F, hence $g \in K_F \circ F$.

On the other hand, the inclusion $I \subset (J :_R K_F)$ comes from the fact that

$$(IK_F) \circ F = I \circ (K_F \circ F) = I \circ I^\perp = 0,$$

hence $IK_F \subset J$. For the reverse inclusion, it is sufficient to see that $(J :_R K_F) \circ I^\perp = 0$. Indeed,

$$(J :_R K_F) \circ I^\perp = (J :_R K_F) \circ (K_F \circ F) = ((J :_R K_F)K_F) \circ F \subset J \circ J^\perp = 0.$$

(ii) Consider the R'-module epimorphism φ defined by

$$\begin{array}{ccccc} R' & \twoheadrightarrow & \langle F \rangle & \twoheadrightarrow & \frac{\langle F \rangle}{I^\perp} \\ a & \mapsto & a \circ F & \mapsto & \overline{a \circ F} \end{array}$$

Its kernel consists of all the elements a in R' such that $a \circ F$ is in $I^\perp$, hence $\ker \varphi = K_F$.
(iii) Since the length of a ring $A = R/I$ coincides with the length of the inverse system of I, then $\ell(G) - \ell(A) = \ell(\langle F \rangle) - \ell(I^\perp)$. We have a short exact sequence

$$0 \longrightarrow I^\perp \longrightarrow J^\perp \longrightarrow J^\perp/I^\perp \longrightarrow 0,$$

hence $\ell(J^\perp) - \ell(I^\perp) = \ell(J^\perp/I^\perp)$. From (ii), we obtain $\ell(J^\perp/I^\perp) = \ell(R'/K_F)$.
(iv) From (i) we get the epimorphism

$$\begin{array}{cccc} \phi : & \frac{K_F}{\mathfrak{m}K_F} & \xrightarrow{\circ F} & \frac{I^\perp}{\mathfrak{m} \circ I^\perp} \\ & a & \mapsto & a \circ F \end{array}$$

with kernel $\ker(\phi) = (\mathfrak{m}K_F + J)/\mathfrak{m}K_F$. Hence

$$\dim_{\mathbf{k}} \frac{I^\perp}{\mathfrak{m} \circ I^\perp} = \dim_{\mathbf{k}} \frac{K_F}{\mathfrak{m}K_F + J}$$

and from [20, Proposition 2.6] we get the right expression for $\tau(A)$. $\square$

THEOREM 2.1.7 Let A be an Artin ring such that $\operatorname{gcl}(A) \leq 2$. If G is a minimal Gorenstein cover of A, then

(i) $\operatorname{embd}(G) = \operatorname{embd}(A)$,

(ii) if $A = R/I$ with $\dim(R) = \operatorname{embd}(G) = \operatorname{embd}(A)$ and F is a generator of $J^\perp$, $G = R/J$, then $I \subset K_F$ and
$$I^2 \subset J \subset I.$$

Moreover, after a linear isomorphism of R we may assume:

$$K_F = \begin{cases} R, & \text{if} \quad \operatorname{gcl}(A) = 0; \\ \mathfrak{m}, & \text{if} \quad \operatorname{gcl}(A) = 1; \\ (x_1, \dots, x_{n-1}, x_n^2), & \text{if} \quad \operatorname{gcl}(A) = 2. \end{cases}$$

Proof: We assume that $A = R/I$ with $R = \mathbf{k}[\![x_1, \dots, x_n]\!]$ where $n = \operatorname{embd}(A)$. In particular, $I \subset \mathfrak{m}^2$.

If $\operatorname{gcl}(A) = 0$ then A is Gorenstein. Hence $G \cong A$ and $K_F = R$. From this we trivially get (i) and $I \subset K_F$.

Assume that $\operatorname{gcl}(A) = 1$. Let $G = R'/J$ be a minimal Gorenstein cover of A with $R' = \mathbf{k}[\![x_1, \dots, x_n, x_{n+1}, \dots, x_{n+t}]\!]$ and $\operatorname{embd}(G) = n + t$. From Proposition 2.1.6.(iii) we get that $K_F = (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+t})$. Let F be a generator of $J^\perp$, then

$$(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+t}) \circ F = I^\perp \subset S = \mathbf{k}[y_1, \dots, y_n].$$

Since $\operatorname{ord}(F) \geq 2$ by Proposition 2.1.4, we get that $F \in S$. Hence we may assume that $t = 0$ and $I \subset K_F \subset R$.

Assume now that $\operatorname{gcl}(A) = 2$. From Proposition 2.1.6.(iii) we get that $K_F = (l_1, \dots, l_{n+t-1}, l_{n+t}^2)$, where $l_1, \dots, l_{n+t}$ are homogeneous linear forms defining a minimal system of generators of the maximal ideal of R', with $\operatorname{embd}(G) = n + t$. We have to consider two cases:

Case I. After a suitable permutation of the linear forms $l_1, \dots, l_{n+t}$, we can assume that $I^\perp$ is contained in $\mathbf{k}[l_1, \dots, l_n]$. Since $\operatorname{ord}(F) \geq 2$ by Proposition 2.1.4, we get that the variables $l_{n+1}, \dots, l_{n+t-1}$ do not appear in F and the monomials l_{n+t}^r, $r \geq 3$, do not appear in F either. Then F can be written as follows

$$F = H(l_1, \dots, l_n) + a l_{n+t}^2$$

with $a \in \mathbf{k}$. Since $\mathbf{k} \subset (l_1, \dots, l_n) \circ H$ we get that

$$I^\perp = (l_1, \dots, l_{n+t-1}, l_{n+t}^2) \circ F = (l_1, \dots, l_n) \circ H.$$

Furthermore, the ring $R/\operatorname{Ann_R}(H)$ is a Gorenstein cover of A. From the structure of F we get $\ell(\langle H \rangle) \leq \ell(\langle F \rangle)$. Since F is a minimal Gorenstein cover, $F = H$ and we deduce (i) and (ii).

Case II. Assume that, after a suitable permutation, $I^\perp \subset S = \mathbf{k}[l_1, \dots, l_{n-1}, l_{n+t}]$. Next we discuss which monomials can appear in F.

II.1 Let us consider a monomial $\underline{l}^L$ of F multiple of $l_i l_j$, with $L \in \mathbb{N}^{n+t}$, $i \in \{1, \dots, n-1\}$ and $j \in \{n, \dots, n+t-1\}$. Since $l_i \in K_F$, the contraction of F by l_i lives in $I^\perp$. In particular, $\underline{l}^L / l_i$ is a monomial of $l_i \circ F \in I^\perp \subset S$ that contains l_j. This is not possible.

II.2 Let us consider a monomial of F of the form $l_j^a l_{n+t}^b$ with $j \in \{n, \dots, n+t-1\}$ and $a, b \geq 1$. If $b \geq 2$, then the contraction of F by $l_{n+t}^2 \in K_F$ gives that $l_j^a l_{n+t}^{b-2}$ is a monomial of $l_{n+t}^2 \circ F \in I^\perp \subset S$. This is not possible, so $b \leq 1$. If $a \geq 2$, after the contraction of F by $l_j \in K_F$ we get that $l_j^{a-1} l_{n+t}^b$ is a monomial of $l_j \circ F \in I^\perp \subset S$. This is not possible, so $a \leq 1$. Hence F is of the form

$$F = H(l_1, \dots, l_{n-1}, l_{n+t}) + l_{n+t}(a_n l_n + \cdots + a_{n+t-1} l_{n+t-1})$$

with $a_n, \dots, a_{n+t} \in \mathbf{k}$.

Assume that $t \geq 1$. Notice that, since $\operatorname{embd}(G) = n+t$, we have that $l_i \in \langle F \rangle = J^\perp$ so $a_i \neq 0$, $i = n, \dots, n+t-1$.

We set $R = \mathbf{k}[\![l_1, \dots, l_{n-1}, l_{n+t}]\!]$ and consider $H' = H + a l_{n+t}^2$ for some $a \in \mathbf{k}$ such that $l_{n+t} \in R \circ H'$. Notice that $R \circ H'$ is the sub-R-module $\langle H' \rangle$ of S generated by H' whereas $\langle F \rangle$ denotes the sub-R'-module of $\mathbf{k}[l_1, \dots, l_{n+t}]$ generated by F.

Claim 1. $I^\perp \subset R \circ H'$.

Recall that $I^\perp = K_F \circ F$, so it is enough to prove that $l_i \circ F \in R \circ H'$, $i = 1, \dots, n+t-1$, and $l_{n+t}^2 \circ F \in R \circ H'$. For all $i = 1, \dots, n-1$, we have

$$l_i \circ F = l_i \circ H = l_i \circ H' \in R \circ H',$$

and for all $i = n, \dots, n+t-1$, we get

$$l_i \circ F = a_i l_{n+t} \in R \circ H'.$$

Finally

$$l_{n+t}^2 \circ F = l_{n+t}^2 \circ H = l_{n+t}^2 \circ H' - a \in R \circ H'.$$

Hence $I^\perp \subset R \circ H'$, that is, $R/\operatorname{Ann_R}(H')$ is a Gorenstein cover of A.

Claim 2. $\ell(R \circ H') \leq \ell(\langle F \rangle) - t$.

It is enough to prove that $\ell(\mathfrak{m} \circ H') \leq \ell(\langle F \rangle) - t - 1$ where $\mathfrak{m}$ is the maximal ideal of R. First we prove that $\mathfrak{m} \circ H' \subset \langle F \rangle$. Indeed, for all $i = 1, \dots, n-1$ we have

$$l_i \circ H' = l_i \circ H = l_i \circ F \in \langle F \rangle;$$

and

$$l_{n+t} \circ H' = l_{n+t} \circ H + a l_{n+t} = l_{n+t} \circ F - \sum_{i=n}^{n+t-1} a_i l_i + a l_{n+t} \in \langle F \rangle.$$

The next step is to prove that

$$\mathfrak{m} \circ H' \cap \langle F, l_n, \dots, l_{n+t-1} \rangle_{\mathbf{k}} = 0.$$

Let us consider

$$\lambda_1 F + \lambda_n l_n + \cdots + \lambda_{n+t-1} l_{n+t-1} \in \mathfrak{m} \circ H'$$

with $\lambda_1, \lambda_n, \dots, \lambda_{n+t-1} \in \mathbf{k}$. Hence

$$\lambda_1 \sum_{i=n}^{n+t-1} a_i l_i l_{n+t} + \lambda_n l_n + \cdots + \lambda_{n+t-1} l_{n+t-1} \in R.$$

Since $a_i \neq 0$ for $i = n, \dots, n+t-1$, we get that $\lambda_1 = \lambda_n = \cdots = \lambda_{n+t-1} = 0$.

Both $\mathfrak{m} \circ H'$ and $\langle F, l_n, \dots, l_{n+t-1} \rangle_{\mathbf{k}}$ are contained in $\langle F \rangle$. Hence a $\mathbf{k}$-vector space dimension computation gives Claim 2.

From previous claims we get that $R/\operatorname{Ann_R}(H')$ is a Gorenstein cover of A of length at most $\ell(A) + 2 - t$. But then $\operatorname{gcl}(A) \leq 1$, which is not possible. Therefore $t = 0$ and we proved that $F \in S = \mathbf{k}[l_1, \dots, l_n]$. From this we get (i) and $I \subset \mathfrak{m}^2 \subset K_F =$

$(l_1, \dots, l_{n-1}, l_n^2)$. Since $I \subset K_F$ we deduce

$$I^2 \circ F \subset (IK_F) \circ F = I \circ (K_F \circ F) = I \circ I^\perp = 0.$$

From this we get $I^2 \subset J$. $\square$

COROLLARY 2.1.8 If $\operatorname{gcl}(A) = 2$, $\operatorname{socdeg}(A) = s \geq 1$ and G is a minimal Gorenstein cover of A, then the socle degree of G is either $s+1$ or $s+2$.

If $\operatorname{socdeg} G = s+2$, then

$$\mathrm{HF}_G(i) = \begin{cases} \mathrm{HF}_A(i), & \text{if} \quad i \leq s; \\ 1, & \text{if} \quad i = s+1, s+2; \\ 0, & \text{if} \quad i \geq s+3. \end{cases}$$

Proof: By Theorem 2.1.7, $I^\perp = K_F \circ F = \langle x_1 \circ F, \dots, x_{n-1} \circ F, x_n^2 \circ F \rangle$, where F generates $J^\perp$ and $G = R/J$ is any minimal Gorenstein cover of A. Since the socle degree of A is s, we get

$$\max_{1 \leq i \leq n-1} \{\deg x_n^2 \circ F, \deg x_i \circ F\} = s.$$

If this maximum is reached by $x_j \circ F$, for some $1 \leq j \leq n-1$, then $\deg F = s+1$. Otherwise, it will be reached by $x_n^2 \circ F$ and hence $\deg F = s+2$. $\square$

We end this section by providing examples that answer several natural questions on Gorenstein covers.

Hilbert functions of minimal Gorenstein covers of a certain Gorenstein colength two ring are not unique in general. We can even have no uniqueness in its socle degree. In the following two examples we consider particular cases of Artin local rings A with $\operatorname{gcl}(A) = 2$ and $\operatorname{socdeg}(A) = 3$. Example 2.1.9 shows a case where both minimal Gorenstein covers of socle degrees 4 and 5 exist. On the contrary, in Example 2.1.10 it is proved that we only have minimal Gorenstein covers of socle degree 4.

EXAMPLE 2.1.9 Consider $A = R/I = \mathbf{k}[\![x_1, x_2]\!]/\left(x_1^2, x_1x_2^2, x_2^4\right)$, with Hilbert function $\{1, 2, 2, 1\}$. $I^\perp = \langle y_1y_2, y_2^3 \rangle$ is contained in the sub-R-modules generated by polynomials $F_1 = y_1y_2^3$ and $F_2 = y_1^2y_2 + y_2^5$. Then both $G_1 = R/\operatorname{Ann}_{\mathrm{R}}(F_1)$ and

$G_2 = R/\operatorname{Ann}_R(F_2)$ are Gorenstein covers of A with Hilbert functions $\{1,2,2,2,1\}$ and $\{1,2,2,1,1,1\}$, respectively, and $\ell(G_1) - \ell(A) = \ell(G_2) - \ell(A) = 2$. Hence, $\operatorname{gcl}(A) \leq 2$. A is clearly not Gorenstein and, by [20, Proposition 4.5], we can also deduce that it is not Teter. Therefore, Gorenstein colength of A is exactly 2 and G_1, G_2 are minimal Gorenstein covers of socle degree 4 and 5, respectively. Note that $K_{F_1} = K_{F_2} = (x_1, x_2^2)$.

EXAMPLE 2.1.10 Take $A = \mathbf{k}[\![x_1,x_2,x_3]\!]/\left(x_1x_2, x_1x_3, x_2x_3, x_2^2, x_3^2 - x_1^3\right)$, with Hilbert function $\{1,3,1,1\}$. Note that $\tau(A) = 2$ and $\operatorname{embd}(A) = 3$. Hence A is not Gorenstein and, by [20, Theorem 3.4], also not Teter since $\tau(A) \neq \operatorname{embd}(A)$. The polynomial $F = y_1^4 + y_1y_3^2 + y_2^2$ generates the inverse system of a Gorenstein cover G of A with $\mathrm{HF}_G = \{1,3,2,1,1\}$. Therefore, $\operatorname{gcl}(A) = \ell(G) - \ell(A) = 2$ and G is a minimal Gorenstein cover of socle degree 4. Note that $K_F = \left(x_1, x_2, x_3^2\right)$.

Let us now assume that there is a minimal Gorenstein cover $G' = R/J$ of A with socle degree 5. According to Corollary 2.1.8, its Hilbert function is $\{1,3,1,1,1,1\}$ and, by [21], G' is isomorphic to $\mathbf{k}[\![x_1,x_2,x_3]\!]/\left(x_1x_2, x_1x_3, x_2x_3, x_2^2 - x_1^5, x_3^2 - x_1^5\right)$. Since $J^\perp = \langle y_1^5 + y_2^2 + y_3^2\rangle$, the only possible choice for a sub-R-module $K^\perp$ of $J^\perp$ such that $\mathrm{HF}_{R/K} = \{1,3,1,1\}$ is

$$K^\perp = \langle 1, y_1, y_2, y_3, y_1^2, y_1^3\rangle_{\mathbf{k}} \subset J^\perp = \langle 1, y_1, y_2, y_3, y_1^2, y_1^3, y_1^4, y_1^5 + y_2^2 + y_3^2\rangle_{\mathbf{k}}.$$

By Proposition 2.6 [20], $\tau(R/K) = \mu(K^\perp) = 3$. But $\tau(A) = 2$ and hence there is no minimal Gorenstein cover G' of A with $\operatorname{socdeg} G' = 5$.

Even in the situation where we have unicity of the Hilbert functions of all minimal Gorenstein covers, such covers are not necessarily unique. The next example shows a ring with two non-isomorphic minimal covers with the same Hilbert function.

EXAMPLE 2.1.11 Consider $A = R/I$ with $R = \mathbf{k}[\![x_1,x_2,x_3]\!]$ and $I = (x_1x_2, x_2x_3, x_3^2)$. Set $\operatorname{char}(\mathbf{k}) = 0$ and note that $HF_A = \{1,3,2\}$. This ring has Cohen-Macaulay type 2, hence it is not Gorenstein not Teter, using the same argument as in previous example. Notice that the following polynomials generate R-modules containing $I^\perp = \langle y_1y_3, y_2y_3\rangle$:

(i) $F_1 = y_1y_2y_3$, $(x_1, x_2, x_3^2) \circ \langle F_1\rangle = I^\perp$;
(ii) $F_2 = y_1y_2y_3 - y_3^3$, $(x_1, x_2, x_3^2) \circ \langle F_2\rangle = I^\perp$.

Set $G_1 = R/\operatorname{Ann}_{\mathrm{R}}(F_1)$ and $G_2 = R/\operatorname{Ann}_{\mathrm{R}}(F_2)$. Since $\ell(G_1) - \ell(A) = \ell(G_2) - \ell(A) = 2$, G_1 and G_2 are minimal Gorenstein covers of A. Both rings have Hilbert function $\{1,3,3,1\}$ but in characteristic zero it is known that they are not isomorphic, [18, Proposition 3.7]. To prove that any minimal Gorenstein cover $G = R/J$ of A must have this Hilbert function, we use the fact that the only other possible Hilbert function is $\{1,3,2,1,1\}$, again by Corollary 2.1.8. This corresponds to an almost stretched **k**-algebra and, by [15], we know what Gorenstein rings with such Hilbert functions look like. Using a similar reasoning as in Example 2.1.10, we get that any ring R/K such that $K^\perp \subset J^\perp$ and $\mathrm{HF}_{R/K} = \{1,3,2\}$ has a Cohen-Macaulay type different than 2.

Let us look at it from the opposite perspective and consider an Artin Gorenstein ring. We can ask ourselves whether it can be a minimal Gorenstein cover of non-isomorphic rings. In the following example we show a Gorenstein ring which is, at the same time, minimal cover of several Artin rings of Gorenstein colength 2 and Teter cover of a Teter ring.

EXAMPLE 2.1.12 Consider the Gorenstein ring $G = R/J$, where $R = \mathbf{k}[\![x_1,x_2,x_3]\!]$ and $J = \left(x_3^2, x_1x_2, x_1x_3, x_2^3, x_1^3 + 3x_2^2x_3\right)$, with Hilbert function $\{1,3,3,1\}$. This ring has inverse system $J^\perp = \langle y_2^2y_3 - y_1^3\rangle$ and contains the following R-modules:

(i) $(x_2 - x_1, x_3, x_2^2) \circ J^\perp = \langle y_1^2 + y_2y_3, y_2^2\rangle = I_1^\perp$;
(ii) $(x_1 + x_2, x_2 + x_3, x_3^2) \circ J^\perp = \langle y_1^2 - y_2y_3, y_2y_3 + y_2^2\rangle = I_2^\perp$;
(iii) $(x_1, x_2, x_3^2) \circ J^\perp = \langle y_1^2, y_2y_3\rangle = I_3^\perp$.

$A_1 = R/I_1$, $A_2 = R/I_2$ and $A_3 = R/I_3$ are non-isomorphic Artin local rings with Hilbert function $\{1,3,2\}$ and Cohen-Macaulay type 2, by the classification provided by Poonen in [40]. Using again the arguments above, A_1, A_2, A_3 are not Gorenstein nor Teter and $\ell(G) - \ell(A_1) = \ell(G) - \ell(A_2) = \ell(G) - \ell(A_3) = 2$. Hence G is a minimal Gorenstein cover to all these rings of Gorenstein colength 2.
Let us now consider $A = R/I$, where $I = \operatorname{Ann}_{\mathrm{R}}\left(\mathfrak{m} \circ J^\perp\right)$. Then

$$I^\perp = \mathfrak{m} \circ J^\perp = \langle y_1^2, y_2y_3, y_2^2\rangle = \langle y_1^2, y_2y_3, y_2^2, y_1, y_2, y_3, 1\rangle_{\mathbf{k}}.$$

Its Hilbert function is $\{1,3,3\}$, hence not Gorenstein. Since $\ell(G) - \ell(A) = 1$, G is a Teter cover of the Teter ring A.

Recall that Artin stretched **k**-algebras are those with Hilbert function of the form $\{1,n,1,\dots,1\}$, see [21] for more details. The next example provides us with a family of Artin stretched rings of Gorenstein colength 2 where we can explicitly compute all its minimal Gorenstein covers.

EXAMPLE 2.1.13 Consider any Artin stretched $\mathbf{k}$-algebra $A = \mathbf{k}[[x_1, \dots, x_n]]/I$ with Cohen-Macaulay type $n-1 > 1$. Set $\operatorname{socdeg} A = s \geq 2$. Note that such A is clearly not Gorenstein and, by [20, Proposition 3.4], also not Teter since $\tau(A) \neq n$. Therefore, to prove that $\operatorname{gcl}(A) = 2$ it is enough to find a Gorenstein cover G of A with $\ell(G) = n + s + 2$. Using the classification theorem of stretched algebras provided by [21], we get $I = \left(\{x_i x_j\}_{1 \leq i < j \leq n}, \{x_i^2\}_{2 \leq i \leq n-1}, x_n^2 - x_1^s\right)$ and it can be proved that its inverse system is $I^\perp = \langle y_2, \dots, y_{n-1}, y_1^s + y_n^2 \rangle$. Choosing $F = y_1^{s+1} + y_1 y_n^2 + y_2^2 + \cdots + y_{n-1}^2$, we obtain $I^\perp = (x_1, \dots, x_{n-1}, x_n^2) \circ J^\perp$ and

$$\operatorname{HF}_G(i) = \begin{cases} 1, & \text{if} \quad i = 0; \\ n, & \text{if} \quad i = 1; \\ 2, & \text{if} \quad i = 2; \\ 1, & \text{if} \quad 3 \leq i \leq s+1; \\ 0, & \text{if} \quad i \geq s+2. \end{cases}$$

Any minimal Gorenstein cover G of A has the Hilbert function above and, in particular, all of them have socle degree $s+1$. Indeed, suppose that exists $G = R/J$ with

$$\operatorname{HF}_G(i) = \begin{cases} 1, & \text{if} \quad i = 0; \\ n, & \text{if} \quad i = 1; \\ 1, & \text{if} \quad 2 \leq i \leq s+2; \\ 0, & \text{if} \quad i \geq s+3. \end{cases}$$

By [21], $J^\perp = \langle y_1^{s+2} + y_2^2 + \cdots + y_n^2 \rangle$ up to analytic isomorphism. If $K^\perp \subset J^\perp$ such that $\operatorname{HF}_{R/K} = \operatorname{HF}_A$, then $K^\perp = \langle y_1^s, y_2, \dots, y_n \rangle$. But $\tau(R/K) = \mu(K^\perp) = n$ and $\tau(A) = n-1$.

If we restrict to the case $\operatorname{char}(\mathbf{k}) = 0$, we can explicitly describe all minimal Gorenstein covers G of A by [15]. Such rings $G = R/J$ are almost stretched $\mathbf{k}$-algebras of type $(s+1, 2)$ and therefore the ideal J is isomorphic to one and only one of the following ideals:

(i) Case $s+1=3$:

$$I_{0,1}=\left(\{x_ix_j\}_{1\leq i<j\leq n,(i,j)\neq(1,2)},x_3^2-x_1^3,x_1^2x_2,x_2^2-x_1x_2-x_1^2\right)$$

$$I_{0,-1/4}=\left(\{x_ix_j\}_{1\leq i<j\leq n,(i,j)\neq(1,2)},x_3^2-x_1^3,x_1^2x_2,x_2^2-x_1x_2-\frac{1}{4}x_1^2\right)$$

(ii) Case $s+1\geq 4$:

$$I_{0,1}=\left(\{x_ix_j\}_{1\leq i<j\leq n,(i,j)\neq(1,2)},\{x_i^2-x_1^{s+1}\}_{3\leq i\leq n},x_1^2x_2,x_2^2-x_1x_2-x_1^s\right)$$

$$I_{\infty}=\left(\{x_ix_j\}_{1\leq i<j\leq n,(i,j)\neq(1,2)},\{x_i^2-x_1^{s+1}\}_{3\leq i\leq n},x_1^2x_2,x_2^2-x_1^s\right)$$

Therefore A has only two minimal non-isomorphic Gorenstein covers. Note that Example 2.1.10 is a particular case of this example.

2.2 On self-dual ideals and Gorenstein colength

In the first part of this section we study the link between the family of ideals $\mathfrak{q}$ of R such that $I^{\perp}\twoheadrightarrow q$ and the family of Gorenstein covers of $A=R/I$. In the second part, we characterize Artin rings of Gorenstein colength two.

PROPOSITION 2.2.1 Let A be an Artin ring.
(i) Let $G=R/J$ be a Gorenstein cover of $A=R/I$. Let F be a generator of $J^{\perp}$. Then there is an $R-$module morphism

$$\delta_F: I^{\perp}\longrightarrow A$$

defined as follows: for all $h\in I^{\perp}$, $\delta_F(h)=\overline{a}$ for any $a\in R$ such that $a\circ F=h$.

It holds

(1) $\ker(\delta_F)=(I\circ F)\cap I^{\perp}$, $\operatorname{im}(\delta_F)=K_F+I/I$, and
(2) $\dim_{\mathbf{k}}(\operatorname{coker}(\delta_F))\leq \ell(G)-\ell(A)$.

(ii) There is a set map

$$\Delta_A: GC(A)\longrightarrow \operatorname{Hom}_A(I^{\perp},A)/A^*$$

such that $\Delta_A(G)=\overline{\delta_F}$ for a (all) generator F of $J^{\perp}$.

Proof: (i) Since $h \in I^\perp \subset J^\perp = \langle F \rangle$, there is an $a \in R$ such that $a \circ F = h$. Let b be an element of R such that $b \circ F = h$, then $(a-b) \circ F = 0$. Hence $a - b \in J \subset I$ and $\overline{a} = \overline{b}$ in A. The map δ_F is a morphism of A-modules. Indeed, let $h \in I^\perp$ and $a \in R$ such that $a \circ F = h$. For all $c \in R$ we get $(ac) \circ F = c \circ (a \circ F) = c \circ h$. Then

$$\delta_F(c \circ h) = \overline{ac} = c\delta_F(h).$$

Consider $h \in \ker(\delta_F)$. If $a \in R$ such that $a \circ F = h$, then $a \in I$. On the other hand, if $h = a \circ F$ with $a \in I$, then $\delta_F(h) = 0$. Hence we deduce that $\ker(\delta_F) = (I \circ F) \cap I^\perp$. Since $\dim_{\mathbf{k}} I^\perp = \dim_{\mathbf{k}} A$ and $\ker(\delta_F) \subset I \circ F$, we get

$$\dim_{\mathbf{k}}(\operatorname{coker}(\delta_F)) = \dim_{\mathbf{k}}(\ker(\delta_F)) \leq \dim_{\mathbf{k}}(I \circ F).$$

The map

$$\begin{array}{ccc} \frac{I}{J} & \longrightarrow & I \circ F \\ \overline{a} & \mapsto & a \circ F \end{array}$$

is an isomorphism, hence we obtain that

$$\dim_{\mathbf{k}}(\operatorname{coker}(\delta_F)) \leq \ell(G) - \ell(A).$$

By the definition of $K_F = \left(I^\perp :_R \langle F \rangle\right)$, we deduce that $\operatorname{im}(\delta_F) = K_F + I/I$.
(ii) Assume that $G = R/J$ is a Gorenstein cover of A. If F_1, F_2 are two generators of $J^\perp$, then there exists an invertible power series $u \in R^*$ such that $F_2 = u \circ F_1$. From this we can prove that $\delta_{F_1} = u\delta_{F_2}$. □

We write $(-)^+ = \operatorname{Hom}_A(-, I^\perp)$. We say that an ideal $\mathfrak{q} \subset A$ is self-dual if $\mathfrak{q} \cong \mathfrak{q}^+$. Let $\mathfrak{q} \subset A$ be a self-dual ideal and consider $\mathfrak{q} \overset{i}{\hookrightarrow} A$. This induces an epimorphism $I^\perp \cong A^+ \overset{i}{\twoheadrightarrow} \mathfrak{q}^+$ and hence a morphism $f : I^\perp \longrightarrow A$ such that $\operatorname{im}(f) = \mathfrak{q}$, see [1, Remark 3.3]. We say that an isomorphism $\phi : \mathfrak{q} \cong \mathfrak{q}^+$ satisfies Teter's condition if $\phi(x)(y) = \phi(y)(x)$ for all $x, y \in \mathfrak{q}$.

The next step is to link the morphisms δ_F to self-dual ideals of A. The following result is [1, Lemma 3.4]. We include it here for readers convenience.

LEMMA 2.2.2 Let $\mathfrak{q}$ be an ideal of A. The following conditions are equivalent:

(i) There is an isomorphism $\phi : \mathfrak{q} \cong \mathfrak{q}^+$,

(ii) There is an epimorphism $f : I^\perp \longrightarrow \mathfrak{q}$ such that $\ker(f) = (0 :_{I^\perp} \mathfrak{q}) = Q^\perp$ where $\mathfrak{q} = Q/I$.

Proof: Applying the functor $(-)^+$ to $0 \longrightarrow \mathfrak{q} \xrightarrow{i} A \longrightarrow A/\mathfrak{q} \longrightarrow 0$, we get the exact sequence of A-modules

$$0 \longrightarrow (0 :_{I^\perp} \mathfrak{q}) \longrightarrow I^\perp \xrightarrow{i^+} \mathfrak{q}^+ \longrightarrow 0.$$

Assuming (i), $f = \phi^{-1} \circ i^+ : I^\perp \twoheadrightarrow \mathfrak{q}$ is an A-module epimorphism satisfying $\ker(f) = \ker(i^+) = (0 :_{I^\perp} \mathfrak{q})$. Conversely, by (ii), we have an exact sequence of A-modules

$$0 \longrightarrow (0 :_{I^\perp} \mathfrak{q}) \longrightarrow I^\perp \xrightarrow{f} \mathfrak{q} \longrightarrow 0.$$

Recall that $0 \longrightarrow (0 :_{I^\perp} \mathfrak{q}) \longrightarrow I^\perp \xrightarrow{i^+} \mathfrak{q}^+ \longrightarrow 0$ is also exact. Hence, $\mathfrak{q} \cong \mathfrak{q}^+$. □

PROPOSITION 2.2.3 Given a Gorenstein cover $G = R/J$ of an Artin ring $A = R/I$, let F be a generator of $J^\perp$ and $\mathfrak{q} = \mathrm{im}(\delta_F)$. Then

(i) the ideal $\mathfrak{q}$ is independent of the generator F of $J^\perp$,

(ii) $\mathfrak{q}$ is a self-dual ideal by means of an isomorphism $\phi : \mathfrak{q} \cong \mathfrak{q}^+$ satisfying Teter's condition.

Proof: (i) Let G be a second generator of $J^\perp$. There is an invertible element $u \in R$ such that $G = u \circ F$. Given $\overline{a} = \delta_G(h) \in \mathrm{im}(\delta_G)$, we know that $h = a \circ G \in I^\perp$. Since $h = (au) \circ F$, we get $\delta_F(h) = \overline{ua} \in \mathrm{im}(\delta_F)$. Hence $\mathrm{im}(\delta_G) \subset \mathrm{im}(\delta_F)$ and by symmetry we get the claim.
(ii) First we prove that $\ker(\delta_F) = (0 :_{I^\perp} \mathfrak{q})$. Given $h \in \ker(\delta_F)$, let a be a series in R such that $h = a \circ F$. Then $\overline{a} = \delta_F(h) = 0$, so $a \in I$. For any $\overline{x} \in \mathfrak{q}$, there is $y \in R$ such that $\overline{x} = \delta_F(y \circ F)$, with $x - y \in I$. Then we have

$$x \circ h = y \circ h = y \circ (a \circ F) = a \circ (y \circ F).$$

Since $y \circ F \in I^\perp$ and $a \in I$, we get $a \circ (y \circ F) = 0$. Hence $\overline{x} \circ h = x \circ h = 0$, that is, $h \in (0 :_{I^\perp} \mathfrak{q})$.

Let us now consider $h \in (0 :_{I^\perp} \mathfrak{q})$, so $\mathfrak{q} \circ h = 0$. Since $h \in I^\perp$, we have that $h = a \circ F$ for some $a \in R$ and then $\delta_F(h) = \overline{a}$. Let $x \circ F$ be a general element of $I^\perp$. Then $\overline{x} = \delta_F(x \circ F) \in \mathfrak{q}$ and hence

$$0 = \overline{x} \circ h = x \circ h = x \circ (a \circ F) = a \circ (x \circ F),$$

Since $a \in R$ annihilates a general element of $I^\perp$, we get that $a \in I$. Then $\delta_F(h) = \overline{a} = 0$ or, in other words, $h \in \ker(\delta_F)$. By Lemma 2.2.2.(ii), $\mathfrak{q}$ is a self-dual ideal.

Next we prove that ϕ satisfies Teter's condition. From Lemma 2.2.2 we get that ϕ is defined as follows. Given $\alpha, \beta \in \mathfrak{q}$ there exist $a, b \in K_F$ such that $\alpha = \overline{a}$ and $\beta = \overline{b}$. Since $h_a = a \circ F$ and $h_b = b \circ F$ are elements in $I^\perp$, then $\delta_F(h_a) = \alpha$ and $\delta_F(h_b) = \beta$. Recall that $\phi : \mathfrak{q} \cong \mathfrak{q}^+ = \operatorname{Hom}_A(\mathfrak{q}, I^\perp)$, so

$$\begin{array}{rccc} \phi(\alpha): & \mathfrak{q} & \longrightarrow & I^\perp \\ & \beta & \mapsto & b \circ h_a. \end{array}$$

By symmetry, we get that

$$\phi(\alpha)(\beta) - \phi(\beta)(\alpha) = b \circ h_a - a \circ h_b = b \circ (a \circ F) - a \circ (b \circ F) = 0.$$

□

LEMMA 2.2.4 Consider a maximal regular sequence $a_1, \dots, a_n$ of $R = \mathbf{k}[\![x_1, \dots, x_n]\!]$ and polynomials $H_1, \dots, H_n$ in $S = \mathbf{k}[y_1, \dots, y_n]$ such that $a_i \circ H_j = a_j \circ H_i$ for any $1 \leq i < j \leq n$. Then exists a polynomial F in S such that $a_i \circ F = H_i$, for any $1 \leq i \leq n$.

Proof: Let us consider the first terms of Koszul's resolution of R defined by the regular sequence $a_1, \dots, a_n$:

$$\mathbb{K}_\bullet : \qquad \cdots \longrightarrow \mathbb{K}_2 = R^{\binom{n}{2}} \xrightarrow{d_2} \mathbb{K}_1 = R^n \xrightarrow{d_1} \mathbb{K}_0 = R \longrightarrow R/(a_1, \dots, a_n) \longrightarrow 0.$$

We consider the natural R-basis $\{e_{i,j}\}_{1 \leq i < j \leq n}$ of $\mathbb{K}_2$ and $\{e_i\}_{1 \leq i \leq n}$ of $\mathbb{K}_1$. Then $d_2(e_{i,j}) = a_j e_i - a_i e_j$ for $1 \leq i < j \leq n$ and $d_1(e_i) = a_i$ for $i = 1, \dots, n$.

Dualizing $\mathbb{K}_\bullet$ we get the exact sequence

$$\mathbb{K}_\bullet^\vee : \qquad 0 \longrightarrow (a_1, \dots, a_n)^\perp \longrightarrow \mathbb{K}_0^\vee = S \xrightarrow{d_1^\vee} \mathbb{K}_1^\vee = S^n \xrightarrow{d_2^\vee} \mathbb{K}_2^\vee = S^{\binom{n}{2}} \longrightarrow \dots$$

where

$$d_1^\vee(F) = (a_1 \circ F, \dots, a_n \circ F)$$

and

$$d_2^\vee(F_1, \dots, F_n) = \sum_{1 \le i < j \le n} (a_i \circ F_j - a_j \circ F_i)\, e_{i,j}.$$

Since $(H_1, \dots, H_n) \in \ker(d_2^\vee)$ there exists $F \in S$ such that $d_1^\vee(F) = (H_1, \dots, H_n)$, i.e. $a_i \circ F = H_i$, $i = 1, \dots, n$. $\square$

Now we give an analogous characterization of Artin rings of Gorenstein colength two in terms of its Macaulay inverse system and we improve the result [1, Theorem 5.5] by weakening the hypothesis $I \subset \mathfrak{m}^6$.

THEOREM 2.2.5 Let $A = R/I$ be an Artin ring with maximal ideal $\mathfrak{n}$ and socle degree $s \ge 1$. We assume that A is neither Gorenstein nor Teter, $I \subset \mathfrak{m}^5$ and $\mathrm{char}(\mathbf{k}) \ne 2$. Then the following conditions are equivalent:

(i) $\mathrm{gcl}(A) = 2$,

(ii) after a linear isomorphism of R there exists a polynomial $F \in S$ of degree $s+1$ or $s+2$ such that $I^\perp = \langle x_1 \circ F, \dots, x_{n-1} \circ F, x_n^2 \circ F \rangle$,

(iii) there exists an epimorphism of A-modules $f : I^\perp \longrightarrow \mathfrak{q}$, where $\mathfrak{q}$ is a self-dual ideal of A by means of an isomorphism satisfying Teter's condition and $\ell(A/\mathfrak{q}) = 2$.

In particular, if $\mathrm{gcl}(A) = 2$ then the Cohen-Macaulay type of A is n.

Proof: Let F be a generator of the inverse system of a minimal cover of A. Let $\mathfrak{q} = Q/I$ be the module $\mathrm{im}(\delta_F)$.

(i) implies (iii). By Theorem 2.1.7 and Proposition 2.2.1 we have that $Q = K_F$, hence $\ell(A/\mathfrak{q}) = \ell(R/K_F) = 2$. If we consider the epimorphism $\delta_F : I^\perp \longrightarrow \mathfrak{q}$, then by Proposition 2.2.3 we get (iii).

(iii) implies (ii). Since $\ell(R/Q) = \ell(A/\mathfrak{q}) = 2$, after an analytic isomorphism of R we may assume that $\mathfrak{q} = Q/I$, where $Q = (x_1, \dots, x_{n-1}, x_n^2)$. Since $\mathfrak{q}$ is self-dual, by Lemma 2.2.2, $\ker f = Q^\perp = \langle y_n \rangle \subset I^\perp$.

Let $G_1, \dots, G_n$ be elements of $I^\perp$ such that $f(G_i) = \overline{x}_i$, with $1 \le i \le n-1$, and $f(G_n) = \overline{x}_n^2$. Consider $\alpha \in I^\perp$, then we have $f(\alpha) = \sum_{i=1}^{n-1} \lambda_i \overline{x}_i + \lambda_n \overline{x}_n^2$. Hence

$f(\alpha - \sum_{i=1}^{n-1} \lambda_i G_i - \lambda_n G_n) = 0$ and $\alpha \in \langle G_1, \dots, G_n \rangle + \ker f$. This implies that $I^\perp = \langle G_1, \dots, G_n \rangle + \ker f$.

Notice that since $I \subset \mathfrak{m}^5$, then $S_{\leq 4} \subset I^\perp$ and $S_{\leq 3} \subset \mathfrak{m} \circ I^\perp$. Since $Q^\perp = \langle y_n \rangle \subset S_{\leq 1} \subset S_{\leq 3}$, then $\ker f \subset \mathfrak{m} \circ I^\perp$ and hence

$$I^\perp = \langle G_1, \dots, G_n \rangle + \mathfrak{m} \circ I^\perp.$$

By Nakayama Lemma we deduce that $G_1, \dots, G_n$ is a minimal system of generators of $I^\perp$.

For all $1 \leq i < j \leq n-1$ it holds

$$f(x_i \circ G_j - x_j \circ G_i) = x_i \overline{x}_j - x_j \overline{x}_i = 0,$$

so there is $\alpha_{i,j} = \lambda_{i,j} + \mu_{i,j} y_n \in Q^\perp$, $\lambda_{i,j}, \mu_{i,j} \in \mathbf{k}$, such that $x_i \circ G_j - x_j \circ G_i = \alpha_{i,j}$. By symmetry, we have $\alpha_{j,i} = -\alpha_{i,j}$. Using the same argument for all $i = 1, \dots, n-1$, there is $\alpha_{i,n} = \lambda_{i,n} + \mu_{i,n} y_n \in Q^\perp$, $\lambda_{i,n}, \mu_{i,n} \in \mathbf{k}$, such that $x_i \circ G_n - x_n^2 \circ G_i = \alpha_{i,n}$. By symmetry again, $\alpha_{n,i} = -\alpha_{i,n}$.

Let us consider the elements of $I^\perp$

$$H_i = G_i - \frac{1}{2} \sum_{l=1}^{i-1} y_l \alpha_{l,i} + \frac{1}{2} \sum_{l=i+1}^{n-1} y_l \alpha_{i,l} + \frac{1}{2} y_n^2 \alpha_{i,n},$$

for $i = 1, \dots, n-1$, and

$$H_n = G_n - \frac{1}{2} \sum_{l=1}^{n-1} y_l \alpha_{l,n}.$$

Since $y_l \alpha_{i,j}, y_n^2 \alpha_{i,n} \in S_{\leq 3} \subset \mathfrak{m} \circ I^\perp$ we get that $H_1, \dots, H_n$ is a minimal system of generators of $I^\perp$ as well.

For all $1 \leq i < j \leq n-1$ we have

$$x_i \circ H_j - x_j \circ H_i = x_i \circ G_j - \frac{1}{2} \alpha_{i,j} - x_j \circ G_i - \frac{1}{2} \alpha_{i,j} = 0$$

and

$$x_i \circ H_n - x_n^2 \circ H_i = x_i \circ G_n - \frac{1}{2} \alpha_{i,n} - x_n^2 \circ G_i - \frac{1}{2} \alpha_{i,n} = 0.$$

Since $x_1, \dots, x_{n-1}, x_n^2$ is a maximal R-sequence of R, by Lemma 2.2.4 there exists

$F \in S$ such that $x_i \circ F = H_i$, $i = 1, \ldots, n$, and $x_n^2 \circ F = H_n$. Hence

$$I^\perp = \langle x_1 \circ F, \ldots, x_{n-1} \circ F, x_n^2 \circ F \rangle.$$

(ii) implies (i). Since $(x_1, \ldots, x_{n-1}, x_n^2) \subset K_F$, by Proposition 2.1.6.(iii) $\ell(G) - \ell(A) = \ell(R/K_F) \leq 2$ and hence $\mathrm{gcl}(A) = 2$.

If $\mathrm{gcl}(A) = 2$, combining (ii) and (iii) we get $\mu(\mathfrak{q}) \leq \mu(I^\perp) \leq n$ and hence the Cohen-Macaulay type of A is n. $\square$

REMARK 2.2.6 Observe that condition $I \subset \mathfrak{m}^5$ of last result is indeed a restriction. In all previous examples, $I \subset \mathfrak{m}^5$ is not satisfied and yet Theorem 2.2.5 still holds except for the Cohen-Macaulay type. The key fact used in the proof to compute the Cohen-Macaulay type of R/I is that $\mu(\mathfrak{q}) \leq \tau(R/I) \leq n$. Recall that $\mathfrak{q} = K_F/I$, $K_F = (x_1, \ldots, x_{n-1}, x_n^2)$ and $I \subset \mathfrak{m}^2$. Therefore $\mu(\mathfrak{q}) = \dim_{\mathbf{k}} K_F/(\mathfrak{m}K_F + I)$ can be either n or $n-1$ depending on whether the ideal I is contained in $\mathfrak{m}K_F$ or not. Under the conditions of the theorem, it is always true that $I \subset \mathfrak{m}K_F$. In fact, $I \subset \mathfrak{m}^3$ is a sufficient - though not necessary - condition to ensure that. On the other hand, it can be checked that self-dual ideals $\mathfrak{q} = K_F/I$ of $A = R/I$ from Example 2.1.10 to Example 2.1.13 are minimally generated by $n-1$ elements. Hence the Cohen-Macaulay type of such rings is allowed to be $n-1$. See Example 2.2.10 for rings that actually are under the conditions of Theorem 2.2.5.

REMARK 2.2.7 It can be proved that for any $\mathbf{k}$-algebra of length less or equal than 6, both conditions $I \subset \mathfrak{m}^5$ and $\mathrm{char}(\mathbf{k}) \neq 2$ can be dropped. Moreover, in Table 5.2 of Chapter 5 we provide a complete list of all analytic types of A such that $\ell(A) \leq 6$ using the classification given by Poonen in [40].

Let us consider again some of the examples of rings of Gorenstein colength 2 we showed at the end of Section 2.1. We explicitly describe the maps δ_F provided by generators F of inverse systems of the minimal Gorenstein covers and compute the corresponding self-dual ideals.

EXAMPLE 2.2.8 (See Example 2.1.9.) Consider the ring $A = \mathbf{k}[\![x_1, x_2]\!]/\left(x_1^2, x_1x_2^2, x_2^4\right)$. Recall that the polynomials of different degree $F_1 = y_1y_2^3$ and $F_2 = y_1^2y_2 + y_2^5$ generate inverse systems of two non-isomorphic minimal covers of A. By Proposition 2.2.1, δ_{F_1} is a morphism of R-modules with $\ker(\delta_{F_1}) = K_{F_1}^\perp = \langle y_2 \rangle$ and $\mathrm{im}\,\delta_{F_1} = K_{F_1}/I$.

Therefore, this is an epimorphism:

$$\begin{array}{rrcl} \delta_{F_1}: & I^\perp = \langle y_1y_2, y_2^3\rangle & \longrightarrow & \mathfrak{q} = (x_1, x_2^2)/I \\ & y_1y_2 & \longmapsto & \overline{x}_2^2 \\ & y_2^3 & \longmapsto & \overline{x}_1 \end{array}$$

By Lemma 2.2.2, $\mathfrak{q} = (x_1, x_2^2)/I$ is a self-dual ideal of A. Also $\ell(A/\mathfrak{q}) = \ell(K_{F_1}^\perp) = 2$ and the same works for δ_{F_2}:

$$\begin{array}{rrcl} \delta_{F_2}: & I^\perp & \longrightarrow & \mathfrak{q} = (x_1, x_2^2)/I \\ & y_1y_2 & \longmapsto & \overline{x}_1 \\ & y_2^3 & \longmapsto & \overline{x}_2^2 \end{array}$$

Note that, despite the fact that F_1 and F_2 are polynomials of different degree, the self-dual ideals given by the images of the corresponding morphisms δ_{F_1} and δ_{F_2} are equal because $K_{F_1} = K_{F_2} = (x_1, x_2^2)$. Observe that the self-dual ideal $\mathfrak{q}$ is minimally generated by $\overline{x}_1, \overline{x}_2^2$ and hence $\mu(\mathfrak{q}) = \tau(A) = 2$.

EXAMPLE 2.2.9 (See Example 2.1.13.) For any Artin stretched algebra A with $\tau(A) = n-1$ we can consider the generator $F = y_1^{s+1} + y_1y_n^2 + y_2^2 + \cdots + y_{n-1}^2$ of the inverse system of a minimal Gorenstein cover G of A. Since $K_F = (x_1, \dots, x_{n-1}, x_n^2)$, then $\ker \delta_F = \langle y_2\rangle$ and $\operatorname{im} \delta_F = (x_1, \dots, x_{n-1}, x_n^2)/I$.

$$\begin{array}{rrcl} \delta_F: & I^\perp = \langle y_1^s + y_n^2, y_2, \dots, y_{n-1}\rangle & \longrightarrow & \mathfrak{q} = (x_1, \dots, x_{n-1}, x_n^2)/I \\ & y_1^{s-1} + y_n^2 & \longmapsto & \overline{x}_1 \\ & y_2 & \longmapsto & \overline{x}_2 \\ & \vdots & & \vdots \\ & y_{n-1} & \longmapsto & \overline{x}_{n-1} \end{array}$$

Note that the self-dual ideal $\mathfrak{q}$ is minimally generated by the $n-1$ elements $\overline{x}_1, \dots, \overline{x}_{n-1}$.

We now provide a family $\{R/I_t\}_{t\geq 3}$ of Artin local $\mathbf{k}$-algebras of Gorenstein colength 2 such that $I_t \subset \mathfrak{m}^t$. For any $t \geq 5$, the ring R/I_t is under the conditions of Theorem 2.2.5.

EXAMPLE 2.2.10 Consider the family of ideals $J_t = (x_1^t, ..., x_n^t)$ of $R = \mathbf{k}[\![x_1, ..., x_n]\!]$, with $t \geq 3$ and $n \geq 2$. $G_t = R/J_t$ is a Gorenstein Artin ring with inverse system $J_t^\perp = \langle y_1^{t-1} \cdots y_n^{t-1} \rangle$, $\operatorname{socdeg} G_t = n(t-1)$ and symmetric Hilbert function. We are only interested in computing the first three terms of its Hilbert function, which do not depend on t, as we will see now. Since $t \geq 3$, all degree 2 polynomials are in $J^\perp$ and hence $\mathrm{HF}_{G_t}(2) = n(n+1)/2$. Therefore, $\mathrm{HF}_{G_t} = \{1, n, n(n+1)/2, \ldots, n(n+1)/2, n, 1\}$.

Contracting by an appropriate ideal $K = (x_1, \ldots, x_{n-1}, x_n^2)$ we obtain a sub-R-module $I_t^\perp$ of $J_t^\perp$ such that $\ell(J_t^\perp) - \ell(I_t^\perp) = 2$. Indeed,

$$I_t^\perp = K \circ J_t^\perp = \langle y_1^{t-2} y_2^{t-1} \cdots y_n^{t-1}, \ldots, y_1^{t-1} \cdots y_{n-2}^{t-1} y_{n-1}^{t-2} y_n^{t-1}, y_1^{t-1} \cdots y_{n-1}^{t-1} y_n^{t-3} \rangle.$$

Hence the Hilbert function of $A_t = R/I_t$ only changes in pieces of degree $n(t-1)-1$ and $n(t-1)$: $\mathrm{HF}_{A_t} = \{1, n, n(n+1)/2, \ldots, n(n+1)/2, n-1\}$. Clearly G_t is a Gorenstein cover of A_t but we want to prove that it is minimal. For any $n \geq 2$, A_t is not Gorenstein because $\tau(A_t) = \mu(I_t^\perp) = n$.

Let us now suppose that A_t is Teter. Then, by [20], there exists a Gorenstein minimal cover G'_t of A_t with Hilbert function $\{1, n, n(n+1)/2, \ldots, n(n+1)/2, n-1, 1\}$ which is no longer symmetric due to the piece of degree $n(t-1)-1$. We will use the shell formula to prove that no Gorenstein ring could have such Hilbert function, see Theorem 1.2.9 for more details. Since $Q(0)$ is an Artin Gorenstein $\mathbf{k}$-algebra with symmetric Hilbert function, all $\mathrm{HF}_{Q(i)}(0)$ must be zero for any $i \geq 1$ and so must be its symmetric pieces. Hence $a_1 = \mathrm{HF}_{Q(0)}(1) = \mathrm{HF}_{Q(0)}(n(t-1)-1) = n-1$ and any possible decomposition must start as in the table below:

i	0	1	2	$\ldots$	$n(t-1)-2$	$n(t-1)-1$	$n(t-1)$
G'_t	1	n	$n(n+1)/2$	$\ldots$	$n(n+1)/2$	$n-1$	1
$Q(0)$	1	$n-1$	a_2	$\ldots$	a_2	$n-1$	1
$Q(1)$	0	b_1	b_2	$\ldots$	b_1	0	0
$Q(2)$	0	c_1	c_2	$\ldots$	0	0	0

Macaulay conditions [5, Theorem 4.2.10] give us the upper bound $\mathrm{HF}_{Q(0)}(1)^{\langle 1 \rangle}$ for $\mathrm{HF}_{Q(0)}(2)$. Hence

$$a_2 = \mathrm{HF}_{Q(0)}(2) \leq \mathrm{HF}_{Q(0)}(1)^{\langle 1 \rangle} = \binom{n}{2} = \frac{n(n-1)}{2}.$$

On the other hand, $b_1 = n(n+1)/2 - a_2 \geq n$. But then $n - 1 + b_1 \geq 2n - 1$ and hence $\mathrm{HF}_{Q(0)}(1) + \mathrm{HF}_{Q(1)}(1) > \mathrm{HF}_{G'_t}(1) = n$ for any $n \geq 2$. Therefore, there is no Gorenstein ring with such Hilbert function and A_t is not Teter. Then $\mathrm{gcl}(A_t) = 2$.

2.3 Higher Gorenstein colength

When dealing with rings of Gorenstein colength higher than 2 we have two main differences with respect to the low colength scenario. Let us focus on the simplest situation to provide some insight into the difficulties we are facing: $\mathrm{gcl}(A) = 3$.

On one hand, since Proposition 2.1.6 holds for arbitrary colength, we have that any K_F corresponding to a minimal Gorenstein cover $G = R/\operatorname{Ann_R} F$ of $A = R/I$ satisfies $\ell(R/K_F) = 3$. Unlike in case $\ell(R/K_F) \leq 2$, now K_F has no longer a unique analytic type. See Appendix B for a formal definition of analytic type. Poonen's classification in [40] provides two different analytic types for any ideal $K_F \subset R$ such that $\ell(R/K_F) = 3$:

$$K_F = \begin{cases} (L_1, \dots, L_{n-1}, L_n^3), \\ (L_1, \dots, L_{n-2}) + (L_{n-1}, L_n)^2, \end{cases}$$

where $L_1, \dots, L_n$ are independent linear forms in R and $n = \dim R$.

On the other hand, we do not know whether the embedding dimension of the minimal Gorenstein covers is preserved or is increased.

Therefore, we pose the following two questions for rings of higher colength:

Question A: Given $\mathrm{gcl}(A) = t$, is there a unique analytic type of ideals $K \subset R$ such that $\ell(R/K) = t$ eligible to be $K = (I^\perp :_R J^\perp)$, where $G = R/J$ is a minimal cover of $A = R/I$?

Question B: Given any Artin ring $A = R/I$, is there a minimal Gorenstein cover $G = R/J$ of A such that $\mathrm{embd}(G) = \mathrm{embd}(A)$?

The answer to the first question is no. We show in the following example that both analytic types of K such that $\ell(R/K) = 3$ can occur as colon ideals of inverse systems:

EXAMPLE 2.3.1 Consider the family of ideals $J_t = (x_1^t, x_2^t)$ of $R = \mathbf{k}[\![x_1, x_2]\!]$, with $t \geq 5$. $G_t = R/J_t$ is a Gorenstein Artin ring with inverse system generated by the polynomial $F = y_1^{t-1}y_2^{t-1}$ and symmetric Hilbert function

$$\mathrm{HF}_{G_t} = \{1, 2, 3, \dots, t-1, t, t-1, \dots, 2, 1\}.$$

Contracting by $K_1 = (x_1^2, x_1x_2, x_2^2)$ we obtain

$$I_{1,t}^{\perp} = K_1 \circ J_t^{\perp} = \langle y_1^{t-3}y_2^{t-1}, y_1^{t-2}y_2^{t-2}, y_1^{t-1}y_2^{t-3}\rangle,$$

with $\mathrm{HF}_{R/I_{1,t}} = \{1, 2, \dots, t-1, t, t-1, \dots, 4, 3\}$.

Contracting by $K_2 = (x_1, x_2^3)$ we get

$$I_{2,t}^{\perp} = K_2 \circ J_t^{\perp} = \langle y_1^{t-2}y_2^{t-1}, y_1^{t-1}y_2^{t-4}\rangle,$$

with $\mathrm{HF}_{R/I_{2,t}} = \{1, 2, \dots, t-1, t, t-1, \dots, 4, 2, 1\}$.

In codimension 2, Theorem 1.2.11 ensures that the Hilbert function h with minimal length that admits a Gorenstein ring and satisfies $\mathrm{HF}_A(i) \leq h(i)$ for any $i \geq 0$ is $h = \{1, 2, \dots, t-1, t, t-1, \dots, 4, 3, 2, 1\}$. Therefore, $\mathrm{gcl}(R/I_{i,t}) = 3$ for $i = 1, 2$.

Summing up, for any $t \geq 5$, there are two non-isomorphic rings $A_{1,t}$ and $A_{2,t}$ of Gorenstein colength 2 that share the same minimal Gorenstein cover G_t:

- $A_{1,t} = R/(x_1^t, x_2^t, x_1^{t-1}x_2^{t-2}, x_1^{t-2}x_2^{t-1})$ is a family of rings with Cohen-Macaulay type 3 and colon ideal K_1.
- $A_{2,t} = R/(x_1^t, x_2^t, x_1^{t-1}x_2^{t-3})$ is a family of rings with Cohen-Macaulay type 2 and colon ideal K_2.

Regarding the embedding dimension of minimal Gorenstein covers, even for Gorenstein colength 3, an analogous argument to the one used in Theorem 2.1.7 only works to prove $\mathrm{embd}(A) = \mathrm{embd}(G)$ for the analytic type $K_F = (L_1, \dots, L_{n-2}) + (L_{n-1}, L_n)^2$.

However, we do have bounds on the embedding dimension that can be deduced directly from Proposition 2.1.6 and therefore hold for any arbitrary colength:

PROPOSITION 2.3.2 Let $G = R/J$ be a minimal Gorenstein cover of $A = R/I$. Then

$$\mathrm{embd}(A) \leq \mathrm{embd}(G) \leq \tau(A) + \mathrm{gcl}(A) - 1.$$

Proof: Set $A = R/I$ and $G = R'/J$ such that $\mathrm{embd}(A) = \dim R$ and $\mathrm{embd}(G) = \dim R'$. We denote by $\mathfrak{m}$ and $\mathfrak{m}'$ the maximal ideals of R and R', respectively. From

Proposition 2.1.6.(i), it is easy to deduce that $K_F/(\mathfrak{m}K_F + J) \simeq I^{\perp}/(\mathfrak{m} \circ I^{\perp})$. Hence $\tau(A) = \dim_{\mathbf{k}} K_F/(\mathfrak{m}K_F + J)$ by [20, Proposition 2.6]. Then

$$\operatorname{embd}(G) + 1 = \dim_{\mathbf{k}} R'/(\mathfrak{m}')^2 \leq \dim_{\mathbf{k}} R'/(\mathfrak{m}K_F + J) = \operatorname{gcl}(A) + \tau(A),$$

where the last equality follows from Proposition 2.1.6.(iii). □

The expression $\tau(A) + \operatorname{gcl}(A) - 1$ can be arbitrarily higher than n. However, no examples of minimal Gorenstein covers of higher embedding dimension are known so far. In Chapter 5 we provide more insight about the bound on $\operatorname{embd}(G)$ for certain families of Artin rings.

Let us now summarize the bounds on Gorenstein colength and socle degree that we do have available in the general case:

PROPOSITION 2.3.3 Let $A = R/I$ be a non-Gorenstein Artin ring with $\operatorname{embd}(A) = \dim R = n$ and $G = R'/J$ a minimal Gorenstein cover of A. Then

(i) $\operatorname{socdeg} A < \operatorname{socdeg} G$,
(ii) $\operatorname{socdeg} G < \operatorname{gcl}(A) + \operatorname{socdeg} A$,
(iii) $\operatorname{gcl}(A) \geq n - \tau(A) + 1$.

Proof: (i) Consider a generator F of $J^{\perp}$. Since A is not Gorenstein, there exists an ideal $K \subset R$ with $\ell(R/K) = \operatorname{gcl}(A) \geq 1$ such that $I^{\perp} = K \circ F$. Then any $H \in I^{\perp}$ has degree at most $\deg F - 1$, hence $\operatorname{socdeg} A < \operatorname{socdeg} G$.
(ii) Following the same notation, since $\ell(R/K) = \operatorname{gcl}(A)$, then $\mathfrak{m}^{\operatorname{gcl}(A)} \subseteq K$ and hence $\mathfrak{m}^{\operatorname{gcl}(A)} \circ F \subseteq I^{\perp}$. Therefore, $\deg F - \operatorname{gcl}(A) \leq \operatorname{socdeg} A$.
(iii) Direct from Proposition 2.3.2. □

Another interesting property that holds for low colength is the inclusion of ideals $I^2 \subset J \subset I$, for any minimal Gorenstein cover $G = R/J$ of $A = R/I$, see Theorem 2.1.7. It is natural to ask whether this also occurs in higher colength. Let us give an equivalent condition:

LEMMA 2.3.4 Consider an Artin local ring $A = R/I$ and a Gorenstein cover $G = R/J$ of A. The following conditions are equivalent:

(i) $I^2 \subset J \subset I$,

(ii) $I \subset K_F$.

Proof: $(i) \Rightarrow (ii)$. I^2 is contained in J, hence the contraction $I^2 \circ J$ vanishes. In other words, $I \circ (I \circ F) = 0$. Thus $I \circ F$ is contained in $I^\perp = K_F \circ F$ and then $I \subset K_F$.

$(ii) \Rightarrow (i)$. From $I \subset K_F$, it follows that $I^2 \subset IK_F$ and hence

$$I^2 \circ F \subseteq (IK_F) \circ F = I \circ (K_F \circ F) = I \circ I^\perp = 0.$$

Therefore, $I^2 \subset J$. $\square$

Again we are able to prove that the inclusion of ideals $I^2 \subset J \subset I$ holds for certain families of rings:

PROPOSITION 2.3.5 Let $A = R/I$ be an Artin ring. In the following cases we have that there exist a minimal Gorenstein cover $G = R/J$ of A such that $I^2 \subset J \subset I$:

(i) $\ell(A) \leq 6$,
(ii) A is stretched,
(iii) $I = \mathfrak{m}^t$ for some $t \geq 1$,
(iv) $\dim(R) = 2$.

Proof:
For the proofs of $(i), (ii), (iii)$, see Chapter 5.
(iv) Assume that $\dim(R) = 2$. From Briançon-Skoda theorem we have $I^2 \subset J$ for all reduction J of I, see [34]. Recall that any minimal reduction of I is a complete intersection, in particular a Gorenstein ideal. Hence we get the claim. $\square$

CHAPTER 3

Variety of minimal Gorenstein covers

In Chapter 2 we provided a characterization of $\mathbf{k}$-algebras of low Gorenstein colength $A = R/I$ in terms of the Macaulay's inverse system of I. Even more, we connected rings A of arbitrary colength t with their minimal Gorenstein covers $G = R/\operatorname{Ann_R} F$ through colon ideals K_F of R of minimal length $\ell(R/K_F) = t$ such that $I^\perp = K_F \circ F$, see Proposition 2.1.6. Two natural questions arise:

Question A: How can we explicitly compute the Gorenstein colength t of a given local Artin $\mathbf{k}$-algebra A?

Question B: Which are all the minimal Gorenstein covers $G = R/\operatorname{Ann_R} F$ of a given local Artin $\mathbf{k}$-algebra A?

In [20, Proposition 4.2], Elias and Silva introduce the notion of Teter variety of A as the set of points $[F] \in \mathbb{P}^N_{\mathbf{k}}$, for a suitable N, such that $G = R/\operatorname{Ann_R} F$ is a minimal Gorenstein cover of A such that $\ell(G) - \ell(A) = 1$. The result in [20, Proposition 4.5] already suggests that a method to explicitly compute such covers is possible.

In this chapter we address questions A and B by extending the idea of Teter variety in Gorenstein colength 1 to the variety of minimal Gorenstein covers $MGC(A)$ where A has arbitrary Gorenstein colength t.

Observe that, given an ideal K of R, we can ask whether it is possible to find a polynomial F defining a cover $G = R/\operatorname{Ann_R} F$ of $A = R/I$ such that $I^\perp = K \circ F$. Our key contribution is the introduction of an inverse operation to contraction of sub-R-modules of S and a recursive procedure to effectively compute the resulting module based on the integration method for inverse systems proposed by Mourrain in [39].

In Section 3.1 we introduce this notion of integral of an R-module M of S with respect to an ideal K of R, denoted by $\int_K M$. By Definition 3.1.1, for any F in S such that $I^\perp = K \circ F$ we have

$$F \in \int_K I^\perp.$$

Therefore, if $\operatorname{gcl}(A) = t$, then all polynomials defining minimal Gorenstein covers of $A = R/I$ lay in some sub-R-module $\int_K I^\perp$ of S, for suitable ideals K with $\ell(R/K) = t$. This approach is exploited in Section 3.2 by identifying the inverse system of a minimal Gorenstein cover $G = R/\operatorname{Ann_R} F$ with the class of any of its generators F in S in the R-module $\int_{\mathfrak{m}^t} I^\perp/I^\perp$. This connection is described in detail in Proposition 3.2.5. The section ends with Theorem 3.2.7, that sets the theoretical background to compute a $\mathbf{k}$-basis of $\int_K I^\perp$ extending Mourrain's integration method.

In Section 3.3, the main result of this chapter, Theorem 3.3.2, proves the existence of a quasi-projective sub-variety $MGC^n(A)$ of $\mathbb{P}_{\mathbf{k}}\left(\int_{\mathfrak{m}^t} I^\perp/I^\perp\right)$ whose set of closed points are associated to polynomials F in S such that $G = R/\operatorname{Ann_R} F$ is a minimal Gorenstein cover of A.

Section 3.4 is devoted to algorithms: explicit methods to compute a $\mathbf{k}$-basis of $\int_{\mathfrak{m}^t} I^\perp$ and $MGC(A)$ for colengths 1 and 2. In this context, we give a precise description of the varieties of minimal Gorenstein covers for rings of low Gorenstein colength:

THEOREM 3.0.1 (See Theorem 3.4.6.) Let $A = R/I$ be a Teter ring with $n \geq 2$, let h be the dimension of $\int_{\mathfrak{m}} I^\perp/I^\perp$ as $\mathbf{k}$-vector space and let $\mathfrak{a}$ be the homogeneous ideal defined in Section 3.4.2 in a polynomial ring with h variables. Then

$$MGC(A) = \mathbb{P}_{\mathbf{k}}^{h-1} \backslash \mathbb{V}_+(\mathfrak{a}).$$

Moreover, for any non-Gorenstein Artin ring A, $\operatorname{gcl}(A) = 1$ if and only if $\mathfrak{a} \neq 0$.

THEOREM (See Corollary 3.4.20.) Let $A = R/I$ be a ring of Gorenstein colength 2 and let h be the dimension of $\int_{\mathfrak{m}^2} I^\perp/I^\perp$ as $\mathbf{k}$-vector space. Let $\mathfrak{b}$ be a homogeneous ideal in the ring of polynomials with h variables and let $\mathfrak{a}$ and $\mathfrak{c}$ be bihomogeneous ideals in the ring of polynomials with $h + n$ variables as defined in Section 3.4.3. Let π_1 be the projection map from $\mathbb{P}_{\mathbf{k}}^{h-1} \times \mathbb{P}_{\mathbf{k}}^{n-1}$ to $\mathbb{P}_{\mathbf{k}}^{h-1}$. Then

$$MGC(A) = \mathbb{V}_+(\mathfrak{b}) \backslash \pi_1 \left(\mathbb{V}_+(\mathfrak{c}) \cap \mathbb{V}_+(\mathfrak{a})\right).$$

Finally, in Section 3.5 we provide several examples of varieties of minimal Gorenstein covers and list the comptutation times of $MGC(A)$ for all analytic types of $\mathbf{k}$-

algebras with $\mathrm{gcl}(A) \leq 2$ appearing in Poonen's classification, see [40].

All algorithms appearing in this chapter have been implemented in *Singular*, [11], by creating a new library, see Appendix A.

Part of the results of this chapter will be published in [17].

3.1 Integrals and inverse systems

Consider an Artin local ring $A = R/I$ and fix an ideal K of R. We want to find, if it exists, a polynomial $F \in S$ such that $K \circ F = I^\perp$. In other words, we want a Gorenstein cover $G = R/\operatorname{Ann}_R F$ such that $K = (I^\perp :_R \langle F \rangle)$. Therefore, it makes sense to think of an inverse operation to contraction:

DEFINITION 3.1.1 Consider an R-submodule M of S. We define the **integral of M with respect to the ideal K**, denoted by $\int_K M$, as

$$\int_K M = \{G \in S \mid K \circ G \subset M\}.$$

Note that the set $N = \{G \in S \mid K \circ G \subset M\}$ is, in fact, a sub-R-submodule N of S equipped with the contraction structure. Indeed, given $G_1, G_2 \in N$ we have $K \circ (G_1 + G_2) = K \circ G_1 + K \circ G_2 \subset M$, hence $G_1 + G_2 \in N$. For all $a \in R$ and $G \in N$ we have $K \circ (a \circ G) = aK \circ G = a \circ (K \circ G) \subset M$, hence $a \circ G \in N$.

PROPOSITION 3.1.2 With the above notations it holds

$$\int_K M = \left(KM^\perp\right)^\perp.$$

Proof: Let $G \in \left(KM^\perp\right)^\perp$. Then $\left(KM^\perp\right) \circ G = 0$, so $M^\perp \circ (K \circ G) = 0$. Hence $K \circ G \subset M$, i.e. $G \in \int_K M$. We have proved that $\left(KM^\perp\right)^\perp \subseteq \int_K M$. Take G in $\int_K M$. By definition, $K \circ G \subset M$, so $M^\perp \circ (K \circ G) = 0$ and hence $\left(M^\perp K\right) \circ G = 0$. Therefore, $G \in \left(M^\perp K\right)^\perp$. $\square$

One of the key results of this paper is the effective computation of $\int_K M$ that we present in Algorithm 1, see Section 3.4.1. Proposition 3.1.2 provides a method to compute the integral of a module consisting of two Macaulay duals. As shown in Section A.1.4, this can be an expensive computation. Therefore, Algorithm 1 is instead based on Mourrain's integration method, see Theorem 3.2.7, that we will explain next.

Before moving on, let us list some of the basic properties of integrals that can be proved directly from the definition of integral:

PROPOSITION 3.1.3 Given ideals K, L of R and sub-R-modules M, N of S, we have

(i) If $K \subset L$, then $\int_L M \subset \int_K M$.
(ii) If $M \subset N$, then $\int_K M \subset \int_K N$.
(iii) $\int_R M = M$.
(iv) $K \circ \int_K M \subset M$.
(v) $M \subset \int_K K \circ M$.

We give now two examples to show that equality does not hold in general for statements (iv) and (v).

EXAMPLE 3.1.4 Let us consider $R = \mathbf{k}[\![x_1, x_2, x_3]\!]$, $K = (x_1, x_2, x_3)$, $S = \mathbf{k}[y_1, y_2, y_3]$, and $M = \langle y_1y_2, y_3^3 \rangle$. We have $\int_K M = \left(KM^\perp\right)^\perp = \langle y_1^2, y_1y_2, y_1y_3, y_2^2, y_2y_3, y_3^4 \rangle$, and $K \circ \int_K M = \mathfrak{m} \circ \langle y_1^2, y_1y_2, y_1y_3, y_2^2, y_2y_3, y_3^4 \rangle = \langle y_1, y_2, y_3^3 \rangle \subsetneq M$.

EXAMPLE 3.1.5 Using the same notation as in Example 3.1.4, we get $K \circ M = \langle y_1, y_2, y_3^2 \rangle$, and

$$\int_K (K \circ M) = \left(K(K \circ M)^\perp\right)^\perp = \langle y_1^2, y_1y_2, y_1y_3, y_2^2, y_2y_3, y_3^2 \rangle \not\subseteq M.$$

In the particular case where we integrate with respect to a principal monomial ideal $K = (x^\alpha)$ in R, the expected equality for integrals

$$x^\alpha \circ \int_{x^\alpha} M = M$$

holds. Indeed, for any $m \in M$, take $G = y^\alpha m$. Since $x^\alpha \circ y^\alpha = 1$, then $x^\alpha \circ y^\alpha m = m$ and the equality is reached.

REMARK 3.1.6 In general we cannot extend the above identity to linear forms. Consider $\mathbf{k} = \mathbb{C}$, $L = x_1 + ix_2$ and $P = y_1 + iy_2$. Then $L \circ \int_L \langle P \rangle \subsetneq \langle P \rangle$.

Let us now consider an even more particular case: the integral of a cyclic module $M = \langle F \rangle$ with respect to the variable x_i. Since the equality $x_i \circ \int_{x_i} M = M$ holds, there exists $G \in S$ such that $x_i \circ G = F$. This polynomial G is not unique because it can have any constant term with respect to x_i, that is $G = y_i F + p(y_1, \ldots, \hat{y}_i, \ldots, y_n)$. However, if we restrict to the non-constant polynomial we can define the following:

DEFINITION 3.1.7 The i-**primitive** of a polynomial $F \in S$ **with respect to contraction** is the polynomial $G \in S$, denoted by $\int_i F$, such that

(i) $x_i \circ G = F$,
(ii) $G|_{y_i=0} = 0$.

This notion of i-primitive of a polynomial $F \in S$ with respect to the variable $x_i \in R$ was provided in [23] using the derivation structure:

DEFINITION 3.1.8 The i-**primitive** of a polynomial $F \in S$ **with respect to derivation** is the polynomial $G \in S$, denoted by $\int_i F$, such that

(i) $\partial_{y_i} G = F$,
(ii) $G|_{y_i=0} = 0$.

Therefore, we can think of the integral of a module with respect to an ideal as a generalization of the i-primitive proposed by Elkadi and Mourrain.

From now on, when we use the notation $\int_{x_i} F$ it refers to the contraction case. Since we are considering the R-module structure given by contraction instead of derivation, the i-primitive is precisely

$$\int_i F = y_i F.$$

Indeed, $x_i \circ (y_i F) = F$ and $(y_i F)\ |_{y_i=0} = 0$, hence (i) and (ii) hold. Uniqueness can be easily proved. Consider G_1, G_2 to be i-primitives of F. Then $x_i \circ (G_1 - G_2) = 0$ and hence $G_1 - G_2 = p(y_1, \ldots, \hat{y}_i, \ldots, y_n)$. Clearly $(G_1 - G_2)|_{y_i=0} = p(y_1, \ldots, \hat{y}_i, \ldots, y_n)$. On the other hand, $(G_1 - G_2)|_{y_i=0} = G_1|_{y_i=0} - G_2|_{y_i=0} = 0$. Hence $p = 0$ and $G_1 = G_2$.

REMARK 3.1.9 Note that, by definition, $x_k \circ \int_k F = F$. Any F can be decomposed in $F = F_1 + F_2$, where the first term is a multiple of y_k and the second has no appearances of this variable. Then

$$\int_k x_k \circ F = \int_k x_k \circ F_1 + \int_k x_k \circ F_2 = F_1 + \int_k 0.$$

Therefore, in general,

$$F_1 = \int_k x_k \circ F \neq x_k \circ \int_k F = F.$$

However for all $l \neq k$

$$\int_l x_k \circ F = \frac{y_l F_1}{y_k} = x_k \circ \int_l F.$$

Now consider an ideal I of R generated by $f_1, \ldots, f_m$. From Definition 1.4.5 we can deduce that polynomials Λ in S that belong to the inverse system $I^\perp$ can be characterized as follows by imposing conditions only on the generators of I:

$$f_i \circ \Lambda = 0, \quad 1 \leq i \leq m. \tag{3.1}$$

Observe that the resulting equations are not linear in general. If we want linear conditions, we can use the equivalent characterization of $I^\perp$ given by Definition 1.4.12:

$$(f \circ \Lambda)(0) = 0, \quad \text{for any } f \in I. \tag{3.2}$$

From the proof of Proposition 1.4.14 we know that it is not enough to impose 3.2 only on $f_1, \ldots, f_m$. The following example shows that there are polynomials $\Lambda \in S$ that satisfy 3.2 but not 3.1 on the generators of I.

EXAMPLE 3.1.10 Consider the ideal $I = (x_1x_2, x_1^3 - x_2^2)$ of $R = \mathbf{k}[\![x_1, x_2]\!]$ and the polynomial $\Lambda = y_1^2 y_2$ of $S = \mathbf{k}[y_1, y_2]$. Note that the condition $(f \circ \Lambda)(0) = 0$ holds for the generators of I:

$$\left(x_1x_2 \circ y_1^2y_2\right)(0) = 0,$$
$$\left(\left(x_1^3 - x_2^2\right) \circ y_1^2y_2\right)(0) = 0.$$

But $x_1x_2 \circ y_1^2y_2 = x$ and $\left(x_1^3 - x_2^2\right) \circ y_1^2y_2 = 0$, hence $\Lambda \notin I$.

To overcome this problem, Elkadi and Mourrain add some extra conditions involving integrals to 3.2. In [23, Theorem 7.36], they characterize the elements Λ of the inverse system $I^\perp$ up to a certain degree d.

We will rewrite both the theorem and the proof using the contraction structure for the sake of completeness. But first, let us make a few comments on the notation we will use and give a technical lemma regarding some properties of the i-primitives that will be needed for the proof. Given an Artin ring $A = R/I$ with $s = \operatorname{socdeg}(A)$,

- $\mathcal{D}_d$ stands for the sub-R-module $I^\perp \cap S_{\leq d}$ of S, for any $1 \leq d \leq s$;
- $b|_{y_j=a}$, where $b \in S$ and $a \in \mathbf{k}$, denotes $b(y_1, \ldots, y_{j-1}, a, y_{j+1}, \ldots, y_n)$.

LEMMA 3.1.11 Consider a polynomial $b \in S$. Then

$$b \mid_{y_{k+1}=\cdots=y_n=0} + \int_{k+1} x_{k+1} \circ b \mid_{y_{k+2}=\cdots=y_n=0} + \cdots + \int_n x_n \circ b = b.$$

Proof: Since any polynomial $b \in S$ can be decomposed as $b^{0,n}+b^n$ such that $b^n = y_n c_n$ and $b^{0,n} \in \mathbf{k}[y_1, \ldots, y_{n-1}]$, then

$$\int_n x_n \circ b = \int_n x_n \circ b^n = \int_n c_n = y_n c_n = b^n.$$

Now decompose $b^{0,n} = b^{0,n-1} + b^{n-1}$ with $b^{n-1} = y_{n-1}c_{n-1}$, where c_{n-1} is a polynomial in $\mathbf{k}[y_1, \ldots, y_{n-1}]$ and $b^{0,n-1} \in \mathbf{k}[y_1, \ldots, y_{n-2}]$. Since $b = b^{0,n-1} + b^{n-1} + b^n$, then

$$\int_{n-1} x_{n-1} \circ b \mid_{y_n=0} = \int_{n-1} x_{n-1} \circ \left(b^{0,n-1} + b^{n-1}\right) = \int_{n-1} x_{n-1} \circ b^{n-1} =$$

$$= \int_{n-1} c_{n-1} = y_{n-1}c_{n-1} = b^{n-1}.$$

By recurrence, we have that for any $k < l \leq n-1$, we can decompose $b^{0,l+1} = b^{0,l} + b^l$, where $b^{0,l} \in \mathbf{k}[y_1, \ldots, y_{l-1}]$ and $b^l = y_l c_l$, with $c_l \in \mathbf{k}[y_1, \ldots, y_l]$. Then

$$b = b^{0,l} + b^l + b^{l+1} + \cdots + b^n,$$

and all the terms in $b^{l+1}, \ldots, b^n$ contain at least one of the variables $y_{l+1}, \ldots, y_n$. Hence

$$\int_l x_l \circ b|_{y_{l+1}=\cdots=y_n=0} = \int_l x_l \circ \left(b^{0,l} + b^l\right) = \int_l x_l \circ b^l = \int_l c_l = y_l c_l = b^l.$$

Finally, $b^{0,k+1} = b^{0,k} + b^k$, with $b^{0,k} \in \mathbf{k}[y_1, \ldots, y_{k-1}]$ and $b^k = y_k c_k$, with c_k in

$\mathbf{k}[y_1, \ldots, y_k]$. Then $b = b^{0,k} + b^k + b^{k+1} + \cdots + b^n$, hence

$$b\mid_{y_{k+1}=\cdots=y_n=0} = b^{0,k} + b^k$$

and we are done. □

We are now in the position to state [23, Theorem 7.36]. Note that, since $\mathcal{D}_s = I^\perp$, this result leads to the algorithm proposed by Mourrain in [39] to effectively compute a $\mathbf{k}$-basis of the inverse system. For the sake of completeness, we rewrite the complete proof using the contraction setting instead of derivation.

THEOREM 3.1.12 (Elkadi-Mourrain) Given an ideal $I = (f_1, \ldots, f_m)$ and $d > 1$. Let $b_1, \ldots, b_{t_{d-1}}$ be a $\mathbf{k}$-basis of $\mathcal{D}_{d-1}$. The polynomials of $\mathcal{D}_d$ with no constant term are of the form

$$\Lambda = \sum_{j=1}^{t_{d-1}} \lambda_j^1 \int_1 b_j|_{y_2=\cdots=y_n=0} + \sum_{j=1}^{t_{d-1}} \lambda_j^2 \int_2 b_j|_{y_3=\cdots=y_n=0} + \cdots + \sum_{j=1}^{t_{d-1}} \lambda_j^n \int_n b_j, \tag{3.3}$$

where $\lambda_j^k \in \mathbf{k}$, such that

$$\sum_{j=1}^{t_{d-1}} \lambda_j^k (x_l \circ b_j) - \sum_{j=1}^{t_{d-1}} \lambda_j^l (x_k \circ b_j) = 0, 1 \le k < l \le n, \tag{3.4}$$

and

$$(f_i \circ \Lambda)(0) = 0, \text{ for } 1 \le i \le m. \tag{3.5}$$

Proof: We will first prove that any element of $\mathcal{D}_d \subset \mathbf{k}[y_1, \ldots, y_n]$ with no constant term can be written in the form of 3.3 and satisfies both 3.4 and 3.5. Consider a polynomial Λ in $\mathcal{D}_d$ with no constant term. There is a unique decomposition

$$\Lambda = \Lambda_1(y_1, \ldots, y_n) + \Lambda_2(y_2, \ldots, y_n) + \cdots + \Lambda(y_n)$$

such that, for any $1 \le i \le n$, all monomials in Λ_i are in $\mathbf{k}[y_i, \ldots, y_n] \backslash \mathbf{k}[y_{i+1}, \ldots, y_n]$, that is, Λ_i is a multiple of y_i. Hence $\Lambda_i\mid_{y_i=0} = 0$ and, by Definition 3.1.7,

$$\int_i x_i \circ \Lambda_i = \Lambda_i.$$

On the other hand, $\mathfrak{m} \circ \mathcal{D}_d \subset \mathcal{D}_{d-1}$. Indeed, $\Lambda \in \mathcal{D}_d = I^\perp \cap S_{\leq d}$ is equivalent to $\Lambda \in I^\perp$ and $\deg \Lambda \leq d$. Because of the R-module structure of $I^\perp$, any contraction of Λ remains in the inverse system and clearly $\deg(x_i \circ \Lambda) \leq d-1$.

In particular, $x_1 \circ \Lambda = x_1 \circ \Lambda_1 \in \mathcal{D}_{d-1} = \langle b_1, \dots, b_{t_{d-1}} \rangle_{\mathbf{k}}$ and hence there exist unique scalars $\lambda_j^1 \in \mathbf{k}$ such that $x_1 \circ \Lambda_1 = \sum_{j=1}^{t_{d-1}} \lambda_j^1 b_j$. Then

$$\Lambda_1 = \int_1 x_1 \circ \Lambda_1 = \int_1 \sum_{j=1}^{t_{d-1}} \lambda_j^1 b_j = \sum_{j=1}^{t_{d-1}} \lambda_j^1 \int_1 b_j.$$

Consider now $x_2 \circ \Lambda = x_2 \circ \Lambda_1 + x_2 \circ \Lambda_2 \in \mathcal{D}_{d-1}$. There exist unique scalars $\lambda_j^2 \in \mathbf{k}$ such that $x_2 \circ \Lambda_2 = \sum_{j=1}^{t_{d-1}} \lambda_j^2 b_j$. Then

$$\Lambda_2 = \int_2 x_2 \circ \Lambda_2 = \int_2 x_2 \circ \Lambda - \int_2 x_2 \circ \Lambda_1 =$$

$$\int_2 \sum_{j=1}^{t_{d-1}} \lambda_j^2 b_j - \int_2 x_2 \circ \Lambda_1 = \sum_{j=1}^{t_{d-1}} \lambda_j^2 \int_2 b_j - \int_2 x_2 \circ \Lambda_1.$$

Let us focus on $\int_2 x_2 \circ \Lambda_1$. Note that it corresponds to the part of Λ_1 that depends on y_2. We want to prove that $\int_2 x_2 \circ \Lambda_1 = \Lambda_1 - \Lambda_1 \mid_{y_2=0}$. Indeed,

(i) $x_2 \circ (\Lambda_1 - \Lambda_1 \mid_{y_2=0}) = x_2 \circ \Lambda_1 - x_2 \circ \Lambda_1 \mid_{y_2=0} = x_2 \circ \Lambda_1$,
(ii) $(\Lambda_1 - \Lambda_1 \mid_{y_2=0}) \mid_{y_2=0} = \Lambda_1 \mid_{y_2=0} - (\Lambda_1 \mid_{y_2=0}) \mid_{y_2=0} = 0$.

Therefore,

$$\Lambda_2 = \sum_{j=1}^{t_{d-1}} \lambda_j^2 \int_2 b_j - (\Lambda_1 - \Lambda_1|_{y_2=0})$$

and

$$\Lambda_1 + \Lambda_2 = \sum_{j=1}^{t_{d-1}} \lambda_j^1 \int_1 b_j + \sum_{j=1}^{t_{d-1}} \lambda_j^2 \int_2 b_j - \left(\sum_{j=1}^{t_{d-1}} \lambda_j^1 \int_1 b_j - \left. \left(\sum_{j=1}^{t_{d-1}} \lambda_j^1 \int_1 b_j \right) \right|_{y_2=0} \right) =$$

$$= \sum_{j=1}^{t_{d-1}} \lambda_j^1 \int_1 b_j|_{y_2=0} + \sum_{j=1}^{t_{d-1}} \lambda_j^2 \int_2 b_j.$$

An analogous computation applied to $x_3 \circ \Lambda$ provides

$$\Lambda_3 = \sum_{j=1}^{t_{d-1}} \lambda_j^3 \int_3 b_j - \left(\sigma_2 - \sigma_2|_{y_3=0}\right),$$

hence

$$\Lambda_1 + \Lambda_2 + \Lambda_3 = \sum_{j=1}^{t_{d-1}} \lambda_j^1 \int_1 b_j|_{y_2=y_3=0} + \sum_{j=1}^{t_{d-1}} \lambda_j^2 \int_2 b_j|_{y_3=0} + \sum_{j=1}^{t_{d-1}} \lambda_j^3 \int_3 b_j.$$

By recurrence, we obtain 3.3:

$$\Lambda = \sum_{j=1}^{t_{d-1}} \lambda_j^1 \int_1 b_j|_{y_2=\cdots=y_n=0} + \sum_{j=1}^{t_{d-1}} \lambda_j^2 \int_2 b_j|_{y_3=\cdots=y_n=0} + \cdots + \sum_{j=1}^{t_{d-1}} \lambda_j^n \int_n b_j.$$

For any $1 \leq l \leq n$, we have

$$\sigma_l = \sum_{j=1}^{t_{d-1}} \lambda_j^1 \int_1 b_j|_{y_2=\cdots=y_l=0} + \sum_{j=1}^{t_{d-1}} \lambda_j^2 \int_2 b_j|_{y_3=\cdots=y_l=0} + \cdots + \sum_{j=1}^{t_{d-1}} \lambda_j^l \int_l b_j, \quad (3.6)$$

where $\sigma_l := \Lambda_1 + \cdots + \Lambda_l$, and

$$\Lambda_l = \sum_{j=1}^{t_{d-1}} \lambda_j^l \int_l b_j - \left(\sigma_{l-1} - \sigma_{l-1}\,|_{y_l=0}\right). \quad (3.7)$$

In order to verify that 3.4 holds, let us first note that, since $\Lambda_l \in \mathbf{k}[y_l, \ldots, y_n]$, then $x_k \circ \Lambda_l = 0$ whenever $1 \leq k < l \leq n$. Hence contracting 3.7 by x_k, with $k < l$, we get

$$\sum_{j=1}^{t_{d-1}} \lambda_j^l \int_l x_k \circ b_j = x_k \circ \left(\sigma_{l-1} - \sigma_{l-1}\,|_{y_l=0}\right).$$

Contracting the previous expression by x_l gives

$$\sum_{j=1}^{t_{d-1}} \lambda_j^l \left(x_l \circ \int_l x_k \circ b_j\right) = x_l \circ \left(x_k \circ \left(\sigma_{l-1} - \sigma_{l-1}\,|_{y_l=0}\right)\right)$$

and it can be rewritten as

$$\sum_{j=1}^{t_{d-1}} \lambda_j^l (x_k \circ b_j) = x_l \circ (x_k \circ \sigma_{l-1}) \,. \tag{3.8}$$

On one hand,

$$x_k \circ \sigma_{l-1} = x_k \circ \sum_{i=1}^{k} \Lambda_i + x_k \circ \sum_{i=k+1}^{l-1} \Lambda_i = x_k \circ \sum_{i=1}^{k} \Lambda_i = x_k \circ \sigma_k,$$

for $k < l$. On the other hand, when contracting 3.6 with $l = k$ by x_k, we get

$$x_k \circ \sigma_k = \sum_{j=1}^{t_{d-1}} \lambda_j^1 \int_1 x_k \circ b_j \mid_{y_2 = \cdots = y_k = 0} + \cdots + \sum_{j=1}^{t_{d-1}} \lambda_j^k \left(x_k \circ \int_k b_j \right) = \sum_{j=1}^{t_{d-1}} \lambda_j^k b_j.$$

Therefore, we can rewrite 3.8 as

$$\sum_{j=1}^{t_{d-1}} \lambda_j^l (x_k \circ b_j) = \sum_{j=1}^{t_{d-1}} \lambda_j^k (x_l \circ b_j),$$

hence 3.4 holds.

Condition 3.5 of the theorem is a direct consequence of $\Lambda \in \mathcal{D}_d \subset I^{\perp}$. Indeed, $f \circ \Lambda = 0$ for any $f \in I$ and, in particular, $(f \circ \Lambda)(0) = 0$.

Conversely, we want to know whether every element of the form 3.3 satisfying 3.4 and 3.5 is in $\mathcal{D}_d$. First of all we will see that it is enough to prove that

$$x_k \circ \Lambda \in \mathcal{D}_{d-1}, \quad 1 \leq k \leq n. \tag{3.9}$$

Indeed, if 3.9 holds, then $x_k \circ \Lambda \in I^{\perp}$ and hence

$$(x_k f_i) \circ \Lambda = f_i \circ (x_k \circ \Lambda) = 0, \quad 1 \leq k \leq n, \, 1 \leq i \leq m.$$

More generally, we have that

$$(\mathfrak{m} I) \circ \Lambda = I \circ (\mathfrak{m} \circ \Lambda) = 0,$$

that is, $\Lambda \in (\mathfrak{m}I)^{\perp}$. Therefore, $(f \circ \Lambda)(0) = 0$ for any $f \in \mathfrak{m}I$. Since

$$I = \langle f_1, \ldots, f_m \rangle_{\mathbf{k}} + \mathfrak{m}I,$$

from 3.5 we deduce that $(f \circ \Lambda)(0) = 0$ for any $f \in I$.

Now let us check that 3.9 is indeed true. Contracting 3.3 by x_k, $1 \leq k \leq n$, we get

$$x_k \circ \Lambda = \sum_{j=1}^{s} \lambda_j^k b_j \mid_{y_{k+1}=\cdots=y_n=0} + \sum_{j=1}^{s} \lambda_j^{k+1} \int_{k+1} x_k \circ b_j \mid_{y_{k+2}=\cdots=y_n=0} +$$

$$+ \cdots + \sum_{j=1}^{s} \lambda_j^n \int_n x_k \circ b_j.$$

The l-primitive of 3.5, for any $k < l \leq n$, gives

$$\sum_{j=1}^{s} \lambda_j^k \int_l x_l \circ b_j = \sum_{j=1}^{s} \lambda_j^l \int_l x_k \circ b_j,$$

hence

$$x_k \circ \Lambda = \sum_{j=1}^{s} \lambda_j^k \left(b_j \mid_{y_{k+1}=\cdots=y_n=0} + \int_{k+1} x_{k+1} \circ b_j \mid_{y_{k+2}=\cdots=y_n=0} + \right.$$

$$\left. + \cdots + \int_n x_n \circ b_j \right).$$

By Lemma 3.1.11, we have $x_k \circ \Lambda = \sum_{j=1}^{s} \lambda_j^k b_j \in \mathcal{D}_{d-1}$, hence 3.9 holds. $\square$

REMARK 3.1.13 From the proof of Theorem 3.1.12, it follows that 3.4 can be replaced by

$$x_k \circ \Lambda \in \mathcal{D}_{d-1}, \quad 1 \leq k \leq n.$$

In other words, what we actually require in 3.4 is $\mathcal{D}_d$ to be stable by contraction.

3.2 Using integrals to obtain Gorenstein covers of Artin rings

Given an Artin $\mathbf{k}$-algebra $A = R/I$, a priori we do not know which are the possible choices for the colon ideal $K_F \subset R$ that provides the relationship between the inverse system $\langle F \rangle$ of a minimal Gorenstein cover $G = R/\operatorname{Ann_R} F$ of A and $I^\perp$. However, once we fix a colength t, we do know a lot about the shape of the ideals K_F associated to a polynomial F such that $\ell(G) - \ell(A) = t$. Rephrasing Proposition 2.1.6, we get the following result:

PROPOSITION 3.2.1 Let $A = R/I$ be a local Artin algebra and $G = R/J$, with $J = \operatorname{Ann_R} F$, a minimal Gorenstein cover of A. Then,

(i) $I^\perp = K_F \circ F$,
(ii) $\operatorname{gcl}(A) = \ell(R/K_F)$.

Moreover,

$$K_F = \begin{cases} R, & \text{if} \quad \operatorname{gcl}(A) = 0; \\ \mathfrak{m}, & \text{if} \quad \operatorname{gcl}(A) = 1; \\ (L_1, \dots, L_{n-1}, L_n^2), & \text{if} \quad \operatorname{gcl}(A) = 2, \end{cases}$$

where $L_1, \dots, L_n$ are suitable independent linear forms in R.

REMARK 3.2.2 If $\operatorname{gcl}(A) = 1$, then any minimal Gorenstein cover $G = R/\operatorname{Ann_R} F$ satisfies

$$F \in \int_{\mathfrak{m}} I^\perp.$$

However, even if $\operatorname{gcl}(A) = 2$, the ideal K_F depends on the particular choice of F. Although it is certainly true that

$$F \in \int_{(L_1,\dots,L_{n-1},L_n^2)} I^\perp,$$

this is not a useful condition to impose in order to find F because the ideal $(L_1, \dots, L_{n-1}, L_n^2)$ already depends on F. For colength higher that 2, things get more complicated since the K_F can even have different analytic type. See Section 2.3 for a detailed description of the situation when $\operatorname{gcl}(A) = 3$.

The dependency of the integral on F can be removed by considering only the condition

$$F \in \int_{\mathfrak{m}^t} I^{\perp},$$

for a suitable integer t, which gives an effective way to compute a (too big) set where all minimal Gorenstein covers live. In Proposition 3.2.5 we provide all the details of this construction. We first need to dig deeper into the structure of the integral of a module with respect to a power of the maximal ideal. The following result permits an iterative approach:

LEMMA 3.2.3 Let M be a finitely generated sub-R-module of S and $d \geq 1$, then

$$\int_{\mathfrak{m}} \left(\int_{\mathfrak{m}^{d-1}} M \right) = \int_{\mathfrak{m}^d} M.$$

Proof: Let us prove first the inclusion $\int_{\mathfrak{m}} \left(\int_{\mathfrak{m}^{d-1}} M \right) \subseteq \int_{\mathfrak{m}^d} M$. Take the polynomial Λ in $\int_{\mathfrak{m}} \left(\int_{\mathfrak{m}^{d-1}} M \right)$, then $\mathfrak{m} \circ \Lambda \subseteq \int_{\mathfrak{m}^{d-1}} M$ and hence $\mathfrak{m}^d \circ \Lambda = \mathfrak{m}^{d-1} \circ (\mathfrak{m} \circ \Lambda) \subseteq M$. Therefore, $\Lambda \in \int_{\mathfrak{m}^d} M$. To prove the reverse inclusion, consider $\Lambda \in \int_{\mathfrak{m}^d} M$, that is, $\mathfrak{m}^{d-1} \circ (\mathfrak{m} \circ \Lambda) = \mathfrak{m}^d \circ \Lambda \subseteq M$. In other words, $\mathfrak{m} \circ \Lambda \subseteq \int_{\mathfrak{m}^{d-1}} M$ and Λ in $\int_{\mathfrak{m}} \left(\int_{\mathfrak{m}^{d-1}} M \right)$. $\square$

Since $\int_{\mathfrak{m}^t} M$ is a finitely dimensional $\mathbf{k}$-vector space that can be obtained by integrating t times M with respect to $\mathfrak{m}$, we can also consider a basis of $\int_{\mathfrak{m}^t} M$ which is constructed by extending the previous basis at each step.

DEFINITION 3.2.4 Let M be a finitely generated sub-R-module of S. Given an integer t, we denote by h_i the dimension of the $\mathbf{k}$-vector space $\int_{\mathfrak{m}^i} M / \int_{\mathfrak{m}^{i-1}} M$, $i = 1, \cdots, t$. An **adapted k-basis** of $\int_{\mathfrak{m}^t} M/M$ is a $\mathbf{k}$-basis $\overline{F}_j^i$, $i = 1, \cdots, t$, $j = 1, \cdots, h_i$, of $\int_{\mathfrak{m}^t} M/M$ such that $F_1^i, \cdots, F_{h_i}^i \in \int_{\mathfrak{m}^i} M$ and their cosets in $\int_{\mathfrak{m}^i} M / \int_{\mathfrak{m}^{i-1}} M$ form a $\mathbf{k}$-basis, $i = 1 \cdots, t$.
Let $A = R/I$ be an Artin ring, we denote by $\mathcal{L}_{A,t}$ the R-module $\int_{\mathfrak{m}^t} I^{\perp}/I^{\perp}$.

The following proposition is meant to overcome the obstacle of non-uniqueness of the ideals K_F.

PROPOSITION 3.2.5 Given a ring $A = R/I$ of Gorenstein colength t and a minimal Gorenstein cover $G = R/\operatorname{Ann}_R F$ of A,

(i) $F \in \int_{\mathfrak{m}^t} I^\perp$;

(ii) for any $H \in \int_{\mathfrak{m}^t} I^\perp$, the condition $I^\perp \subset \langle H \rangle$ does not depend on the representative of the class $\overline{H}$ in $\mathcal{L}_{A,t}$.

In particular, any $F' \in \int_{\mathfrak{m}^t} I^\perp$ such that $\overline{F'} = \overline{F}$ in $\mathcal{L}_{A,t}$ defines the same minimal Gorenstein cover $G = R/\operatorname{Ann}_R F$.

Proof: (i) By Proposition 2.1.6, we have $\operatorname{gcl}(A) = \ell(R/K_F)$, where $K_F \circ F = I^\perp$ for any polynomial F that generates a minimal Gorenstein cover $G = R/\operatorname{Ann}_R F$ of A. From the definition of integral we have $F \in \int_{K_F} I^\perp$. Since $\ell(R/K_F) = t$, then $\operatorname{socdeg}(R/K_F) \leq t-1$. Indeed, the extremal case corresponds to the most expanded Hilbert function $\{1, 1, \dots, 1\}$, that is, a stretched algebra (see Appendix B). Then $\operatorname{HF}_{R/K_F}(i) = 0$, for any $i \geq t$, regardless of the particular form of K_F, and hence $\mathfrak{m}^t \subset K_F$. Therefore,

$$F \in \int_{K_F} I^\perp \subset \int_{\mathfrak{m}^t} I^\perp.$$

(ii) Consider a polynomial $H \in \int_{\mathfrak{m}^t} I^\perp$ such that $I^\perp \subset \langle H \rangle$. By Proposition 2.1.6, $K_H \circ H = I^\perp$. Consider $H' \in \int_{\mathfrak{m}^t} I^\perp$ such that $\overline{H} = \overline{H'}$ in $\mathcal{L}_{A,t}$, so $H = H' + G$ for some $G \in I^\perp$. We want to prove that

$$K_H \circ H' + \mathfrak{m} \circ I^\perp = K_H \circ H + \mathfrak{m} \circ I^\perp = I^\perp. \tag{3.10}$$

The second equality is direct from $K_H \circ H = I^\perp$. Let us check the first. Take $h \circ H' + \mathfrak{m} \circ I^\perp \in K_H \circ H' + \mathfrak{m} \circ I^\perp$, with $h \in K_H \subset \mathfrak{m}$,

$$h \circ H' + \mathfrak{m} \circ I^\perp = h \circ H - h \circ G + \mathfrak{m} \circ I^\perp = h \circ H + \mathfrak{m} \circ I^\perp \subset K_H \circ H + \mathfrak{m} \circ I^\perp.$$

The same argument holds for the reverse inclusion. Therefore, 3.10 holds and we can apply Nakayama's lemma to get $K_H \circ H' = I^\perp$. Hence $I^\perp \subset \langle H' \rangle$. In particular, $\langle H' \rangle = \langle H \rangle$. Indeed, since $H' = H - G$ and $\langle G \rangle \subset \langle I^\perp \rangle \subset \langle H \rangle$, then $H' \in \langle H \rangle + \langle G \rangle = \langle H \rangle$ and a similar argument gives $H \in \langle H' \rangle$. □

Observe that the proposition says that, although not all F in $\int_{\mathfrak{m}^t} I^\perp$ correspond to covers $G = R/\operatorname{Ann}_R F$ of $A = R/I$, if F is actually a cover, then any F' in $\int_{\mathfrak{m}^t} I^\perp$

such that $\overline{F'} = \overline{F} \in \mathcal{L}_{A,t}$ provides the exact same cover. That is, $R/\operatorname{Ann}_{\mathrm{R}}(F) = R/\operatorname{Ann}_{\mathrm{R}}(F')$.

COROLLARY 3.2.6 Let $A = R/I$ be an Artin ring of Gorenstein colength t and consider an adapted $\mathbf{k}$-basis $\{\overline{F}_j^i\}_{1\leq i\leq t, 1\leq j\leq h_i}$ of $\mathcal{L}_{A,t}$. Given a minimal Gorenstein cover $G = R/J$ there is a generator F of $J^\perp$ such that F can be written as

$$F = a_1^1 F_1^1 + \cdots + a_{h_1}^1 F_{h_1}^1 + \cdots + a_1^t F_1^t + \cdots + a_{h_t}^t F_{h_t}^t \in \int_{\mathfrak{m}^t} I^\perp,\ a_i^j \in \mathbf{k}.$$

Moreover, $\deg F \leq s + t$, where $s = \operatorname{socdeg} A$.

Proof: In $\mathcal{L}_{A,t}$ we have $\overline{F} = \sum_{i=1}^t \sum_{j=1}^{h_j} a_j^i \overline{F_j^i}$ and hence $F = \sum_{i=1}^t \sum_{j=1}^{h_i} a_j^i F_j^i + G$ with $G \in I^\perp$. By Proposition 3.2.5, any representative of the class $\overline{F}$ provides the same Gorenstein cover. In particular, we can take $G = 0$ and we are done.

Now consider $F \in \int_{\mathfrak{m}} I^\perp$. By definition, $\mathfrak{m} \circ F \in I^\perp$. Since any polynomial in $I^\perp$ has degree at most the socle degree of A, then $\deg(x_i \circ F) \leq s$ for any $1 \leq i \leq n$. Therefore, $\deg F \leq s + 1$ and at each integral with respect to $\mathfrak{m}$, the degree can only be increased by 1. □

Our goal now is to compute the integral of the inverse system with respect to a power of the maximal ideal. Assume we have a $\mathbf{k}$-basis of $I^\perp$ and we want to find a $\mathbf{k}$-basis of $\int_{\mathfrak{m}} I^\perp$. Consider an element Λ in $\int_{\mathfrak{m}} I^\perp$. By definition, it must satisfy

$$\mathfrak{m} \circ \Lambda \subset I^\perp = \mathcal{D}_s, \tag{3.11}$$

where $s = \operatorname{socdeg}(A)$ and $D_d = I^\perp \cap S_{\leq d}$, as defined in Section 3.1. Thanks to Remark 3.1.13, we know that 3.11 is equivalent to condition 3.4 of Elkadi-Mourrain result for inverse systems, Theorem 3.1.12. Condition 3.5 of Theorem 3.1.12 is no longer needed because we do not require Λ to be in $I^\perp$ anymore.

Therefore, the most natural approach to find all elements Λ of $\int_{\mathfrak{m}} I^\perp$ is to apply the procedure of Theorem 3.1.12 to a $\mathbf{k}$-basis of $\mathcal{D}_s = I^\perp$ removing the condition of orthogonality with respect to the generators of the ideal I.

The theorem below tells us what the elements of $\int_{\mathfrak{m}} M$ look like, for any sub-R-module M of S. It sets the theoretical ground for an algorithm that effectively computes a $\mathbf{k}$-basis of the integral of a module with respect to any power of the maximal ideal, see Algorithm 1. Since the proof we present is very similar to the one given in Theorem 3.1.12, we only emphasize the parts that differ, consult it for complete details.

THEOREM 3.2.7 Consider a sub-R-module M of S and let $b_1, \dots, b_s$ be a $\mathbf{k}$-basis of M. Let $\Lambda \in S$ be a polynomial with no constant terms. Then $\Lambda \in \int_{\mathfrak{m}} M$ if and only if

$$\Lambda = \sum_{j=1}^{s} \lambda_j^1 \int_1 b_j|_{y_2=\cdots=y_n=0} + \sum_{j=1}^{s} \lambda_j^2 \int_2 b_j|_{y_3=\cdots=y_n=0} + \cdots + \sum_{j=1}^{s} \lambda_j^n \int_n b_j, \quad \lambda_j^k \in \mathbf{k}, \tag{3.12}$$

such that

$$\sum_{j=1}^{s} \lambda_j^k (x_l \circ b_j) - \sum_{j=1}^{s} \lambda_j^l (x_k \circ b_j) = 0, 1 \leq k < l \leq n. \tag{3.13}$$

Proof: To prove that any element Λ in $\int_{\mathfrak{m}} M$ is of the form 3.12 and satisfies condition 3.4, we just have to note that, by definition, $\mathfrak{m} \circ \Lambda \subset M = \langle b_1, \dots, b_s \rangle_{\mathbf{k}}$. Everything else follows exactly as in Theorem 3.1.12.

Conversely, we want to know if every element of the form 3.12 satisfying 3.4 is in $\int_{\mathfrak{m}} M$. By definition, $\Lambda \in \int_{\mathfrak{m}} M$ if and only if $\mathfrak{m} \circ \Lambda \subset M$. Therefore, it is enough to prove that $x_k \circ \Lambda \in M$ for any $1 \leq k \leq n$. As in Theorem 3.1.12, contracting and integrating with respect to appropriate variables we obtain $x_k \circ \Lambda = \sum_{j=1}^{s} \lambda_j^k b_j \in M$ and we are done. □

From the previous theorem and Lemma 3.2.3 the next corollary follows directly.

COROLLARY 3.2.8 Consider a sub-R-module M of S and $d \geq 1$. Let $b_1, \dots, b_{t_{d-1}}$ be a $\mathbf{k}$-basis of $\int_{\mathfrak{m}^{d-1}} M$ and let Λ be a polynomial with no constant terms. Then $\Lambda \in \int_{\mathfrak{m}^d} M$ if and only if it is of the form

$$\Lambda = \sum_{j=1}^{t_{d-1}} \lambda_j^1 \int_1 b_j|_{y_2=\cdots=y_n=0} + \sum_{j=1}^{t_{d-1}} \lambda_j^2 \int_2 b_j|_{y_3=\cdots=y_n=0} + \cdots + \sum_{j=1}^{t_{d-1}} \lambda_j^n \int_n b_j, \quad \lambda_j^k \in \mathbf{k}, \tag{3.14}$$

such that

$$\sum_{j=1}^{t_{d-1}} \lambda_j^k (x_l \circ b_j) - \sum_{j=1}^{t_{d-1}} \lambda_j^l (x_k \circ b_j) = 0, 1 \leq k < l \leq n. \tag{3.15}$$

REMARK 3.2.9 It can be proved that

$$\mathcal{D}_d = I^\perp \cap \int_{\mathfrak{m}} \mathcal{D}_{d-1},$$

for any $1 < d \leq s$. Indeed, by Theorem 3.1.12, $\mathcal{D}_d$ is stable by contraction, hence $\mathcal{D}_d$ is contained in $I^\perp \cap \int_{\mathfrak{m}} \mathcal{D}_{d-1}$. Conversely, any element Λ in $\left(\int_{\mathfrak{m}} \mathcal{D}_{d-1}\right) \cap I^\perp$ satisfies $\mathfrak{m} \circ \Lambda \subseteq \mathcal{D}_{d-1} = I^\perp \cap S_{\leq d-1}$. Then $\deg \Lambda \leq d$ and hence $\Lambda \in I^\perp \cap S_{\leq d} = \mathcal{D}_d$.

We end this section by considering the low Gorenstein colength cases.

3.2.1 Teter rings

Let us remind that Teter rings are those $A = R/I$ such that $A \cong G/\operatorname{soc}(G)$ for some Gorenstein ring G. According to [20, Proposition 2.1.3], Teter rings can be characterized as rings of Gorenstein colength 1, whenever their embedding dimension is equal or greater than 2. Otherwise, if A is a Teter ring with $\operatorname{embd}(A) = 1$, then A is Gorenstein. Rings of Gorenstein colength 1 are a special case to deal with because the K_F associated to any generator $F \in S$ of a minimal cover is always the maximal ideal. We provide some additional criteria to characterize such rings:

PROPOSITION 3.2.10 Let $A = R/I$ be a non-Gorenstein local Artin ring of socle degree $s \geq 1$ and let $\{\overline{F}_j\}_{1\leq j\leq h}$ be an adapted $\mathbf{k}$-basis of $\mathcal{L}_{A,1}$. Then $\operatorname{gcl}(A) = 1$ if and only if there exist a polynomial $F = \sum_{j=1}^h a_j F_j \in \int_{\mathfrak{m}} I^\perp$, $a_j \in \mathbf{k}$, such that $\dim_{\mathbf{k}}(\mathfrak{m} \circ F) = \dim_{\mathbf{k}} I^\perp$.

Proof: The first implication is straightforward from Corollary 3.2.6 and Teter rings characterization in [20, Proposition 2.1.3]. Reciprocally, if $F \in \int_{\mathfrak{m}} I^\perp$, then $\mathfrak{m} \circ F \subset I^\perp$ by definition, and from the equality of dimensions, it follows that $\mathfrak{m} \circ F = I^\perp$. Therefore, $0 < \operatorname{gcl}(A) \leq \ell(R/\mathfrak{m}) = 1$ and we are done. □

EXAMPLE 3.2.11 Recall Example 3.1.4 with $I^\perp = \langle y_1y_2, y_3^3 \rangle$ and

$$\int_{\mathfrak{m}} I^\perp = \langle y_1^2, y_1y_2, y_1y_3, y_2^2, y_2y_3, y_3^4 \rangle.$$

Then $\overline{y}_1^2, \overline{y}_1\overline{y}_3, \overline{y}_2^2, \overline{y}_2\overline{y}_3, \overline{y}_3^4$ is a $\mathbf{k}$-basis of $\mathcal{L}_{A,1}$. As a consequence of Proposition 3.2.10,

A is Teter if and only if there exists a polynomial

$$F = a_1y_1^2 + a_2y_1y_3 + a_3y_2^2 + a_4y_2y_3 + a_5y_3^4$$

such that $\mathfrak{m} \circ F = I^\perp$. But $\mathfrak{m} \circ F = \langle a_1y_1 + a_2y_3, a_3y_2 + a_4y_3, a_2y_1 + a_4y_2 + a_5y_3^3\rangle$ and clearly y_1y_2 does not belong here. Therefore, $\mathrm{gcl}(A) > 1$.

3.2.2 Gorenstein colength 2

By Theorem 2.2.5, we know that A is of Gorenstein colength 2 if and only if there exists a polynomial F of degree $s+1$ or $s+2$ such that $K_F \circ F = I^\perp$, where $K_F = (L_1, \ldots, L_{n-1}, L_n^2)$, where L_i are suitable independent linear forms.

Observe that a completely analogous characterization to the one we did for Teter rings is not possible. If $A = R/I$ has Gorenstein colength 2, by Corollary 3.2.6, there exists $F = \sum_{i=1}^2 \sum_{j=1}^{h_i} a_j^i F_j^i \in \int_{\mathfrak{m}^2} I^\perp$, where $\{\overline{F}_j^i\}_{1\leq i\leq 2, 1\leq j\leq h_i}$ is a $\mathbf{k}$-basis of $\mathcal{L}_{A,2}$, that generates a minimal Gorenstein cover of A and then trivially $I^\perp \subset \langle F\rangle$. However, the reverse implication is not true.

EXAMPLE 3.2.12 Consider $A = R/\mathfrak{m}^3$, where R is the ring of power series in 2 variables, and consider $F = y_1^2y_2^2$. It is easy to see that $F \in \int_{\mathfrak{m}^2} I^\perp = S_{\leq 4}$ and $I^\perp \subset \langle F\rangle$. However, it can be proved that $\mathrm{gcl}(A) = 3$ with [2, Corollary 3.3]. Note that $K_F = \mathfrak{m}^2$ and hence $\ell(R/K_F) = 3$.

Therefore, given $F \in \int_{\mathfrak{m}^2} I^\perp$, the condition $I \subset \langle F\rangle$ is not sufficient to ensure that $\mathrm{gcl}(A) = 2$. We must require that $\ell(R/K_F) = 2$ as well.

PROPOSITION 3.2.13 Given a non-Gorenstein non-Teter local Artin ring $A = R/I$, $\mathrm{gcl}(A) = 2$ if and only if there exist a polynomial $F = \sum_{i=1}^2 \sum_{j=1}^{h_i} a_j^i F_j^i \in \int_{\mathfrak{m}^2} I^\perp$ such that $\{\overline{F}_j^i\}_{1\leq i\leq 2, 1\leq j\leq h_i}$ is an adapted $\mathbf{k}$-basis of $\mathcal{L}_{A,2}$ and $(L_1, \ldots, L_{n-1}, L_n^2) \circ F = I^\perp$ for suitable independent linear forms $L_1, \ldots, L_n$.

Proof: We will only prove that if F satisfies the required conditions, then $\mathrm{gcl}(A) = 2$. By definition of K_F, if $(L_1, \ldots, L_{n-1}, L_n^2) \circ F = I^\perp$, then $(L_1, \ldots, L_{n-1}, L_n^2)$ is contained in K_F. By Proposition 2.1.6, $\mathrm{gcl}(A) \leq \ell(R/K_F)$ and hence $\mathrm{gcl}(A)$ is equal or less than $\ell\left(R/(L_1, \ldots, L_{n-1}, L_n^2)\right) = 2$. Since $\mathrm{gcl}(A) \geq 2$ by hypothesis, then $\mathrm{gcl}(A) = 2$. □

EXAMPLE 3.2.14 Recall the ring $A = R/I$ in Example 3.2.11. Since

$$\int_{\mathfrak{m}^2} I^\perp = \langle y_1^3, y_1^2y_2, y_1y_2^2, y_2^3, y_1^2y_3, y_1y_2y_3, y_2^2y_3, y_1y_3^2, y_2y_3^3, y_3^5\rangle$$

and $\operatorname{gcl}(A) > 1$, its Gorenstein colength is 2 if and only if there exist some F in

$$\langle y_1^2, y_1y_2, y_1y_3, y_2^2, y_2y_3, y_3^4, y_1^3, y_1^2y_2, y_1y_2^2, y_2^3, y_1^2y_3, y_1y_2y_3, y_2^2y_3, y_1y_3^2, y_2y_3^3, y_3^5\rangle_{\mathbf{k}}$$

such that $(L_1, \ldots, L_{n-1}, L_n^2) \circ F = I^\perp$. Consider $F = y_3^4 + y_1^2y_2$, then

$$(x_1, x_2^2, x_3) \circ F = \langle x_1 \circ F, x_2^2 \circ F, x_3 \circ F\rangle = \langle y_1y_2, y_3^3\rangle$$

and hence $\operatorname{gcl}(A) = 2$.

3.3 The variety of minimal Gorenstein covers

We are now interested in providing a geometric interpretation of the set of all minimal Gorenstein covers $G = R/J$ of a given local Artin $\mathbf{k}$-algebra $A = R/I$. From now on, we will assume that $\mathbf{k}$ is an algebraically closed field. The following result is well known and it is an easy linear algebra exercise.

LEMMA 3.3.1 Let $\varphi_i : \mathbf{k}^a \longrightarrow \mathbf{k}^b$, $i = 1 \cdots, r$, be a family of Zariski continuous maps. Then the function $\varphi^* : \mathbf{k}^a \longrightarrow \mathbb{N}$ defined by $\varphi^*(z) = \dim_{\mathbf{k}}\langle \varphi_1(z), \cdots, \varphi_r(z)\rangle_{\mathbf{k}}$ is lower semicontinous, i.e. for all $z_0 \in \mathbf{k}^a$ there is a Zariski open set $z_0 \in U \subset \mathbf{k}^a$ such that for all $z \in U$ it holds $\varphi^*(z) \geq \varphi^*(z_0)$.

THEOREM 3.3.2 Let $A = R/I$ be an Artin ring of Gorenstein colength t. There exists a quasi-projective sub-variety $MGC^n(A)$, $n = \dim(R)$, of $\mathbb{P}_{\mathbf{k}}(\mathcal{L}_{A,t})$ whose set of closed points are the points $[\overline{F}]$, $\overline{F} \in \mathcal{L}_{A,t}$, such that $G = R/\operatorname{Ann_R} F$ is a minimal Gorenstein cover of A.

Proof: Let E be a sub-$\mathbf{k}$-vector space of $\int_{\mathfrak{m}^t} I^\perp$ such that

$$\int_{\mathfrak{m}^t} I^\perp \cong E \oplus I^\perp,$$

we identify $\mathcal{L}_{A,t}$ with E. From Proposition 3.2.5, for all minimal Gorenstein covers $G = R/\operatorname{Ann_R} F$ we may assume that $F \in E$. From Corollary 3.2.6, we also know that $\deg F \leq s+t$. Given $F \in E$, the quotient $G = R/\operatorname{Ann_R} F$ is a minimal cover of A if and only if

(1) $\dim_{\mathbf{k}}\langle F\rangle = \dim_{\mathbf{k}}(A) + t$, and
(2) $\dim_{\mathbf{k}}(I^{\perp} + \langle F\rangle) = \dim_{\mathbf{k}}\langle F\rangle$.

Define the family of Zariski continuous maps $\{\varphi_\alpha\}_{|\alpha|\leq s+t}$, $\alpha \in \mathbb{N}^n$, where

$$\begin{array}{rccc} \varphi_\alpha : & E & \longrightarrow & E \\ & F & \longmapsto & x^\alpha \circ F \end{array}$$

In particular, $\varphi_0 = Id_R$. We write

$$\begin{array}{rccc} \varphi^* : & E & \longrightarrow & \mathbb{N} \\ & F & \longmapsto & \dim_{\mathbf{k}}\langle x^\alpha \circ F : |\alpha| \leq s+t\rangle_{\mathbf{k}} \end{array}$$

Note that $\varphi^*(F) = \dim_{\mathbf{k}}\langle F\rangle$ and, by Lemma 3.3.1, φ^* is a lower semicontinuous map. Hence $U_1 = \{F \in E \mid \dim_{\mathbf{k}}\langle F\rangle \geq \dim_{\mathbf{k}} A + t\}$ is an open Zariski set in E. Using the same argument, $U_2 = \{F \in E \mid \dim_{\mathbf{k}}\langle F\rangle \geq \dim_{\mathbf{k}} A + t + 1\}$ is also an open Zariski set in E and hence $Z_1 = E\backslash U_2$ is a Zariski closed set such that $\dim_{\mathbf{k}}\langle F\rangle \leq \dim_{\mathbf{k}} A + t$ for any $F \in Z_1$. Then $Z_1 \cap U_1 = \{F \in E \mid \dim_{\mathbf{k}}\langle F\rangle = \dim_{\mathbf{k}} A + t\}$ is a locally closed set.

Let $G_1, \cdots, G_r$ be a **k**-basis of $I^{\perp}$ and consider the constant map

$$\begin{array}{rccc} \psi_i : & E & \longrightarrow & E \\ & F & \longmapsto & G_i \end{array}$$

for any $i = 1, \cdots, r$. By Lemma 3.3.1,

$$\begin{array}{rccc} \psi^* : & E & \longrightarrow & \mathbb{N} \\ & F & \longmapsto & \dim_{\mathbf{k}}\left(\langle F\rangle + I^{\perp}\right) = \dim_{\mathbf{k}}\langle \{x^\alpha \circ F\}_{|\alpha|\leq s+t}, G_1, \ldots, G_r\rangle_{\mathbf{k}} \end{array}$$

is a lower semicontinuous map.

Using an analogous argument, we can prove that $T = \{F \in E \mid \dim_{\mathbf{k}}(I^{\perp} + \langle F \rangle) = \dim_{\mathbf{k}} A + t\}$ is a locally closed set. Therefore,

$$W = (Z_1 \cap U_1) \cap T = \{F \in E \mid \dim_{\mathbf{k}} A + t = \dim_{\mathbf{k}}(I^{\perp} + \langle F \rangle) = \dim_{\mathbf{k}} \langle F \rangle\}$$

is a locally closed subset of E whose set of closed points can be identified with polynomials F in E satisfying (1) and (2), that is, $F \in S$ such that $G = R/\operatorname{Ann}_{\mathrm{R}} F$ is a minimal Gorenstein cover of A.

Moreover, since $\langle F \rangle = \langle \lambda F \rangle$ for any $\lambda \in \mathbf{k}^*$, conditions (1) and (2) are invariant under the multiplicative action of $\mathbf{k}^*$ on F and hence

$$MGC^n(A) = \mathbb{P}_{\mathbf{k}}(W) \subset \mathbb{P}_{\mathbf{k}}(E) = \mathbb{P}_{\mathbf{k}}\left(\mathcal{L}_{A,t}\right).$$

□

Recall that we have the upper bound $\tau(A) + \operatorname{gcl}(A) - 1$ for the embedding dimension of any minimal Gorenstein cover given by Proposition 2.3.2.

DEFINITION 3.3.3 Given an Artin ring $A = R/I$, the variety $MGC(A) = MGC^n(A)$, with $n = \tau(A) + \operatorname{gcl}(A) - 1$, is called the **minimal Gorenstein cover variety** associated to A.

REMARK 3.3.4 In Theorem 2.1.7 we proved that for low Gorenstein colength of A, i.e. $\operatorname{gcl}(A) \leq 2$, then $\operatorname{embd}(G) = \operatorname{embd}(A)$ for any minimal Gorenstein cover G of A. In this situation we can define $MGC(A)$ as the variety $MGC^n(A)$ with $n = \operatorname{embd}(A)$.

Observe that this notion of minimal Gorenstein cover variety generalizes the definition of Teter variety introduced in [20], which applies only to rings of Gorenstein colength 1, to any arbitrary colength.

3.4 Computing $MGC(A)$ for low Gorenstein colength

In this section we provide algorithms and examples to compute the variety of minimal Gorenstein covers of a given ring A whenever its Gorenstein colength is 1 or 2. These algorithms can also be used to decide whether a ring has colength greater than 2, since it will correspond to empty varieties.

To start with, we provide an auxiliary algorithm to compute the integral of $I^\perp$ with respect to the t-th power of the maximal ideal of R. If there exist polynomials defining minimal Gorenstein covers of colength t, they must belong to this integral.

3.4.1 Computing integrals of modules

Let $\mathbf{b}$ a $\mathbf{k}$-basis $b_1, \ldots, b_t$ of a finitely generated sub-R-module M of S and consider $x_k \circ b_i = \sum_{j=1}^t a_j^i b_j$, for any $1 \leq i \leq t$ and $1 \leq k \leq n$. Let us define matrices $U_k = (a_j^i)_{1\leq j,i\leq t}$ for any $1 \leq k \leq n$. Note that

$$(x_k \circ b_1 \cdots x_k \circ b_t) = (b_1 \cdots b_t) \begin{pmatrix} a_1^1 & \ldots & a_1^t \\ \vdots & & \vdots \\ a_t^1 & \ldots & a_t^t \end{pmatrix}.$$

Now consider any element $h \in M$. Then

$$x_k \circ h = x_k \circ \sum_{i=1}^t h_i b_i = \sum_{i=1}^t (x_k \circ h_i b_i) = \sum_{i=1}^t (x_k \circ b_i) h_i =$$

$$= (x_k \circ b_1 \cdots x_k \circ b_t) \begin{pmatrix} h_1 \\ \vdots \\ h_t \end{pmatrix} = (b_1 \cdots b_t)\, U_k \begin{pmatrix} h_1 \\ \vdots \\ h_t \end{pmatrix},$$

where $h_1, \ldots, h_t \in \mathbf{k}$.

DEFINITION 3.4.1 Let U_k, $1 \leq k \leq n$, be the square matrix of order t such that

$$x_k \circ h = \mathbf{b}\, U_k\, \mathbf{h}^t,$$

where $\mathbf{h} = (h_1, \ldots, h_t)$ for any $h \in M$, with $h = \sum_{i=1}^t h_i b_i$. We call U_k the **contraction matrix** of M with respect to x_k associated to a $\mathbf{k}$-basis $\mathbf{b}$ of M.

REMARK 3.4.2 Since $x_k x_l \circ h = x_l x_k \circ h$ for any $h \in M$, we have $U_k U_l = U_l U_k$, with $1 \leq k < l \leq n$.

In [39], Mourrain provides an effective algorithm based on Theorem 3.1.12 that computes, along with a $\mathbf{k}$-basis of the inverse system $I^\perp$ of an ideal I of R, the contraction matrices $U_1, \dots, U_n$ of $I^\perp$ associated to that basis.

EXAMPLE 3.4.3 Consider $A = R/I$, with $R = \mathbf{k}[\![x_1, x_2]\!]$ and $I = \mathfrak{m}^2$. Then $1, y_1, y_2$ is a $\mathbf{k}$-basis of $I^\perp$ and U_1, U_2 are its contraction matrices with respect to x_1, x_2, respectively:

$$U_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We present Algorithm 1, based on Theorem 3.2.7, which computes the integral of a finitely generated sub-R-module M with respect to the maximal ideal. The algorithm can use the output of Mourrain's integration method as initial data: a $\mathbf{k}$-basis of $I^\perp$ and the contraction matrices associated to this basis.

REMARK 3.4.4 Observe that the classes in $\int_{\mathfrak{m}} M/M$ of the output $b_{t+1}, \dots, b_{t+h}$ of Algorithm 1 form a $\mathbf{k}$-basis of $\int_{\mathfrak{m}} M/M$. Moreover, since the algorithm returns the contraction matrices of $\int_{\mathfrak{m}} M$, we can iterate the procedure in order to obtain a $\mathbf{k}$-basis of $\int_{\mathfrak{m}^k} M$ for any $k \geq 1$. By construction, the elements of this $\mathbf{k}$-basis that do not belong to M form an adapted $\mathbf{k}$-basis of $\int_{\mathfrak{m}^k} M/M$.

EXAMPLE 3.4.5 Consider $A = R/I$, with $R = \mathbf{k}[\![x_1, x_2]\!]$ and $I = \mathfrak{m}^2$. Then 1, y_1, y_2, y_2^2, $y_1 y_2$, y_1^2 is a $\mathbf{k}$-basis of $\int_{\mathfrak{m}} I^\perp = S_{\leq 2}$ with the following contraction matrices:

$$U_1' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad U_2' = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Algorithm 1 Compute a $\mathbf{k}$-basis of $\int_{\mathfrak{m}} M$ and its contraction matrices

Input: $D = b_1, \dots, b_t$ $\mathbf{k}$-basis of M;

$U_1, \dots, U_n$ contraction matrices of M associated to the $\mathbf{k}$-basis D.

Output: $D = b_1, \dots, b_t, b_{t+1}, \dots, b_{t+h}$ $\mathbf{k}$-basis of $\int_{\mathfrak{m}} M$;

$U'_1, \dots, U'_n$ contraction matrices of $\int_{\mathfrak{m}} M$ associated to the $\mathbf{k}$-basis D.

Steps:

(i) Set $\lambda_i = (\lambda_1^i \ \cdots \ \lambda_t^i)^t$, for any $1 \leq i \leq n$. Solve the system of equations

$$U_k \, \lambda_l - U_l \, \lambda_k = 0 \text{ for any } 1 \leq k < l \leq n. \tag{3.16}$$

(ii) Consider a system of generators $\mathbf{H}_1, \dots, \mathbf{H}_m$ of the solutions of 3.16.

(iii) For any $\mathbf{H}_i = [\lambda_1, \dots, \lambda_n]$, $1 \leq i \leq m$, define the associated polynomial

$$\Lambda_{\mathbf{H}_i} = \sum_{k=1}^{n} \left(\sum_{j=1}^{t} \lambda_j^k \int_k b_j|_{y_{k+1}=\cdots=y_n=0} \right).$$

(iv) If $\Lambda_{\mathbf{H}_1} \notin \langle D \rangle_{\mathbf{k}}$, then $b_{t+1} := \Lambda_{\mathbf{H}_1}$ and $D = D, b_{t+1}$. Repeat the procedure for $\Lambda_{\mathbf{H}_2}, \dots, \Lambda_{\mathbf{H}_m}$.

(v) Set h as the number of new elements in D.

(vi) Define square matrices U'_k of order $t + h$ and set $U'_k[i] = U_k[i]$ for $1 \leq i \leq t$.

(vii) Compute $x_k \circ b_i = \sum_{j=1}^{t} \mu_j^i b_j$ for $t + 1 \leq i \leq t + h$ and set

$$U'_k[i] = \begin{pmatrix} \mu_1^i & \cdots & \mu_t^i & 0 & \cdots & 0 \end{pmatrix}^t.$$

3.4.2 Computing $MGC(A)$ for Teter rings

Let us consider a non-Gorenstein local Artin ring $A = R/I$ of socle degree s. Fix a $\mathbf{k}$-basis $b_1, \dots, b_t$ of $I^\perp$ and consider a polynomial $F = \sum_{j=1}^{h} a_j F_j \in \int_{\mathfrak{m}} I^\perp$, where $\overline{F}_1, \dots, \overline{F}_h$ is an adapted $\mathbf{k}$-basis of $\mathcal{L}_{A,1}$. According to Proposition 3.2.10, F corresponds to a minimal Gorenstein cover if and only if $\dim_{\mathbf{k}}(\mathfrak{m} \circ F) = t$. Therefore, we want to know for which values of $a_1, \dots, a_h$ this equality holds.

Note that $\deg F \leq s + 1$ and $x_k x_l \circ F = x_l x_k \circ F$. Then $\mathfrak{m} \circ F = \langle x^\alpha \circ F : 1 \leq |\alpha| \leq s + 1 \rangle_{\mathbf{k}}$. Each $x^\alpha \circ F \in I^\perp$, hence $x^\alpha \circ F = \sum_{j=1}^{t} \mu_\alpha^j b_j$ for some $\mu_\alpha^j \in \mathbf{k}$.

Consider the matrix $A = (\mu_\alpha^j)_{1\leq|\alpha|\leq s+1,\,1\leq j\leq t}$, whose rows are the contractions $x^\alpha \circ F$ expressed in terms of the $\mathbf{k}$-basis $b_1,\dots,b_t$ of $I^\perp$. The rows of A are a system of generators of $\mathfrak{m}\circ F$ as $\mathbf{k}$-vector space, hence $\dim_{\mathbf{k}}(\mathfrak{m}\circ F) < t$ if and only if all order t minors of A vanish. Let $\mathfrak{a}$ be the ideal generated by all order t minors $p_1,\dots,p_r$ of A. Note that the entries of matrix A are homogeneous polynomials of degree 1 in $\mathbf{k}[a_1,\dots,a_h]$. Hence $\mathfrak{a}$ is generated by homogeneous polynomials of degree t in $\mathbf{k}[a_1,\dots,a_h]$. Therefore, we can view the projective algebraic set

$$\mathbb{V}_+(\mathfrak{a}) = \{[a_1:\dots:a_h]\in\mathbb{P}^{h-1}_{\mathbf{k}} \mid p_i(a_1,\dots,a_h)=0,\ 1\leq i\leq r\},$$

as the set of all points that do not correspond to Teter covers. We just proved the following result:

THEOREM 3.4.6 Let $A = R/I$ be an Artin ring with $\operatorname{gcl}(A)=1$, $h=\dim_{\mathbf{k}}\mathcal{L}_{A,1}$ and $\mathfrak{a}$ be the ideal of minors previously defined. Then

$$MGC(A) = \mathbb{P}^{h-1}_{\mathbf{k}}\backslash\mathbb{V}_+(\mathfrak{a}).$$

Moreover, for any non-Gorenstein Artin ring A, $\operatorname{gcl}(A)=1$ if and only if $\mathfrak{a}\neq 0$.

Proof: The first part is already proved. On the other hand, if $\mathfrak{a}=0$, then $\mathbb{V}_+(\mathfrak{a})$ is the whole $\mathbb{P}^{h-1}_{\mathbf{k}}$ and $MGC(A)=\emptyset$. In other words, there exist no Teter covers, hence $\operatorname{gcl}(A)>1$. □

Algorithm 2 provides a method based on Theorem 3.4.6 to decide whether a non-Gorenstein ring $A=R/I$ has colength 1 and, if this is the case, it explicitly computes its $MGC(A)$.

With the following example we show how to interpret the output of the algorithm:

EXAMPLE 3.4.7 Consider $A=R/I$, with $R=\mathbf{k}[\![x_1,x_2]\!]$ and $I=\mathfrak{m}^2$ [20, Example 4.3]. From Example 3.4.5 we gather all the information we need for the input of Algorithm 2:
Input: $b_1=1, b_2=y_1, b_3=y_2$ $\mathbf{k}$-basis of $I^\perp$; $F_1=y^2, F_2=y_1y_2, F_3=y_1^2$ adapted $\mathbf{k}$-basis of $\mathcal{L}_{A,1}$; U_1', U_2' contraction matrices of $\int_{\mathfrak{m}} I^\perp$.
Output: $\operatorname{rad}(\mathfrak{a}) = a_2^2 - a_1a_3$.
We consider points $(a_1:a_2:a_3)\in\mathbb{P}^2$. Then $MGC(A)=\mathbb{P}^2\backslash\{a_2^2-a_1a_3=0\}$ and any minimal Gorenstein cover $G=R/\operatorname{Ann}_{\mathrm{R}}F$ of A is given by a polynomial $F=a_1y_2^4+a_2y_1y_2+a_3y_1^2$ such that $a_2^2-a_1a_3\neq 0$.

Algorithm 2 Compute the Teter variety of $A = R/I$ with $n \geq 2$

Input: s socle degree of $A = R/I$;
$b_1, \dots, b_t$ $\mathbf{k}$-basis of $I^\perp$;
$F_1, \dots, F_h$ adapted $\mathbf{k}$-basis of $\mathcal{L}_{A,1}$;
$U_1, \dots, U_n$ contraction matrices of $\int_{\mathfrak{m}} I^\perp$.
Output: ideal $\mathfrak{a}$ such that $MGC(A) = \mathbb{P}_{\mathbf{k}}^{h-1} \backslash \mathbb{V}_+(\mathfrak{a})$.
Steps:

(i) Set $F = a_1 F_1 + \cdots + a_h F_h$ and $\mathbf{F} = (a_1, \dots, a_h)^t$, where $a_1, \dots, a_h$ are variables in $\mathbf{k}$.

(ii) Build matrix $A = \left(\mu_j^\alpha\right)_{1 \leq |\alpha| \leq s+1, 1 \leq j \leq t}$, where

$$U^\alpha \mathbf{F} = \sum_{j=1}^{t} \mu_j^\alpha b_j, \quad U^\alpha = U_1^{\alpha_1} \cdots U_n^{\alpha_n}.$$

(iii) Compute the ideal $\mathfrak{a}$ generated by all minors of order t of the matrix A.

3.4.3 Computing $MGC(A)$ in colength 2

Consider a local Artin ring $A = R/I$ with $\operatorname{gcl}(A) > 1$, a $\mathbf{k}$-basis $b_1, \dots, b_t$ of $I^\perp$ and an adapted $\mathbf{k}$-basis $\overline{F}_1, \dots, \overline{F}_{h_1}, \overline{G}_1, \dots, \overline{G}_{h_2}$ of $\mathcal{L}_{A,2}$ (see Definition 3.2.4) such that

- $b_1, \dots, b_t, F_1, \dots, F_{h_1}$ is a $\mathbf{k}$-basis of $\int_{\mathfrak{m}} I^\perp$,
- $b_1, \dots, b_t, F_1, \dots, F_{h_1}, G_1, \dots, G_{h_2}$ is a $\mathbf{k}$-basis of $\int_{\mathfrak{m}^2} I^\perp$.

If a minimal Gorenstein cover $G = R/\operatorname{Ann_R} H$ of A such that $\ell(G) - \ell(A) = 2$ exists, then, by Corollary 3.2.6, H is a polynomial of the form

$$H = \sum_{i=1}^{h_1} \alpha_i F_i + \sum_{i=1}^{h_2} \beta_i G_i, \quad \alpha_i, \beta_i \in \mathbf{k}.$$

We want to obtain conditions on the α's and β's under which H actually generates a minimal Gorenstein cover of colength 2. By definition, H is in $\int_{\mathfrak{m}^2} I^\perp$, hence $x_k \circ H$ is in $\mathfrak{m} \circ \int_{\mathfrak{m}} \left(\int_{\mathfrak{m}} I^\perp\right) \subseteq \int_{\mathfrak{m}} I^\perp$ and

$$x_k \circ H = \sum_{j=1}^{t} \mu_k^j b_j + \sum_{j=1}^{h_1} \rho_k^j F_j, \quad \mu_k^j, \rho_k^j \in \mathbf{k}.$$

Set matrices $A_H = (\mu_k^j)$ and $B_H = (\rho_k^j)$. Let us describe matrix B_H explicitly. We have

$$x_k \circ H = \sum_{i=1}^{h_1} \alpha_i (x_k \circ F_i) + \sum_{i=1}^{h_2} \beta_i (x_k \circ G_i).$$

Note that each $x_k \circ G_i$, for any $1 \leq i \leq h_2$, is in $\int_{\mathfrak{m}} I^\perp$ and hence it can be decomposed as

$$x_k \circ G_i = \sum_{j=1}^{t} \lambda_j^{k,i} b_j + \sum_{j=1}^{h_1} a_j^{k,i} F_j, \quad \lambda_j^{k,i}, a_j^{k,i} \in \mathbf{k}.$$

Then

$$x_k \circ H = \sum_{i=1}^{h_1} \alpha_i (x_k \circ F_i) + \sum_{i=1}^{h_2} \beta_i \left(\sum_{j=1}^{t} \lambda_j^{k,i} b_j + \sum_{j=1}^{h_1} a_j^{k,i} F_j \right) =$$

$$= b + \sum_{j=1}^{h_1} \left(\sum_{i=1}^{h_2} \beta_i a_j^{k,i} \right) F_j,$$

where $b := \sum_{i=1}^{h_1} \alpha_i (x_k \circ F_i) + \sum_{i=1}^{h_2} \beta_i \left(\sum_{j=1}^{t} \lambda_j^{k,i} b_j \right) \in I^\perp$. Observe that

$$\rho_k^j = \sum_{i=1}^{h_2} a_j^{k,i} \beta_i, \tag{3.17}$$

hence the entries of matrix B_H can be regarded as polynomials in variables $\beta_1, \dots, \beta_{h_2}$ with coefficients in $\mathbf{k}$.

LEMMA 3.4.8 Consider the matrix $B_H = (\rho_k^j)$ as previously defined and let $B'_H = (\varrho_k^j)$ be the matrix of the coefficients of $\overline{L_k \circ H} = \sum_{j=1}^{h_1} \varrho_k^j \overline{F}_j \in \mathcal{L}_{A,1}$ where $L_1, \dots, L_n$ are independent linear forms. Then,

(i) $\operatorname{rk} B_H = \dim_{\mathbf{k}} \left(\dfrac{\mathfrak{m} \circ H + I^\perp}{I^\perp} \right)$,

(ii) $\operatorname{rk} B'_H = \operatorname{rk} B_H$.

Proof: Note that $\overline{x_k \circ H} = \sum_{j=1}^{h_1} \rho_k^j \overline{F}_j \in \mathcal{L}_{A,1}$ and

$$\langle \overline{x_1 \circ H}, \dots, \overline{x_n \circ H} \rangle_{\mathbf{k}} = (\mathfrak{m} \circ H + I^\perp)/I^\perp.$$

Since $\overline{F}_1, \dots, \overline{F}_{h_1}$ is a $\mathbf{k}$-basis of $\mathcal{L}_{A,1}$ and $(\mathfrak{m} \circ H + I^\perp)/I^\perp \subseteq \mathcal{L}_{A,1}$, then (i) holds.

For (ii) it will be enough to prove that

$$\langle \overline{x_1 \circ H}, \dots, \overline{x_n \circ H} \rangle_{\mathbf{k}} = \langle \overline{L_1 \circ H}, \dots, \overline{L_n \circ H} \rangle_{\mathbf{k}}.$$

Indeed, since $L_i = \sum_{j=1}^{n} \lambda_j^i x_j$ for any $1 \leq i \leq n$, then $\overline{L_i \circ H} = \sum_{j=1}^{n} \lambda_j^i (\overline{x_j \circ H})$ in $\langle \overline{x_1 \circ H}, \dots, \overline{x_n \circ H} \rangle_{\mathbf{k}}$. The reverse inclusion comes from the fact that $(L_1, \dots, L_n)$ is the maximal ideal, hence x_i can be expressed as a linear combination of $L_1, \dots, L_n$. □

LEMMA 3.4.9 With the previous notation, consider a polynomial $H \in \int_{\mathfrak{m}^2} I^\perp$ with coefficients $\beta_1, \dots, \beta_{h_2}$ of $G_1, \dots, G_{h_2}$, respectively, and its corresponding matrix B_H. Then the following are equivalent:

(i) $B_H \neq 0$,
(ii) $\mathfrak{m} \circ H \not\subseteq I^\perp$,
(iii) $(\beta_1, \dots, \beta_{h_2}) \neq (0, \dots, 0)$.

Proof: (i) implies (ii). If $B_H \neq 0$, by Lemma 3.4.8, $(\mathfrak{m} \circ H + I^\perp)/I^\perp \neq 0$ and hence $\mathfrak{m} \circ H \not\subseteq I^\perp$.
(ii) implies (iii). If $\mathfrak{m} \circ H \not\subseteq I^\perp$, by definition $H \notin \int_{\mathfrak{m}} I^\perp$ and hence H belongs to $\int_{\mathfrak{m}^2} I^\perp \setminus \int_{\mathfrak{m}} I^\perp$. Therefore, some β_i must be non-zero.
(iii) implies (i). Since $G_i \in \int_{\mathfrak{m}^2} I^\perp \setminus \int_{\mathfrak{m}} I^\perp$ for any $1 \leq i \leq h_2$ and, by hypothesis, there is some non-zero β_i, we have that $H \in \int_{\mathfrak{m}^2} I^\perp \setminus \int_{\mathfrak{m}} I^\perp$. We claim that $x_k \circ H$ is in $\int_{\mathfrak{m}} I^\perp \setminus I^\perp$ for some $1 \leq k \leq n$. Suppose the claim is not true. Then $x_k \circ H \in I^\perp$ for any $1 \leq k \leq n$, or equivalently, $\mathfrak{m} \circ H \subseteq I^\perp$. But, by definition, this means that $H \in \int_{\mathfrak{m}} I^\perp$, which is a contradiction. Since

$$x_k \circ H = b + \sum_{j=1}^{h_1} \left(\sum_{i=1}^{h_2} \beta_i a_j^{k,i} \right) F_j \in \int_{\mathfrak{m}} I^\perp \setminus I^\perp, \quad b \in I^\perp,$$

for some $1 \leq k \leq n$, then $\rho_k^j \neq 0$, for some $j \in \{1, \dots, h_1\}$. Therefore, $B_H \neq 0$. □

LEMMA 3.4.10 Consider the previous setting. If $B_H = 0$, then either $\operatorname{gcl}(A) = 0$ or $\operatorname{gcl}(A) = 1$ or $R/\operatorname{Ann_R} H$ is not a cover of A.

Proof: If $B_H = 0$, then $\mathfrak{m} \circ H \subseteq I^\perp$ and hence $\ell(H) - 1 \leq \ell(I^\perp)$. If $I^\perp \subseteq \langle H \rangle$, then

$G = R/\operatorname{Ann}_R H$ is a Gorenstein cover of A such that $\ell(G) - \ell(A) \leq 1$. Therefore, either $\operatorname{gcl}(A) \leq 1$ or G is not a cover. □

We already have techniques to check whether A has colength 0 or 1. Therefore, we can assume $\operatorname{gcl}(A) \geq 2$. The previous two lemmas allow us to take into consideration only those polynomials H such that $(\beta_1, \dots, \beta_{h_2}) \neq 0$ or, equivalently, $B_H \neq 0$. According to Proposition 3.2.13, $\operatorname{gcl}(A) = 2$ if and only if

$$(L_1, \dots, L_{n-1}, L_n^2) \circ H = I^\perp$$

for some H of the previously stated form and some independent linear forms $L_1, \dots, L_n$.

PROPOSITION 3.4.11 Assume that $B_H \neq 0$. Then $\operatorname{rk} B_H = 1$ if and only if

$$(L_1, \dots, L_{n-1}, L_n^2) \circ H \subseteq I^\perp$$

for some independent linear forms $L_1, \dots, L_n$.

Proof: Recall that, since we are under the assumption that $B_H \neq 0$, there exists k such that $x_k \circ H \notin I^\perp$. Without loss of generality, we can assume that $x_n \circ H \notin I^\perp$. If $\operatorname{rk} B_H = 1$, then any other row of B_H must be a multiple of row n. Therefore, for any $1 \leq i \leq n-1$, there exists $\lambda_i \in \mathbf{k}$ such that $(x_i - \lambda_i x_n) \circ H \in I^\perp$. Take $L_n := x_k$ and $L_i := x_i - \lambda_i x_n$. It is clear that $L_1, \dots, L_n$ are linearly independent and that $L_i \circ H \in I^\perp$ for any $1 \leq i \leq n-1$. Moreover, $L_n^2 \circ H = x_k^2 \circ H \in \mathfrak{m}^2 \circ \int_{m^2} I^\perp \subseteq I^\perp$.

Reciprocally, let $B'_H = (\varrho_k^j)$ be the matrix of the coefficients of

$$\overline{L_k \circ H} = \sum_{j=1}^{h_1} \varrho_k^j \overline{F}_j \in \mathcal{L}_{A,1}.$$

By Lemma 3.4.8, since $B_H \neq 0$, then $B'_H \neq 0$. We are assuming that $\overline{L_1 \circ H} = \dots = \overline{L_{n-1} \circ H} = 0$ but, since $B'_H \neq 0$, then $\overline{L_n \circ H} \neq 0$. It is clear that $\operatorname{rk} B'_H = 1$ and hence, again by Lemma 3.4.8, $\operatorname{rk} B_H = 1$. □

Recall that $\langle H \rangle = \langle \lambda H \rangle$ for any $\lambda \in \mathbf{k}^*$. Therefore, as pointed out in Theorem 3.3.2, for any $H \neq 0$, a Gorenstein ring $G = R/\operatorname{Ann}_R H$ can be identified with a point $[H]$ in $\mathbb{P}_\mathbf{k}(\mathcal{L}_{A,2})$ by taking coordinates $(\alpha_1 : \dots : \alpha_{h_1} : \beta_1 : \dots : \beta_{h_2})$. Observe that $\mathbb{P}_\mathbf{k}(\mathcal{L}_{A,2})$ is a projective space over $\mathbf{k}$ of dimension $h_1 + h_2 - 1$, we denote it by $\mathbb{P}_\mathbf{k}^{h_1+h_2-1}$.

On the other hand, from the expression 3.17 we can deduce that any minor of the matrix $B_H = (\rho_k^j)$ is a homogeneous polynomial in variables $\beta_1, \dots, \beta_{h_2}$. Therefore, we can consider the homogeneous ideal $\mathfrak{b}$ generated by all order-2-minors of B_H in the polynomial ring $\mathbf{k}[\alpha_1, \dots, \alpha_{h_1}, \beta_1, \dots, \beta_{h_2}]$. Hence $\mathbb{V}_+(\mathfrak{b})$ is the projective variety consisting of all points $[H] \in \mathbb{P}_{\mathbf{k}}^{h_1+h_2-1}$ such that $\operatorname{rk} B_H \leq 1$.

REMARK 3.4.12 In this section we will use the notation $MGC_2(A)$ to denote the set of points $[H] \in \mathbb{P}_{\mathbf{k}}^{h_1+h_2-1}$ such that $G = R/\operatorname{Ann}_R H$ is a Gorenstein cover of A with $\ell(G) - \ell(A) = 2$. Since we are considering rings such that $\operatorname{gcl}(A) > 1$, we can characterize rings of higher colength than 2 as those such that $MGC_2(A) = \emptyset$. On the other hand, $\operatorname{gcl}(A) = 2$ if and only if $MGC_2(A) \neq \emptyset$, hence in this case $MGC_2(A) = MGC(A)$, see Definition 3.3.3 and Remark 3.3.4.

COROLLARY 3.4.13 Let $A = R/I$ be an Artin ring such that $\operatorname{gcl}(A) = 2$. Then

$$MGC_2(A) \subseteq \mathbb{V}_+(\mathfrak{b}) \subseteq \mathbb{P}_{\mathbf{k}}^{h_1+h_2-1}.$$

Proof: By Proposition 2.1.6.(ii), points $[H] \in MGC_2(A)$ correspond to Gorenstein covers $G = R/\operatorname{Ann}_R H$ of A such that $I^\perp = (L_1, \dots, L_{n-1}, L_n^2) \circ H$ for some $L_1, \dots, L_n$. Since $B_H \neq 0$ by Lemma 3.4.10, then we can apply Proposition 3.4.11 to deduce that $\operatorname{rk} B_H = 1$. $\square$

Note that the conditions on the rank of B_H do not provide any information about which particular choices of independent linear forms $L_1, \dots, L_n$ satisfy the inclusion $(L_1, \dots, L_{n-1}, L_n^2) \circ H \subseteq I^\perp$. In fact, it will be enough to understand which are the L_n that meet the requirements.

To that end, we fix $L_n = v_1 x_1 + \dots + v_n x_n$, where $v = (v_1, \dots, v_n) \neq 0$. We can choose linear forms $L_i = \lambda_1^i x_1 + \dots + \lambda_n^i x_n$, where $\lambda_i = (\lambda_1^i, \dots, \lambda_n^i) \neq 0$, for $1 \leq i \leq n-1$, such that $L_1, \dots, L_n$ are linearly independent and $\lambda_i \cdot v = 0$. It is a linear algebra exercise to check that the k-vector space generated by $L_1, \dots, L_{n-1}$ can be expressed in terms of $v_1, \dots, v_n$. Indeed,

$$\langle L_1, \dots, L_{n-1} \rangle_{\mathbf{k}} = \langle v_l x_k - v_k x_l \,:\, 1 \leq k < l \leq n \rangle_{\mathbf{k}}.$$

Let us now add the coefficients of L_n to matrix B_H by defining the following matrix depending both on H and v:

$$C_{H,v} := \begin{pmatrix} \rho_1^1 & \dots & \rho_1^{h_1} & v_1 \\ \vdots & & \vdots & \vdots \\ \rho_n^1 & \dots & \rho_n^{h_1} & v_n \end{pmatrix}.$$

PROPOSITION 3.4.14 Assume $B_H \neq 0$ and consider $L_1, \dots, L_n$ linearly independent linear forms such that $L_n = v_1 x_1 + \cdots + v_n x_n$, where $v = (v_1, \dots, v_n) \neq 0$. Then $\operatorname{rk} C_{H,v} = 1$ if and only if $(L_1, \dots, L_{n-1}, L_n^2) \circ H \subseteq I^\perp$.

Proof: If $\operatorname{rk} C_{H,v} = 1$, then all 2-minors of $C_{H,v}$ vanish and, in particular,

$$v_l \rho_k^j - v_k \rho_l^j = 0 \text{ for any } 1 \leq k < l \leq n \text{ and } 1 \leq j \leq h_1. \tag{3.18}$$

Recall from 3.17 that

$$(v_l x_k - v_k x_l) \circ H = b + \sum_{j=1}^{h_1} \left(v_l \rho_k^j - v_k \rho_l^j \right) F_j, \text{ where } b \in I^\perp, \tag{3.19}$$

hence $(v_l x_k - v_k x_l) \circ H \in I^\perp$. Therefore, $L_i \circ H \in I^\perp$ for $1 \leq i \leq n-1$. Moreover, $L_n^2 \circ H \in \mathfrak{m}^2 \circ \int_{\mathfrak{m}^2} I^\perp \subseteq I^\perp$.

Conversely, if $(L_1, \dots, L_{n-1}, L_n^2) \circ H \subseteq I^\perp$, then $\operatorname{rk} B_H = 1$ by Proposition 3.4.11. Hence $\operatorname{rk} C_{H,v} = 1$ if and only if 3.18 holds. Since $L_i \circ H \in I^\perp$ for any $1 \leq i \leq n-1$, then $(v_l x_k - v_k x_l) \circ H \in I^\perp$ and we deduce from 3.19 that 3.18 is indeed satisfied. $\square$

DEFINITION 3.4.15 We say that $v = (v_1, \dots, v_n)$ is an **admissible vector** of H if $v \neq 0$ and $v_l \rho_k^j - v_k \rho_l^j = 0$ for any $1 \leq k < l \leq n$ and $1 \leq j \leq h_1$.

LEMMA 3.4.16 Given a polynomial H of the previous form such that $\operatorname{rk} B_H = 1$:

(i) there always exists an admissible vector $v \in \mathbf{k}^n$ of H;
(ii) if $w \in \mathbf{k}^n$ such that $w = \lambda v$, with $\lambda \in \mathbf{k}^*$, then w is an admissible vector of H;
(iii) the admissible vector of H is unique up to multiplication by elements of $\mathbf{k}^*$.

Proof: (i) Since $\mathrm{rk}_H\, B = 1$, Proposition 3.4.11 ensures the existence of linearly independent linear forms $L_1, \ldots, L_n$ such that $(L_1, \ldots, L_{n-1}, L_n^2) \circ H \subseteq I^{\perp}$. By Proposition 3.4.14, the vector whose components are the coefficients of L_n is admissible.
(ii) Since v is admissible, $w = \lambda v \neq 0$ and $w_l \rho_k^j - w_k \rho_l^j = \lambda(v_l \rho_k^j - v_k \rho_l^j) = 0$.
(iii) Since $B_H \neq 0$, there exists $\rho_k^j \neq 0$ for some $1 \leq j \leq h_1$ and $1 \leq k \leq n$. We will first prove that $v_k \neq 0$. Suppose that $v_k = 0$. By Definition 3.4.15, there exists $v_i \neq 0$, $i \neq k$, and $v_i \rho_k^j - v_k \rho_i^j = 0$. Then $v_i \rho_k^j = 0$ and we reach a contradiction.

Consider now $w = (w_1, \ldots, w_n)$ admissible with respect to H. From $\rho_k^j v_l - \rho_l^j v_k = 0$ and $\rho_k^j w_l - \rho_l^j w_k = 0$, we get $v_l = \left(\rho_l^j / \rho_k^j\right) v_k$ and $w_l = \left(\rho_l^j / \rho_k^j\right) w_k$. Set $\lambda_l := \rho_l^j / \rho_k^j$. For any $1 \leq l \leq n$, with $l \neq k$, from $v_l = \lambda_l v_k$ and $w_l = \lambda_l w_k$, we deduce that $w_l = (w_k / v_k)\, v_l$. Hence $w = \lambda v$, where $\lambda = w_k / v_k$, and any two admissible vectors of H are linearly dependent. □

We now want to provide a geometric interpretation of pairs of polynomials and admissible vectors and describe the variety where they lay. Let us first note that whenever $B_H = 0$, any $v \neq 0$ is an admissible vector. With this observation and Lemma 3.4.16, for any polynomial H such that $\mathrm{rk}\, B_H \leq 1$, we can consider its admissible vectors v as points $[v]$ in the projective space $\mathbb{P}_{\mathbf{k}}^{n-1}$ by taking homogeneous coordinates $(v_1 : \cdots : v_n)$.

Let us consider the ideal generated in $\mathbf{k}[\alpha_1, \ldots, \alpha_{h_1}, \beta_1, \ldots, \beta_{h_2}, v_1, \ldots, v_n]$ by polynomials of the form

$$\rho_k^j \rho_m^l - \rho_k^l \rho_m^j, \quad 1 \leq k < m \leq n, 1 \leq j < l \leq h_1; \tag{3.20}$$

$$v_l \rho_k^j - v_k \rho_l^j, \quad 1 \leq k < l \leq n, 1 \leq j \leq h_1. \tag{3.21}$$

It can be checked that all these polynomials are bihomogeneous polynomials in the sets of variables $\alpha_1, \ldots, \alpha_{h_1}, \beta_1, \ldots, \beta_{h_2}$ and $v_1, \ldots, v_n$. Therefore, this ideal defines a variety in $\mathbb{P}_{\mathbf{k}}^{h_1+h_2-1} \times \mathbb{P}_{\mathbf{k}}^{n-1}$ the points of which satisfy equations 3.20 and 3.21.

We denote by $\mathfrak{c}$ the ideal in $\mathbf{k}[\alpha_1, \ldots, \alpha_{h_1}, \beta_1, \ldots, \beta_{h_2}, v_1, \ldots, v_n]$ generated by all order 2 minors of $C_{H,v}$. We denote by $\mathbb{V}_+(\mathfrak{c})$ the variety defined by $\mathfrak{c}$ in $\mathbb{P}_{\mathbf{k}}^{h_1+h_2-1} \times \mathbb{P}_{\mathbf{k}}^{n-1}$.

LEMMA 3.4.17 With the previous definitions, the set of points of $\mathbb{V}_+(\mathfrak{c})$ is

$$\left\{ ([H], [v]) \in \mathbb{P}_{\mathbf{k}}^{h_1+h_2-1} \times \mathbb{P}_{\mathbf{k}}^{n-1} \mid [H] \in \mathbb{V}_+(\mathfrak{b}) \text{ and } v \text{ admissible with respect to } H \right\}.$$

Proof: It follows from 3.20 and 3.21. □

LEMMA 3.4.18 Let π_1 be the projection map from $\mathbb{P}_{\mathbf{k}}^{h_1+h_2-1} \times \mathbb{P}_{\mathbf{k}}^{n-1}$ to $\mathbb{P}_{\mathbf{k}}^{h_1+h_2-1}$. Then $\pi_1(\mathbb{V}_+(\mathfrak{c})) = \mathbb{V}_+(\mathfrak{b})$. Moreover, π_1 is a bijection when restricted to the subset of $\mathbb{V}_+(\mathfrak{c})$ where $\operatorname{rk} B_H = 1$.

Proof: Any element of $\mathbb{V}_+(\mathfrak{c})$ is of the form $([H],[v])$ described in Lemma 3.4.17. Then $\pi_1([H],[v]) = [H] \in \mathbb{V}_+(\mathfrak{b})$. Conversely, given an element $[H] \in \mathbb{V}_+(\mathfrak{b})$, we have $\operatorname{rk} B_H \leq 1$. If $B_H = 0$, then any $v \neq 0$ satisfies $([H],[v]) \in \mathbb{V}_+(\mathfrak{c})$. If $\operatorname{rk} B = 1$, by Lemma 3.4.16, there exists a unique admissible v up to scalar multiplication, hence $([H],[v])$ is the unique point in $\mathbb{V}_+(\mathfrak{c})$ such that $\pi_1([H],[v]) = [H]$. □

From Corollary 3.4.13, we know that all covers $G = R/\operatorname{Ann_R} H$ of $A = R/I$ colength 2 correspond to points $[H] \in \mathbb{V}_+(\mathfrak{b})$ but, in general, not all points of $\mathbb{V}_+(\mathfrak{b})$ correspond to such covers. Therefore, we need to identify and remove those $[H]$ such that $(L_1, \dots, L_{n-1}, L_n^2) \circ H \subsetneq I^\perp$.

As $\mathbf{k}$-vector space, $(L_1, \dots, L_{n-1}, L_n^2) \circ H$ is generated by

- $(v_l x_k - v_k x_l) \circ H$, $1 \leq k < l \leq n$;
- $x^\theta \circ H$, $2 \leq |\theta| \leq s+2$.

Since $(L_1, \dots, L_{n-1}, L_n^2) \circ H \subseteq I^\perp$, we can provide an explicit description of these generators with respect to the $\mathbf{k}$-basis $b_1, \dots, b_t$ of $I^\perp$ as follows:

$$(x_k v_l - x_l v_k) \circ H =$$

$$= \sum_{j=1}^{t} \left(v_l \sum_{i=1}^{h_1} \alpha_i \mu_j^{k,i} - v_k \sum_{i=1}^{h_1} \alpha_i \mu_j^{l,i} + v_l \sum_{i=1}^{h_2} \beta_i \lambda_j^{k,i} - v_k \sum_{i=1}^{h_2} \beta_i \lambda_j^{l,i} \right) b_j,$$

for $1 \leq l < k \leq n$, with $x_k \circ F_i = \sum_{j=1}^{t} \mu_j^{k,i} b_j$ and $x_k \circ G_i = \sum_{j=1}^{t} \lambda_j^{k,i} b_j + \sum_{j=1}^{h_1} a_j^{k,i} F_j$, where $\mu_j^{k,i}, \lambda_j^{k,i}, a_j^{k,i}$ are in $\mathbf{k}$;

$$x^\theta \circ H = \sum_{j=1}^{t} \left(\sum_{i=1}^{h_1} \mu_j^{\theta,i} \alpha_i + \sum_{i=1}^{h_2} \lambda_j^{\theta,i} \beta_i \right) b_j,$$

where $2 \leq |\theta| \leq s+2$, $x^\theta \circ F_i = \sum_{j=1}^{t} \mu_j^{\theta,i} b_j$ and

$$x^\theta \circ G_i = \sum_{j=1}^{t} \lambda_j^{\theta,i} b_j,$$

with $\mu_j^{\theta,i}$ and $\lambda_j^{\theta,i}$ in $\mathbf{k}$.

We now define matrix $U_{H,v}$ such that its rows are the coefficients of each generator of $(L_1, \ldots, L_{n-1}, L_n^2) \circ H$ with respect to the $\mathbf{k}$-basis $b_1, \ldots, b_t$ of $I^\perp$:

Ţ	b_1	$\cdots$	b_t
$(x_2 v_1 - x_1 v_2) \circ H$	$\varrho_{1,2}^1$	$\cdots$	$\varrho_{1,2}^t$
$\vdots$	$\vdots$		$\vdots$
$(x_n v_{n-1} - x_{n-1} v_n) \circ H$	$\varrho_{n-1,n}^1$	$\cdots$	$\varrho_{n-1,n}^t$
$x_1^2 \circ H$	$\varsigma_{(2,0,\ldots,0)}^1$	$\cdots$	$\varsigma_{(2,0,\ldots,0)}^t$
$x_1 x_2 \circ H$	$\varsigma_{(1,1,0,\ldots,0)}^1$	$\cdots$	$\varsigma_{(1,1,0,\ldots,0)}^t$
$\vdots$	$\vdots$		$\vdots$
$x_n^2 \circ H$	$\varsigma_{(0,\ldots,0,2)}^1$	$\cdots$	$\varsigma_{(0,\ldots,0,2)}^t$
$\vdots$	$\vdots$		$\vdots$
$x_n^{s+2} \circ H$	$\varsigma_{(0,\ldots,0,s+2)}^1$	$\cdots$	$\varsigma_{(0,\ldots,0,s+2)}^t$

where

$$\varrho_{l,k}^j := v_l \sum_{i=1}^{h_1} \alpha_i \mu_j^{k,i} - v_k \sum_{i=1}^{h_1} \alpha_i \mu_j^{l,i} + v_l \sum_{i=1}^{h_2} \beta_i \lambda_j^{k,i} - v_k \sum_{i=1}^{h_2} \beta_i \lambda_j^{l,i}$$

and

$$\varsigma_\theta^j := \sum_{i=1}^{h_1} \mu_j^{\theta,i} \alpha_i + \sum_{i=1}^{h_2} \lambda_j^{\theta,i} \beta_i.$$

It can be easily checked that the entries of this matrix are either bihomogeneous polynomials $\varrho_{l,k}^j$ in variables $((\alpha,\beta),v)$ of bidegree $(1,1)$ or homogeneous polynomials ς_θ^j in variables (α,β) of degree 1. Let $\mathfrak{a}$ be the ideal in $\mathbf{k}[\alpha_1, \ldots, \alpha_{h_1}, \beta_1, \ldots, \beta_{h_2}, v_1, \ldots, v_n]$ generated by all minors of $U_{H,v}$ of order $t = \dim_{\mathbf{k}} I^\perp$.

It can be checked that $\mathfrak{a}$ is a bihomogeneous ideal in variables $((\alpha,\beta),v)$, hence we can think of $\mathbb{V}_+(\mathfrak{a})$ as the following variety in $\mathbb{P}^{h_1+h_2-1}\times\mathbb{P}^{n-1}$:

$$\mathbb{V}_+(\mathfrak{a}) = \{([H],[v]) \in \mathbb{P}^{h_1+h_2-1}\times\mathbb{P}^{n-1} \mid \operatorname{rk} U_{H,v} < t\}.$$

PROPOSITION 3.4.19 Assume $\operatorname{gcl}(A) > 1$. Consider a point $([H],[v]) \in \mathbb{V}_+(\mathfrak{c})$ in $\mathbb{P}^{h_1+h_2-1}\times\mathbb{P}^{n-1}$. Then

$$[H] \in MGC_2(A) \iff ([H],[v]) \notin \mathbb{V}_+(\mathfrak{a}),$$

Proof: From Corollary 3.4.13 we deduce that if $[H]$ is a point in $MGC_2(A)$, then $\operatorname{rk} B_H \le 1$. The same is true for any point $([H],[v]) \in \mathbb{V}_+(\mathfrak{c})$. Let us consider these two cases:

Case $B_H = 0$. Since $\operatorname{gcl}(A) > 1$, then $R/\operatorname{Ann}_{\mathrm{R}} H$ is not a Gorenstein cover of A by Lemma 3.4.10, hence $[H] \notin MGC_2(A)$. On the other hand, as stated in the proof of Lemma 3.4.18, $([H],[v]) \in \mathbb{V}_+(\mathfrak{c})$ for any $v \neq 0$. By Lemma 3.4.9 and $\operatorname{gcl}(A) \neq 1$, it follows that

$$(L_1,\dots,L_{n-1},L_n^2)\circ H \subseteq \mathfrak{m}\circ H \subsetneq I^\perp$$

for any $L_1,\dots,L_n$ linearly independent linear forms, where $L_n = v_1x_1+\cdots+v_nx_n$. Therefore, the rank of matrix $U_{H,v}$ is always strictly smaller than $\dim_{\mathbf{k}} I^\perp$. Hence $([H],[v]) \in \mathbb{V}_+(\mathfrak{a})$ for any $v \neq 0$.

Case $\operatorname{rk} B_H = 1$. If $[H] \in MGC_2(A)$, then there exist $L_1,\dots,L_n$ such that $(L_1,\dots,L_{n-1},L_n^2)\circ H = I^\perp$. Take v as the vector of coefficients of L_n, it is an admissible vector by definition. By Lemma 3.4.18, $([H],[v]) \in \mathbb{V}_+(\mathfrak{c})$ is unique and $\operatorname{rk} U_{H,v} = \dim_{\mathbf{k}} I^\perp$. Therefore, $([H],[v]) \notin \mathbb{V}_+(\mathfrak{a})$.

Conversely, if $([H],[v]) \in \mathbb{V}_+(\mathfrak{c})\cap\mathbb{V}_+(\mathfrak{a})$, then $\operatorname{rk} U_{H,v} < \dim_{\mathbf{k}} I^\perp$ and hence $(L_1,\dots,L_{n-1},L_n^2)\circ H \subsetneq I^\perp$, where $L_n = v_1x_1+\cdots+v_nx_n$. By unicity of v, no other choice of $L_1,\dots,L_n$ satisfies the inclusion $(L_1,\dots,L_{n-1},L_n^2)\circ H \subset I^\perp$, hence $[H] \notin MGC_2(A)$. $\square$

COROLLARY 3.4.20 Assume $\operatorname{gcl}(A) > 1$. With previous definitions,

$$MGC_2(A) = \mathbb{V}_+(\mathfrak{b})\backslash\pi_1\left(\mathbb{V}_+(\mathfrak{c})\cap\mathbb{V}_+(\mathfrak{a})\right).$$

Proof: It follows from Lemma 3.4.18 and Proposition 3.4.19. $\square$

Finally, let us recall the following result for bihomogeneous ideals, see [10]:

LEMMA 3.4.21 Let ideals $\mathfrak{a}, \mathfrak{c}$ be as previously defined, $\mathfrak{d} = \mathfrak{a} + \mathfrak{c}$ be the sum ideal and $\pi_1 : \mathbb{P}_{\mathbf{k}}^{h_1+h_2-1} \times \mathbb{P}_{\mathbf{k}}^{n-1} \longrightarrow \mathbb{P}_{\mathbf{k}}^{h_1+h_2-1}$ be the projection map. Let $\widehat{\mathfrak{d}}$ be the projective elimination of the ideal $\mathfrak{d}$ with respect to variables $v_1, \ldots, v_n$. Then,

$$\pi_1(\mathbb{V}_+(\mathfrak{a}) \cap \mathbb{V}_+(\mathfrak{c})) = \mathbb{V}_+(\widehat{\mathfrak{d}}).$$

Algorithm 3 Compute $MGC_2(A)$ of $A = R/I$ with $n \geq 2$ and $\mathrm{gcl}(A) > 1$

Input: s socle degree of $A = R/I$; $b_1, \ldots, b_t$ $\mathbf{k}$-basis of the inverse system $I^\perp$; $F_1, \ldots, F_{h_1}, G_1, \ldots, G_{h_2}$ an adapted $\mathbf{k}$-basis of $\mathcal{L}_{A,2}$; $U_1, \ldots, U_n$ contraction matrices of $\int_{\mathfrak{m}^2} I^\perp$.

Output: ideals $\mathfrak{b}$ and $\widehat{\mathfrak{d}}$ such that $MGC_2(A) = \mathbb{V}_+(\mathfrak{b}) \backslash \mathbb{V}_+(\widehat{\mathfrak{d}})$.

Steps:

(i) Set $H = \alpha_1 F_1 + \cdots + \alpha_{h_1} F_{h_1} + \beta_1 G_1 + \cdots + \beta_{h_2} G_{h_2}$, where α, β are variables in $\mathbf{k}$. Set column vectors $\mathbf{H} = (0, \ldots, 0, \alpha, \beta)^t$ and $v = (v_1, \ldots, v_n)^t$ in $R = \mathbf{k}[\alpha, \beta, v]$, where the first t components of $\mathbf{H}$ are zero.

(ii) Build matrix $B_H = (\rho_i^j)_{1 \leq i \leq n,\, 1 \leq j \leq h_1}$, where $U_i \mathbf{H}$ is the column vector $(\mu_i^1, \ldots, \mu_i^t, \rho_i^1, \ldots, \rho_i^{h_1}, 0, \ldots, 0)^t$.

(iii) Build matrix $C_{H,v} = \left(\begin{array}{c|c} B_H & v \end{array} \right)$ as an horizontal concatenation of B_H and the column vector v.

(iv) Compute the ideal $\mathfrak{c} \subseteq R$ generated by all minors of order 2 of B_H.

(v) Build matrix $U_{H,v}$ as a vertical concatenation of matrices $(\varrho_{l,k}^j)_{1 \leq j \leq h_1,\, 1 \leq l < k \leq n}$ and $(\varsigma_\theta^j)_{2 \leq |\theta| \leq s+2,\, 1 \leq j \leq h_1}$, such that $(v_l U_k - v_k U_l)\mathbf{H} = (\varrho_{l,k}^1, \cdots, \varrho_{l,k}^{h_1}, 0, \cdots, 0)^t$ and $U^\theta \mathbf{H} = (\varsigma_\theta^1, \cdots, \varsigma_\theta^{h_1}, 0, \cdots, 0)^t$, with $1 \leq k < l \leq n$ and $2 \leq |\theta| \leq s+2$.

(vi) Compute the ideal $\mathfrak{a} \subseteq R$ generated by all minors of order t of $U_{H,v}$ and the ideal $\mathfrak{d} = \mathfrak{a} + \mathfrak{c} \subseteq R$.

(vii) Compute $\widehat{\mathfrak{d}} \subseteq R' = \mathbf{k}[\alpha, \beta]$, where $\widehat{\cdot}$ denotes the projective elimination of the ideal in R with respect to variables $v_1, \ldots, v_n$.

(viii) Compute the ideal $\mathfrak{b} := \widehat{\mathfrak{c}} \subseteq R'$.

Algorithm 3 effectively computes $MGC_2(A)$ for any ring $A = R/I$ with Gorenstein colength strictly higher than 1. Its output can be interpreted as $MGC_2(A) = \mathbb{V}_+(\mathfrak{b})\backslash\mathbb{V}_+(\widehat{\mathfrak{d}})$. Moreover, any point $[\alpha_1 : \cdots : \alpha_{h_1} : \beta_1 : \cdots : \beta_{h_2}]$ in $MGC_2(A)$ corresponds to a minimal Gorenstein cover $G = R/\operatorname{Ann_R} H$ of colength 2 of A, where $H = \alpha_1 F_1 + \cdots + \alpha_{h_1} F_{h_1} + \beta_1 G_1 + \cdots + \beta_{h_2} G_{h_2}$. If $MGC_2(A) \neq \emptyset$, then $\operatorname{gcl}(A) = 2$ and hence $MGC(A) = MGC_2(A)$. Otherwise, $\operatorname{gcl}(A) > 2$.

EXAMPLE 3.4.22 Consider $A = R/I$, with $R = \mathbf{k}[\![x_1, x_2]\!]$ and $I = (x_1^2, x_1x_2^2, x_2^4)$. Applying Algorithm 1 twice we get the necessary input for Algorithm 3:
Input: $b_1 = 1, b_2 = y_1, b_3 = y_2, b_4 = y_2^2, b_5 = y_1y_2, b_6 = y_2^3$ $\mathbf{k}$-basis of $I^\perp$; $F_1 = y_2^4, F_2 = y_1y_2^2, F_3 = y_1^2, G_1 = y_1^2y_2, G_2 = y_1y_2^3, G_3 = y_2^5, G_4 = y_1^3$ adapted $\mathbf{k}$-basis of $\mathcal{L}_{A,2}$; U_1, U_2 contraction matrices of $\int_{\mathfrak{m}^2} I^\perp$.
Output: $\mathfrak{b} = (b_3b_4, b_2b_4)$, $\widehat{\mathfrak{d}} = (b_3b_4, b_2b_4, b_2^2 - b_1b_3)$.
$MGC_2(A) = \mathbb{V}_+(b_3b_4, b_2b_4)\backslash\mathbb{V}_+(b_3b_4, b_2b_4, b_2^2 - b_1b_3) = \mathbb{V}_+(b_3b_4, b_2b_4)\backslash\mathbb{V}_+(b_2^2 - b_1b_3)$. Note that if $b_3b_4 = b_2b_4 = 0$ and $b_4 \neq 0$, then both b_2 and b_3 are zero and condition $b_2^2 - b_1b_3 = 0$ always holds. Therefore, $\operatorname{gcl}(A) = 2$ and hence

$$MGC(A) = \mathbb{V}_+(b_4)\backslash\mathbb{V}_+(b_2^2 - b_1b_3) \simeq \mathbb{P}^5\backslash\mathbb{V}_+(b_2^2 - b_1b_3),$$

where $(a_1 : a_2 : a_3 : b_1 : b_2 : b_3)$ are the coordinates of the points in $\mathbb{P}^5$. Moreover, any minimal Gorenstein cover is of the form $G = R/\operatorname{Ann_R} H$, where

$$H = a_1y_2^4 + a_2y_1y_2^2 + a_3y_1^2 + b_1y_1^2y_2 + b_2y_1y_2^3 + b_3y_2^5$$

satisfies $b_2^2 - b_1b_3 \neq 0$. All such covers admit (x_1, x_2^2) as the corresponding K_H.

3.5 Computations

The first aim of this section is to provide a wide range of examples of the computation of the minimal Gorenstein cover variety of a local ring A. In [40], Poonen provides a complete classification of local algebras over an algebraically closed field of length equal or less than 6. Note that, for higher lengths, the number of isomorphism classes is no longer finite. We will go through all algebras of Poonen's list and restrict, for the sake of simplicity, to fields of characteristic zero.

On the other hand, we also intend to test the efficiency of the algorithms by collecting the computation times. We have implemented algorithms 1, 2 and 3 of Section 3.4 in the commutative algebra software *Singular* [11]. The computer we use runs into the operating system Microsoft Windows 10 Pro and its technical specifications are the following: Surface Pro 3; Processor: 1.90 GHz Intel Core i5-4300U 3 MB SmartCache; Memory: 4GB 1600MHz DDR3.

3.5.1 Teter varieties

In this first part of the section we are interested in the computation of Teter varieties, that is, the $MGC(A)$ variety for local algebras A of Gorenstein colength 1. All the results are obtained by running Algorithm 2 in *Singular*.

EXAMPLE 3.5.1 Consider $A = R/I$, with $R = \mathbf{k}[\![x_1, x_2, x_3]\!]$ and $I = (x_1^2, x_1x_2, x_1x_3, x_2x_3, x_2^3, x_3^3)$. Note that $\mathrm{HF}_A = \{1, 3, 2\}$ and $\tau(A) = 3$. The output provided by our implementation of the algorithm in *Singular* [11] is the following:

```
F;
a(4)*x(2)^3+a(1)*x(3)^3+a(6)*x(1)^2+a(5)*x(1)*x(2)
+a(3)*x(1)*x(3)+a(2)*x(2)*x(3)
radical(a);
a(1)*a(4)*a(6)
```

We consider points with coordinates $(a_1 : a_2 : a_3 : a_4 : a_5 : a_6) \in \mathbb{P}^5$. Therefore, $MGC(A) = \mathbb{P}^5 \backslash \mathbb{V}_+(a_1a_4a_6)$ and any minimal Gorenstein cover is of the form $G = R/\operatorname{Ann_R} H$, where $H = a_1y_3^3 + a_2y_2y_3 + a_3y_1y_3 + a_4y_2^3 + a_5y_1y_2 + a_6y_1^2$ with $a_1a_4a_6 \neq 0$.

In Table 3.1 below we show the computation time (in seconds) of all isomorphism classes of local $\mathbf{k}$-algebras A of $\mathrm{gcl}(A) = 1$ appearing in Poonen's classification [40]. In this table, we list the Hilbert function of $A = R/I$, the expression of the ideal I up to linear isomorphism, the dimension $h - 1$ of the projective space $\mathbb{P}^{h-1}$ where the variety $MGC(A)$ lies and the computation time. Note that our implementation of Algorithm 2 includes also the computation of the $\mathbf{k}$-basis of $\int_{\mathfrak{m}} I^\perp$, hence the computation time corresponds to the total amount of time.

$\mathrm{HF}_{R/I}$	I	$h-1$	$t(s)$
$1,2$	$(x_1,x_2)^2$	2	0,06
$1,2,1$	x_1x_2, x_2^2, x_1^3	2	0,06
$1,3$	$(x_1,x_2,x_3)^2$	5	0,13
$1,2,1,1$	x_1^2, x_1x_2, x_2^4	2	0,23
$1,2,2$	x_1x_2, x_1^3, x_2^3	2	0,11
	$x_1x_2^2, x_1^2, x_2^3$	2	0,05
$1,3,1$	$x_1x_2, x_1x_3, x_2x_3, x_2^2, x_3^2, x_1^3$	5	0,16
$1,4$	$(x_1,x_2,x_3,x_4)^2$	9	2,30
$1,2,1,1,1$	x_1x_2, x_1^5, x_2^2	2	0,17
$1,2,2,1$	x_1x_2, x_1^3, x_2^4	2	0,09
	$x_1^2+x_2^3, x_1x_2^2, x_2^4$	2	0,1
$1,3,1,1$	$x_1x_2, x_1x_3, x_2x_3, x_2^2, x_3^2, x_1^4$	5	3,05
$1,3,2$	$x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3^2, x_3^3$	5	0,33
	$x_1^2, x_1x_2, x_1x_3, x_2x_3, x_2^3, x_3^3$	5	0,23
$1,4,1$	$x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_2^2, x_3^2, x_4^2, x_1^3$	9	3,21
$1,5$	$(x_1,x_2,x_3,x_4,x_5)^2$	14	1,25

TABLE 3.1 Computation times of $MGC(A)$ for $A=R/I$ with $\ell(A)\leq 6$ and $\mathrm{gcl}(A)=1$.

See Appendix C for an explicit description of $MGC(A)$ for all the ideals represented in Table 3.1.

3.5.2 Minimal Gorenstein covers variety in colength 2

Now we want to compute $MGC(A)$ for $\mathrm{gcl}(A)=2$. All the examples are obtained by running Algorithm 3 in *Singular*.

EXAMPLE 3.5.2 Consider $A = R/I$, with $R = \mathbf{k}[\![x_1,x_2,x_3]\!]$ and $I = (x_1^2,\ x_2^2,\ x_3^2,\ x_1x_2,\ x_1x_3)$. Note that $\mathrm{HF}_A = \{1,3,1\}$ and $\tau(A)=2$. The output provided by our implementation of the algorithm in *Singular* [11] is the following:

```
H;
b(10)*x(1)^3+b(7)*x(1)^2*x(2)+
+b(8)*x(1)*x(2)^2+b(9)*x(2)^3+
+b(1)*x(1)^2*x(3)+b(2)*x(1)*x(2)*x(3)+
+b(3)*x(2)^2*x(3)+b(4)*x(1)*x(3)^2+
+b(6)*x(2)*x(3)^2+b(5)*x(3)^3+
+a(5)*x(1)^2+a(4)*x(1)*x(2)+
+a(3)*x(2)^2+a(2)*x(1)*x(3)+
+a(1)*x(3)^2
radical(b);
_[1]=b(8)^2-b(7)*b(9)
_[2]=b(7)*b(8)-b(9)*b(10)
_[3]=b(6)*b(8)-b(4)*b(9)
_[4]=b(3)*b(8)-b(2)*b(9)
_[5]=b(2)*b(8)-b(1)*b(9)
_[6]=b(1)*b(8)-b(3)*b(10)
_[7]=b(7)^2-b(8)*b(10)
_[8]=b(6)*b(7)-b(4)*b(8)
_[9]=b(4)*b(7)-b(6)*b(10)
_[10]=b(3)*b(7)-b(1)*b(9)
_[11]=b(2)*b(7)-b(3)*b(10)
_[12]=b(1)*b(7)-b(2)*b(10)
_[13]=b(3)*b(6)-b(5)*b(9)
_[14]=b(2)*b(6)-b(5)*b(8)
_[15]=b(1)*b(6)-b(5)*b(7)
_[16]=b(2)*b(5)-b(4)*b(6)
_[17]=b(4)^2-b(1)*b(5)
_[18]=b(3)*b(4)-b(5)*b(8)
_[19]=b(2)*b(4)-b(5)*b(7)
_[20]=b(1)*b(4)-b(5)*b(10)
_[21]=b(2)*b(3)-b(4)*b(9)
_[22]=b(1)*b(3)-b(4)*b(8)
_[23]=b(2)^2-b(4)*b(8)
_[24]=b(1)*b(2)-b(6)*b(10)
_[25]=b(1)^2-b(4)*b(10)
_[26]=b(3)*b(5)*b(10)-b(6)^2*b(10)
_[27]=b(3)^2*b(10)-b(6)*b(9)*b(10)
_[28]=b(4)*b(6)^2-b(5)^2*b(8)
_[29]=b(6)^3*b(10)-b(5)^2*b(9)*b(10)
radical(d);
_[1]=b(8)^2-b(7)*b(9)
_[2]=b(7)*b(8)-b(9)*b(10)
_[3]=b(6)*b(8)-b(4)*b(9)
_[4]=b(3)*b(8)-b(2)*b(9)
_[5]=b(2)*b(8)-b(1)*b(9)
_[6]=b(1)*b(8)-b(3)*b(10)
_[7]=b(7)^2-b(8)*b(10)
_[8]=b(6)*b(7)-b(4)*b(8)
_[9]=b(4)*b(7)-b(6)*b(10)
_[10]=b(3)*b(7)-b(1)*b(9)
_[11]=b(2)*b(7)-b(3)*b(10)
_[12]=b(1)*b(7)-b(2)*b(10)
_[13]=b(3)*b(6)-b(5)*b(9)
_[14]=b(2)*b(6)-b(5)*b(8)
_[15]=b(1)*b(6)-b(5)*b(7)
_[16]=b(2)*b(5)-b(4)*b(6)
_[17]=b(4)^2-b(1)*b(5)
_[18]=b(3)*b(4)-b(5)*b(8)
_[19]=b(2)*b(4)-b(5)*b(7)
_[20]=b(1)*b(4)-b(5)*b(10)
_[21]=b(2)*b(3)-b(4)*b(9)
_[22]=b(1)*b(3)-b(4)*b(8)
_[23]=b(2)^2-b(4)*b(8)
_[24]=b(1)*b(2)-b(6)*b(10)
_[25]=b(1)^2-b(4)*b(10)
_[26]=b(3)*b(5)*b(10)-b(6)^2*b(10)
_[27]=b(3)^2*b(10)-b(6)*b(9)*b(10)
_[28]=b(4)*b(6)^2-b(5)^2*b(8)
_[29]=a(5)*b(3)*b(5)-a(5)*b(6)^2
_[30]=a(5)*b(3)^2-a(5)*b(6)*b(9)
_[31]=b(6)^3*b(10)-b(5)^2*b(9)*b(10)
_[32]=a(5)*b(6)^3-a(5)*b(5)^2*b(9)
```

We can simplify the output by using the primary decomposition $\bigcap_{i=1}^{k} \mathfrak{b}_i$ of the ideal $\mathfrak{b}$. Then,

$$MGC(A) = \left(\bigcup_{i=1}^{k} \mathbb{V}_+(\mathfrak{b}_i)\right) \backslash \mathbb{V}_+(\widehat{\mathfrak{d}}) = \bigcup_{i=1}^{k} \left(\mathbb{V}_+(\mathfrak{b}_i) \backslash \mathbb{V}_+(\widehat{\mathfrak{d}})\right).$$

Singular [11] provides a primary decomposition $\mathfrak{b} = \mathfrak{b}_1 \cap \mathfrak{b}_2$ that satisfies

$$\mathbb{V}_+(\mathfrak{b}_2) \backslash \mathbb{V}_+(\widehat{\mathfrak{d}}) = \emptyset.$$

Therefore, we get

$$MGC(A) = \mathbb{V}_+(b_1, b_2, b_4, b_7, b_8, b_{10}, b_3 b_6 - b_5 b_9) \backslash \left(\mathbb{V}_+(a_5) \cup \mathbb{V}_+(\mathfrak{d})\right) \subset \mathbb{P}^{14},$$

where $\mathfrak{d} = (-b_6^3 + b_5^2 b_9, b_3 b_5 - b_6^2, b_3^2 - b_6 b_9)$. We can eliminate some of the variables and consider $MGC(A)$ to be the following variety:

$$MGC(A) = \mathbb{V}_+(b_3b_6 - b_5b_9) \setminus \left(\mathbb{V}_+(a_5) \cup \mathbb{V}_+(b_5^2b_9 - b_6^3, b_3b_5 - b_6^2, b_3^2 - b_6b_9)\right) \subset \mathbb{P}^8.$$

Therefore, any minimal Gorenstein cover is of the form $G = R/\operatorname{Ann}_R H$, where

$$H = a_1y_3^2 + a_2y_1y_3 + a_3y_2^2 + a_4y_1y_2 + a_5y_1^2 + b_3y_2^2y_3 + b_5y_3^3 + b_6y_2y_3^2 + b_9y_2^3$$

satisfies $b_3b_6 - b_5b_9 = 0$ and $a_5 \neq 0$ and at least one of the following conditions: $b_5^2b_9 - b_6^3 \neq 0$, $b_3b_5 - b_6^2 \neq 0$ or $b_3^2 - b_6b_9 \neq 0$.

Moreover, note that $\mathbb{V}_+(\mathfrak{c})\setminus\mathbb{V}_+(\mathfrak{a}) = \mathbb{V}_+(\mathfrak{c}_1)\setminus\mathbb{V}_+(\mathfrak{a})$, where $\mathfrak{c} = \mathfrak{c}_1 \cap \mathfrak{c}_2$ is the primary decomposition of $\mathfrak{c}$ and $\mathfrak{c}_1 = \mathfrak{b}_1 + (v_1, v_2b_5 - v_3b_6, v_2b_3 - v_3b_9)$. Hence, any K_H such that $K_H \circ H = I^\perp$ will be of the form $K_H = (L_1, L_2, L_3^2)$, where L_1, L_2, L_3 are independent linear forms in R such that $L_3 = v_2x_2 + v_3x_3$, with $v_2b_5 - v_3b_6 = v_2b_3 - v_3b_9 = 0$.

EXAMPLE 3.5.3 Consider $A = R/I$, with $R = \mathbf{k}[\![x_1, x_2, x_3]\!]$ and $I = (x_1x_2,\ x_1x_3,\ x_2x_3,\ x_2^2,\ x_3^2 - x_1^3)$. Note that $\mathrm{HF}_A = \{1, 3, 1, 1\}$ and $\tau(A) = 2$. The output provided by our implementation of the algorithm in *Singular* [11] is the following:

```
H;
-b(10)*x(1)^4+b(9)*x(1)^2*x(2)+
+b(7)*x(1)*x(2)^2+b(8)*x(2)^3+
+b(6)*x(1)^2*x(3)+b(1)*x(1)*x(2)*x(3)+
+b(2)*x(2)^2*x(3)+b(3)*x(1)*x(3)^2+
+b(4)*x(2)*x(3)^2+b(5)*x(3)^3+
+a(5)*x(1)*x(2)+a(4)*x(2)^2+
+a(3)*x(1)*x(3)+a(2)*x(2)*x(3)+
+a(1)*x(3)^2
radical(b);
_[1]=b(8)*b(10)
_[2]=b(7)*b(10)
_[3]=b(4)*b(10)
_[4]=b(2)*b(10)
_[5]=b(1)*b(10)
_[6]=b(6)*b(8)-b(2)*b(9)
_[7]=b(7)^2-b(8)*b(9)
_[8]=b(6)*b(7)-b(1)*b(9)
_[9]=b(4)*b(7)-b(3)*b(8)
_[10]=b(3)*b(7)-b(4)*b(9)
_[11]=b(2)*b(7)-b(1)*b(8)
_[12]=b(1)*b(7)-b(2)*b(9)
_[13]=b(4)*b(6)-b(5)*b(9)
_[14]=b(2)*b(6)-b(4)*b(9)
_[15]=b(1)*b(6)-b(3)*b(9)
_[16]=b(4)^2-b(2)*b(5)
_[17]=b(3)*b(4)-b(1)*b(5)
_[18]=b(2)*b(4)-b(5)*b(8)
_[19]=b(1)*b(4)-b(5)*b(7)
_[20]=b(3)^2-b(5)*b(6)+b(3)*b(10)
_[21]=b(2)*b(3)-b(5)*b(7)
_[22]=b(1)*b(3)-b(5)*b(9)
_[23]=b(2)^2-b(4)*b(8)
_[24]=b(1)*b(2)-b(3)*b(8)
_[25]=b(1)^2-b(4)*b(9)
_[26]=b(5)*b(9)*b(10)
_[27]=b(3)*b(9)*b(10)
radical(d);
_[1]=b(8)*b(10)
_[2]=b(7)*b(10)
_[3]=b(4)*b(10)
_[4]=b(2)*b(10)
_[5]=b(1)*b(10)
_[6]=b(6)*b(8)-b(2)*b(9)
_[7]=b(7)^2-b(8)*b(9)
_[8]=b(6)*b(7)-b(1)*b(9)
_[9]=b(4)*b(7)-b(3)*b(8)
_[10]=b(3)*b(7)-b(4)*b(9)
_[11]=b(2)*b(7)-b(1)*b(8)
_[12]=b(1)*b(7)-b(2)*b(9)
_[13]=b(4)*b(6)-b(5)*b(9)
_[14]=b(2)*b(6)-b(4)*b(9)
_[15]=b(1)*b(6)-b(3)*b(9)
_[16]=b(4)^2-b(2)*b(5)
_[17]=b(3)*b(4)-b(1)*b(5)
_[18]=b(2)*b(4)-b(5)*b(8)
_[19]=b(1)*b(4)-b(5)*b(7)
_[20]=b(3)^2-b(5)*b(6)+b(3)*b(10)
_[21]=b(2)*b(3)-b(5)*b(7)
_[22]=b(1)*b(3)-b(5)*b(9)
_[23]=b(2)^2-b(4)*b(8)
_[24]=b(1)*b(2)-b(3)*b(8)
_[25]=b(1)^2-b(4)*b(9)
_[26]=b(5)*b(9)*b(10)
_[27]=b(3)*b(9)*b(10)
_[28]=a(4)*b(5)*b(10)
_[29]=a(4)*b(3)*b(10)
```

Singular provides a primary decomposition $\mathfrak{b} = \mathfrak{b}_1 \cap \mathfrak{b}_2 \cap \mathfrak{b}_3$ such that $\mathbb{V}_+(\mathfrak{b})\setminus\mathbb{V}_+(\widehat{\mathfrak{d}}) = \mathbb{V}_+(\mathfrak{b}_2)\setminus\mathbb{V}_+(\widehat{\mathfrak{d}})$. Therefore, $MGC(A)$ corresponds to

$$\mathbb{V}_+(b_1, b_2, b_4, b_7, b_8, b_9, b_3^2 - b_5b_6 + b_3b_{10}) \setminus (\mathbb{V}_+(a_4) \cup \mathbb{V}_+(b_{10}) \cup \mathbb{V}_+(b_3, b_5)) \subset \mathbb{P}^{14}.$$

We can eliminate some of the variables and consider $MGC(A)$ to be the following variety:

$$MGC(A) = \mathbb{V}_+(b_3^2 - b_5b_6 + b_3b_{10}) \setminus (\mathbb{V}_+(a_4) \cup \mathbb{V}_+(b_{10}) \cup \mathbb{V}_+(b_3, b_5)) \subset \mathbb{P}^8.$$

Therefore, any minimal Gorenstein cover is of the form $G = R/\operatorname{Ann}_{\mathrm{R}} H$, where

$$H = a_1y_3^2 + a_2y_2y_3 + a_3y_1y_3 + a_4y_2^2 + a_5y_1y_2 + b_3y_1y_3^2 + b_5y_3^3 + b_6y_1^2y_3 - b_{10}y_1^4$$

satisfies $b_3^2 - b_5b_6 + b_3b_{10} = 0$, $a_4 \neq 0$, $b_{10} \neq 0$ and either $b_3 \neq 0$ or $b_5 \neq 0$ (or both).

Moreover, note that $\mathbb{V}_+(\mathfrak{c}) \setminus \mathbb{V}_+(\mathfrak{a}) = \mathbb{V}_+(\mathfrak{c}_2) \setminus \mathbb{V}_+(\mathfrak{a})$, where $\mathfrak{c} = \mathfrak{c}_1 \cap \mathfrak{c}_2 \cap \mathfrak{c}_3$ is the primary decomposition of $\mathfrak{c}$ and $\mathfrak{c}_2 = \mathfrak{b}_2 + (v_2, v_1b_5 - v_3b_3 - v_3b_{10}, v_1b_3 - v_3b_6)$. Hence, any K_H such that $K_H \circ H = I^\perp$ will be of the form $K_H = (L_1, L_2, L_3^2)$, where L_1, L_2, L_3 are independent linear forms in R such that $L_3 = v_1x_1 + v_3x_3$, with $v_1b_5 - v_3b_3 - v_3b_{10} = v_1b_3 - v_3b_6 = 0$.

EXAMPLE 3.5.4 Consider $A = R/I$, with $R = \mathbf{k}[\![x_1, x_2, x_3]\!]$ and $I = (x_1^2, x_2^2, x_3^2, x_1x_2)$. Note that $\mathrm{HF}_A = \{1, 3, 2\}$ and $\tau(A) = 2$. Doing analogous computations to the previous examples, *Singular* provides the following variety:

$$MGC(A) = \mathbb{P}^7 \setminus \mathbb{V}_+(b_2^2 - b_1b_3).$$

The coordinates of points in $MGC(A)$ are of the form $(a_1 : \cdots : a_4 : b_1 : b_2 : b_3 : b_4) \in \mathbb{P}^7$ and they correspond to a polynomial

$$H = b_1y_1^2y_3 + b_2y_1y_2y_3 + b_3y_2^2y_3 + b_4y_3^3 + a_1y_3^2 + a_2y_2^2 + a_3y_1y_2 + a_4y_1^2$$

such that $b_2^2 - b_1b_3 \neq 0$. Any $G = R/\operatorname{Ann}_{\mathrm{R}} H$ is a minimal Gorenstein cover of colength 2 of A and all such covers admit (x_1, x_2, x_3^2) as the corresponding K_H.

EXAMPLE 3.5.5 Consider $A = R/I$, with $R = \mathbf{k}[\![x_1, x_2, x_3, x_4]\!]$ and $I = (x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4)$. Note that $\mathrm{HF}_A = \{1, 4, 1\}$ and $\tau(A) = 3$. Doing analogous computations to the previous examples, *Singular* provides the following variety:

$$MGC(A) = \mathbb{V}_+(b_6b_{10} - b_9b_{16}) \setminus (\mathbb{V}_+(\mathfrak{d}_1) \cup \mathbb{V}_+(\mathfrak{d}_2)) \subset \mathbb{P}^{12},$$

where $\mathfrak{d}_1 = (a_7a_9 - a_8^2)$ and $\mathfrak{d}_2 = (b_9^2b_{16} - b_{10}^3, b_6b_9 - b_{10}^2, b_6^2 - b_{10}b_{16})$. The coordinates of points in $MGC(A)$ are of the form $[H] = (a_1 : \cdots : a_9 : b_6 : b_9 : b_{10} : b_{16}) \in \mathbb{P}^{12}$, where

$$H = b_{16}y_3^3 + b_6y_3^2y_4 + b_{10}y_3y_4^2 + b_9y_4^3 + a_9y_1^2 + a_8y_1y_2 + a_7y_2^2 +$$

$$+a_6y_1y_3 + a_5y_2y_3 + a_4y_3^2 + a_3y_1y_4 + a_2y_2y_4 + a_1y_4^2.$$

Then $G = R/\operatorname{Ann}_R H$ is a minimal Gorenstein cover of colength 2 of A if and only if $[H] \in MGC(A)$. Moreover, any K_H such that $K_H \circ H = I^\perp$ will be of the form $K_H = (L_1, L_2, L_3, L_4^2)$, where L_1, L_2, L_3, L_4 are independent linear forms in R such that $L_4 = v_3x_3 + v_4x_4$, with $v_3b_9 - v_4b_{10} = v_3b_6 - v_4b_{16} = 0$.

As in the case of colength 1, we now provide a table with the computation times of $MGC(A)$ for all analytic types of local $\mathbf{k}$-algebras A of length equal or less than 6 such that $\operatorname{gcl}(A) = 2$.

$\mathrm{HF}_{R/I}$	I	$t(s)$
$1,3,1$	$x_1x_2, x_1x_3, x_1^2, x_2^2, x_3^2$	0,42
$1,2,2,1$	$x_1^2, x_1x_2^2, x_2^4$	0,18
$1,3,1,1$	$x_1x_2, x_1x_3, x_2x_3, x_2^2, x_3^2 - x_1^3$	3,56
$1,3,2$	$x_1x_2, x_2x_3, x_3^2, x_2^2 - x_1x_3, x_1^3$	4,4
	$x_1x_2, x_3^2, x_1x_3 - x_2x_3, x_1^2 + x_2^2 - x_1x_3$	1254,34
	$x_1x_2, x_1x_3, x_2^2, x_3^2, x_1^3$	3,33
	$x_1x_2, x_1x_3, x_2x_3, x_1^2 + x_2^2 - x_3^2$	4,61
	$x_1^2, x_1x_2, x_2x_3, x_1x_3 + x_2^2 - x_3^2$	4,09
	$x_1^2, x_1x_2, x_2^2, x_3^2$	0,45
$1,4,1$	$x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4$	242,28

TABLE 3.2 Computation times of $MGC(A)$ for $A = R/I$ with $\ell(A) \leq 6$ and $\operatorname{gcl}(A) = 2$.

See Appendix C for an explicit description of $MGC(A)$ for all the ideals represented in Table 3.2.

CHAPTER 4

Gorenstein colength in codimension two

In this chapter we focus on codimension two local Artin rings $A = R/I$, hence we can assume that $R = \mathbf{k}[\![x, y]\!]$ and $I \subset (x, y)^2$.

Our goal is again to find minimal Gorenstein covers $G = R/J$ of $A = R/I$ using tools that are only available in codimension two. For instance, by Hilbert-Burch theorem, see [12, Theorem 20.15], any minimal free resolution of R/K is of the form

$$0 \longrightarrow R^{t-1} \xrightarrow{M} R^t \longrightarrow R \longrightarrow R/K \longrightarrow 0$$

and $K = I_t(M)$, where $I_t(M)$ stands for the ideal generated by the maximal minors of the matrix M.

DEFINITION 4.0.1 A monomial ideal $L \in R$ is called a **lex-segment ideal** with respect to x if it is minimally generated by elements $x^t, x^{t-1}y^{m_1}, \dots, y^{m_t}$ for some $t \geq 1$ and a succession of integers $0 = m_0 < m_1 < \dots < m_t$.

By Macaulay's theorem [35], given any homogeneous ideal H in $P = \mathbf{k}[x, y]$ with Hilbert function h, there exists a unique lex-segment ideal $L = \mathrm{Lex}(h)$ with the same Hilbert function. From Proposition 1.5.10, given an ideal K of R with Hilbert function h, it follows that

$$\mathrm{HF}_{R/K} = \mathrm{HF}_{P/K^*} = \mathrm{HF}_{P/L}\,.$$

Given a Hilbert function $h = \{1, 2, \ldots, t, h_{t+1}, \ldots, h_s\}$, the minimal free graded resolution of $P/\operatorname{Lex}(h)$ is

$$0 \longrightarrow \bigoplus_{j\geq 0} P^{e_{t+j}}(-t-j-1) \longrightarrow P^{e_{t+1}}(-t) \bigoplus_{j\geq 1} P^{e_{t+j}}(-t-j) \longrightarrow$$

$$\longrightarrow P \longrightarrow P/L \longrightarrow 0,$$

where $e_j := |h_j - h_{j-1}|$ for every $j > 0$.

Rossi and Sharifan prove in [41] that for each sequence of zero or negative consecutive cancellations on the previous resolution, an ideal K of R with this resulting resolution can be realized. This procedure allows us, whenever the Hilbert function h admits it, to explicitly construct Gorenstein rings $G = R/J$ such that $\operatorname{HF}_G = h$. See Theorem 1.2.11 for a characterization of which Hilbert functions admit Gorenstein rings in codimension 2.

However, to decide whether a Gorenstein cover of A with a given Hilbert function h exists, we need not only some but all Gorenstein rings G with $\operatorname{HF}_G = h$. A natural question arises:

Question A: Can we build all Gorenstein ideals with a given Hilbert function h via deformations of $\operatorname{Lex}(h)$? And, more generally, can we build any ideal with a given Hilbert function in this way?

In [8], Conca and Valla parametrize all ideals K in $P = \mathbf{k}[x, y]$ that share the same leading term ideal with respect to the lexicographical order in terms of a certain canonical Hilbert Burch matrix of K. In [9], Constantinescu provides an analogous parametrization for the degree lexicographical order whenever the leading term ideal is a lex-segment ideal.

The first section of this chapter is devoted to the extension of this result to the local setting for the local order $\overline{\tau}$ induced by the lexicographical order. In Section 4.1.1 we review the parametrizations given by Conca-Valla and Constantinescu. The main result of the chapter is given in Section 4.1.2:

THEOREM 4.0.2 (See Theorem 4.1.24 for more details.) Given a lex-segment ideal L in R with canonical Hilbert-Burch matrix H, the set $V(L) = \{K \subset R : \operatorname{Lt}_{\overline{\tau}}(K) = L\}$ is an affine space parametrized by the bijection

$$\begin{array}{rccl}\Psi: & \mathcal{M}(L) & \longrightarrow & V(L)\\ & N & \longmapsto & I_t(H+N),\end{array}$$

where $\mathcal{M}(L)$ is the set of matrices from Definition 4.1.21.

Note that Ψ associates to each ideal in $V(L)$ a canonical Hilbert-Burch matrix $H+N$. In particular, the coordinates of the affine space $\mathbb{A}^N$ correspond to the coefficients of the polynomials in $\mathbf{k}[y]$ that can occur as entries of the matrix N in $\mathcal{M}(L)$.

More relevantly, observe that Ψ parametrizes any $\mathfrak{m}$-primary ideal K of R with a given Hilbert function h up to a generic change of coordinates, since $\mathrm{Gin}(K) = \mathrm{Lex}(h)$. Observe that the realization given in [41, Remark 4.7] of an ideal with a resolution obtained via zero and negative cancellation of the resolution of $\mathrm{Lex}(h)$ is a particular deformation of Hilbert-Burch matrices of $\mathrm{Lex}(h)$, whereas Ψ gives all possible deformations.

Moreover, we can answer Question A: all Gorenstein ideals J such that $\mathrm{HF}_{R/J} = h$ can be obtained as a deformation of the canonical Hilbert-Burch matrix H of $\mathrm{Lex}(h)$ by adding a suitable matrix $N \in \mathcal{M}$, again up to a generic change of coordinates.

However, when we look for a Gorenstein cover $G = R/J$ of a given ring $A = R/I$ we also ask for J to be contained in I. In general, this inclusion property is not preserved after a generic change of coordinates on the generators of J. Therefore, for the purpose of seeking covers it is not enough to parametrize deformations of the lex-segment ideal $\mathrm{Lex}(h)$. The question we need to ask then is the following:

Question B: Can we build all Gorenstein covers G of A with a given Hilbert function h via similar deformations of all monomial ideals E such that $\mathrm{HF}_{R/E} = h$?

For a general $\mathfrak{m}$-primary monomial ideal E of R, we give the following result on the set $V(E)$:

PROPOSITION 4.0.3 (See Proposition 4.1.9 for more details.) Consider a monomial $\mathfrak{m}$-primary ideal E in R with canonical Hilbert-Burch matrix H, let $V(E)$ be the set of ideals $\{K \subset R : \mathrm{Lt}_{\overline{\tau}}(K) = E\}$ and let $\mathcal{N}(E)$ be the set of matrices defined in Definition 4.1.8. The map

$$\begin{array}{rccl}\varphi: & \mathcal{N}(E) & \longrightarrow & V(E)\\ & N & \longmapsto & I_t(H+N),\end{array}$$

is surjective.

In Section 4.2 we address the problem of obtaining Gorenstein covers of a given ring. Despite the lack of injectivity in Proposition 4.1.9, the map φ provides Hilbert-Burch matrices $N+H$ for all ideals in $V(E)$. This allows us to scan through all the ideals in $V(E)$ in search of Gorenstein ideals J with $\mathrm{Lt}_{\overline{\tau}}(J) = E$. Consider a ring $A = R/I$ and a monomial ideal E such that $\mathrm{HF}_A(i) \leq \mathrm{HF}_{R/E}(i)$ for any $i \geq 0$, we are interested in determining which matrices $N \in \mathcal{N}(E)$ define Gorenstein covers $J = I_t(N+H)$ of A such that $\mathrm{Lt}_{\overline{\tau}}(J) = E$.

First, we give to the subset $V_G(L)$ of Gorenstein ideals in $V(L)$ a structure quasi-affine variety.

COROLLARY 4.0.4 (See Corollary 4.2.9.) Let L be a lex-segment ideal. The set $V_G(L)$ of Gorenstein ideals J such that $\mathrm{Lt}_{\overline{\tau}}(J) = L$ is a quasi-affine variety.

Even more, we completely describe the subset $V_{GC(A)}(L)$ of ideals that correspond to Gorenstein covers of A:

COROLLARY 4.0.5 (See Corollary 4.2.11.) Let $A = R/I$ be an Artin ring. Consider a Hilbert-function h such that $\mathrm{HF}_A(i) \leq h(i)$ for any $i \geq 0$. If $\mathrm{Lex}(h) \subset \mathrm{Lt}_{\overline{\tau}}(I)$, then the set of Gorenstein covers $G = R/J$ of A such that $\mathrm{Lt}_{\overline{\tau}}(J) = \mathrm{Lex}(h)$ is a quasi-affine variety parametrized by points p_J in

$$\mathbb{V}(p_1, \ldots, p_r) \backslash \mathbb{V}(c^0_{3,1} c^0_{4,2} \cdots c^0_{t+1,t-1}),$$

where $c^k_{i,j}$ are the coefficients of the entries of matrices N in $\mathcal{M}(\mathrm{Lex}(h))$ and p_i are polynomials in variables $c^k_{i,j}$ that occur as coefficients of the reduction of J modulo I.

On the other hand, for any monomial ideal E, Algorithm 4 helps us to compute the set $V_G(E)$ of Gorenstein ideals J such that $\mathrm{Lt}_{\overline{\tau}}(J) = E$. We get as output the quasi-affine variety $\mathbb{A}^{\mathbf{N}}_{\mathbf{k}} \backslash \mathbb{V}(\mathfrak{a})$ whose points correspond to Gorenstein ideals J in $V(E)$, even though different points might correspond to the same ideal.

Using computational tools to determine inclusion of ideals, we can ensure that the quasi-affine variety $\mathbb{V}(p_1, \ldots, p_r) \backslash \mathbb{V}(\mathfrak{a})$, where $p_1, \ldots, p_r$ is built as in Corollary 4.2.9, consists of all points that correspond to Gorenstein covers $G = R/J$ of $A = R/I$. Again, this is not a parametrization but it allows us to all determine Gorenstein covers with a given leading term ideal.

Note that given a Hilbert function h, as the length increases, the amount of monomial ideals E such that $\mathrm{HF}_{R/E} = h$ is extremely large. As a closure of the chapter, we pose the problem of determining which particular monomial ideals E we must deform to obtain Gorenstein covers of A with Hilbert function h. This question remains open but we suggest an interesting direction to follow by showing several examples.

The first part of this chapter is a result of a collaboration with Anna-Lena Winz, from Freie Universität Berlin, under the supervision of Maria Evelina Rossi.

4.1 Parametrization of ideals in $\mathbf{k}[[x,y]]$

In this section, our goal is to parametrize $\mathfrak{m}$-primary ideals J of $R = \mathbf{k}[\![x,y]\!]$ with a given leading term ideal E. Therefore, we need to fix a local ordering: consider the local degree ordering $\overline{\tau}$ induced by the lexicographic order $\tau = \mathrm{lex}$ where $x > y$. Note that $\overline{\tau}$ coincides with local ordering induced by deglex. Moreover, in 2 variables, the lexicographical order and reverse-lexicographical also coincide.

DEFINITION 4.1.1 Given an $\mathfrak{m}$-primary monomial E ideal in R, we denote by $V(E)$ the set of ideals $J \subset R$ such that $\mathrm{Lt}_{\overline{\tau}}(J) = E$.

REMARK 4.1.2 Note that if $\mathrm{Lt}_{\overline{\tau}}(J)$ is an $\mathfrak{m}$-primary ideal, then J is also an $\mathfrak{m}$-primary ideal of R. The converse is also true.

Any $\mathfrak{m}$-primary monomial ideal E of R must contain pure powers x^a, y^b for some $a, b \geq 1$. Moreover, E is minimally generated by elements of the form x^a, $x^{a_1}y^{b_1}, \dots,$ $x^{a_r}y^{b_r}$, y^b with $a > a_1 > \cdots > a_r$ and $b_1 < \cdots < b_r < b$ for some $r \geq 0$.

It is always possible to extend this minimal system of generators to a lex-segment-like system of generators $x^t, x^{t-1}y^{m_1}, \dots, x^{t-i}y^{m_i}, \dots, y^{m_t}$ with $0 = m_0 \leq m_1 \leq \cdots \leq m_t$. Indeed, we set $t = a$, $m_t = b$ and, whenever a power $t-i$ of x is missing, we add the monomial $x^{t-i}y^{b_j}$, where $1 \leq j \leq r$ such that $a_{j-1} > t-i > a_j$.

REMARK 4.1.3 Note that E is a lex-segment ideal if and only if $0 < m_1 < \cdots < m_t$.

DEFINITION 4.1.4 We call **canonical Hilbert-Burch matrix** of the monomial ideal $E = (x^t, \dots, x^{t-i}y^{m_i}, \dots, y^{m_t})$ the Hilbert-Burch matrix of E of the form

$$H = \begin{pmatrix} y^{d_1} & 0 & \cdots & 0 \\ -x & y^{d_2} & \cdots & 0 \\ 0 & -x & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & y^{d_t} \\ 0 & 0 & \cdots & -x \end{pmatrix},$$

where $d_i = m_i - m_{i-1}$ for any $1 \leq i \leq t$.

DEFINITION 4.1.5 The **degree matrix** U of E is the $(t+1) \times t$ matrix with integer entries $u_{i,j} = m_j - m_{i-1} + i - j$, for $1 \leq i \leq t+1$ and $1 \leq j \leq t$.

It follows from the definition that $u_{i,i} = d_i$ and $u_{i+1,i} = 1$, for $1 \leq i \leq t$.

4.1.1 Parametrizations of ideals in $\mathbf{k}[x, y]$

Let us now recall the parametrization of ideals in the polynomial ring with the same leading term ideal with respect to the lexicographical order, given by Conca and Valla in [8]. Let $P = \mathbf{k}[x, y]$ be the polynomial ring in two variables and let $\mathfrak{m} = (x, y)$ the maximal ideal generated by the variables. Given a monomial ideal $E = (x^t, \ldots, x^{t-i}y^{m_i}, \ldots, y^{m_t})$ in P, denote by $V_2(E)$ the set of $\mathfrak{m}$-primary ideals $J \subset P$ such that $\mathrm{Lt}_{\mathrm{lex}}(J) = E$. Consider the canonical Hilbert-Burch matrix H of E defined as in Definition 4.1.4. We denote by $T_2(E)$ the set of matrices N of size $(t+1) \times t$ with entries in $\mathbf{k}[y]$ such that

- $n_{i,j} = 0$ for any $i \leq j$,
- $\deg(n_{i,j}) < d_j$ for any $i > j$,
- $\mathrm{ord}(n_{i,j}) \geq 1$ whenever $d_j > 0$ and $j + 1 \leq i \leq k + 1$, where

$$k = \min\{v : j \leq v \leq t,\, m_v = m_j\}.$$

THEOREM 4.1.6 [8, Theorem 3.3, Corollary 3.1] Given a monomial ideal $E = (x^t, \ldots,$

$x^{t-i}y^{m_i},\dots,y^{m_t})$ in $P=\mathbf{k}[x,y]$ with canonical Hilbert-Burch matrix H, the map

$$\begin{array}{rccc}\Phi: & T_2(E) & \longrightarrow & V_2(E)\\ & N & \longmapsto & I_t(N+H)\end{array}$$

is a bijection. In particular, $V_2(E)$ is an affine space of dimension $\dim_{\mathbf{k}}\mathbf{k}[x,y]/E-\min\{j : x^j\in E\}$.

Observe that this theorem allows us to define the canonical Hilbert-Burch matrix of any $\mathfrak{m}$-primary ideal J of P as $H+\Phi^{-1}(J)$, where H is the canonical Hilbert-Burch matrix of the monomial ideal $\mathrm{Lt}_{\mathrm{lex}}(J)$ as defined in Definition 4.1.4.

On the other hand, Constantinescu parametrizes in [9] the variety $V_{\mathrm{deglex}}(E)=\{J\subset P : \mathrm{Lt}_{\mathrm{deglex}}(J)=E\}$, where the leading term ideals are considered with respect to the degree-lexicographical order, for E lex-segment. Let us denote by $\mathcal{A}(E)$ the set of f $(t+1)\times t$ matrices $A=(a_{i,j})_{1\le i\le t+1,\,1\le j\le t}$ with entries in $\mathbf{k}[y]$ such that all its non-zero entries satisfy

$$\deg(a_{i,j})\ge\begin{cases}u_{i,j}+1, & i\le j;\\ u_{i,j}, & i>j.\end{cases}$$

and $u_{i,j}$ are the entries of the degree matrix U of E.

THEOREM 4.1.7 [9, Theorem 3.1] Given a lex-segment ideal L in $P=\mathbf{k}[x,y]$ with canonical Hilbert-Burch matrix H, the map

$$\begin{array}{rccc}\Phi: & \mathcal{A}(L) & \longrightarrow & V_{\mathrm{deglex}}(L)\\ & A & \longmapsto & I_t(A+H)\end{array}$$

is a bijection.

The proof of well-definition and surjectivity of Φ holds for any monomial ideal $E=(x^t,\dots,x^{t-i}y^{m_i},\dots,y^{m_t})$, non necessarily lex-segment. However, the lex-segment hypothesis cannot be dropped because it is needed to prove injectivity.

4.1.2 Canonical Hilbert-Burch matrices of $\mathfrak{m}$-primary ideals in the local ring $\mathbf{k}[[x, y]]$

We want to obtain canonical Hilbert-Burch matrices for $\mathfrak{m}$-primary ideals in $\mathbf{k}[[x, y]]$ in an analogous way to Section 4.1.1. Therefore, we look for a parametrization of the ideals in $V(E)$ as in Theorem 4.1.6 and Theorem 4.1.7. Let us start by defining a sets of matrices whose maximal minors generate all the ideals with the same leading term ideal with respect to the local order $\overline{\tau}$.

DEFINITION 4.1.8 We define the set $\mathcal{N}(E)$ of $(t+1)\times t$ matrices $N = (n_{i,j})$ with entries in $\mathbf{k}[y]$ such that all its non-zero entries satisfy

$$\max\{d_j, 1\} \geq \deg n_{i,j} \geq \operatorname{ord}(n_{i,j}) \geq \begin{cases} u_{i,j} + 1, & i \leq j; \\ u_{i,j}, & i > j. \end{cases}$$

PROPOSITION 4.1.9 Given a monomial ideal $E = (x^t, \dots, x^{t-i}y^{m_i}, \dots, y^{m_t})$ in R with canonical Hilbert-Burch matrix H and degree matrix U, let $V(E)$ be the set of ideals $\{K \subset R : \mathrm{Lt}_{\overline{\tau}}(K) = E\}$ and let $\mathcal{N}(E)$ be the set of matrices defined in Definition 4.1.8. The map

$$\begin{array}{cccc} \varphi: & \mathcal{N}(E) & \longrightarrow & V(E) \\ & N & \longmapsto & I_t(H+N) \end{array}$$

is surjective.

We prove this proposition in two steps: well-definition in Lemma 4.1.10 and surjectivity in Lemma 4.1.12.

LEMMA 4.1.10 The map φ is well-defined.

Proof: Given a monomial ideal $E = (x^t, x^{t-1}y^{m_1}, \dots, y^{m_t})$ with canonical Hilbert-Burch matrix H and associated degree matrix U, we want to prove that the leading term ideal $\mathrm{Lt}_{\overline{\tau}}(I_t(H+N))$ is the mononomial ideal E for any matrix $N \in \mathcal{N}(E)$.

Let us consider the matrix

$$M = H + N = \begin{pmatrix} y^{d_1} + n_{1,1} & n_{1,2} & \cdots & n_{1,t} \\ -x + n_{2,1} & y^{d_2} + n_{2,2} & \cdots & n_{2,t} \\ \vdots & \vdots & & \vdots \\ n_{t,1} & n_{t,2} & \cdots & y^{d_t} + n_{t,t} \\ n_{t+1,1} & n_{t+1,2} & \cdots & -x + n_{t+1,t} \end{pmatrix}.$$

From the order bounds on the polynomials $n_{i,j}$, for $1 \leq i \leq t$ and $1 \leq j \leq t+1$, we have $\operatorname{ord}(m_{i,i}) = u_{i,i}$, $\operatorname{ord}(m_{i+1,i}) = u_{i+1,i}$ and

$$\operatorname{ord}(m_{i,j}) \geq \begin{cases} u_{i,j} + 1, & i < j; \\ u_{i,j}, & i > j - 1. \end{cases}$$

Set $f_i = \det[M]_{i+1}$, for any $0 \leq i \leq t$, where $[M]_{i+1}$ is the square matrix that we get after removing row $i+1$ of M. It has the following shape:

$$\begin{pmatrix} y^{d_1} + n_{1,1} & n_{1,2} & \cdots & n_{1,i} & n_{1,i+1} & \cdots & n_{1,t} \\ -x + n_{2,1} & y^{d_2} + n_{2,2} & \cdots & n_{2,i} & n_{2,i+1} & \cdots & n_{2,t} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ n_{i,1} & n_{i,2} & \cdots & y^{d_i} + n_{i,i} & n_{i,i+1} & \cdots & n_{i,t} \\ n_{i+2,1} & n_{i+2,2} & \cdots & n_{i+2,i} & -x + n_{i+2,i+1} & \cdots & n_{i+2,t} \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ n_{t+1,1} & n_{t+1,2} & \cdots & n_{t+1,i} & n_{t+1,i+1} & \cdots & -x + n_{t+1,t} \end{pmatrix}.$$

Since $f_i = \sum_{\sigma \in S_t} \operatorname{sgn}(\sigma) \prod_{1 \leq k \leq t+1,\, k \neq i+1} m_{k,\sigma(k)}$, we focus on the study of the leading terms of polynomials of the form $h = \prod_{1 \leq k \leq t+1,\, k \neq i+1} m_{k,\sigma(k)}$.

If h is the product of elements of the main diagonal, then $\operatorname{Lt}_{\overline{\tau}}(h) = y^{d_1} \cdots y^{d_i} x^{t-i} = x^{t-i} y^{m_i}$. We claim that any other $h \neq 0$ satisfies $\operatorname{Lt}_{\overline{\tau}}(h) <_{\overline{\tau}} x^{t-i} y^{m_i}$. Indeed, since

$$\operatorname{Lt}_{\overline{\tau}}(h) = \prod_{1 \leq k \leq t+1,\, k \neq i+1} \operatorname{Lt}_{\overline{\tau}}(m_{k,\sigma(k)}),$$

then

$$\operatorname{ord}(h) = \sum_{1 \le k \le t+1,\, k \ne i+1} \operatorname{ord}(m_{k,\sigma(k)}) \ge \sum_{1 \le k \le t+1,\, k \ne i+1} u_{k,\sigma(k)}.$$

To reach the equality $\operatorname{ord}(h) = \sum_{1 \le k \le t+1,\, k \ne i+1} u_{k,\sigma(k)}$ it is necessary that each $m_{i,\sigma(i)}$ is either in the lower triangle of M, its main diagonal or right above its main diagonal from row $i+1$ onwards. However, this forces

$$h = \prod_{k=1}^{i} (y^{d_k} + n_{k,k}) \prod_{k=i+1}^{t+1} m_{k,\sigma(k)},$$

hence the maximal power of x is only reached at the main diagonal. Thus, any $h \ne 0$ different from the main diagonal satisfies $\operatorname{Lt}_{\overline{\tau}}(h) <_{\overline{\tau}} x^{t-i} y^{m_i}$ and, therefore, $\operatorname{Lt}_{\overline{\tau}}(f_i) = x^{t-i} y^{m_i}$.

Now we need to show that $f_0, \dots, f_t$ form a $\overline{\tau}$-enhanced standard basis of $I_t(M)$. By [4, Theorem 1.11] it is enough to show that $\operatorname{ht}((\operatorname{Lt}_{\overline{\tau}}(f_0), \dots, \operatorname{Lt}_{\overline{\tau}}(f_t))) = 2$, which is clear because this ideal contains pure powers x^t and y^{m_t}. Therefore, $\operatorname{Lt}_{\overline{\tau}}(I_t(M)) = E$. □

The proof of surjectivity of the map φ follows the essential ideas of Conca, Valla and Constantinescu. However, we must use other tools, such as Grauert's division or homogenization, to deal with the local order $\overline{\tau}$ in an analogous way as the authors dealt with the lexicographical or degree lexicographical orders in [8] and [9]. Let us pose the following definiton that links the order and the degree of a polynomial and will be essential in the homogenization process:

DEFINITION 4.1.11 We define the **ecart** of $f \in P$ as the difference between the degree and the order of the polynomial f, that is, $\operatorname{ecart}(f) := \deg f - \deg \operatorname{Lt}_{\overline{\tau}}(f)$.

LEMMA 4.1.12 φ is surjective.

Proof: Using the same notation as in the previous proof, we will show that any ideal $J \subset R$ such that $\operatorname{Lt}_{\overline{\tau}}(J) = E$ is of the form $J = I_t(M)$, where $M = H + N$ for some $N \in \mathcal{N}(E)$.

Since $\operatorname{Lt}_{\overline{\tau}}(J) = (x^t, x^{t-1}y^{m_1}, \dots, y^{m_t})$, there exist a $\overline{\tau}$-enhanced standard basis $f_0, \dots, f_t$ of J such that $\operatorname{Lt}_{\overline{\tau}}(f_i) = x^{t-i} y^{m_i}$ for any $0 \le i \le t$. We can assume that all these elements have leading coefficient 1 and, by Proposition 1.5.18, that they are polynomials of degree at most s, where $s = \operatorname{socdeg} R/J + 1$.

By Grauert's division theorem (see Theorem 1.5.12), we can also assume that the monomials in the support of the f_i's are not divisible by x^t, except for $\mathrm{Lt}_{\overline{\tau}}(f_0)$.

For any $1 \leq j \leq t$, consider the S-polynomials $S_j := S(f_{j-1}, f_j) = y^{d_j} f_{j-1} - x f_j$. If $2 \leq j \leq t$, monomials in the support of S_j are not divisible by x^t. In $S_1 = y^{d_1} f_0 - x f_1$, the term $x^t = \mathrm{Lt}_{\overline{\tau}}(f_0)$ only appears multiplied by y^{d_1}. Therefore, no monomial in $\mathrm{Supp}(S_j)$ is divisible by x^{t+1} for any $1 \leq j \leq t$.

We claim that under the previous conditions,

$$S_j = \sum_{i=0}^{t} q_{i,j} f_i,$$

for some $q_{i,j} \in \mathbf{k}[y]$ such that $\mathrm{Lt}_{\overline{\tau}}(q_{i,j} f_i) \leq \mathrm{Lt}_{\overline{\tau}}(S_j)$. In fact, we will prove that this holds for any $f \in J$ such that x^{t+1} does not divide any monomial in $\mathrm{Supp}(f)$. Consider such an f, then $\mathrm{Lt}_{\overline{\tau}}(f) = x^s y^r$ for some $0 \leq s \leq t$. On the other hand, from the fact that $\mathrm{Lt}_{\overline{\tau}}(f)$ belongs to $\mathrm{Lt}_{\overline{\tau}}(J)$, it follows that $x^{t-i} y^{m_i}$ must divide $\mathrm{Lt}_{\overline{\tau}}(f)$ for some $0 \leq i \leq t$. Then $t - i \leq s$ and $m_i \leq r$, hence $m_{t-s} \leq m_i \leq r$.

Now consider the homogenization f^h of f with respect to a new variable z. On the set of monomials in variables z, x, y, we can define the following global order:

$$z^p x^a y^b >_h z^q x^c y^d$$

if either $p + a + b > q + c + d$ or $p + a + b = q + c + d$ and $x^a y^b >_{\overline{\tau}} x^c y^d$.

See [27, Definition 1.2.4] for a definition of global ordering and [27, Algorithm 1.7.6] for more information on this construction. It can be proved that $\mathrm{Lt}_{\mathrm{h}}(f^h) = t^{\mathrm{ecart}(f)} \mathrm{Lt}_{\overline{\tau}}(f)$, see [27, Definition 1.7.5]. Hence $\mathrm{Lt}_{\mathrm{h}}(f^h) = z^\alpha x^s y^r$, where $\alpha = \mathrm{ecart}(f)$. We define a new polynomial in the following way:

$$g^h = f^h - \mathrm{LC}_{\mathrm{h}}(f^h) z^{\alpha - \alpha_{t-s}} y^{r - m_{t-s}} f^h_{t-s}.$$

Note that $\mathrm{Lt}_{\mathrm{h}}(f^h_{t-s}) = z^{\alpha_{t-s}} x^s y^{m_{t-s}}$, where $\alpha_i = \mathrm{ecart}(f_i)$, and hence

$$\mathrm{Lt}_{\mathrm{h}}(z^{\alpha - \alpha_{t-s}} y^{r - m_{t-s}} f^h_{t-s}) = z^{\alpha - \alpha_{t-s}} y^{r - m_{t-s}} z^{\alpha_{t-s}} x^s y^{m_{t-s}} = z^\alpha x^s y^r.$$

Therefore, the leading monomials of f^h and $\mathrm{LC}_\mathrm{h}(f^h)z^{\alpha-\alpha_{t-s}}y^{r-m_{t-s}}f^h_{t-s}$ with respect to the ordering h cancel and hence $\mathrm{Lt}_\mathrm{h}(g^h) <_h \mathrm{Lt}_\mathrm{h}(f^h)$. By construction, the monomials of $g = g^h\mid_{z=1}$ are not divisible by x^{t+1}, so g satisfies the same properties as f and we can apply the same procedure. Since h is a global ordering, after repeating this process finitely many times, it will reduce to zero. This provides an expression

$$f^h = \sum_{i=0}^{t} P_i f_i^h,$$

where $P_i \in \mathbf{k}[z,y]$. By specializing to $z = 1$, we get

$$f = \sum_{i=0}^{t} q_i f_i,$$

where $q_i \in \mathbf{k}[y]$ and $q_i^h = P_i$.

For any $0 \le i \le t$, set $\beta_i = \mathrm{ord}(q_i)$, then we have $\mathrm{Lt}_{\overline{\tau}}(q_i f_i) = \mathrm{Lt}_{\overline{\tau}}(q_i)\,\mathrm{Lt}_{\overline{\tau}}(f_i) = x^{t-i}y^{m_i+\beta_i}$. Therefore, the power of x is different at each $\mathrm{Lt}_{\overline{\tau}}(q_i f_i)$ and hence they cannot cancel each other. This yields

$$\mathrm{Lt}_{\overline{\tau}}(f) = \mathrm{Lt}_{\overline{\tau}}\left(\sum_{i=0}^{t} q_i f_i\right) = \max_{\overline{\tau}}\{\mathrm{Lt}_{\overline{\tau}}(q_i f_i) : 0 \le i \le t\},$$

hence $\mathrm{Lt}_{\overline{\tau}}(f) \ge_{\overline{\tau}} \mathrm{Lt}_{\overline{\tau}}(q_i f_i)$.

Set $n_{i,j} = -q_{i+1,j}$, for any $1 \le i \le t+1$, $1 \le j \le t$. Then

$$y^{d_j} f_{j-1} - x f_j + \sum_{i=1}^{t+1} n_{i,j} f_{i-1} = 0, \text{ for } 1 \le j \le t. \tag{4.1}$$

Note that the expressions in 4.1 are liftings of $y^{d_j}\,\mathrm{Lt}_{\overline{\tau}}(f_{j-1}) - x\,\mathrm{Lt}_{\overline{\tau}}(f_j) = 0$. Writing the later expressions in a matrix shape gives the canonical Hilbert-Burch matrix H associated to the monomial ideal E. The columns $\sigma_1, \ldots, \sigma_t$ of H are a homogeneous system of generators of $\mathrm{Syz}(\mathrm{Lt}_{\overline{\tau}}(J))$.

Since 4.1 can be translated into a matrix $M = H + N$ with column j

$$m_j = \begin{pmatrix} n_{1,j} \\ n_{2,j} \\ \vdots \\ y^{d_j} + n_{j,j} \\ -x + n_{j+1,j} \\ \vdots \\ n_{t+1,j} \end{pmatrix},$$

by Theorem 1.5.19, $m_1, \dots, m_t$ generate the module of syzygies of J. Since we are in codimension 2, by Hilbert-Burch theorem, J is generated by the maximal minors of the matrix M that has as columns the generators of the module of syzygies of J, i.e. $I_t(M) = J$.

The order bounds on the entries of N are obtained from $\mathrm{Lt}_{\overline{\tau}}(n_{i,j}f_{i-1}) \leq_{\overline{\tau}} \mathrm{Lt}_{\overline{\tau}}(S_j)$. Indeed, from $\mathrm{Lt}_{\overline{\tau}}(n_{i,j}f_i) \leq \mathrm{Lt}_{\overline{\tau}}(y^{d_j}f_{j-1} - xf_j)$ it follows that

$$x^{t-i+1}y^{m_{i-1}+\beta_{i,j}} <_{\overline{\tau}} \max_{\overline{\tau}}\{\mathrm{Lt}_{\overline{\tau}}(y^{d_j}f_{j-1}), \mathrm{Lt}_{\overline{\tau}}(xf_j)\} = x^{t-j+1}y^{m_j},$$

where $\mathrm{Lt}_{\overline{\tau}}(n_{i,j}) = y^{\beta_{i,j}}$. By definition,

$$x^{t-i+1}y^{m_{i-1}+\beta_{i,j}} <_{\overline{\tau}} x^{t-j+1}y^{m_j}$$

if and only if $\beta_{i,j} + t - i + 1 + m_{i-1} > t - j + 1 + m_j$ or

$$\beta_{i,j} + t - i + 1 + m_{i-1} = t - j + 1 + m_j \text{ and } t - i + 1 < t - j + 1.$$

Note that the previous inequalities mean that either

$$\beta_{i,j} > i - j + m_j - m_{i-1} = u_{i,j}$$

or

$$\beta_{i,j} = i - j + m_j - m_{i-1} = u_{i,j},$$

which can only occur if $t - i < t - j$, that is, $i > j$.

Now let us prove that we can slightly modify, if needed, polynomials $f_0, \dots, f_t$ in order to make sure that the matrix N generated by the previous construction satisfies the degree bounds on the entries of N and hence $N \in \mathcal{N}(E)$. Define new polynomials

$$f_i' = f_i + x^{t-i}y^{b_i}, \quad t - i + b_i = s + 1.$$

Note that $(f_0', \dots, f_t') = (f_0, \dots, f_t)$, $\mathrm{Lt}_{\overline{\tau}}(f_i') = \mathrm{Lt}_{\overline{\tau}}(f_i)$ and the monomials in $\mathrm{Supp}(f_i')$ are still not divisible by x^t. The only exceptions are x^t and $x^t y^{b_0}$, but they only occur in $\mathrm{Supp}(f_0')$. Therefore, all the previous steps of the proof apply and the matrix N' built from $f_0', \dots, f_t'$ satisfies the order bounds on the entries and provides $J = I_t(N' + H)$. Let us now rename f_i' as f_i.

Consider the degree lexicographical order τ'. By construction, $\mathrm{Lt}_{\tau'}(f_i) = x^{t-i}y^{b_i}$ and the leading terms with respect to τ' do not cancel each other. Hence

$$y^{d_j} f_{j-1} - x f_j + \sum_{i=1}^{t+1} n_{i,j} f_{i-1} = 0$$

yields

$$\mathrm{Lt}_{\tau'}(S_j) = \max_{\tau'}\{\mathrm{Lt}_{\tau'}(n_{i,j} f_{i-1}) : 1 \le i \le t+1\}. \tag{4.2}$$

On the other hand, since $\mathrm{Lt}_{\tau'}(y^{d_j} f_{j-1}) = x^{t-j+1}y^{b_{j-1}+d_j}$ and $\mathrm{Lt}_{\tau'}(x f_j) = x^{t-j+1}y^{b_j}$, then

$$\deg \mathrm{Lt}_{\tau'}(S_j) \le \max\{\deg \mathrm{Lt}_{\tau'}(y^{d_j} f_{j-1}), \deg \mathrm{Lt}_{\tau'}(x f_j)\} = \max\{s + 1 + d_j, s + 2\}. \tag{4.3}$$

Note that $\deg(S_j) = \deg \mathrm{Lt}_{\tau'}(S_j)$, hence from 4.2 and 4.3 we deduce that $\deg n_{i,j} f_{i-1} \le \max\{s + 1 + d_j, s + 2\}$. Since $\deg f_i = s + 1$ by construction, then $\deg n_{i,j} \le \max\{d_j, 1\}$. □

However, as we can see in the following example, φ is not injective even in the lex-segment case:

EXAMPLE 4.1.13 Consider the lex-segment ideal $L = (x^3, x^2y, xy^3, y^5)$. Let H be its canonical Hilbert-Burch matrix and U its degree matrix:

$$H = \begin{pmatrix} y & 0 & 0 \\ -x & y^2 & 0 \\ 0 & -x & y^2 \\ 0 & 0 & -x \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{pmatrix}.$$

Hence

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ y & 0 & 0 \end{pmatrix} \in \mathcal{N}(L)$$

and $I_3(H+N) = L$. Then $\varphi(0) = \varphi(N)$, hence φ is not injective.

REMARK 4.1.14 Observe that if L is a lex-segment ideal, then the sets $\mathcal{N}(L)$ and $T_2(L)$ are very similar.

On one hand, any matrix $N \in \mathcal{N}(L)$ is a lower triangular matrix with main diagonal 0. First of all, we deduce from $0 = m_0 < m_i < \cdots < m_t$ that $d_j \geq 1$ and $\{m_i - i\}_{0 \leq i \leq t}$ is a monotonously increasing sequence. For any $i \leq j$, it holds that

$$u_{i,j} = m_j - m_{i-1} + i - j = d_j + (m_{j-1} - (j-1)) - (m_{i-1} - (i-1)) \geq d_j,$$

hence

$$d_j < u_{i,j} + 1 \leq \operatorname{ord}(n_{i,j}) \leq \deg n_{i,j} \leq d_j.$$

Therefore, $n_{i,j} = 0$ for any $i \leq j$.

On the other hand, any matrix in $T_2(L)$ is always a lower triangular matrix with main diagonal 0. But if L is lex-segment, the condition on the order can be translated to $\operatorname{ord}(n_{i+1,i}) \geq 1$ for any $1 \leq i \leq t$. This is always true for matrices in $\mathcal{N}(L)$.

Therefore, the two sets only differ in two things:

- $\deg n_{i,j} < d_j$ in $T_2(L)$ whereas $\deg n_{i,j} \leq d_j$ in $\mathcal{N}(L)$,
- the entries in $\mathcal{N}(L)$ have lower bounds on $\operatorname{ord} n_{i,j}$ for $i \geq j+2$ whereas in $T_2(L)$ there are no such bounds.

It is reasonable to think that the degree of the entries of matrices in $\mathcal{N}(E)$ can by dropped by one. However, we only have been able to prove it in the lex-segment case so far. The proof uses a very strong fact: for any ideal J whose leading term ideal is a lex-segment ideal L, there exists a τ-enhanced standard basis of J that it is also a Gröbner basis with respect to the lexicographical order. This implies that we can use Conca and Valla's parametrization of $V_2(L)$ in this scenario.

Let us start by determining under which conditions a τ-enhanced standard basis of a lex-segment ideal L is also a lex-Gröbner basis.

LEMMA 4.1.15 Let J be an ideal in R such that $\mathrm{Lt}_{\overline{\tau}}(J) = E$, where E is the monomial ideal $(x^t, x^{t-1}y^{m_1}, \dots, y^{m_t})$. If $f_0, \dots, f_t$ is a $\overline{\tau}$-enhanced standard basis of J such that $\mathrm{Lt}_{\overline{\tau}}(f_i) = \mathrm{Lt}_{\mathrm{lex}}(f_i) = x^{t-i}y^{m_i}$, then $f_0, \dots, f_t$ is a Gröbner basis of $J = (f_0, \dots, f_t)\mathbf{k}[x, y]$ with respect to the lexicographic term order and $\mathrm{Lt}_{\mathrm{lex}}(J) = E$.

Proof: By Lemma 4.1.12, there exist polynomials $n_{i,j} \in \mathbf{k}[y]$ such that

$$y^{d_i} f_{i-1} - x f_i + \sum_{j=0}^{t} n_{i,j} f_j = 0. \tag{4.4}$$

For any $n_{i,j} \neq 0$, if we can prove that $\mathrm{Lt}_{\mathrm{lex}}(n_{i,j} f_j) \leq_{\mathrm{lex}} \mathrm{Lt}_{\mathrm{lex}}(y^{d_i} f_{i-1} - x f_i)$, then it means that the S-polinomials $S_i = y^{d_i} f_{i-1} - x f_i$ reduce to zero. Hence $f_0, \dots, f_t$ is a Gröbner basis of $(f_0, \dots, f_t)\mathbf{k}[x, y]$ with respect to the lexicographical order. Indeed, setting $\mathrm{Lt}_{\mathrm{lex}}(n_{i,j}) = y^{\beta_{i,j}}$, where $\beta_{i,j} = \deg(n_{i,j})$, we have

$$\mathrm{Lt}_{\mathrm{lex}}(n_{i,j} f_j) = \mathrm{Lt}_{\mathrm{lex}}(n_{i,j}) x^{t-j} y^{m_j} = x^{t-j} y^{m_j + \beta_{i,j}}.$$

Note that, by hypothesis, each $\mathrm{Lt}_{\mathrm{lex}}(n_{i,j} f_j)$ has a different power of x, hence they cannot cancel each other:

$$\mathrm{Lt}_{\mathrm{lex}}(y^{d_i} f_{i-1} - x f_i) = \mathrm{Lt}_{\mathrm{lex}}\left(-\sum_{i=0}^{t} n_{i,j} f_j\right) = \max_{0 \leq i \leq t}\{\mathrm{Lt}_{\mathrm{lex}}(n_{i,j} f_j)\},$$

hence $\mathrm{Lt}_{\mathrm{lex}}(n_{i,j} f_j) \leq_{\mathrm{lex}} \mathrm{Lt}_{\mathrm{lex}}(y^{d_i} f_{i-1} - x f_i)$. $\square$

We prove now that all ideals with lex-segment leading term ideal satisfy this property:

LEMMA 4.1.16 Let J be an ideal in R such that $\mathrm{Lt}_{\overline{\tau}}(J) = L$, where L is the lex-segment ideal $(x^t, x^{t-1}y^{m_1}, \dots, y^{m_t})$ and $f_0, \dots, f_t$ is the reduced $\overline{\tau}$-enhanced standard basis of J. Then $\mathrm{Lt}_{\mathrm{lex}}(f_i) = \mathrm{Lt}_{\overline{\tau}}(f_i) = x^{t-i}y^{m_i}$ for any $0 \leq i \leq t$.

Proof: Since L is a lex-segment ideal, $x^t, x^{t-1}y^{m_1}, \ldots, y^{m_t}$ is a minimal system of generators of L. The unique reduced $\overline{\tau}$-enhanced standard basis $f_0, \ldots, f_t$ of J must satisfy $\mathrm{Lt}_{\overline{\tau}}(f_i) = x^{t-i}y^{m_i}$ after reordering the elements appropriately. Let $\mathtt{tail}(f_i)$ be the tail of f_i with respect to the local order, that is, $\mathtt{tail}(f_i) = f_i - x^{t-i}y^{m_i}$. Let us suppose that $\mathrm{Lt}_{\mathrm{lex}}(f_i) = x^k y^l \neq x^{t-i}y^{m_i}$. Since $x^{t-i}y^{m_i} \in \mathrm{Supp}(f_i)$, then

$$x^k y^l >_{\mathrm{lex}} x^{t-i}y^{m_i}$$

and hence there are two possible situations:
Case I: $k = t - i$ and $l > m_i$. $\mathrm{Lt}_{\mathrm{lex}}(f_i) = x^{t-i}y^l$ is in the support of $\mathtt{tail}(f_i)$ but $x^{t-i}y^l$ is in $(x^{t-i}y^{m_i}) = (\mathrm{Lt}_{\overline{\tau}}(f_i)) \subset \mathrm{Lt}_{\overline{\tau}}(J)$ and this contradicts the reducedness hypothesis on $f_0, \ldots, f_t$.
Case II: $k > t - i$. Then we can set $k = t - j$ for some $j < i$. Since $\mathrm{Lt}_{\mathrm{lex}}(f_i) = x^{t-j}x^l$ and $\mathrm{Lt}_{\overline{\tau}}(f_i) = x^{t-i}y^{m_i}$, then

$$t - i + m_i = \deg(x^{t-i}y^{m_i}) \leq \deg(x^{t-j}y^l) = t - j + l.$$

If there is an equality, the local order is equal to the lex order and then $\mathrm{Lt}_{\overline{\tau}}(f_i) = x^{t-j}y^l$, which contradicts $\mathrm{Lt}_{\overline{\tau}}(f_i) = x^{t-i}y^{m_i}$. Therefore, we have

$$t - i + m_i < t - j + l. \tag{4.5}$$

If $l \geq m_j$, the argument of Case I holds. Otherwise, if $l < m_j$, then

$$t - j + l < t - j + m_j = t + (m_j - j).$$

Since L is a lex-segment ideal, $m_i - i$ is monotonously increasing (see Remark 4.1.14) and hence

$$t - j + l < t + m_j - j \leq t + m_i - i < t - j + l,$$

where the last inequality comes from 4.5. □

REMARK 4.1.17 Note that a $\overline{\tau}$-enhanced standard basis $f_0, \ldots, f_t$ of J with leading terms $\mathrm{Lt}_{\overline{\tau}}(f_i) = x^{t-i}y^{m_i}$ can only be reduced if J is a lex-segment ideal. Otherwise it is not reduced because condition (ii) of Definition 1.5.13 always fails.

In general, we lose the property $\mathrm{Lt}_{\mathrm{lex}}(f_i) = \mathrm{Lt}_{\overline{\tau}}(f_i)$ if we remove the assumption of E lex-segment, but it is not an if and only if. It is easy to prove that equality on the leading terms with respect to both local and global orders has other equivalences:

LEMMA 4.1.18 Let $J \subset R$ be an ideal such that $\mathrm{Lt}_{\overline{\tau}}(J) = E$. The following are equivalent:

(i) there exists a lower triangular matrix $N \in \mathcal{N}(E)$ such that $J = I_t(N + H)$,

(ii) there exists a $\overline{\tau}$-enhanced standard basis $f_0, \dots, f_t$ of J such that $\mathrm{Lt}_{\overline{\tau}}(f_i) = \mathrm{Lt}_{\mathrm{lex}}(f_i)$,

(iii) there exists a $\overline{\tau}$-enhanced standard basis $f_0, \dots, f_t$ of J such that x^{t-i} does not divide any monomials in the tail of f_i.

Let us show an example of an ideal J with $\mathrm{Lt}_{\overline{\tau}}(J) = E$ not lex-segment where we can build a $\overline{\tau}$-enhanced standard basis $f_0, \dots, f_t$ of J such that $\mathrm{Lt}_{\mathrm{lex}}(f_i) = \mathrm{Lt}_{\overline{\tau}}(f_i)$.

EXAMPLE 4.1.19 Consider $J = (x^6, xy^2 - y^5, y^8)$, then $E = \mathrm{Lt}_{\overline{\tau}}(J) = x^6$, x^5y^2, x^4y^2, x^3y^2, x^2y^2, xy^2, $y^8)$ and $d_1 = 2, d_2 = d_3 = d_4 = d_5 = 0, d_6 = 6$, hence $\mathrm{Lt}_{\overline{\tau}}(J)$ is not lex-segment. The reduced $\overline{\tau}$-enhanced standard basis x^6, $xy^2 - y^5$, y^8 of J satisfies that its leading terms are the same with respect both local and global order. Then the most natural way to build a $\overline{\tau}$-enhanced standard basis $f_0, \dots, f_6$ with the same property is completing it with $f_{j-1} = xf_j$ whenever $d_j = 0$. Indeed,

$$
\begin{aligned}
f_0 &= x^6 \\
f_1 &= x^5y^2 - x^4y^5 \\
f_2 &= x^4y^2 - x^3y^5 \\
f_3 &= x^3y^2 - x^2y^5 \\
f_4 &= x^2y^2 - xy^5 \\
f_5 &= xy^2 - y^5 \\
f_6 &= y^8
\end{aligned}
$$

is a $\overline{\tau}$-enhanced standard basis of J such that $\mathrm{Lt}_{\mathrm{lex}}(f_i) = \mathrm{Lt}_{\overline{\tau}}(f_i)$.

By Lemma 4.1.15, any ideal satisfying the equivalent conditions of Lemma 4.1.18 can be generated by a Gröbner basis with respect to the lexicographical order and, therefore, it can be obtained via Φ from a matrix in $T_2(E)$, see Theorem 4.1.6.

PROPOSITION 4.1.20 If J is under any of the equivalent conditions of Lemma 4.1.18, then there exists a unique matrix $N \in \mathcal{N}(E) \cap T_2(E)$ such that $J = I_t(N + H)$.

Proof: The syzygies of the leading terms are exactly the same with respect to both τ and $\overline{\tau}$:

$$y_j^d \operatorname{Lt}_{\text{lex}}(f_{j-1}) - x \operatorname{Lt}_{\text{lex}}(f_j) = y_j^d \operatorname{Lt}_{\overline{\tau}}(f_{j-1}) - x \operatorname{Lt}_{\overline{\tau}}(f_j) = 0.$$

They can be lifted to syzygies of the generators of J with respect to both orders. Lifting with respect to $\overline{\tau}$ provides a matrix N such that $J = I_t(N + H)$ and, by Lemma 4.1.12, $N \in \mathcal{N}(E)$. Lifting with respect to τ provides a matrix A such that $J = I_t(A + H)$ and, by [8], $A \in T_2(E)$. Denote by $C_1, \dots, C_t$ the columns of the matrix $N+H$ and $C'_1, \dots, C'_t$ the columns of the matrix $A+H$. These columns provide two systems of generators of the module of syzygies of J, hence $\operatorname{Syz}(J) = \langle C_1, \dots, C_t \rangle = \langle C'_1, \dots, C'_t \rangle$. We know the explicit shape of the columns:

$$C_i = \begin{pmatrix} n_{1,j} \\ n_{2,j} \\ \vdots \\ y^{d_j} + n_{j,j} \\ -x + n_{j+1,j} \\ \vdots \\ n_{t+1,j} \end{pmatrix} \quad \text{and} \quad C'_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ y^{d_j} \\ -x + a_{j+1,j} \\ \vdots \\ a_{t+1,j} \end{pmatrix}.$$

Every C_i must be described by an R-linear combination $c_1^i C'_1 + \cdots + c_t^i C'_t$ with $c_j^i \in R$. Start with $C_1 = c_1^1 C'_1 + \cdots + c_t^1 C'_t$. Then

$$\begin{aligned}
y^{d_1} + n_{1,1} &= c_1^1 y^{d_1}, && \text{hence } c_1^1 = 1, n_{1,1} = 0; \\
-x + n_{2,1} &= -x + a_{2,1} + c_2^1 y^{d_2}, && \text{hence } n_{2,1} = a_{2,1} + c_2^1 y^{d_2}; \\
n_{3,1} &= a_{3,1} + c_2^1(-x + a_{2,1}) + c_3^1 y^{d_3}, && \text{hence } c_2^1 = 0, n_{3,1} = a_{3,1} + c_3^1 y^{d_3}; \\
n_{4,1} &= a_{4,1} + c_3^1(-x + a_{3,1}) + c_4^1 y^{d_4}, && \text{hence } c_3^1 = 0, n_{4,1} = a_{4,1} + c_4^1 y^{d_4}; \\
&\vdots \\
n_{t+1,t} &= a_{t+1,1} + c_t^1(-x + a_{t+1,t}), && \text{hence } c_t^1 = 0.
\end{aligned}$$

To ensure that $c_i^1 = 0$ for $2 \leq i \leq t$, we perform the following reasoning: assume $c_i^1 \neq 0$, then the only way of cancelling x is by having $a_{i,i-1} \in R^*$ and setting c_i^1 to be the inverse element of $-x + a_{1,i-1}$. But we know that $-x + a_{1,i-1}$ is a polynomial, hence c_i^1 must be a series and then $n_{i,1} = a_{i,i-1} + c_i^1 y^{d_i}$ would also be a series, which is a contradiction. Therefore, $c_i^1 = 0$. Repeating the same procedure we obtain that $n_{i,j} = a_{i,j}$ for all $1 \leq i \leq t+1$, $1 \leq j \leq t$. Hence $N = A \in \mathcal{N}(E) \cap T_2(E)$.

Uniqueness follows by the injectivity of Φ given by [8]. □

Therefore, we can extend the definition of canonical Hilbert-Burch matrix from monomial ideals E to any ideal J under the equivalent conditions in Lemma 4.1.18. Let us define the smaller set of matrices that comes out from the proposition:

DEFINITION 4.1.21 We define the set $\mathcal{M}(E) = \mathcal{N}(E) \cap T_2(E)$.

DEFINITION 4.1.22 Given an ideal J that admits a $\overline{\tau}$-enhanced standard basis $f_0, \ldots, f_t$ such that $\mathrm{Lt}_{\overline{\tau}}(f_i) = \mathrm{Lt}_{\mathrm{lex}}(f_i)$ for any $1 \leq i \leq t$, we define the **canonical Hilbert-Burch matrix** of J as the unique matrix N in the set $\mathcal{M}(E)$ such that $J = I_t(N + H)$, where H is the canonical Hilbert-Burch matrix of $E = \mathrm{Lt}_{\overline{\tau}}(J)$ as defined in Definition 4.1.4.

We show an example on how to use Proposition 4.1.20 and the construction of matrices N in $\mathcal{N}(E)$ from Lemma 4.1.12 to obtain the canonical Hilbert-Burch matrix in $\mathcal{M}(E)$.

EXAMPLE 4.1.23 Consider again the ideal $J = (x^6, xy^2 - y^5, y^8)$. From Example 4.1.19 we know that J is under the conditions of Lemma 4.1.18, hence Proposition 4.1.20 applies and there exists a canonical Hilbert-Burch matrix $H + N$ of J with $N \in \mathcal{M}(E)$. Let $S_j = y^{d_j} f_{j-1} - x f_j$ be the S-polynomial $S(f_{j-1}, f_j)$. Recall that, by Lemma 4.1.12, the entries $n_{i,j}$ in $N \in \mathcal{N}(E)$ correspond to $S_j = -\sum_{i=1}^{t+1} n_{i,j} f_{i-1}$. Consider the $\overline{\tau}$-enhanced standard basis from Example 4.1.19, then $S_2 = \cdots = S_5 = 0$. Note that $S_6 = y^{d_6} f_5 - x f_6 = -y^{11} = -y^3 f_6$. Since $\deg y^3 < d_6 = 6$, $n_{7,6} = y^3$ is under the conditions of $N = (n_{i,j}) \in \mathcal{M}(E)$. But $S_1 = y^{d_1} f_0 - x f_1 = x^5 y^5$, which is not possible to describe as a combination of $f_0, \ldots, f_6$ multiplied by polynomials in $\mathbf{k}[y]$ of degree strictly less that $d_1 = 2$. Then we can modify f_0 by adding terms of higher degree that are already in J:

$$\begin{aligned} f_0 &= x^6 - x^5y^3 \\ f_1 &= x^4f_5 = x^5y^2 - x^4y^5 \\ f_2 &= x^3f_5 = x^4y^2 - x^3y^5 \\ f_3 &= x^2f_5 = x^3y^2 - x^2y^5 \\ f_4 &= xf_5 = x^2y^2 - xy^5 \\ f_5 &= xy^2 - y^5 \\ f_6 &= y^8 \end{aligned}$$

Note that now $S_1 = y^{d_1}f_0 - xf_1 = 0$. Therefore,

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y^3 \end{pmatrix} \in \mathcal{M}(E)$$

and $J = I_6(N + H)$.

Despite the existence of canonical Hilbert-Burch matrices beyond the lex-segment case, we still need the assumption of E lex-segment to make sure that any J with leading term ideal $\mathrm{Lt}_{\overline{\tau}}(J) = E$ actually has a canonical Hilbert-Burch matrix. Therefore, we finally state the parametrization of affine spaces $V(L)$:

THEOREM 4.1.24 Let $L = (x^t, \dots, x^{t-i}y^{m_i}, \dots, y^{m_t})$ be a lex-segment ideal with canonical Hilbert-Burch matrix H. Then

$$\begin{aligned} \Psi : \quad \mathcal{M}(L) &\longrightarrow V(L) \\ N &\longmapsto I_t(H + N) \end{aligned}$$

is a bijection.

Proof: The map ψ is the restriction of φ to the set $\mathcal{M}(L) = \mathcal{N}(L) \cap T_2(L)$, hence Lemma 4.1.10 ensures that ψ is well-defined. Moreover, since L is lex-segment, then Lemma 4.1.16 ensures that all J such that $\mathrm{Lt}_{\overline{\tau}}(J) = L$ are under the conditions of Lemma 4.1.18. Hence Proposition 4.1.20 ensures there exists a unique matrix $N \in \mathcal{M}$ such that $J = I_t(N + H)$. □

Note that when L is a lex-segment ideal, then the set $\mathcal{M}(L)$ has a simple description. It is formed by matrices of size $(t+1) \times t$ with entries in $\mathbf{k}[y]$ such that

$$n_{i,j} = \begin{cases} 0, & i \leq j; \\ c_{i,j}^{u_{i,j}} y^{u_{i,j}} + c_{i,j}^{u_{i,j}+1} y^{u_{i,j}+1} + \cdots + c_{i,j}^{d_j-1} y^{d_j-1}, & i > j. \end{cases} \tag{4.6}$$

COROLLARY 4.1.25 Let L be the lex-segment ideal $(x^t, x^{t-1}y^{m_1}, \ldots, y^{m_t})$ with degree matrix $U = (u_{i,j})_{1 \leq i \leq t+1, 1 \leq j \leq t}$ and $d_j = m_j - m_{j-1}$ for any $1 \leq j \leq t$. Then $V(L)$ is an affine space of dimension $\mathbf{N}$, where

$$\mathbf{N} = \sum_{2 \leq j+1 \leq i \leq t+1} (d_j - u_{i,j}) \,.$$

Proof: By Theorem 4.1.24, each ideal J in $V(L)$ is uniquely associated to a matrix N in $\mathcal{M}(L)$ with entries in $\mathbf{k}[y]$. Then we can identify J with a point p_J in the affine space $\mathbb{A}^{\mathbf{N}}_{\mathbf{k}}$, for a suitable $\mathbf{N}$, by taking as coordinates the coefficients $c_{i,j}^k$ of the polynomials $n_{i,j}$ in 4.6 for any $i \geq j+1$:

$$p_J = (c_{2,1}^{u_{2,1}}, c_{2,1}^{u_{2,1}+1}, \ldots, c_{2,1}^{d_1-1}, c_{3,1}^{u_{3,1}}, \ldots, c_{3,1}^{d_1-1}, \ldots, c_{t+1,1}^{u_{t+1,1}}, \ldots, c_{t+1,1}^{d_1-1},$$

$$c_{3,2}^{u_{3,2}}, \ldots, c_{3,2}^{d_2-1}, \ldots, c_{t+1,2}^{u_{t+1,2}}, \ldots, c_{t+1,2}^{d_2-1}, \ldots, c_{t+1,t}^{u_{t+1,t}}, \ldots, c_{t+1,t}^{d_t-1})$$

In particular, the dimension of the affine space is the total number of coefficients $c_{i,j}^k$ for $2 \leq j+1 \leq i \leq t+1$ and $u_{i,j} \leq k \leq d_j - 1$. □

Let us show the details of the parametrization of $V(L)$ as an affine space $\mathbb{A}^{\mathbf{N}}_{\mathbf{k}}$ with an example:

EXAMPLE 4.1.26 Consider the lex-segment ideal $L = (x^3, x^2y, xy^3, y^5)$ from Example 4.1.13. By Theorem 4.1.24, any Hilbert-Burch matrix $M = H + N$, with N in $\mathcal{M}(L)$, associated to an ideal J in $V(L)$ is of the form

$$M = \begin{pmatrix} y & 0 & 0 \\ -x & y^2 & 0 \\ c^0_{3,1} & -x + c^1_{3,2}y & y^2 \\ c^0_{4,1} & c^0_{4,2} + c^1_{4,2}y & -x + c^1_{4,3}y \end{pmatrix}.$$

We identify any ideal $J = I_3(M)$ with the point

$$p_J = (c^0_{3,1}, c^0_{4,1}, c^1_{3,2}, c^0_{4,2}, c^1_{4,2}, c^1_{4,3}) \in \mathbb{A}^6_{\mathbf{k}}.$$

In other words, $V(L)$ can be identified with the affine space $\mathbb{A}^6_{\mathbf{k}}$. Note that the point at the origin in $\mathbb{A}^6_{\mathbf{k}}$ corresponds to the lex-segment ideal L.

4.2 Obtaining Gorenstein covers via Hilbert-Burch matrices

The goal of this section is to obtain all Gorenstein covers $G = R/J$ of $A = R/I$ with a given Hilbert function h such that $\mathrm{HF}_A(i) \leq h(i)$, for $i \geq 0$. Theorem 1.2.11 determines when a Hilbert function h admits Gorenstein rings, that is, there exist any Gorenstein ring $G = R/J$ with $\mathrm{HF}_G = h$.

By Proposition 4.1.9, we know that any $\mathfrak{m}$-primary ideal J of R is generated by the maximal minors of the matrix $H + N$, where H is the canonical Hilbert-Burch matrix of $E = \mathrm{Lt}_{\overline{\tau}}(J)$ and N is a matrix in the set $\mathcal{N}(E)$, see Definition 4.1.8. All ideals J such that $\mathrm{Lt}_{\overline{\tau}}(J) = E$ can be generated with this procedure, although the systems of generators are not unique. Recall that when $\mathrm{Lt}_{\overline{\tau}}(J)$ is the lex-segment ideal $L = \mathrm{Lex}(h)$, then any ideal J is uniquely generated by the maximal minors of $H+N$ with N in $\mathcal{M}(L)$, see Theorem 4.1.24 and Definition 4.1.21.

We now focus on determining which matrices N in $\mathcal{N}(E)$ define Gorenstein covers $J = I_t(N + H)$ of $A = R/I$. This means we impose two unrelated conditions on J:

- $G = R/J$ is Gorenstein.
- $J \subset I$.

REMARK 4.2.1 A brief comment on notation: throughout the section we denote by E the leading term ideal $(x^t, x^{t-1}y^{m_1}, \ldots, y^{m_t})$ of J in the general case and we denote it by L whenever it corresponds to the lex-segment ideal. In any case, H denotes the canonical Hilbert-Burch matrix of $\mathrm{Lt}_{\overline{\tau}}(J)$ and U its degree matrix.

Let us recall the link between the minimal number of generators of the ideal J with a system of generators of its syzygies, see [4, Lemma 2.1]:

PROPOSITION 4.2.2 Let J be an ideal of R and let M in $\mathrm{Mat}_{(t+1)\times t}(R)$ be a matrix whose columns $C_1, \ldots, C_t$ are a system of generators of $\mathrm{Syz}(J)$. Denote by $\overline{M}$ the matrix having as entries the classes in $R/\mathfrak{m}$ of the corresponding entries in M. Then

$$\mu(J) = t + 1 - \mathrm{rk}(\overline{M}).$$

Therefore, we can characterize Gorenstein ideals $J \subset R$ in terms of the rank of any of its Hilbert-Burch matrices $N + H$, where $N \in \mathcal{N}(E)$, as follows:

COROLLARY 4.2.3 Let J be an ideal of R and consider any matrix $M = N + H$ such that $J = I_t(M)$, where $N \in \mathcal{N}(E)$. Then J is Gorenstein if and only if $\mathrm{rk}(\overline{M}) = t - 1$.

Proof: In codimension 2, J is Gorenstein if and only if it is minimally generated by 2 elements. Since M is under the conditions of Proposition 4.2.2, then $\mathrm{rk}(\overline{M}) = t - 1$. □

According to Theorem 4.1.24, whenever J has a lex-segment leading term ideal $\mathrm{Lt}_{\overline{\tau}}(J) = L$, we can chose M to be of the form

$$M = \begin{pmatrix} y^{d_1} & 0 & 0 & \cdots & 0 & 0 \\ -x + n_{2,1} & y^{d_2} & 0 & \cdots & 0 & 0 \\ n_{3,1} & -x + n_{3,2} & y^{d_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ n_{t,1} & n_{t,2} & n_{t,3} & \cdots & n_{t,t-1} & y^{d_t} \\ n_{t+1,1} & n_{t+1,2} & n_{t+1,3} & \cdots & n_{t+1,t-1} & -x + n_{t+1,t} \end{pmatrix},$$

where $u_{i,j} \leq \mathrm{ord}(n_{i,j}) \leq \deg(n_{i,j}) < d_j$ and $u_{i,j}$ is the (i, j)-entry of the degree matrix U of H.

Since $u_{i+1,i} = 1$, considering the entries of M in $R/\mathfrak{m}$ and the notation from 4.6, we get

$$\overline{M} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ c^0_{3,1} & 0 & 0 & \cdots & 0 & 0 \\ c^0_{4,1} & c^0_{4,2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c^0_{t,1} & c^0_{t,2} & c^0_{t,3} & \cdots & 0 & 0 \\ c^0_{t+1,1} & c^0_{t+1,2} & c^0_{t+1,3} & \cdots & c^0_{t+1,t-1} & 0 \end{pmatrix}.$$

Define

$$\overline{M}' = \begin{pmatrix} c^0_{3,1} & 0 & 0 & \cdots & 0 \\ c^0_{4,1} & c^0_{4,2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ c^0_{t,1} & c^0_{t,2} & c^0_{t,3} & \cdots & 0 \\ c^0_{t+1,1} & c^0_{t+1,2} & c^0_{t+1,3} & \cdots & c^0_{t+1,t-1} \end{pmatrix}.$$

Then $\mathrm{rk}(\overline{M}) = t-1$ if and only if $\det(\overline{M}') = c^0_{3,1}c^0_{4,2}\cdots c^0_{t+1,t-1} \neq 0$. Therefore, J is Gorenstein if and only if $n_{i+2,i}$ is a polynomial with non-zero constant term for any $1 \leq i \leq t-1$. In particular, this holds for the lex-segment ideal case, see Lemma 4.1.16:

PROPOSITION 4.2.4 Let L be a lex-segment ideal with canonical Hilbert-Burch matrix H and let J be an ideal with $\mathrm{Lt}_{\overline{\tau}}(J) = L$. Then J is Gorenstein if and only if $n_{3,1}, n_{4,2}, \ldots, n_{t+1,t-1}$ are polynomials in y with non-zero constant terms, where $N = (n_{i,j})$ is the unique matrix in $\mathcal{M}(L)$ such that $J = I_t(H+N)$.

Even more, the entry $n_{i+2,i}$ of N admits a non-zero constant term only if the order of $n_{i+2,i}$ is zero. But this is only possible if $u_{i+2,i} \leq 0$.

COROLLARY 4.2.5 Let L be a lex-segment ideal with associated degree matrix $U = (u_{i,j})_{1\leq i\leq t+1, 1\leq j\leq t}$. A Gorenstein ideal J such that $\mathrm{Lt}_{\overline{\tau}}(J) = L$ exists if and only if $u_{i+2,i} \leq 0$ for any $1 \leq i \leq t-1$.

REMARK 4.2.6 The characterization of Gorenstein-admissible Hilbert functions in Theorem 1.2.11 can be reproved using Corollary 4.2.5. See [41, Corollary 4.6] for more details. Note that a Hilbert function h is Gorenstein-admissible if and only if we can obtain Gorenstein ideals from a deformation of $\mathrm{Lex}(h)$.

Consider a Hilbert function h that admits a Gorenstein ring. In [41, Remark 4.7], Rossi and Sharifan give a procedure to explicitly construct a Gorenstein ring J whose resolution is obtained by consecutive and zero cancellation of the resolution of $L = \mathrm{Lex}(h)$, hence its Hilbert function is preserved. The method consists in taking the ideal of maximal minors of the canonical Hilbert-Burch matrix H of L with 1's added in all entries in position $(i+2, i)$.

Let us show how this procedure allows us to obtain a Gorenstein deformation of the lex-segment ideal:

EXAMPLE 4.2.7 Consider the lex-segment ideal $L = (x^3, x^2y, xy^3, y^5)$ associated to the Hilbert function $h = \{1, 2, 3, 2, 1\}$. Its canonical Hilbert-Burch matrix H is computed in Example 4.1.13. Set

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then $J = I_3(H + N) = (x^3 - 2xy^2, x^2y - y^3, xy^3, y^5)$ is a Gorenstein ideal with Hilbert function $(1, 2, 3, 2, 1)$. Note that $N \in \mathcal{M}(L)$.

Observe that the matrix with 1's in the second main diagonal always belongs to the set of matrices $\mathcal{M}(L)$, see Definition 4.1.21. Using the parametrization in Theorem 4.1.24 and the rank criteria given by Proposition 4.2.4, we can provide the explicit description of all Gorenstein ideals J such that $\mathrm{Lt}_{\overline{\tau}}(J) = L$.

Let us broaden Example 4.2.7 and give a parametrization of all the Gorenstein ideals in $V(L)$:

EXAMPLE 4.2.8 Consider again the lex-segment ideal $L = (x^3, x^2y, xy^3, y^5)$. In Example 4.1.26 we showed the general form of canonical Hilbert-Burch matrices $M = H + N$ associated to ideals $J = I_3(M)$ in $V(L)$. Then the matrix whose entries are the class of entries in M modulo $R/\mathfrak{m}$ is

$$\overline{M} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c^0_{3,1} & 0 & 0 \\ c^0_{4,1} & c^0_{4,2} & 0 \end{pmatrix}.$$

By Proposition 4.2.4, $J = I_3(M)$ is Gorenstein if an only if $c^0_{3,1}c^0_{4,2} \neq 0$. Then the set of Gorenstein ideals J with $\mathrm{Lt}_{\overline{\tau}}(J) = L$ can be identified with $\mathbb{A}^6_{\mathbf{k}} \backslash \mathbb{V}(c^0_{3,1}c^0_{4,2})$.

COROLLARY 4.2.9 Let L be a lex-segment ideal. The set $V_G(L)$ of Gorenstein ideals J such that $\mathrm{Lt}_{\overline{\tau}}(J) = L$ is a quasi-affine variety.

Proof: By Corollary 4.1.25, $V(L)$ can be identified with $\mathbb{A}^{\mathbf{N}}_{\mathbf{k}}$ for a suitable $\mathbf{N}$ by taking coordinates $c^k_{i,j}$. By Proposition 4.2.4, J is a Gorenstein ideal if and only if coordinates $c^0_{3,1}, \dots, c^0_{t+1,t-1}$ of the point p_J in $\mathbb{A}^{\mathbf{N}}_{\mathbf{k}}$ are all non-zero. Hence J is Gorenstein if and only if

$$p_J \in \mathbb{A}^{\mathbb{N}}_{\mathbf{k}} \backslash \mathbb{V}(c^0_{3,1} \cdots c^0_{t+1,t}).$$

□

At this stage, we have only considered the Gorenstein condition. Now we want to introduce the inclusion condition $J = I_t(M) \subset I$ for a given $A = R/I$ in order to determine Gorenstein covers.

Let us show through an example how we can find such Gorenstein covers using the canonical Hilbert-Burch matrices provided by Theorem 4.1.24:

EXAMPLE 4.2.10 Consider the ideal $I = (x^3 - 2xy^2, x^2y - 2y^3, y^3)$.

i	0	1	2	3	4
$\mathrm{HF}_{R/I}$	1	2	3	1	
$\mathrm{HF}_{R/J}$	1	2	3	2	1

By Theorem 1.2.11, $h = \{1, 2, 3, 2, 1\}$ is the smallest Hilbert function such that $\mathrm{HF}_A(i) \leq h(i)$, for any $i \geq 0$, and admits Gorenstein rings. We want to know whether the lex-segment ideal $L = \mathrm{Lex}(h)$ can be deformed into a Gorenstein cover of A. Note that $L = (x^3, x^2y, xy^3, y^5)$ is the ideal from Example 4.2.8, hence we know that

$$V_G(L) \simeq \mathbb{A}^6_{\mathbf{k}} \backslash \mathbb{V}(c^0_{3,1}c^0_{4,2}).$$

For any ideal $J = I_3(M)$ in $V(L)$, the $\overline{\tau}$-enhanced standard basis obtained from its maximal minors is the following:

$$\begin{aligned}
f_0 &= x^3 - c^1_{4,2}xy^3 - c^0_{4,1}y^4 - (c^1_{3,2} + c^1_{4,3})x^2y - (c^0_{3,1} - c^1_{3,2}c^1_{4,3} + c^0_{4,2})xy^2 + c^0_{3,1}c^1_{4,3}y^3 \\
f_1 &= x^2y - c^1_{4,2}y^4 - (c^1_{3,2} + c^1_{4,3})xy^2 - (c^0_{4,2} - c^1_{3,2}c^1_{4,3})y^3 \\
f_2 &= xy^3 - c^1_{4,3}y^4 \\
f_3 &= y^5
\end{aligned}$$

Consider a standard basis S of I, let us compute the normal forms of f_0, f_1, f_2, f_3 of $J = I_3(M)$ with respect to S:

$$\begin{aligned}
\mathrm{NF}(f_0 \mid S) &= (-c^0_{3,1} + c^1_{3,2}c^1_{4,3} - c^0_{4,2} + 2)xy^2 \\
\mathrm{NF}(f_1 \mid S) &= (c^1_{3,1} + c^1_{4,3})xy^2 \\
\mathrm{NF}(f_2 \mid S) &= 0 \\
\mathrm{NF}(f_3 \mid S) &= 0
\end{aligned}$$

Note that the inclusion of J in I depends on the vanishing of the expressions in variables $c^k_{i,j}$ that appear in the computation of the normal forms. Hence any point p_J in the affine variety $\mathbb{V}(-c^0_{3,1} + c^1_{3,2}c^1_{4,3} - c^0_{4,2} + 2, c^1_{3,1} + c^1_{4,3})$ satisfies the inclusion property $J \subset I$.

Therefore, J is a Gorenstein cover of A if and only if

$$p_J \in \mathbb{V}(-c^0_{3,1} + c^1_{3,2}c^1_{4,3} - c^0_{4,2} + 2, c^1_{3,1} + c^1_{4,3}) \backslash \mathbb{V}(c^0_{3,1}c^0_{4,2}).$$

Note that it is not empty since p_J with $c^0_{3,1} = c^0_{4,2} = 1$ and $c^0_{4,1} = c^1_{3,2} = c^1_{4,2} = c^1_{4,3} = 0$, belongs to the quasi-affine variety. The point $(1,0,0,1,0,0)$ in $\mathbb{A}^6_{\mathbf{k}}$ corresponds to the Gorenstein cover $G = R/J$, where $J = (x^2y - y^3, x^3 - 2xy^2)$. In particular, we proved that $\mathrm{gcl}(A) = 2$.

COROLLARY 4.2.11 Let $A = R/I$ be an Artin ring. Consider a Hilbert-function h such that $\mathrm{HF}_A(i) \leq h(i)$ for any $i \geq 0$. If $\mathrm{Lex}(h) \subset \mathrm{Lt}_{\overline{\tau}}(I)$, then the set of Gorenstein covers $G = R/J$ of A such that $\mathrm{Lt}_{\overline{\tau}}(J) = \mathrm{Lex}(h)$ is a quasi-affine variety parametrized by points p_J in

$$\mathbb{V}(p_1, \ldots, p_r) \backslash \mathbb{V}(c^0_{3,1} c^0_{4,2} \cdots c^0_{t+1,t-1}),$$

where $c^k_{i,j}$ are the coefficients of the entries of matrices N in $\mathcal{M}(\mathrm{Lex}(h))$ and p_i are polynomials in variables $c^k_{i,j}$ that occur as coefficients of the reduction of J modulo I.

REMARK 4.2.12 Observe that the condition $\mathrm{Lex}(h) \subset \mathrm{Lt}_{\overline{\tau}}(I)$ is a necessary but not sufficient condition to ensure that $J = I_t(M) \subset I$.

If I is a monomial ideal, then $I = \mathrm{Lt}_{\overline{\tau}}(I)$ and the condition $\mathrm{Lex}(h) \subset \mathrm{Lt}_{\overline{\tau}}(I)$ can be translated to $\mathrm{Lex}(h) \subset I$. In this situation, there always exists a choice of $c^k_{i,j}$ that ensures the inclusion of some $J = I_t(M)$ in I. Indeed, the trivial case of taking zeros corresponds to the lex-segment ideal $L = \mathrm{Lex}(h)$ and $\mathrm{Lex}(h) \subset I$ holds by assumption.

However, if I is not monomial, then we are no longer sure that some choice of $c^k_{i,j}$ ensures that I is contained in some $J = I_t(M)$. In this situation, when we reduce J modulo I, we might obtain coefficients on the polynomials that do not depend on the variables $c^k_{i,j}$. If this is the case, inclusion of I in $I_t(M)$ will never occur. But then some p_i will be a constant polynomial, hence $\mathbb{V}(p_1, \ldots, p_r) = \emptyset$, which is consistent with the idea that no Gorenstein covers exist.

REMARK 4.2.13 We can add to Corollary 4.2.11 the hypothesis that h corresponds to a Hilbert function that admits Gorenstein ideals. However, if h does not admit Gorenstein rings, then

$$\mathbb{V}(c^0_{3,1} c^0_{4,2} \cdots c^0_{t+1,t-1}) = \mathbb{A}^{\mathbf{N}}_{\mathbf{k}}.$$

Now that we know how to obtain all Gorenstein covers $G = R/J$ of A with $\mathrm{Lt}_{\overline{\tau}}(J) = \mathrm{Lex}(h)$, a natural question arises: if a ring $A = R/I$ has Gorenstein covers with Hilbert function h, can we always find at least one such cover $G = R/J$ such that J is a deformation of $L = \mathrm{Lex}(h)$?

The answer is no. In general, there is no reason why the inclusion condition $J \subset I$ should hold when we deform $\mathrm{Lex}(h)$. Let us illustrate it in the following example:

EXAMPLE 4.2.14 Consider the ideal $I = (x^3, xy^2, y^3)$.

i	0	1	2	3	4
$\mathrm{HF}_{R/I}$	1	2	3	1	
$\mathrm{HF}_{R/J}$	1	2	3	2	1

The numerical sequence $h = \{1, 2, 3, 2, 1\}$ corresponds to the minimal Gorenstein-admissible Hilbert function satisfying $\mathrm{HF}_{R/I}(i) \leq h(i)$, for any $i \geq 0$. We want to see if there exist Gorenstein covers R/J of R/I such that $\mathrm{HF}_{R/J} = h$.

Any ideal $J \subset I$ satisfies $\mathrm{Lt}_{\overline{\tau}}(J) \subset \mathrm{Lt}_{\overline{\tau}}(I) = I$. Therefore, we must consider Gorenstein deformations of all monomial ideals E contained in I with Hilbert function of R/E equal to h. For each one, we have two conditions to check:

(1) $\mathrm{rk}(\overline{M}) = t - 1$ for some values of the entries of $\overline{M}$.
(2) $I_t(M) \subset I$.

There are three possibilities:
Case I. $E = (x^3, y^3)$. E is itself a Gorenstein cover of I but we want to find all the deformations of E that still give a cover.
(1) The Cohen-Macaulay type of a deformation of E cannot increase, therefore any $I_3(M)$ is Gorenstein, for $M = H + N$, where $N \in \mathcal{N}(E)$.
(2) The Hilbert-Burch matrix of any ideal with leading term ideal E is of the form

$$M = \begin{pmatrix} y^3 & 0 & 0 \\ -x + c_{2,1}^1 y + c_{2,1}^2 y^2 + c_{2,1}^3 y^3 & 1 + c_{2,2}^1 y & c_{2,3}^0 + c_{2,3}^1 y \\ c_{3,1}^2 y^2 + c_{3,1}^3 y^3 & -x + c_{3,2}^1 y & 1 + c_{3,3}^1 y \\ c_{4,1}^3 y^3 & 0 & -x + c_{4,3}^1 y \end{pmatrix}.$$

It can be checked that $I_3(M) \subset I$ if and only if $c_{2,1}^1 + c_{3,2}^1 + c_{4,3}^1 = 0$.
Case II. $E = (x^3, xy^2, y^5)$.
(1) The Hilbert-Burch matrix of any ideal with leading term ideal E is of the form

$$M = \begin{pmatrix} y^2 & 0 & 0 \\ -x + c_{2,1}^1 y + c_{2,1}^2 y^2 & 1 + c_{2,2}^1 y & c_{2,3}^3 y^3 \\ c_{3,1}^2 y^2 & -x + c_{3,2}^1 y & y^3 \\ c_{4,1}^0 + c_{4,1}^1 y + c_{4,1}^2 y^2 & c_{4,2}^0 + c_{4,2}^1 y & -x + c_{4,3}^1 y + c_{4,3}^2 y^2 + c_{4,3}^3 y^3 \end{pmatrix}.$$

Since

$$\overline{M} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ c^0_{4,1} & c^0_{4,2} & 0 \end{pmatrix},$$

then $J = I_3(M)$ is Gorenstein if and only if $c^0_{4,1} \neq 0$.
(2) $J = I_3(M) \subset I$ if and only if $c^1_{2,1} + c^1_{3,2} + c^1_{4,3} = 0$.
Case III. $E = (x^5, x^3y, xy^2, y^3)$. Since

$$\overline{M} = \begin{pmatrix} 0 & 0 & 0 & c^0_{1,4} & c^0_{1,5} \\ 0 & 1 & 0 & c^0_{2,4} & c^0_{2,5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has rank either 2 or 3, $J = I_5(M)$ will never be Gorenstein.

Observe that the lex-segment ideal $L = (x^3, x^2y, xy^3, y^5)$ is not contained in I and hence none of its deformations will. Therefore, although there exist Gorenstein covers of I with this Hilbert function, it is not possible to obtain them by deforming the lex-segment ideal.

Since not all Gorenstein covers $G = R/J$ of $A = R/I$ with Hilbert function h can be obtained from a deformation of the lex-segment ideal $L = \mathrm{Lex}(h)$, we need to consider Gorenstein deformations J of all monomial ideals E with Hilbert function h such that $\mathrm{Lt}_{\overline{\tau}}(J) = E \subset \mathrm{Lt}_{\overline{\tau}}(I)$.

However, if E is not a lex-segment ideal, we do not have an easy criteria such as Proposition 4.2.4 to determine whether E admits Gorenstein deformations. Hence we must use Corollary 4.2.3 directly. A simple algorithm can be implemented in order to obtain the conditions, if possible, under which any monomial ideal $E = (x^t, x^{t-1}y^{m_1}, \ldots, y^{m_t})$ can be deformed into a Gorenstein ideal. With this purpose, we present here Algorithm 4.

Algorithm 4 Compute Gorenstein ideals J such that $\mathrm{Lt}_{\overline{\tau}}(J) = E$

Input: $(m_1, \dots, m_t)$ integer vector.

Output: matrix M such that $\mathrm{Lt}_{\overline{\tau}}(I_t(M)) = E$, ideal $\mathfrak{a}$.

Steps:

(i) Compute $d_j = m_j - m_{j-1}$, where $1 \le j \le t$ and $m_0 = 0$.

(ii) Build canonical Hilbert-Burch matrix

$$H = \begin{pmatrix} y^{d_1} & 0 & \cdots & 0 \\ -x & y^{d_2} & \cdots & 0 \\ 0 & -x & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & y^{d_t} \\ 0 & 0 & \cdots & -x \end{pmatrix},$$

(iii) Compute degree matrix U of H.

(iv) If $m_1 < m_2 < \cdots < m_t$, then build matrix $N = (n_{i,j})$ with

$$n_{i,j} = \begin{cases} 0, & i \le j; \\ c_{i,j}^{u_{i,j}} y^{u_{i,j}} + \cdots + c_{i,j}^{d_j - 1} y^{d_j - 1}, & i > j. \end{cases}$$

Else

a) Build matrix of order bounds $OB = (a_{i,j})$ and degree bounds $DB = (b_{i,j})$ as

$$a_{i,j} = \begin{cases} \max\{u_{i,j} + 1, 0\}, & i \le j; \\ \max\{u_{i,j}, 0\}, & i > j. \end{cases}$$

and $b_{i,j} = \max\{d_j, 1\}$.

b) Built matrix N, where $N = (n_{i,j})$ has entries in $\mathbf{k}[y]$ such that

$$n_{i,j} = c_{i,j}^{a_{i,j}} y^{a_{i,j}} + \cdots + c_{i,j}^{b_{i,j}} y^{b_{i,j}},$$

where $c_{i,j}^k$ are variables in $\mathbf{k}$.

(v) Define $M = H + N$.

(vi) Compute matrix $\overline{M}$, whose entries are the class of the entries of M in $R/\mathfrak{m}$.

(vii) Compute the ideal $\mathfrak{a}$ of $(t-1)$-minors of $\overline{M}$.

Let us interpret the output of Algorithm 4: matrix M and ideal $\mathfrak{a}$. On one hand, matrix M, whose entries have parameters $c_{i,j}^k$, is the general form of a Hilbert-Burch matrix of any ideal J such that $\mathrm{Lt}_{\overline{\tau}}(J) = E$. Hence J can always be expressed as $I_t(M)$ for some such M. Note that the choice of M is not necessarily unique when E is not lex-segment.

On the other hand, $\mathfrak{a}$ is an ideal in the ring of polynomials with variables $c_{i,j}^k$. If $\mathfrak{a} = 0$, then the rank of $\overline{M}$ is always strictly smaller than $t-1$. Therefore, E is deformable to Gorenstein if and only if $\mathfrak{a} \neq 0$.

REMARK 4.2.15 If the input of Algorithm 4 is a lex-segment ideal L, then the output provides the quasi-affine variety defined in Corollary 4.2.9.

REMARK 4.2.16 Emulating Corollary 4.2.9, consider the quasi-affine variety $\mathbb{A}_{\mathbf{k}}^{\mathbf{N}} \backslash \mathbb{V}(\mathfrak{a})$, where $\mathbf{N}$ is the total number of parameters $c_{i,j}^k$. Any point p_J in this set corresponds to a Gorenstein ideal $J = I_t(M)$ with $\mathrm{Lt}_{\overline{\tau}}(J) = E$. However, $\mathbb{A}_{\mathbf{k}}^{\mathbf{N}} \backslash \mathbb{V}(\mathfrak{a})$ is not isomorphic to $V_G(E)$ because different points might correspond to the same ideal.

Consider an Artin ring $A = R/I$ admitting Gorenstein covers with Hilbert function h. We are interested in studying from which monomial ideals E such that $\mathrm{HF}_{R/E} = h$ we obtain Gorenstein deformations J that satisfy the inclusion $I \subset J$.

Question: Given an Artin ring $A = R/I$, can we find some criteria to know which particular monomial ideals E with Hilbert function h are sufficient to check in order to determine whether a Gorenstein cover G of A with $\mathrm{HF}_G = h$ exists or not?

This question remains open. Our guess so far is that it is enough to check those monomial ideals E with minimal number of generators $\mu(E)$ amongst all Gorenstein-deformable ideals E such that $E \subset \mathrm{Lt}_{\overline{\tau}}(I)$. In rest of the chapter we provide several examples that support our claim.

We start with monomial ideals I, that is, $\mathrm{Lt}_{\overline{\tau}}(I) = I$. The first two examples show different reasons why the minimal cover does not come from a deformation of a monomial ideal E minimally generated by two elements. In Example 4.2.17, there exist no such ideals E with $\mu(E) = 2$. In Example 4.2.18, it exists, but the inclusion condition $E \subset I$ is not satisfied.

EXAMPLE 4.2.17 Consider the ideal $I = (x^5, xy, y^2)$, $\mathrm{HF}_{R/I} = \{1,2,1,1,1\}$. We know that $\mathrm{gcl}(A) = 1$, hence the Hilbert function of any minimal Gorenstein cover is $h = \{1, 2, 1, 1, 1, 1\}$. It can be checked that the minimal monomial Gorenstein cover of I is $J = (x^5, y^2)$, hence it is clearly not a minimal Gorenstein cover.

Let us consider all monomials E with Hilbert function h.

Case $L = \mathrm{Lex}(h) = (x^2, xy, y^6)$. Lex-segment ideal with respect to x. Since $L \not\subseteq I$, it will not provide any cover.

Case $E = (x^6, xy, y^2)$. Note that it is the lex-segment ideal with respect to y. Applying Algorithm 4 with $m = (1, 1, 1, 1, 1, 2)$, we obtain the matrix M defined by

$$\begin{pmatrix} y & c^1_{1,2}y & c^0_{1,3}+c^1_{1,3}y & c^0_{1,4}+c^1_{1,4}y & c^0_{1,5}+c^1_{1,5}y & c^0_{1,6}+c^1_{1,6}y \\ -x+c^1_{2,1}y & 1+c^1_{2,2}y & c^0_{2,3}+c^1_{2,3}y & c^0_{2,4}+c^1_{2,4}y & c^0_{2,5}+c^1_{2,5}y & c^0_{2,6}+c^1_{2,6}y \\ 0 & -x+c^1_{3,2}y & 1+c^1_{3,3}y & c^0_{3,4}+c^1_{3,4}y & c^0_{3,5}+c^1_{3,5}y & c^0_{3,6}+c^1_{3,6}y \\ 0 & 0 & -x+c^1_{4,3}y & 1+c^1_{4,4}y & c^0_{4,5}+c^1_{4,6}y & c^0_{4,6}+c^1_{4,6}y \\ 0 & 0 & 0 & -x+c^1_{5,4}y & 1+c^1_{5,5}y & c^1_{5,6}y \\ 0 & 0 & 0 & 0 & -x+c^1_{6,5}y & y \\ 0 & 0 & 0 & 0 & c^1_{7,5}y & -x+c^1_{7,6}y \end{pmatrix}$$

and $K = (c^0_{1,3}c^0_{3,4}c^0_{4,6} - c^0_{1,3}c^0_{3,6} - c^0_{1,4}c^0_{4,6} + c^0_{1,6})$. Since it can be easily checked that $J = I_6(M) \subset I$ if and only if $c^0_{1,3} = c^0_{1,4} = 0$, then $J = I_6 M$ is a Gorenstein cover of I if and only if $c^0_{1,6} \neq 0$.

There is no other monomial ideal with Hilbert function E. In particular, there exists no monomial ideals E with Hilbert function h such that $\mu(E) = 2$.

EXAMPLE 4.2.18 Consider the ideal $I = (x^3, xy, y^3)$, $\mathrm{HF}_{R/I} = \{1, 2, 2\}$. The Gorenstein colength of $A = R/I$ is 1, hence the Hilbert function of any of its minimal Gorenstein covers is $h = (1, 2, 2, 1)$. Let us consider all monomials E with Hilbert function h. As opposed to the previous example, the generators of I have symmetric roles of the variables involved, hence we consider only one ideal in each case:

Case $L = \mathrm{Lex}(h) = (x^2, xy^2, y^4)$. Since $L \not\subseteq I$, it will not provide any cover.

Case $E = (x^3, xy, y^4)$. Algorithm 4 with $m = (1,1,4)$ provides

$$M = \begin{pmatrix} y & c_{1,2}^1 y & c_{1,3}^1 y \\ -x + c_{2,1}^1 y & 1 + c_{2,2}^1 y & c_{2,3}^1 y \\ 0 & -x + c_{3,2}^1 y & y^3 \\ c_{4,1}^0 + c_{4,1}^1 y & c_{4,2}^0 + c_{4,2}^1 y & -x + c_{4,3}^1 y + c_{4,3}^2 y^2 + c_{4,3}^3 y^3 \end{pmatrix}$$

and $K = (c_{4,1}^0)$. Since $J = I_3(M) \subset I$ holds if and only if $c_{4,3}^1 = 0$, then $J = I_3(M)$ is a Gorenstein cover if and only if $c_{4,1}^0 \neq 0$ and $c_{4,3}^1 = 0$.

Case $E = (x^2, y^3)$. Since $E \not\subseteq I$, we can ensure that no cover comes from a deformation of the monomial ideal with less generators.

In the following example we want to show that having $E \subset I$ and E deformable to Gorenstein is not enough to ensure that a Gorenstein cover J such that $\mathrm{Lt}_{\overline{\tau}}(J) = E$ actually exists:

EXAMPLE 4.2.19 Now consider $I = (x^3, x^2y, xy^2, y^4)$, $\mathrm{HF}_{R/I} = \{1,2,3,1\}$. The Hilbert function $h = \{1,2,3,2,1\}$ is the smallest Gorenstein-admissible Hilbert function such that $\mathrm{HF}_{R/I}(i) \leq h(i)$ for any $i \geq 0$. Recall that in Example 4.2.14 we where also looking for Gorenstein ideals with this Hilbert function. It turns out that there are only 3 monomial ideals with this particular Hilbert function that can be deformed to Gorenstein:

- $E_1 = (x^3, y^3)$, with $\mu(E) = 2$ and $m = (3,3,3)$.
- $E_2 = (x^3, xy^2, y^5)$, with $\mu(E) = 3$ and $m = (2,2,5)$.
- $E_3 = \mathrm{Lex}(h) = (x^3, x^2y, xy^3, y^5)$, with $\mu(E) = 4$ and $m = (1,3,5)$.

To check whether R/I has a Gorenstein cover with Hilbert function h we only have to check the last 2 ideals because $E_1 \not\subseteq I$.

Case $E_2 = (x^3, xy^2, y^5)$. Algorithm 4 with $m = (2,2,5)$ provides the matrix M

$$\begin{pmatrix} y^2 & 0 & 0 \\ -x+c^1_{2,1}y+x^2_{2,1}y^2 & 1+c^1_{2,2}y & c^3_{2,3}y^3 \\ c^2_{3,1}y^2 & -x+c^1_{3,2}y & y^3 \\ c^0_{4,1}+c^1_{4,1}y+c^2_{4,1}y^2 & c^0_{4,2}+c^1_{4,2}y & -x+c^1_{4,3}y+c^2_{4,3}y^2+c^3_{4,3}y^3 \end{pmatrix}$$

and $K = (c^0_{4,1})$. $J = I_3(M) \subset I$ if and only if $c^0_{4,3} = c^0_{4,1} = 0$, hence J cannot be Gorenstein and a cover of I simultaneously.

Case $E_3 = (x^3, x^2y, xy^3, y^5)$. Algorithm 4 with $m = (1, 3, 5)$ provides

$$M = \begin{pmatrix} y & 0 & 0 \\ -x & y^2 & 0 \\ c^0_{3,1} & -x+c^1_{3,2}y & y^2 \\ c^0_{4,1} & c^0_{4,2}+c^1_{4,2}y & -x+c^1_{4,3}y \end{pmatrix}$$

and $K = (c^0_{3,1}c^0_{4,2})$. $J = I_3(M) \subset I$ if and only if $c^0_{4,2} - c^1_{3,2}c^1_{4,3} = c^1_{4,3}c^0_{3,1} = 0$. But

$$\mathbb{V}(c^0_{4,2} - c^1_{3,2}c^1_{4,3}, c^1_{4,3}c^0_{3,1}) \backslash \mathbb{V}(c^0_{3,1}c^0_{4,2}) = \emptyset,$$

hence J cannot satisfy simultaneously the inclusion and the Gorenstein property.

Therefore, $\mathrm{gcl}(R/I) > 2$. If there exists G such that $\ell(G) = 10$, then its Hilbert function must be $\mathrm{HF}_G = \{1, 2, 3, 2, 1, 1\}$. Let us consider all the monomial ideals E that can be deformed to Gorenstein ideals:

(i) $E_1 = (x^3, xy^2, y^6)$, with $\mu(E) = 3$ and $m = (2, 2, 6)$.
(ii) $E_2 = (x^6, x^2y, y^3)$, with $\mu(E) = 3$ and $m = (1, 1, 1, 1, 1, 3)$.
(iii) $E_3 = (x^3, x^2y, xy^3, y^6) = \mathrm{Lex}(\mathrm{HF}_G)$, with $\mu(E) = 4$ and $m = (2, 2, 6)$.
(iv) $E_4 = (x^6, x^2y, xy^2, y^4)$, with $\mu(E) = 4$ and $m = (1, 1, 1, 1, 2, 4)$.

Since $E_2 \not\subseteq I$, we only have to check the other 3. In E_4 the conditions $I_6(M)$ Gorenstein and $I_6(M) \subset I$ are not compatible. However, both E_1 and E_3 provide Gorenstein deformations that are covers. Again, a cover is obtained by deformation of a minimally generated monomial ideal, which is E_1.

Moreover, we just proved that $\mathrm{gcl}(A) = 3$.

Finally, we provide an example of two non-monomial ideals I_1 and I_2 that also supports our claim. This example is particularly interesting because I_1 and I_2 have the same leading term ideal but I_1 has Gorenstein covers with a certain Hilbert function h whereas I_2 has not. Let E be the monomial ideal with Hilbert function h with less minimal generators such that admits Gorenstein deformation. Although $E \subset \mathrm{Lt}_{\overline{\tau}}(I_2)$, none of the Gorenstein deformations are included in I_2. This turns out to be enough to determine that there are no Gorenstein covers G of R/I_2 such that $\mathrm{HF}_G = h$.

EXAMPLE 4.2.20 Let us consider the ideals $I_1 = (x^4 - y^4, x^2y^2, xy^4 + y^5)$ and $I_2 = (xy^4 - y^5, x^2y^2 - 2xy^3, x^4 - 2x^3y)$. Observe that both of them have 3 minimal generators and share the leading term ideal: $\mathrm{Lt}_{\overline{\tau}}(I_1) = \mathrm{Lt}_{\overline{\tau}}(I_2) = (x^4, x^2y^2, xy^4, y^6)$. In particular, they both have Hilbert function $\{1, 2, 3, 4, 3, 1\}$.

Consider the symmetric numerical function $h = \{1, 2, 3, 4, 3, 2, 1\}$. We list all monomial ideals E with $\mathrm{HF}_{R/E} = h$ that can be deformed to a Gorenstein ideal classified by $\mu(E)$:

(i) $\mu(E) = 2$: $m = (4, 4, 4, 4)$.
(ii) $\mu(E) = 3$: $m = (3, 3, 3, 7)$, $m = (2, 2, 6, 6)$ and $m = (1, 5, 5, 5)$.
(iii) $\mu(E) = 4$: $m = (2, 2, 5, 7)$, $m = (1, 4, 4, 7)$ and $m = (1, 3, 6, 6)$.
(iv) $\mu(E) = 5$: $m = (1, 3, 5, 7)$.

Let us check the smallest minimally generated E such that $E \subset \mathrm{Lt}_{\overline{\tau}}(I)$:
$E = (x^4, x^2y^2, y^6)$, with $m = (2, 2, 6, 6)$.
Then the matrix M is

$$\begin{pmatrix} y^2 & 0 & 0 & 0 \\ -x + c_{2,1}^1 y + c_{2,1}^2 & 1 + c_{2,2}^1 y & c_{2,3}^4 y^4 & 0 \\ c_{3,1}^2 y^2 & -x + c_{3,2}^1 y & y^4 & 0 \\ c_{4,1}^0 + c_{4,1}^1 y + c_{4,1}^2 y^2 & c_{4,2}^0 + c_{4,2}^1 y & -x + c_{4,3}^1 y + c_{4,3}^2 y^2 + c_{4,3}^3 y^3 + c_{4,3}^4 y^4 & 1 + c_{4,4}^1 y \\ c_{5,1}^0 + c_{5,1}^1 y + c_{5,1}^2 y^2 & c_{5,2}^0 + c_{5,2}^1 y & c_{5,3}^2 y^2 + c_{5,3}^3 y^3 + c_{5,3}^4 y^4 & -x + c_{5,5}^1 y \end{pmatrix},$$

It can be checked that there exist Gorenstein covers $J_1 = I_4(M)$ of I_1 whereas no Gorenstein ideal generated by maximal minors of M is included in I_2, that is, the inclusion condition $J \subset I_2$ is not compatible with $J = I_4(M)$ being Gorenstein.

Using our method to compute the variety of covers of Gorenstein colength 2, we can make sure that $\mathrm{gcl}(R/I_1) = 2$ and $\mathrm{gcl}(R/I_2) > 2$. Hence this is consistent with the conjecture: if the cover does not appear in the monomial ideal generated by less generators, then there is no Gorenstein cover with this Hilbert function.

Let us now focus on I_2. We first list the possible Hilbert functions of Gorenstein rings G such that $\ell(G) - \ell(R/I_2) \leq 4$:

length	i	0	1	2	3	4	5	6	7	8	9
14	A	1	2	3	4	3	1				
16	G	1	2	3	4	3	2	1			
17		1	2	3	4	3	2	1	1		
18		1	2	3	4	3	2	1	1	1	
		1	2	3	4	3	2	2	1		

For $h = (1, 2, 3, 4, 3, 2, 1, 1)$, we only need to check $m = (2, 2, 5, 8)$ because it is the only monomial ideal E satisfying $E \subseteq \mathrm{Lt}_{\overline{\tau}}(I)$. It turns out that Gorenstein and inclusion are not compatible. Again, for $h = (1, 2, 3, 4, 3, 2, 1, 1, 1)$, we only need to check $m = (2, 2, 5, 9)$ for the same reason and with the same outcome.

For $h = (1, 2, 3, 4, 3, 2, 2, 1)$, there are two monomial ideals E such that $E \subset \mathrm{Lt}_{\overline{\tau}}(I)$:

- $E_1 = (x^4, x^2y^2, xy^6, y^8)$, with $\mu(E) = 4$ and $m = (2, 2, 6, 8)$.
- $E_2 = (x^4, x^2y^2, y^7)$, with $\mu(E) = 3$ and $m = (2, 2, 7, 7)$.

We check first the ideal minimally generated by 3 elements. We are able to make sure the existence of Gorenstein covers and provide a particular solution. It can be checked that the ideal $J = (x^2y^2 - 2xy^3 - xy^4 + y^5, x^4 - 2x^3y - x^3y^2 + x^2y^3 + 2y^5)$ satisfies both $\mathrm{Lt}_{\overline{\tau}}(J) = E_2$ and $J \subset I$, hence it is a Gorenstein cover. Again, this result is consistent with the conjecture. Since we discarded the existence of any other cover with smaller Hilbert function, we just proved that $\mathrm{gcl}(R/I_2) = 4$.

Moreover, E_1 also provides Gorenstein covers. Take as an example $J = (x^2y^2 - 2xy^3, x^4 - 2x^3y - xy^4 + y^5)$.

CHAPTER 5

Gorenstein colength of special families

Along the previous chapters we become aware of the high difficulty to compute how far a given Artin $\mathbf{k}$-algebra $A = R/I$ is from being Gorenstein. In Chapter 2 we give a characterization of rings with $\operatorname{gcl}(A) \leq 2$ in terms of their inverse systems but we already point out in Section 2.3 which are the obstacles to providing analogous characterizations for higher Gorenstein colength. Two main questions arise there:

Question A: Given any Artin ring $A = R/I$, is there a minimal Gorenstein cover $G = R/J$ of A such that $\operatorname{embd}(G) = \operatorname{embd}(A)$?

Question B: Given any Artin ring $A = R/I$, is there a minimal Gorenstein cover $G = R/J$ of A such that $I^2 \subset J \subset I$?

A stronger version of those two questions would be to ask whether this is true for all minimal Gorenstein covers of A.

On the other hand, thanks to the results in Chapter 3, we have algorithms to decide whether A is at distance 0, 1 or 2, and we can compute the explicit expressions of its Gorenstein covers G. In Chapter 4, we obtain more insight in codimension two and even give a constructive approach to study the Gorenstein colength but it gets more and more inefficient as the colength increases because of the amount of Hilbert functions we have to check. So, generally speaking, we do not have many tools available to compute the Gorenstein colength of an arbitrary Artin ring A.

This chapter is devoted to the study of the Gorenstein colength of certain families of rings such as stretched $\mathbf{k}$-algebras or monomial rings. We address the questions posed at the beginning by computing $\operatorname{gcl}(A)$ and studying its minimal Gorenstein covers.

In Section 5.1 we study stretched $\mathbf{k}$-algebras, see Appendix B, and quotients of powers of maximal ideals. In characteristic zero, we get explicit formulas for their Gorenstein colength in terms of invariants of the ring:

PROPOSITION 5.0.1 (See Proposition 5.1.2.) Let $A = R/I$ be a non-Gorenstein Artin stretched ring. Then,

$$\operatorname{gcl}(A) = \operatorname{embd}(A) - \tau(A) + 1.$$

PROPOSITION 5.0.2 (See [2, Theorem 3.1].) Let $A = R/\mathfrak{m}^t$, then

$$\operatorname{gcl}(A) = \binom{n+t-2}{t-2}.$$

In Section 5.2 we do a deep study of all analytic types of $\mathbf{k}$-algebras A with length equal or less than 6, taking Poonen's classification of such rings as a starting point, see [40]. Regarding questions A and B, let us summarize the obtained results:

PROPOSITION 5.0.3 Let $A = R/I$ be an Artin ring. In the following cases we have that there exists a minimal Gorenstein cover $G = R/J$ of A such that $\operatorname{embd}(G) = \operatorname{embd}(A)$ and $I^2 \subset J \subset I$:

(i) $\ell(A) \leq 6$,

(ii) A is stretched,

(iii) $I = \mathfrak{m}^t$ for some $t \geq 1$.

Moreover, for stretched rings all minimal Gorenstein covers preserve the embedding dimension of the base ring.

Finally, Section 5.3 is dedicated to monomial ideals. In particular, we study when minimal Gorenstein covers are monomial rings.

The examples in this chapter illustrate the complexity of the problem and provide ways to construct examples of any given Gorenstein colength. All the computations have been done using the *Singular* libraries **InverseSyst.lib** and **GorensteinCovers.lib**. See Appendix A for a review of the latter.

5.1 Some general families

In this section, we will assume that $\mathbf{k}$ is a field of characteristic zero.

5.1.1 Stretched $\mathbf{k}$-algebras

We now compute the Gorenstein colength of stretched $\mathbf{k}$-algebras. Recall that an Artin ring $A = R/I$ is stretched if $\mathrm{HF}_A(2) = 1$. Stretched rings were defined and classified by Sally, see [42], [21]. See Appendix B for definition and structure theorems regarding stretched rings.

PROPOSITION 5.1.1 Let $A = R/I$ be an Artin stretched $\mathbf{k}$-algebra with $I \subseteq \mathfrak{m}^2$. Let $\tau = \tau(A)$ be the Cohen Macaulay type of A and $s \geq 2$ its socle degree. Then,

$$I^\perp = \langle y_2, \ldots, y_\tau, y_1^s + y_{\tau+1}^2 + \cdots + y_n^2 \rangle.$$

Proof: By [21, Theorem 3.1],

$$I = \begin{cases} \left(\{x_i x_j\}_{1 \leq i < j \leq n}, \{x_j^2\}_{2 \leq j \leq \tau}, \{x_i^2 - x_1^s\}_{\tau+1 \leq i \leq n}\right), & \text{if } \tau(A) < n; \\ \left(\{x_1 x_j\}_{2 \leq j \leq n}, \{x_i x_j\}_{2 \leq i \leq j \leq n}, x_1^{s+1}\right), & \text{if } \tau(A) = n. \end{cases}$$

Since $I \subseteq \mathfrak{m}^2$, then $S_{\leq 1} \subseteq I^\perp$. To prove the inclusion

$$\langle y_2, \ldots, y_\tau, y_1^s + y_{\tau+1}^2 + \cdots + y_n^2 \rangle \subseteq I^\perp,$$

it is enough to check that $I \circ \left(y_1^s + y_{\tau+1}^2 + \cdots + y_n^2\right) = 0$. Equality follows from the fact that $\dim_{\mathbf{k}} I^\perp = \ell(A) = n + s$. $\square$

PROPOSITION 5.1.2 Let $A = R/I$ be a non-Gorenstein Artin stretched ring. Let $\tau(A)$ be the Cohen Macaulay type of A and s its socle degree with $I \subset \mathfrak{m}^2$. Then,

$$\mathrm{gcl}(A) = \mathrm{embd}(A) - \tau(A) + 1.$$

If $n = \operatorname{embd}(A)$, then $G = R/\operatorname{Ann}_R F$, where

$$F = y_1^{s+1} + y_1 y_{\tau+1}^2 + \cdots + y_1 y_n^2 + y_2^2 + \cdots + y_\tau^2,$$

is a minimal Gorenstein cover of A. Moreover,

$$K_F = \left(I^\perp :_R F\right) = \left(x_1, \ldots, x_\tau, x_{\tau+1}^2, \ldots, x_n^2, \{x_i x_j\}_{\tau+1 \leq i < j \leq n}\right).$$

In particular, G is a minimal Gorenstein cover of A with Hilbert function $\mathrm{HF}_G = \{1, n, n - \tau(A) + 1, 1, \ldots, 1\}$ and $\operatorname{socdeg}(G) = s + 1$.

Proof: Set $\dim(R) = \operatorname{embd}(A) = n$ and $\tau = \tau(A)$. Consider the sub-R-module $J^\perp = \langle F \rangle$, where F is the polynomial of the statement, and the ideal $K = (x_1, \ldots, x_\tau, x_{\tau+1}^2, \ldots, x_n^2, \{x_i x_j\}_{\tau+1 \leq i < j \leq n})$ of R. Then

$$K \circ J^\perp = \langle y_2, \ldots, y_\tau, y_1^s + y_{\tau+1}^2 + \cdots + y_n^2 \rangle.$$

By Proposition 5.1.1, $I^\perp = K \circ J^\perp$, hence $G = R/J$ is a Gorenstein cover of the Artin stretched ring $A = R/I$. As $\mathbf{k}$-vector space, the inverse system of J can be written as

$$J^\perp = \langle 1, y_1, \ldots, y_n, y_1^2, y_1 y_{\tau+1}, \ldots, y_1 y_n, y_1^3, \ldots, y_1^s, F \rangle_{\mathbf{k}}.$$

Therefore, $\ell(G) = 2n - \tau + s + 1 = \ell(A) + n - \tau + 1$ and $\operatorname{gcl}(A) \leq n - \tau + 1$. From Proposition 2.3.2 we know that $\operatorname{gcl}(A) \geq \operatorname{embd}(A) - \tau(A) + 1$, hence we get the equality $\operatorname{gcl}(A) = n - \tau + 1$. Then $G = R/\operatorname{Ann}_R F$ is a minimal Gorenstein with Hilbert function $\{1, n, n - \tau(A) + 1, 1, \ldots, 1\}$.

Moreover, $K \subseteq K_F = \left(I^\perp :_R J^\perp\right)$ and equality holds if $\ell(R/K) = \ell(R/K_F)$. Indeed,

$$\ell(R/K) = \dim_{\mathbf{k}} \frac{\langle 1, x_1, \ldots, x_n \rangle_{\mathbf{k}} + \mathfrak{m}^2}{\langle x_1, \ldots, x_\tau \rangle_{\mathbf{k}} + \mathfrak{m}^2} = \dim_{\mathbf{k}} \langle \overline{1}, \overline{x}_{\tau+1}, \ldots, \overline{x}_n \rangle_{\mathbf{k}} = n - \tau + 1$$

coincides with $\ell(R/K_F) = n - \tau + 1$ by 2.1.6. $\square$

COROLLARY 5.1.3 If A is an Artin stretched ring, any minimal Gorenstein cover G of A satisfies $\operatorname{embd}(G) = \operatorname{embd}(A)$.

Proof: By Proposition 2.3.2, $n = \mathrm{embd}(A) \leq \mathrm{embd}(G) \leq \mathrm{gcl}(A) + \tau(A) - 1 = n$. □

REMARK 5.1.4 Observe that from Proposition 5.1.2 we can deduce that the embedding dimension of any minimal cover G is the same as the embedding dimension of A. This is not enough to claim that any G has Hilbert function $\{1, n, n - \tau + 1, 1, \dots, 1\}$. However, the examples we have studied suggest unicity. See the study of unicity of $I = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_2^2, x_3^2 - x_1^2, x_4^2)$, with $\mathrm{HF}_{R/I} = \{1, 4, 1\}$, in Table 5.2, Case 22 of $\ell(A) = 6$.

COROLLARY 5.1.5 If $A = R/I$ is an Artin stretched ring, then there exists a minimal Gorenstein cover $G = R/J$ of A such that $I^2 \subset J \subset I$.

Proof: By Proposition 5.1.2, $K_F = \left(x_1, \dots, x_\tau, x_{\tau+1}^2, \dots, x_n^2, \{x_ix_j\}_{\tau+1 \leq i < j \leq n}\right)$. It can be easily checked that $I \subset K_F$. By Lemma 2.3.4, $I^2 \subset J \subset I$. □

5.1.2 Powers of the maximal ideal

In [44, Corollary 2.2], Teter shows that $A = R/\mathfrak{m}^2$ is a Teter ring. Later on, in [20, Proposition 3.6], Elias and Silva prove that any quotient of a power of the maximal ideal $A = R/\mathfrak{m}^t$ is Teter if and only if either $t \leq 2$ or its embedding dimension is 1. Combining this result with [20, Proposition 3.7], we get the following characterization:

PROPOSITION 5.1.6 Consider the ring $A = R/\mathfrak{m}^t$, where $t \geq 2$ and n is the embedding dimension of A. Then

(i) $\mathrm{gcl}(A) = 0$ if and only if $n = 1$.
(ii) $\mathrm{gcl}(A) = 1$ if and only if $n \geq 2$ and $t = 2$.

In [2, Theorem 3.1], Ananthnarayan provides an explicit formula to compute the Gorenstein colength of such rings $A = R/\mathfrak{m}^t$ for arbitrary values of n and t. In particular, he proves that $\mathrm{gcl}(A) = \ell(R/\mathfrak{m}^{t-1})$. From this we deduce the following result:

PROPOSITION 5.1.7 Let $A = R/\mathfrak{m}^t$, for some $t \geq 2$. Then $G = R/\operatorname{Ann}_{\mathrm{R}} F$ where $F = (y_1 + \cdots + y_n)^{2t-2}$, is a minimal Gorenstein cover of A and

$$\operatorname{gcl}(A) = \begin{pmatrix} n+t-2 \\ t-2 \end{pmatrix}.$$

Proof: The Gorenstein colength follows from [2, Theorem 3.8]. Hence we only need to prove that $G = R/J$ has the right length and $I^\perp \subset J^\perp = \langle F \rangle$, where $I = \mathfrak{m}^t$ and $F = (y_1 + \cdots + y_n)^{2t-2}$.

G has a symmetric Hilbert function with respect to piece of degree $t-1$:

$$\operatorname{HF}_G(i) = \begin{cases} \begin{pmatrix} n+i-1 \\ i \end{pmatrix}, & 0 \leq i \leq t-1; \\ \begin{pmatrix} n+k-1 \\ k \end{pmatrix}, & t \leq i = 2t-k-2 \leq 2t-2, \quad 0 \leq k \leq t-1; \\ 0, & i \geq 2t-1. \end{cases}$$

Adding up the dimension of each piece, we check that $\ell(G) = \ell(A) + \operatorname{gcl}(A)$. For any $0 \leq i \leq t-1$, each piece $J_i^\perp$ has maximal dimension $\binom{n+i-1}{i}$, hence $S_{\leq t-1} \subset J^\perp$. Since $(\mathfrak{m}^t)^\perp = S_{\leq t-1}$, we just proved that G is a minimal Gorenstein cover of A. □

COROLLARY 5.1.8 Let $A = R/\mathfrak{m}^t$, for some $t \geq 2$. Then there exists a minimal Gorenstein cover $G = R/J$ such that $I^2 \subset J \subset I$, where $I = \mathfrak{m}^t$.

Proof: $I^\perp = (\mathfrak{m}^t)^\perp = S_{\leq t-1} \subset J^\perp = \langle F \rangle \subset S_{\leq 2t-1} = (\mathfrak{m}^{2t})^\perp = (I^2)^\perp$. □

REMARK 5.1.9 Note that we proved that there always exists at least one minimal Gorenstein cover with the same embedding dimension as the base ring, but we could not prove that all of them must preserve the embedding dimension. Observe that the upper bound on the embedding dimension provided by Proposition 2.3.2 gives

$$\operatorname{embd}(G) \leq \binom{n+t-1}{t-1} - 1 \tag{5.1}$$

on Gorenstein covers G of rings $A = R/\mathfrak{m}^t$, which is bigger than $\operatorname{embd}(A)$ in general.

However, we did not find any example where $\operatorname{embd}(G) > \operatorname{embd}(A)$. On the contrary, we do have examples where the previous bound on the embedding dimension of G is not reached. For $A = \mathbf{k}[\![x_1, x_2]\!]/\mathfrak{m}^3$, the inequality 5.1 gives $\operatorname{embd}(G) \leq 5$, whereas we prove in Section 5.2 that $\operatorname{embd}(G) = 2$ for any minimal Gorenstein cover. See Case 9, $\ell(A) = 6$ of Table 5.2 for more details.

5.2 k-algebras of rank equal or less than 6

In [40], Poonen provides a complete list of all the analytic types of Artin local algebras over an algebraically closed field $\mathbf{k}$ of length less or equal than 6. His classification holds for any characteristic of the ground field. However, for $\operatorname{char}(\mathbf{k}) = 2$ or 3, some extra analytic types must be added.

The goal of this section is to compute the Gorenstein colength and describe minimal Gorenstein covers of all finitely many analytic types of Artin local $\mathbf{k}$-algebras $A = R/I$ with $\ell(A) \leq 6$. As Poonen already recalls in his paper, Suprunenko proved in [43] that the number of isomorphism classes is infinite when $\ell(A) \geq 7$. Therefore, it is reasonable to consider rings up to length 6. In Table 5.2 we present a complete list of all the analytic types of such rings together with several invariants of both the base ring $A = R/I$ and its minimal covers $G = R/\operatorname{Ann}_{\mathrm{R}} F$.

Let us start with posing some natural questions for $\ell(A) \leq 6$:

Question 1: How can we effectively compute $I^\perp$?

The first issue we need to address is the computation of the inverse system of the ideal $I \subset R$ for any characteristic of the field $\mathbf{k}$. Using *Singular* we can compute $I^\perp$ in both zero and positive characteristic. See Appendix A for information on the different methods available and how to use them. However, we are interested in an expression of the inverse system which is valid for any characteristic of the residue field. Therefore, we will perform the computations in characteristic zero and generalize the results to arbitrary characteristic afterwards.

Observe in [40] that given an ideal $I = (f_1, \dots, f_m)$ from Poonen's list, its inverse system $I_0^\perp$ in characteristic 0 is minimally generated by polynomials $F_1, \dots, F_r$ such that all its coefficients are 1. On one hand, for any $F \in I_0^\perp$, since

$$f_i \circ F = 0, \quad 1 \leq i \leq m,$$

is true in characteristic zero, then it holds in arbitrary characteristic. Hence $\langle F_1, \dots, F_r \rangle$ is contained in $I_p^\perp$. On the other hand, the sub-module $\langle F_1, \dots, F_r \rangle$ of $\mathbf{k}[y_1, \dots, y_n]$, where $\operatorname{char}(\mathbf{k}) = p$, is again minimally generated by $F_1, \dots, F_m$. Therefore, since $\ell(A) = \ell(I_0^\perp) = \ell(I_p^\perp)$, then $I_p^\perp = \langle F_1, \dots, F_r \rangle$.

Question 2: How can we effectively compute $\operatorname{gcl}(A)$?

We will now use alternative arguments to avoid, at this stage, the computation of the variety of minimal Gorenstein covers and stick to the study of the inverse system. Only by looking at the minimal number of generators of $I^\perp$ as R-module, we obtain a lot of information on the Gorenstein colength. Let n be the embedding dimension of A and recall that $\mu(I^\perp) = \tau(A)$, see Proposition 1.4.19. The relationship between the embedding dimension and the Cohen-Macaulay type in rings of low Gorenstein colength comes from the characterizations of such rings. In the Teter case, Theorem 2.0.4 by Elias and Silva determines that $\tau(A) = n$. We obtain the same Cohen-Macaulay type for rings $A = R/I$ of Gorenstein colength 2 in Theorem 2.2.5 whenever $I \subset \mathfrak{m}^5$. But notice that none of rings in Table 5.2 are under the conditions of the theorem. As it was already noted in Remark 2.2.6, in this case $\tau(A)$ is either $n-1$ or n. Summarizing:

- If $\mu(I^\perp) = 1$, then $\operatorname{gcl}(A) = 0$.
- If $\mu(I^\perp) \neq 1, n$, then $\operatorname{gcl}(A) > 1$.
- If $\mu(I^\perp) \neq 1, n-1, n$, then $\operatorname{gcl}(A) > 2$.

Therefore, if we find a Gorenstein cover $G = R/\operatorname{Ann}_{\mathrm{R}} F$ reaching the lowest possible colength according to the previous outline, we are done. This is also the case for rings A such that $\operatorname{gcl}(A) > 2$:

<u>Case 9 of $\ell(A) = 6$.</u> $A = R/I$, with $I^\perp = \langle y_1^2, y_1y_2, y_2^2 \rangle$ and $n = 2$, satisfies $\mu(I^\perp) \neq 1, 2$. Then $\operatorname{gcl}(A) > 2$. $F = y_1^2y_2^2$ gives a Gorenstein cover $G = R/\operatorname{Ann}_{\mathrm{R}} F$ of A such that $\ell(G) - \ell(A) = 3$, hence $\operatorname{gcl}(A) = 3$.

<u>Case 22 of $\ell(A) = 6$.</u> $A = R/I$, with $I^\perp = \langle y_2, y_1^2 + y_3^2 + y_4^2 \rangle$ and $n = 2$, satisfies

$\mu(I^\perp) \neq 1, 3, 4$. Then $\operatorname{gcl}(A) > 2$. $F = y_1^3 + y_1 y_3^2 + y_1 y_4^2 + y_2^2$ gives a Gorenstein cover $G = R/\operatorname{Ann_R} F$ of A such that $\ell(G) - \ell(A) = 3$, hence $\operatorname{gcl}(A) = 3$.

REMARK 5.2.1 Observe that Case 9 of $\ell(A) = 6$ is a quotient by a power of the maximal ideal, hence $\operatorname{gcl}(A) = \ell\left(A/\mathfrak{m}^2\right) = 3$ by Proposition 5.1.7. On the other hand, Case 22 of $\ell(A) = 6$ is stretched, hence $\operatorname{gcl}(A) = n - \tau(A) + 1 = 3$ by Proposition 5.1.2. The advantage of the argument about the Cohen-Macaulay type is that it works regardless of the characteristic of the field whereas the other one depends on the structure theorem of stretched $\mathbf{k}$-algebras and Ananthnarayan's work on quotients by powers of maximal ideals, see Section 5.1.

However, if we are not able to find F, it requires a deeper study in order to prove that no such polynomial exists. Only one analytic type in Poonen's classification deserves this special treatment.

Case 7 of $\ell(A) = 6$. Consider $A = \mathbf{k}[\![x_1, x_2]\!]/I$, $I^\perp = \langle y_1 y_2, y_2^3 \rangle$. By [20, Proposition 4.5], A is Teter if and only if exists a non-singular matrix $C = (c_{ij})_{1 \le i,j \le 2}$, with c_{ij} in $\mathbf{k}[y_1, y_2]$ and $\deg c_{ij} \le 3$ such that

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \circ \begin{pmatrix} y_1 y_2 \\ y_2^3 \end{pmatrix}$$

satisfies $x_2 \circ H_1 = x_1 \circ H_2$.

The Schwartz condition implies that $c_{11} \circ y_1 + c_{12} \circ y_2^2 = c_{21} \circ y_2$. For any $1 \le i, j \le 2$, consider $c_{ij} = c_{ij}^0 + c_{ij}^1 + c_{ij}^2 + c_{ij}^3$. Let us pay attention to what occurs in each degree of this equality:

Degree 2: $c_{12}^0 \circ y_2^2 = 0$, hence $c_{12}^0 = 0$.

Degree 1: $c_{11}^0 \circ y_1 + c_{12}^1 \circ y_2^2 = c_{21}^0 \circ y_2$, hence $c_{11}^0 = 0$.

Then $\begin{vmatrix} c_{11}^0 & c_{12}^0 \\ c_{21}^0 & c_{22}^0 \end{vmatrix} = 0$, hence C is singular and $\operatorname{gcl}(A) > 1$. The polynomial $F = y_1 y_2^3$ gives a Gorenstein cover $G = R/\operatorname{Ann_R} F$ of A such that $\ell(G) - \ell(A) = 2$, hence $\operatorname{gcl}(A) = 2$.

REMARK 5.2.2 Note that this procedure is a primitive version of Algorithm 2 to compute Teter varieties, that is, the minimal Gorenstein cover variety for rings of Gorenstein colength 1.

Question 3: Which are the possible Hilbert functions of a minimal Gorenstein cover G of A?

Hilbert functions of Teter covers are unique regardless of the characteristic of the field $\mathbf{k}$. From Theorem 2.0.4 we can deduce that, given a Teter ring A of socle degree s, any minimal Gorenstein cover G of A satisfies

$$\mathrm{HF}_G(i) = \begin{cases} \mathrm{HF}_A(i), & \text{if } i \leq s; \\ 1, & \text{if } i = s+1; \\ 0, & \text{otherwise.} \end{cases}$$

See [20] for more details.

In Gorenstein colength 2, according to Theorem 2.2.5, the socle degree of a minimal Gorenstein cover G of A could be either $s+1$ or $s+2$. Hence, a priori, we cannot ensure unicity of the Hilbert function. In fact, as shown in Example 2.1.9, in Case 7 of $\ell(A) = 6$ there are minimal covers with two different associated Hilbert functions.

From now on, we will assume that $\mathrm{char}(\mathbf{k}) = 0$ in order to use structure theorems of stretched and almost stretched $\mathbf{k}$-algebras, see Appendix B.

In characteristic zero, it can be proved that Hilbert functions of minimal Gorenstein covers of A such that $\mathrm{gcl}(A) = 2$ and $\ell(A) \leq 6$ are unique except for the so called Case 7 of $\ell(A) = 6$. One approach to prove this uniqueness is to study the degree of the polynomials associated to the $MGC(A)$ variety, since it provides a bound on the socle degree of G. See Section C.2 for more details. Another strategy is to study whether Gorenstein rings with appropriate Hilbert functions are indeed covers of A, as done in Example 2.1.10.

In Gorenstein colength 3, determining which are the possible Hilbert functions of minimal covers becomes specially relevant since, in particular, it addresses the problem of embedding dimension in higher colength posed in Section 2.3.

For rings A in Table 5.2 such that $\mathrm{gcl}(A) = 3$ (cases 9 and 22 of $\ell(A) = 6$), we study all the Gorenstein rings G such that $\ell(G) - \ell(A) = 3$ with $\mathrm{HF}_G(i) \geq \mathrm{HF}_A(i)$ for any $i \geq 0$.

Case 9 of $\ell(A) = 6$. $A = R/\mathfrak{m}^3$ is a quotient by a power of the maximal ideal. By Remark 5.1.9 we have the upper bound $\binom{n+t-1}{t-1} - 1$ for the embedding dimension of any minimal Gorenstein cover G of A. Since $t = 3$ and $n = 2$, $\mathrm{embd}(G) \leq 5$.

<table>
<tr><th>i</th><th>0</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th></th></tr>
<tr><td>A</td><td>1</td><td>2</td><td>3</td><td></td><td></td><td></td><td></td></tr>
<tr><td rowspan="5">G</td><td rowspan="5">1</td><td rowspan="2">2</td><td rowspan="2">3</td><td>1</td><td>1</td><td>1</td><td>(1)</td></tr>
<tr><td>2</td><td>1</td><td></td><td>(2)</td></tr>
<tr><td rowspan="2">3</td><td>3</td><td>1</td><td>1</td><td></td><td>(3)</td></tr>
<tr><td>4</td><td>1</td><td></td><td></td><td>(4)</td></tr>
<tr><td>4</td><td>3</td><td>1</td><td></td><td></td><td>(5)</td></tr>
</table>

In the table above we list all possible Hilbert functions of rings $G = R/J$ with multiplicity 9 ending in 1 such that $\mathrm{HF}_G(i) \geq \mathrm{HF}_A(i)$, $i \geq 0$.

First we recall that $F = y_1^2 y_2^2$ generates a minimal Gorenstein cover $G = R/\operatorname{Ann_R} F$ with symmetric Hilbert function $\{1, 2, 3, 2, 1\}$.

Next we list those Hilbert functions that do not correspond to Gorenstein rings:

(1) $\{1, 2, 3, 1, 1, 1\}$, since in codimension 2 Gorenstein rings only correspond to Hilbert functions with jumps of at most 1, see Theorem 1.2.11.

(4) $\{1, 3, 4, 1\}$, since there is no Q-decomposition, see Example 1.2.10.

Finally, Hilbert functions (3) and (5) do admit Gorenstein rings but we want to prove that they can never be covers of A. To do so, we first give a lemma that will help dealing with covers of higher embedding dimension:

LEMMA 5.2.3 Let $A = R/I$ be a ring with embedding dimension n and a Gorenstein cover $G = R'/J$ of A of embedding dimension $n+k$, where $R' = \mathbf{k}[\![x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k}]\!]$. Then $I^\perp \subset J^\perp$ in $S' = \mathbf{k}[y_1, \dots, y_n, y_{n+1}, \dots, y_{n+k}]$.

Proof: Consider a system of generators $f_1, \dots, f_m$ of the ideal I in R and define I' as the ideal generated by $f_1, \dots, f_m, x_{n+1}, \dots, x_{n+k}$ in R'. Since $A = R/I \cong R'/I'$, then $(I')^\perp \subset J^\perp$ by definition of Gorenstein cover.

Next we will prove that $(I')^\perp = I^\perp$. Note that any $F \in (I')^\perp \subset S'$, satisfies $x_{n+i} \circ F = 0$ for any $1 \leq i \leq k$. Then F must be a polynomial in variables $x_1, \dots, x_n$. But F also satisfies $f_i \circ F = 0$ for any $1 \leq i \leq m$, hence $F \in I^\perp$. Therefore $(I')^\perp \subseteq I^\perp$ and equality follows from $\ell(R/I) = \ell(R'/I') = \ell(A)$. $\square$

Let us see why Gorenstein rings G with Hilbert functions (3) and (5) cannot be covers of A:

(3) Consider a Gorenstein cover G of A with $\mathrm{HF}_G = \{1,3,3,1,1\}$. Let F, with $\deg F = 4$, be a generator of $J^\perp$. Since $\mathrm{gcl}(A) = 3$, then there are two possible analytic types for K_F, hence either

$$I^\perp = \langle l_1 \circ F, l_2 \circ F, l_3^3 \circ F \rangle$$

or

$$I^\perp = \langle l_1 \circ F, l_2^2 \circ F, l_2 l_3 \circ F, l_3^2 \circ F \rangle.$$

Elements in $I^\perp$ have, at most, degree 2. To prove that such G can never be a cover of A it is enough to show that, if elements in $K_F \circ F$ have degree at most 2, then $\mathrm{HF}_A(2) \leq 2$. Indeed, this contradicts $\mathrm{HF}_A(2) = 3$.

Case $K_F = (l_1, l_2, l_3^3)$: Since $\deg F = 4$, then $l_3^3 \circ F \leq 1$ and hence $\mathrm{HF}_A(2) \leq 2$.

Case $K_F = (l_1) + (l_2, l_3)^2$: Since $\mathrm{HF}_G(3) = 1$, there is essentially only one polynomial in degree 3. Then either $l_2 \circ F = \lambda(l_3 \circ F)$, for some $\lambda \neq 0$, or we can assume that $\deg l_2 \circ F \leq 2$.
In the first scenario, both $l_2^2 \circ F$ and $l_3^2 \circ F$ are multiples of $l_2 l_3 \circ F$, hence $\mathrm{HF}_A(2) \leq 2$. In the second case, we get that the degrees of both $l_2^2 \circ F$ and $l_2 l_3 \circ F$ are strictly less than two, hence again $\mathrm{HF}_A(2) \leq 2$.

(5) Consider a Gorenstein cover G of A with $\mathrm{HF}_G = \{1,4,3,1\}$. Since $\dim_{\mathbf{k}}(I^\perp)_2 = 3$, from Lemma 5.2.3 it follows that $J^\perp$ must contain three algebraically independent polynomials F_1, F_2, F_3 of degree 2 in variables l_1, l_2, where l_1, l_2 are linear forms in $\mathbf{k}[y_1, y_2, y_3, y_4]$. Using same notation as in [6], we give in Table 5.1 a representative of a generator of $J^\perp$ for every analytic type of a Gorenstein ring G such that $\mathrm{HF}_G = \{1,4,3,1\}$.
Now consider the ring homomorphism

$$\begin{array}{rccc} \phi: & \mathbf{k}[a,b,c] & \longrightarrow & \mathbf{k}[x,y,z] \\ & a & \longmapsto & F_1 \\ & b & \longmapsto & F_2 \\ & c & \longmapsto & F_3 \end{array}$$

TABLE 5.1 Generators of the inverse system of analytic types of Gorenstein rings with Hilbert function $1, 4, 3, 1$.

	F	$x_1 \circ F$	$x_2 \circ F$	$x_3 \circ F$	$x_4 \circ F$
$J_{1,\alpha} = \left(A_{4,3}^{1,\alpha^2}\right)^{\perp}$	$y_4^2 + y_1^3 - y_1 y_2^2 + \alpha^2 y_2^3 + y_2 y_3^2$	$y_1^2 - y_2^2$	$-y_1 y_2 + \alpha^2 y_2^2 + y_3^2$	$y_2 y_3$	y_4
$J_2 = \left(A_{4,3}^{2,0}\right)^{\perp}$	$y_4^2 + y_1 y_2 y_3 - 2y_3^3$	$y_2 y_3$	$y_1 y_3$	$-2y_3^2 + y_1 y_2$	y_4
$J_3 = \left(A_{4,3}^{3,0}\right)^{\perp}$	$y_4^2 + y_1 y_2 y_3$	$y_2 y_3$	$y_1 y_3$	$y_1 y_2$	y_4
$J_4 = \left(A_{4,3}^{4,0}\right)^{\perp}$	$y_4^2 + y_1^3 + y_2^3 + y_3^3$	y_1^2	y_2^2	y_3^2	y_4
$J_5 = \left(A_{4,3}^{5,0}\right)^{\perp}$	$y_4^2 + y_2^3 + y_1 y_3^2 + y_3^3$	y_3^2	y_2^2	$y_1 y_3 + y_3^2$	y_4
$J_6 = \left(A_{4,3}^{6,0}\right)^{\perp}$	$2y_2^2 y_3 - y_4^2 - y_1 y_3^2$	$-y_3^2$	$2y_2 y_3$	$2y_2^2 - y_1 y_3$	$-y_4$

with $\operatorname{im}\phi = \mathbf{k}[F_1, F_2, F_3]$. Hence

$$\dim \mathbf{k}[F_1, F_2, F_3] = \dim \mathbf{k}[a, b, c]/\ker\phi.$$

If $\ker\phi = 0$, there is no linear R-isomorphism such that $\mathbf{k}[F_1, F_2, F_3] \cong \mathbf{k}[l_1, l_2]$, that is, no suitable change of variables. We checked with *Singular* that this is always the case for $F_i = x_i \circ F$, $i = 1, 2, 3$, for any analytic type from Table 5.1. Therefore, $I^\perp \not\subseteq J^\perp$ and hence G is not a cover of a ring with Hilbert function $\{1, 2, 3\}$.

Summing up, any minimal Gorenstein cover G of A has Hilbert function $\{1, 2, 3, 2, 1\}$. In particular, $\operatorname{embd}(G) = \operatorname{embd}(A) = 2$ and the upper bound provided by Proposition 2.3.2 is clearly not reached.

$l(A) = 6$, case 22: Since A is a stretched $\mathbf{k}$-algebra, by Proposition 5.1.2 $\operatorname{embd}(A) = \operatorname{embd}(G)$ and exists a cover G with Hilbert function $\{1, 4, 3, 1\}$. Let us list all possible Hilbert functions of minimal Gorenstein covers of A:

i	0	1	2	3	4	5
A	1	4	1			
G	1	4	1	1	1	1
			2	1	1	
			3	1		

- A Gorenstein ring $G = R/J$ with $\mathrm{HF}_G = \{1, 4, 1, 1, 1, 1\}$ is a stretched $\mathbf{k}$-algebra with $s = 5$, $n = 4$ and $\tau = 1$. By Proposition 5.1.1, $F = y_1^5 + y_2^2 + y_3^2 + y_4^2$ is a representative of the generator of $J^\perp$ of the unique analytic type of such G. If $K^\perp$ is a subset of $J^\perp$ such that $\mathrm{HF}_{R/K} = \{1, 4, 1\}$, then

$$K^\perp = \langle 1, y_1, y_2, y_3, y_4, y_1^2\rangle \subset \langle 1, y_1, y_2, y_3, y_4, y_1^2, y_1^3, y_1^4, F\rangle = J^\perp.$$

  Hence G can only be a cover of rings of Cohen-Macaulay type 4 but $\tau(A) = 2$.
- A Gorenstein ring $G = R/J$ with $\mathrm{HF}_G = \{1, 4, 2, 1, 1\}$ is an almost stretched $\mathbf{k}$-algebra with $s = 4$, $t = 2$ and $n = 4$. By Theorem B.3.2, there are only two

analytic types for J: either $J \cong I_{0,1}$ or $J \cong I_\infty$, see Definition B.3.1. Consider $K^\perp \subset J^\perp$ such that $\mathrm{HF}_{R/K} = \{1,4,1\}$,

(i) if $K^\perp \subset I_{0,1}^\perp = \langle y_3^2 + y_4^2 + y_1 y_2^2 + y_2^3 + y_1^4 \rangle$, then $\tau(R/K) = 3, 4$;
(ii) if $K^\perp \subset I_\infty^\perp = \langle y_3^2 + y_4^2 + y_1 y_2^2 + y_1^4 \rangle$, again $\tau(R/K) = 3, 4$.

Therefore, any minimal Gorenstein cover G of A has Hilbert function $\{1,4,3,1\}$.

5.2.1 Poonen's classification

We will now provide a set of tables listing all the analytic types of $A = R/I$ together with several details of both the base ring and its minimal covers $G = R/\operatorname{Ann}_{\mathrm{R}} F$:

- Hilbert function HF_A,
- Cohen-Macaulay type $\tau(A)$,
- the representative I of the analytic type provided by Poonen's list,
- the inverse system $I^\perp$,
- a polynomial F such that $G = R/\operatorname{Ann}_{\mathrm{R}} F$ is a minimal Gorenstein cover of A,
- all possible Hilbert functions HF_G of a minimal Gorenstein cover of A when $\operatorname{char}(\mathbf{k}) = 0$,
- the Gorenstein colength of A.

REMARK 5.2.4 In Table 5.2, cases 2.2 in $\ell(A) = 4$, 7.2 in $\ell(A)$ and 5.2, 10.2, 14.2, 15.2, 18.2, 21.2, 23.2 in $\ell(A) = 6$ have been computed in characteristic 2. Case 5.3 has been computed in characteristic 3. The remaining cases have been computed in characteristic zero but are still valid in arbitrary characteristic.

REMARK 5.2.5 For any A such that $\ell(A) \leq 6$, the property $I^2 \subset J \subset I$ holds for minimal Gorenstein covers $G = R/J$ of A listed in Table 5.2. Hence Proposition 2.3.5.(i) is proved. Moreover, in characteristic zero, we can check that it is true for all minimal Gorenstein cover G as long as the Hilbert function of G corresponds to rings whose analytic types have been widely studied. This is the case for stretched (see [21] or Theorem B.2.1), almost stretched (see [15] or Theorem B.3.2), $\mathrm{HF}_G = \{1,3,3,1\}$ (see [18, Proposition 3.7]) and $\mathrm{HF}_G = \{1,4,3,1\}$ (see [6]).

REMARK 5.2.6 Note that the Gorenstein colength never exceeds $\ell(A)/2$.

TABLE 5.2 Gorenstein colength and minimal Gorenstein covers of rings $A = R/I$ such that $\ell(A) \leq 6$.

$\ell(A)$	Case	HF_A	$\tau(A)$	I	$I^\perp$	$J^\perp$	HF_G	$\mathrm{gcl}(A)$
2	1	1,1	1	x_1^2	y_1		1,1	0
3	1	1,1,1	1	x_1^3	y_1^2		1,1,1	0
	2	1,2	2	$(x_1, x_2)^2$	y_1, y_2	$y_1^2 + y_2^2$	1,2,1	1
4	1	1,1,1,1	1	x_1^4	y_1^3		1,1,1,1	0
	2	1,2,1	1	x_1^2, x_2^2	$y_1 y_2$		1,2,1	0
	2.2		1	$x_1 x_2, x_1^2 + x_2^2$	$y_1^2 + y_2^2$		1,2,1	0
	3		2	$x_1 x_2, x_2^2, x_1^3$	y_1^2, y_2	$y_1^3 + y_2^2$	1,2,1,1	1
	4	1,3	3	$(x_1, x_2, x_3)^2$	y_1, y_2, y_3	$y_1^2 + y_2^2 + y_3^2$	1,3,1	1
5	1	1,1,1,1,1	1	x_1^5	y_1^4		1,1,1,1,1	0
	2	1,2,1,1	1	$x_1^2 + x_2^3, x_1 x_2$	$y_1^2 + y_2^3$		1,2,1,1	0
	3		2	$x_1^2, x_1 x_2, x_2^4$	y_1, y_2^3	$y_1^2 + y_2^4$	1,2,1,1,1	1
	4	1,2,2	2	$x_1 x_2, x_1^3, x_2^3$	y_1^2, y_2^2	$y_1^3 + y_2^3$	1,2,2,1	1
	5		2	$x_1^2, x_1 x_2^2, x_2^3$	$y_1 y_2, y_2^2$	$y_1 y_2^2 + y_2^3$	1,2,2,1	1
	6	1,3,1	1	$x_1 x_2, x_1 x_3, x_2 x_3,$ $x_2^2 - x_1^2, x_3^2 - x_1^2$	$y_1^2 + y_2^2 + y_3^2$		1,3,1	0
	7		2	$x_1 x_2, x_1 x_3, x_1^2, x_2^2, x_3^2$	$y_1, y_2 y_3$	$y_1^2 + y_2 y_3^2$	1,3,2,1	2
	7.2		2	$x_1 x_2, x_1 x_3, x_2 x_3, x_2^2,$ $x_1^2 + x_3^2$	$y_2, y_1^2 + y_3^2$	$y_1^3 + y_1 y_3^2 + y_2^2$	1,3,2,1	2
	8		3	$x_1 x_2, x_1 x_3, x_2 x_3,$ x_2^2, x_3^2, x_1^3	y_2, y_3, y_1^2	$y_1^3 + y_2^2 + y_3^2$	1,3,1,1	1
	9	1,4	4	$(x_1, x_2, x_3, x_4)^2$	y_1, y_2, y_3, y_4	$y_1^2 + y_2^2 + y_3^2 + y_4^2$	1,4,1	1

$\ell(A)$	Case	HF_A	$\tau(A)$	I	$I^{\perp}$	$J^{\perp}$	HF_G	$\mathrm{gcl}(A)$
6	1	1,1,1,1,1,1	1	x_1^6	y_1^5		1,1,1,1,1,1	0
	2	1,2,1,1,1	1	$x_1x_2, x_2^2 - x_1^4$	$y_1^4 + y_2^2$		1,2,1,1,1	0
	3		2	x_1x_2, x_2^2, x_1^5	y_2, y_1^4	$y_1^5 + y_2^2$	1,2,1,1,1,1	1
	4	1,2,2,1	1	$x_1x_2, x_1^3 + x_2^3$	$y_1^3 + y_2^3$		1,2,2,1	0
	5		1	x_1^2, x_2^3	$y_1y_2^2$		1,2,2,1	0
	5.2		1	$x_1^2 + x_1x_2^2, x_2^3$	$y_1^2 + y_1y_2^2$		1,2,2,1	0
	5.3		1	$x_1^2, x_1x_2^2 + x_2^3$	$y_1y_2^2 - y_3$		1,2,2,1	0
	6		2	x_1x_2, x_1^3, x_2^4	y_1^2, y_2^3	$y_1^3 + y_2^4$	1,2,2,1,1	1
	7		2	$x_1^2, x_1x_2^2, x_2^4$	y_1y_2, y_2^3	$y_1y_2^3$	1,2,2,2,1	2
						$y_1^2y_2 + y_2^5$	1,2,2,1,1,1	
	8		2	$x_1^2 + x_2^3, x_1x_2^2, x_2^4$	$y_1^2 - y_2^3, y_1y_2$	$y_1^2y_2 - y_2^4$	1,2,2,1,1	1
	9	1,2,3	3	$x_1^3, x_2^3, x_1^2x_2, x_1x_2^2$	y_1^2, y_1y_2, y_2^2	$y_1^2y_2^2$	1,2,3,2,1	3
	10	1,3,1,1	1	$x_1x_2, x_1x_3, x_2x_3+x_1^3,$ x_2^2, x_3^2, x_1^4	$y_2y_3 + y_1^3$		1,3,1,1	0
	10.2		1	$x_1x_2, x_1x_3, x_2x_3,$ $x_2^2 + x_1^3, x_3^2 + x_1^3, x_1^4$	$y_1^3 + y_2^2 + y_3^2$		1,3,1,1	0
	11		2	$x_1x_2, x_1x_3, x_2x_3,$ $x_2^2, x_3^2 - x_1^3$	$y_2, y_1^3 + y_3^2$	$y_1^4 + y_1y_3^2 + y_2^2$	1,3,2,1,1	2
	12		3	$x_1x_2, x_1x_3, x_2x_3,$ x_2^2, x_3^2, x_1^4	y_2, y_3, y_1^3	$y_1^4 + y_2^2 + y_3^2$	1,3,1,1,1	1
	13	1,3,2	2	x_1x_2, x_2x_3, x_3^2 $x_2^2 - x_1x_3, x_1^3$	$y_1^2, y_2^2 + y_1y_3$	$y_2^3 + y_1^2y_3$	1,3,3,1	2
	14		2	$x_1x_2, x_3^2, x_1x_3 - x_2x_3,$ $x_1^2 + x_2^2 - x_1x_3$	$y_1^2 - y_2^2, y_2^2 + y_1y_3 + y_2y_3$	$y_1^2y_3 - y_2^2y_3 - y_2^3$	1,3,3,1	2

$\ell(A)$	Case	HF_A	$\tau(A)$	I	$I^\perp$	$J^\perp$	HF_G	$\mathrm{gcl}(A)$
6	14.2	1,3,2	2	$x_1^2, x_3^2, x_2^2 - x_1x_3, x_2x_3$	$y_1y_2, y_2^2 + y_1y_3$	$y_1y_2^2 + y_1^2y_3$	1,3,3,1	2
	15		2	$x_1x_2, x_1x_3, x_2^2, x_3^2, x_1^3$	y_1^2, y_2y_3	$y_1^3 + y_2^2y_3$	1,3,3,1	2
	15.2		2	$x_1x_2, x_1x_3, x_2x_3, x_2^2 + x_3^2, x_1^3$	$y_1^2, y_2^2 + y_3^2$	$y_1^3 + y_2^3 + y_2y_3^2$	1,3,3,1	2
	16		2	$x_1x_2, x_1x_3, x_2x_3, x_1^2 + x_2^2 - x_3^2$	$y_1^2 - y_2^2, y_2^2 + y_3^2$	$y_1^3 + y_2^3 + y_3^3$	1,3,3,1	2
	17		2	$x_1^2, x_1x_2, x_2x_3, x_1x_3 + x_2^2 - x_3^2$	$y_2^2 - y_1y_3, y_1y_3 + y_3^2$	$y_1y_3^2 + y_2^3 + y_3^3$	1,3,3,1	2
	18		2	$x_1^2, x_1x_2, x_2^2, x_3^2$	y_1y_3, y_2y_3	$y_1^2y_3 + y_2^2y_3$	1,3,3,1	2
	18.2		2	$x_1^2, x_1x_2, x_2^2, x_3^2 - x_1x_3$	$y_1y_3 + y_3^2, y_2y_3$	$y_1^2y_3 + y_1y_3^2 + y_2^2y_3$	1,3,3,1	2
	19		3	$x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3^2, x_3^3$	y_1, y_2y_3, y_3^2	$y_1^2 + y_2y_3^2$	1,3,2,1	1
	20		3	$x_1^2, x_1x_2, x_1x_3, x_2x_3, x_2^3, x_3^3$	y_1, y_2^2, y_3^2	$y_1^2 + y_2^3 + y_3^3$	1,3,2,1	1
	21	1,4,1	1	$x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2 - x_3x_4, x_1x_3, x_1x_4, x_2x_3, x_2x_4$	$y_1y_2 + y_3y_4$		1,4,1	0
	21.2		1	$x_1^2 + x_2^2, x_1^2 + x_3^2, x_1^2 + x_4^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4$	$y_1^2 + y_2^2 + y_3^2 + y_4^2$		1,4,1	0
	22		2	$x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_2^2, x_3^2 - x_1^2, x_4^2 - x_1^2$	$y_2, y_1^2 + y_3^2 + y_4^2$	$y_1^3 + y_1y_3^2 + y_1y_4^2 + y_2^2$	1,4,3,1	3
	23		3	$x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4$	y_1, y_2, y_3y_4	$y_1^2 + y_2^2 + y_3^2y_4$	1,4,2,1	2
	23.2		3	$x_1^2, x_2^2, x_3^2 + x_4^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4$	$y_1, y_2, y_3^2 + y_4^2$	$y_1^2 + y_2^2 + y_3^3 + y_3y_4^2$	1,4,2,1	2
	24		4	$x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_2^2, x_3^2, x_4^2, x_1^3$	y_2, y_3, y_4, y_1^2	$y_1^3 + y_2^2 + y_3^2 + y_4^2$	1,4,1,1	1
	25	1,5	5	$(x_1, x_2, x_3, x_4, x_5)^2$	y_1, y_2, y_3, y_4, y_5	$y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2$	1,5,1	1

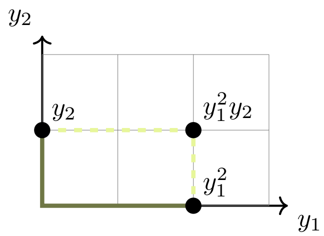
5.3 Monomial ideals

Monomial ideals I of R have a lot of nice properties, see [29]. For instance, their inverse system $I^\perp$ is also generated by monomials as an R-module and its generators as a $\mathbf{k}$-vector space coincide with a $\mathbf{k}$-basis of R/I, see Corollary A.1.6 for more details. Also the Gorenstein property has a nice translation in the monomial situation:

PROPOSITION 5.3.1 [29, A.6.5] Let $A = R/I$ be an Artin ring and I a monomial ideal. Then A is Gorenstein if and only if A is a complete intersection. If any of the equivalent conditions hold, then I is generated by pure powers of the variables.

It is natural to ask whether the computation of Gorenstein colength and minimal Gorenstein covers is simpler for monomial rings. The first relevant observation is that, in general, monomial rings do not have monomial minimal Gorenstein covers.

EXAMPLE 5.3.2 Consider the monomial ideal $I = (x_1^3, x_2^2, x_1x_2)$ in $R = \mathbf{k}[\![x_1, x_2]\!]$, Case 3 of $\ell(A) = 4$ in Table 5.2. In the chart below, we represent $I^\perp = \langle y_1^2, y_2\rangle$ in dark green and $J^\perp = \langle y_1^2 y_2\rangle$ in light green.



Note that $y_1^2y_2$ is the monomial of lowest degree that generates an inverse system containing $I^\perp$. But $\ell(R/J) - \ell(R/I) = 2$ and $\operatorname{gcl}(A) = 1$, hence none of the minimal Gorenstein covers of A are monomial rings.

However, we can always consider the minimal monomial Gorenstein cover:

DEFINITION 5.3.3 Given a monomial ring $A = R/I$, we say that G is a **minimal monomial Gorenstein cover** if

(i) $G = R/J$ is a Gorenstein cover,
(ii) J is monomial,
(iii) $\ell(G)$ is minimal amongst the G satisfying (i) and (ii).

PROPOSITION 5.3.4 Let $A = R/I$ be a monomial ring. Then $G = R/J$, where J is the monomial ideal $(x_1^{a_1}, \ldots, x_n^{a_n})$ such that $I \cap \mathbf{k}[\![x_i]\!] = (x^{a_i})$, is a minimal monomial Gorenstein cover of A.

Proof: Assume that I is a monomial ideal. Then there exist monomials x^{a_i} with a_i in $\mathbb{N}^n$ and such that, $i = 1, \cdots, n$, $I \cap \mathbf{k}[\![x_i]\!] = (x_i^{a_i})$. $J = (x_1^{a_1}, \cdots, x_n^{a_n})$ is clearly contained in I and it is Gorenstein by Proposition 5.3.1. By constuction, $G = R/J$ has the minimal length amongst all monomial Gorenstein covers. $\square$

COROLLARY 5.3.5 If $A = R/I$ is a monomial ring with monomial minimal Gorenstein cover, then

$$\operatorname{gcl}(A) = \prod_{i=1}^{n} (a_i - 1) - \ell(A),$$

where a_i is the smallest integer such that $x_i^{a_i} \in I$.

Proof: Take $J = (x_1^{a_1}, \ldots, x_n^{a_n})$ as in Proposition 5.3.4. Then a_i is the smallest integer such that $x_i^{a_i}$ is in I and $G = R/J$ is also a minimal Gorenstein cover. It is easy to check that $\ell(G) = \prod_{i=1}^{n}(a_i - 1)$. $\square$

In this section we will provide some examples and partial results for monomial rings in codimension 2 and review which monomial rings of Poonen's classification in Table 5.2 have monomial minimal Gorenstein covers.

5.3.1 Monomial rings in codimension 2

In Chapter 4, monomial ideals take a leading role in the search for Gorenstein covers of $A = R/I$. We deform monomial ideals E with an appropriate Hilbert function into ideals J such that $\mathrm{Lt}_{\overline{\tau}}(J) = E$, preserving their Hilbert function. In this way, we provide a constructive procedure where we eventually find Gorenstein ideals J such that $J \subset I$. However, the effectivity of the method decreases dramatically as the colength increases because of the combinatorics of both the admissible Hilbert functions and the associated monomial ideals.

In this section we want to give expressions of the Gorenstein colength in terms of the exponents of the generators of the ideal. Recall that $\tau(A) = n$ for Teter rings and $\tau(A) = n-1, n$ in colength 2. Therefore, if $n = 2$, non-Gorenstein rings of low colength have Cohen-Macaulay type 2. By Proposition 1.4.19, I must be minimally generated by 3 elements in R. Hence we will restrict our study to monomial rings $I = (x_1^t, x_2^s, x_1^a x_2^b)$, with $1 \leq a \leq t-1$ and $1 \leq b \leq s-1$.

5.3.1.1 Teter rings

Let us consider an ideal $I = (x^t, y^s, x^a y^b)$, with $1 \leq a \leq t-1$ and $1 \leq b \leq s-1$. Note that its inverse system is $I^\perp = \langle x^{t-1}y^{b-1}, x^{a-1}y^{s-1}\rangle$.

By [20], A is Teter if and only if exist polynomials H_1, H_2 of degree at most the maximum between $t+b-2$ and $a+s-2$ such that $y \circ H_1 = x \circ H_2$, where

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \circ \begin{pmatrix} x^{a-1}y^{s-1} \\ x^{t-1}y^{b-1} \end{pmatrix}$$

and $C = \{c_{ij}\}_{1 \leq i,j \leq 2}$ is a matrix with polynomial entries such that $\det C_0 \neq 0$.

$$\begin{cases} H_1 = c_{11} \circ x^{a-1}y^{s-1} + c_{12} \circ x^{t-1}y^{b-1} \\ H_2 = c_{21} \circ x^{a-1}y^{s-1} + c_{22} \circ x^{t-1}y^{b-1} \end{cases}$$

Since $y \circ H_1 = x \circ H_2$, then

$$c_{11} \circ x^{a-1}y^{s-2} + c_{12} \circ x^{t-1}y^{b-2} = c_{21} \circ x^{a-2}y^{s-1} + c_{22} \circ x^{t-2}y^{b-1}.$$

If $a = b = 1$ we are considering an inverse system of the form $I^\perp = \langle x^{t-1}, y^{s-1}\rangle$. In such case, $F = x^t + y^s$ generates the inverse system of a minimal Gorenstein cover of A. Indeed, $(x,y) \circ F = \langle x^{t-1}, y^{s-1}\rangle$ and a dimension computation of $I^\perp$ as $\mathbf{k}$-vector space gives that $\ell(A) = t+s-1$ and $\ell(R/\operatorname{Ann_R}(F)) = t+s$.

We can now assume that either $a > 1$ or $b > 1$.

- Case $b + t > a + s$.

  In maximum degree $b+t-3$:

  $$c_{12}^0 x^{t-1}y^{b-2} = c_{22}^0 x^{t-2}y^{b-1}.$$

  If $b > 1$, then $c_{12}^0 = c_{22}^0 = 0$ and $\det C_0 = \begin{vmatrix} c_{11}^0 & 0 \\ c_{21}^0 & 0 \end{vmatrix} = 0$. Hence $\operatorname{gcl}(A) > 1$.

If $b = 1$, then in maximum degree $t-2$ we get $c^0_{22}x^{t-2} = 0$ and hence $c^0_{22} = 0$. In degree $a+s-3$ we have

$$c^0_{11}x^{a-1}y^{s-2} = c^0_{21}x^{a-2}y^{s-1} + c^k_{22} \circ x^{t-2},$$

with $c^k_{22} = a^k_{22}x^k$, $k = t-a-s+1$ and $a^k_{22} \in \mathbf{k}$. Therefore,

$$c^0_{11}x^{a-1}y^{s-2} = c^0_{21}x^{a-2}y^{s-1} + a^k_{22}x^{a+s-3}.$$

Since $a > 1$, then $c^0_{11} = c^0_{21} = a^k_{22} = 0$ and $\mathrm{gcl}(A) > 1$.

- Case $b+t = a+s$.
  In maximum degree $b+t-3 = a+s-3$:

$$c^0_{11}x^{a-1}y^{s-2} + c^0_{12}x^{t-1}y^{b-2} = c^0_{21}x^{a-2}y^{s-1} + c^0_{22}x^{t-2}y^{b-1}.$$

  - $a, b > 1$.
    If $a-1 < t-2$ or $b-1 < s-2$, then $c^0_{11} = c^0_{12} = c^0_{21} = c^0_{22} = 0$ and hence $\mathrm{gcl}(A) > 1$.

    If $a-1 = t-2$ and $b-1 = s-2$, then $c^0_{12} = c^0_{21} = 0$ and $c^0_{11} = c^0_{22}$. In this case, $I^\perp = \langle x^{t-2}y^{s-1}, x^{t-1}y^{s-2}\rangle$ and, taking $C = Id$, we get

$$\begin{cases} H_1 = x \circ F = x^{t-2}y^{s-1} \Rightarrow F = x^{t-1}y^{s-1} + p(y) \\ H_2 = y \circ F = x^{t-1}y^{s-2} \Rightarrow F = x^{t-1}y^{s-1} + p(x) \end{cases}$$

    Hence $J^\perp = \langle x^{t-1}y^{s-1}\rangle$ is a minimal Teter cover of A.
  - $a = 1$ or $b = 1$. Since a and b have symmetric roles in the expression above, we can assume that $a = 1$ and $b > 1$.

$$c^0_{11}y^{s-2} + c^0_{12}x^{t-1}y^{b-2} = c^0_{22}x^{t-2}y^{b-1}.$$

    If $t = 2$, then

$$c^0_{11}y^{s-2} + c^0_{12}xy^{b-2} = c^0_{22}y^{b-1}.$$

    Hence
    * If $b-1 < s-2$, then $c^0_{12} = c^0_{22} = 0$. Hence if $a = 1$, $t = 2$ and $1 < b < s-1$, then $\mathrm{gcl}(A) > 1$.
    * If $b = s-1$, then $c^0_{11}y^{b-1} + c^0_{12}xy^{b-2} = c^0_{22}y^{b-1}$ only gives $c^0_{12} = 0$.

It can be checked that $J^{\perp} = \langle xy^{s-1} \rangle$ provides a Teter cover of A.

If $t > 2$, then $c^0_{12} = c^0_{22} = 0$ and hence $\mathrm{gcl}(A) > 1$.

Summing up, if $I^{\perp}$ is of the form $\langle x^{t-1}y^{s-2}, x^{t-2}y^{s-1} \rangle$ for any $s, t \geq 2$, then it is Teter.

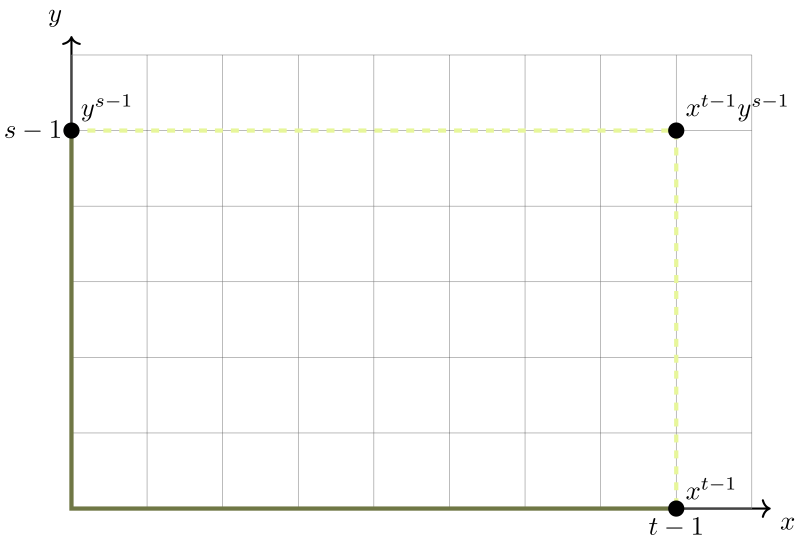
- Case $b + t < a + s$.
  Since x and y are symmetric, this is the same situation as in case $b + t > a + s$.

Therefore, monomial Teter rings in codimension 2 are of the following forms:

(I) $A = \mathbf{k}[[x, y]]/(x^t, y^s, x^{t-1}y^{s-1})$.

(II) $A = \mathbf{k}[[x, y]]/(x^t, y^s, xy)$.

Note that rings of type (I) and (II) coincide when $s = t = 2$. Also observe that (I) always admits a monomial Teter cover $G = \mathbf{k}[[x, y]]/(x^t, y^s)$. (II) admits the Teter cover $G = \mathbf{k}[[x, y]]/(xy, y^s - x^t)$ but has no monomial minimal covers unless $t = s = 2$, as the following picture clearly illustrates:



REMARK 5.3.6 In Gorenstein colength 2, an analogous argument can be performed but the conditions we obtain on the exponents are not as elegant as in the Teter case.

5.3.1.2 Ideals with 3 minimal generators

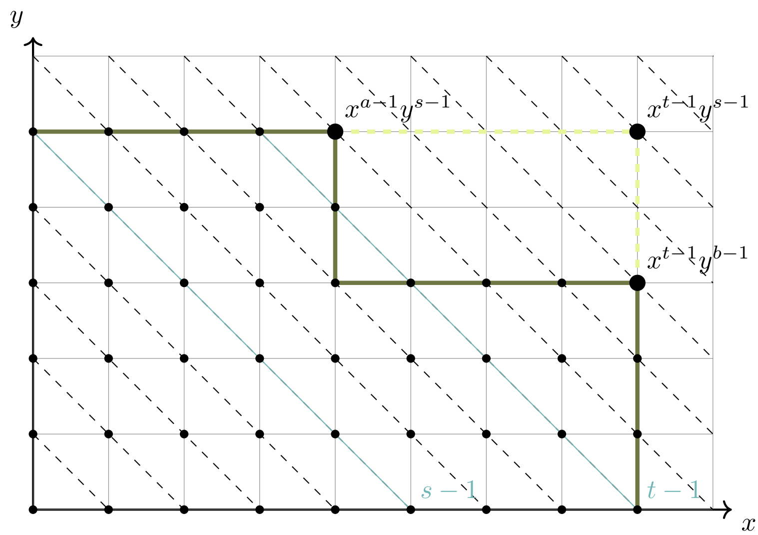
PROPOSITION 5.3.7 Let $I = (x_1^t, x_2^s, x_1^a x_2^b)$, with $1 \leq a \leq t-1$ and $1 \leq b \leq s-1$, be a monomial ideal of $R = \mathbf{k}[\![x_1, x_2]\!]$ with $\mu(I) = 3$ such that R/I is an Artin ring of codimension 2. If $\max\{s,t\} \leq a+b$, then

$$\operatorname{gcl}(R/I) = (t-a)(s-b)$$

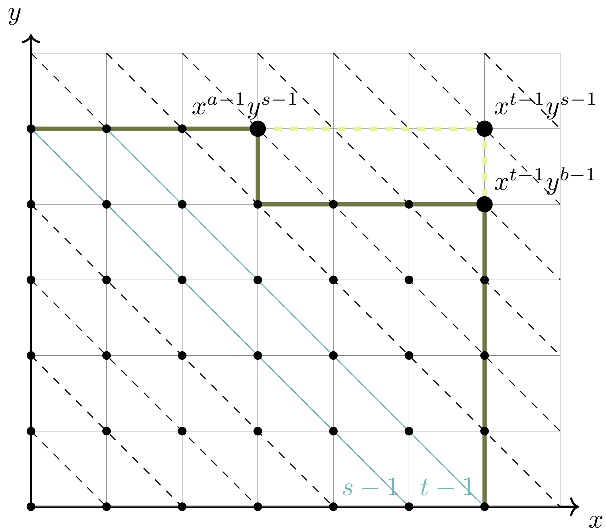
and $G = \mathbf{k}[\![x_1, x_2]\!]/(x_1^t, x_2^s)$ is a minimal Gorenstein cover of R/I.

Proof: Recall that $I^\perp = \langle y_1^{t-1} y_2^{b-1}, y_1^{a-1} y_2^{s-1} \rangle$. By symmetry of the roles of x and y, we can assume without loss of generality that $t \geq s$. Since $t \geq s$ and $\max\{s,t\} \leq a+b$, then $s-1 \leq t-1 \leq a+b-1$.

Let us picture the extremal case $s-1 \leq t-1 = a+b-1$:



Next we represent the general case $s-1 \leq t-1 \leq a+b-2$:



From the representations above we can deduce that, whenever $s-1 \leq t-1 \leq a+b-1$, the Hilbert function of $A = R/I$ is

$$HF_A(i) = \begin{cases} i+1, & 0 \leq i \leq s-2; \\ s, & s-1 \leq i \leq t-1; \\ h_i, & t \leq i \leq t+s-3; \\ 0, & i \geq t+s-2, \end{cases}$$

where $h_i \leq s+t-(i+1)$ for any $t \leq i \leq t+s-3$.

By Theorem 1.2.11, the Gorenstein-admissible Hilbert function with minimal length starting by the sequence $\{1, 2, 3, \dots, s, \dots, s\}$ is the symmetric numerical function $\{1, 2, 3, \dots, s, \dots, s, s-1, s-2, \dots, 2, 1\}$. The ring $G = \mathbf{k}[\![x_1, x_2]\!]/(x_1^s, x_2^t)$ happens to have exactly this Hilbert function:

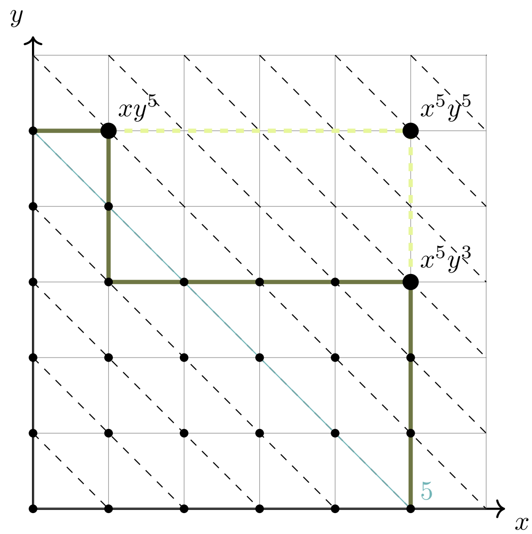
$$HF_G(i) = \begin{cases} i+1, & i \leq s-1; \\ s, & s \leq i \leq t-1; \\ s+t-(i+1), & t \leq i \leq t+s-2; \\ 0, & i \geq t+s-1. \end{cases}$$

Since $I^\perp \subset \langle y_1^{t-1} y_2^{s-1} \rangle$ and $\ell(G) - \ell(A)$ is minimal, G is a minimal cover of R/I.

Finally, it is easy to check that $\ell(G) - \ell(A) = (s-b)(t-a)$ because it corresponds to the upper right rectangle that belongs to $J^{\perp}$ but not to $I^{\perp}$. Hence $\mathrm{gcl}(A) = (s-b)(t-a)$.$\square$

Let us give an example which is under the conditions of Proposition 5.3.7 and show how its Gorenstein colength can be alternatively deduced from its Hilbert function:

EXAMPLE 5.3.8 $t = s = 6$, $a = 2$, $b = 4$. By Proposition 5.3.7, $\mathrm{gcl}(A) = 8$.



Observe that the numerical function HF_G from the chart below is the minimal Hilbert function admitting Gorenstein rings such that $\mathrm{HF}_G(i) \geq \mathrm{HF}_A(i)$, for $i \geq 0$. Then $R/(x_1^6, x_2^6)$ is a Gorenstein cover of A with Hilbert function HF_G, hence it is minimal.

length	i	0	1	2	3	4	5	6	7	8	9	10
28	HF_A	1	2	3	4	5	6	4	2	1	0	0
36	HF_G	1	2	3	4	5	6	5	4	3	2	1

Finally we provide a monomial ideal which is not under the conditions of Proposition 5.3.7, where we can easily determine that it has no monomial minimal Gorenstein covers even if we cannot compute the Gorenstein colength.

EXAMPLE 5.3.9 $t = 17, s = 7, a = 3, b = 4$, $\ell(A) = 77$. If a monomial minimal Gorenstein cover exists, it must be $G = \mathbf{k}[\![x_1, x_2]\!]/(x^{17}, y^7)$. It can be checked that $G' = \mathbf{k}[\![x_1, x_2]\!]/\operatorname{Ann}_{\mathrm{R}}(x^2y^{10}+x^{19}y^3)$ is also a Gorenstein cover of A. But $\ell(G) = 119$ and $\ell(G') = 89$, hence G is clearly not minimal. Observe that we cannot claim that G' is minimal, we merely proved that $\operatorname{gcl}(A) \leq \ell(G') - \ell(A) = 12$.

5.3.2 Monomial rings of length equal or less than 6

In this last part of the chapter, we review Poonen's classification in Table 5.2 and pay special attention to non-Gorenstein monomial rings. We study whether they admit or not monomial minimal Gorenstein covers.

REMARK 5.3.10 Observe that Case 7 of $\ell(A) = 6$ only admits monomial minimal covers in one of the two different Hilbert functions of its minimal Gorenstein covers: $\mathrm{HF}_G = \{1, 2, 2, 2, 1\}$.

TABLE 5.3 Non-Gorenstein monomial rings $\ell(A) \leq 6$. In turquoise, rings admitting monomial minimal Gorenstein covers.

Case	HF_A	$I^\perp$	$J^\perp$	$\mathrm{gcl}(A)$
$\ell(A) = 3, 2$	1,2	y_1, y_2	y_1y_2	1
$\ell(A) = 4, 3$	1,2,1	y_2, y_1^2	$y_1^3 + y_2^2$	1
$\ell(A) = 4, 4$	1,3	y_1, y_2, y_3	$y_1^2 + y_2^2 + y_3^2$	1
$\ell(A) = 5, 3$	1,2,1,1	y_1, y_2^3	$y_1^2 + y_2^4$	1
$\ell(A) = 5, 4$	1,2,2	y_1^2, y_2^2	$y_1^3 + y_2^3$	1
$\ell(A) = 5, 5$		y_1y_2, y_2^2	$y_1y_2^2$	1
$\ell(A) = 5, 8$	1,3,1	y_2, y_3, y_1^2	$y_1^3 + y_2^2 + y_3^2$	1
$\ell(A) = 5, 9$	1,4	y_1, y_2, y_3, y_4	$y_1^2 + y_2^2 + y_3^2 + y_4^2$	1
$\ell(A) = 6, 3$	1,2,1,1,1	y_2, y_1^4	$y_1^5 + y_2^2$	1
$\ell(A) = 6, 6$		y_1^2, y_2^3	$y_1^3 + y_2^4$	1
$\ell(A) = 6, 7$		y_1y_2, y_2^3	$y_1^2y_2 + y_2^5$	2
			$y_1y_2^3$	
$\ell(A) = 6, 9$	1,2,3	y_1^2, y_1y_2, y_2^2	$y_1^2y_2^2$	3
$\ell(A) = 6, 12$	1,3,1,1	y_2, y_3, y_1^3	$y_1^4 + y_2^2 + y_3^2$	1
$\ell(A) = 6, 15$	1,3,2	y_1^2, y_2y_3	$y_1^3 + y_2^2y_3$	2
$\ell(A) = 6, 18$		y_1y_3, y_2y_3	$y_1y_2y_3$	2
$\ell(A) = 6, 19$		y_1, y_2y_3, y_3^2	$y_1^2 + y_2y_3^2$	1
$\ell(A) = 6, 20$		y_1, y_2^2, y_3^2	$y_1^2 + y_2^3 + y_3^3$	1
$\ell(A) = 6, 24$	1,4,1	y_2, y_3, y_4, y_1^2	$y_1^3 + y_2^2 + y_3^2 + y_4^2$	1
$\ell(A) = 6, 25$	1,5	y_1, y_2, y_3, y_4, y_5	$y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2$	1

Appendices

APPENDIX A

Singular manual for computing minimal Gorenstein covers

All the algorithms presented in this thesis have been implemented with the commutative algebra software *Singular*, [11]. This appendix is a review of the *Singular* library **GorensteinCovers.lib**, which has been specifically created to do most of the computations that appear in the previous chapters.

The purpose of this library is, as its name suggests, to help with the computation of Gorenstein covers $G = R/J$ of a given ring $A = R/I$, where I is an $\mathfrak{m}$-primary ring of $R = \mathbf{k}[\![x_1, \dots, x_n]\!]$.

The main procedures contained in it can be classified into three blocks:

(i) Computation of the inverse system $I^\perp$ of I.
(ii) Computation of the integral of a module M with respect to an ideal K, i.e. $\int_K M$.
(iii) Computation of the variety $MGC(A)$ for low-colength rings A, i.e. $\operatorname{gcl}(A) = 1, 2$.

The *Singular* library **InverseSyst.lib** by Elias will also be needed, see [13] for a review of its contents.

Next we provide some important general remarks on how to use this library:

REMARK A.0.1 Since we are dealing with a local scenario, note that the ground ring should be defined with a local ordering, that is

```
ring r=p,(x(1..n)),ord;
```

where p is the characteristic, n is an integer, and ord is a local ordering (ds, ls or Ds).

REMARK A.0.2 The structure of $S = k[y_1, ..., y_n]$ as $R = \mathbf{k}[\![x_1, \dots, x_n]\!]$-module is by contraction. In **InverseSyst.lib** both derivation and contraction structures of S are taken into account, hence we will only use the commands with ending `NC` (no coefficients) or `IHNC`(injective hull with no coefficients).

REMARK A.0.3 A sub-R-module of S generated by $F_1, ..., F_r$ is handled in this LIB as an ideal generated by $F_1, ..., F_r$, keeping with Elias's treatment in **InverseSyst.lib**. His library provides very useful procedures to operate with these ideals as R-modules.

A.1 Methods to compute inverse systems

Inverse systems are a useful tool to deal with local Artin $\mathbf{k}$-algebras and, in a more general setting, to study isolated points in a variety. Some properties of ideals in R that have a difficult computational approach have a particularly nice translation into inverse systems: quotient ideals, elimination of variables or even differential equations. See [23, Sections 7.1.5-7.1.8] for more details.

Here we describe 3 different methods to compute a $\mathbf{k}$-basis of the inverse system of an $\mathfrak{m}$-primary ideal I of $R = \mathbf{k}[\![x_1, \dots, x_n]\!]$. In all three situations, once we obtain a $\mathbf{k}$-basis, we can use Elias' procedure `minGensIHNC` to obtain a minimal system of generators of $I^\perp$ as an R-module.

A.1.1 Method 1: system of equations

This method is implemented by Elias in the procedure `invSystNC` of the *Singular* library **InverseSyst.lib**, see [13]. Given an ideal $I = (f_1, \dots, f_m) \subseteq R$, we can compute its inverse system by solving the system of equations

$$f_i \circ F = 0, \qquad \text{for any } 1 \leq i \leq m \tag{A.1}$$

for enough polynomials $F \in S = \mathbf{k}[y_1, \dots, y_n]$.

EXAMPLE A.1.1 Consider $I = (x_1^4, x_1^2 - x_2) \subset \mathbf{k}[\![x_1, x_2]\!]$, set $f_1 = x_1^4$ and $f_2 = x_1^2 - x_2$. Consider the reverse-degree reverse lexicographical order (`ds` in *Singular*). The Artin ring R/I has socle degree 3, hence all polynomials in $I^\perp$ have degree at most 3. We denote by $S_{\leq 3}$ the sub-R-module of $\mathbf{k}[y_1, y_2]$ formed by polynomials of degree equal or less than 3 and denote by $(y^\alpha)_\alpha$ the elements of the monomial $\mathbf{k}$-basis $y_2^3, y_1y_2^2, y_2^2, y_1^2y_2, y_1y_2, y_2, y_1^3, y_1^2, y_1, 1$ of $S_{\leq 3}$. Consider the linear map

$$\begin{array}{rccc} \varphi: & S_{\leq 3} & \longrightarrow & S_{\leq 3} \times S_{\leq 3} \\ & y^\alpha & \longmapsto & (f_1 \circ y^\alpha, f_2 \circ y^\alpha) \end{array}$$

The matrix associated to φ is the following:

	$\varphi(y_2^3)$	$\varphi(y_1y_2^2)$	$\varphi(y_2^2)$	$\varphi(y_1^2y_2)$	$\varphi(y_1y_2)$	$\varphi(y_2)$	$\varphi(y_1^3)$	$\varphi(y_1^2)$	$\varphi(y_1)$	$\varphi(1)$
y_1^3	0	0	0	0	0	0	0	0	0	0
$y_1y_2^2$	0	0	0	0	0	0	0	0	0	0
y_2^2	0	0	0	0	0	0	0	0	0	0
$y_1^2y_2$	0	0	0	0	0	0	0	0	0	0
y_1y_2	0	0	0	0	0	0	0	0	0	0
y_2	0	0	0	0	0	0	0	0	0	0
y_1^3	0	0	0	0	0	0	0	0	0	0
y_1^2	0	0	0	0	0	0	0	0	0	0
y_1	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
y_2^3	0	0	0	0	0	0	0	0	0	0
$y_1y_2^2$	0	0	0	0	0	0	0	0	0	0
y_2^2	-1	0	0	0	0	0	0	0	0	0
$y_1^2y_2$	0	0	0	0	0	0	0	0	0	0
y_1y_2	0	-1	0	0	0	0	0	0	0	0
y_2	0	0	-1	1	0	0	0	0	0	0
y_1^3	0	0	0	0	0	0	0	0	0	0
y_1^2	0	0	0	-1	0	0	0	0	0	0
y_1	0	0	0	0	-1	0	1	0	0	0
1	0	0	0	0	0	-1	0	1	0	0

The $\mathbf{k}$-basis of the kernel of φ is precisely a $\mathbf{k}$-basis of those polynomials $F \in S_{\leq 3}$ such that $f_1 \circ F = f_2 \circ F = 0$. *Singular* provides the following $\mathbf{k}$-basis of $\ker \varphi$:

$$\begin{aligned} g_1 &= (0,0,0,0,0,0,0,0,0,1), \\ g_2 &= (0,0,0,0,0,0,0,0,1,0), \\ g_3 &= (0,0,0,0,0,1,0,1,0,0), \\ g_4 &= (0,0,0,0,1,0,1,0,0,0). \end{aligned}$$

Therefore, we can retrieve $I^\perp = \langle 1, y_1, y_2 + y_1^2, y_1y_2 + y_1^3 \rangle_{\mathbf{k}}$.

Algorithm 5 Computation of $I^\perp$ via system of equations

Input: $f_1, \dots, f_m$ generators of the ideal I.

Output: $b_1, \dots, b_t$ $\mathbf{k}$-basis of $I^\perp$.

Steps:

(i) Define $s = \operatorname{socdeg}(R/I)$.

(ii) Set $M = (y^\alpha)_{\alpha \le s}$. Note that M is a $\mathbf{k}$-basis of $S_{\le s}$.

(iii) Define the linear map

$$\begin{array}{rccl} \varphi: & S_{\le s} & \longrightarrow & S_{\le s} \times S_{\le s} \times \cdots \times S_{\le s} \\ & y^\alpha & \longmapsto & (f_1 \circ y^\alpha, f_2 \circ y^\alpha, \dots, f_m \circ y^\alpha) \end{array}$$

(iv) Compute a $\mathbf{k}$-basis of the kernel of φ.

(v) Define $b_1, \dots, b_t$ as the elements of the $\mathbf{k}$-basis of $\ker \varphi$ in polynomial notation.

REMARK A.1.2 The implementation of `kinvSystNC` in **GorensteinCovers.lib** follows the idea of `invSystNC` in **Inverse-Syst.lib** but we remove the computation of the generators of $I^\perp$ as R-module. In this way, all three algorithms provide a $\mathbf{k}$-basis and we are able to compare them.

Next we show a sample session in *Singular* on how to use this procedure:

```
//load library
> LIB "GorensteinCovers.lib";
//define ring with local order
> ring r=0,(x,y),ds;
//define ideal
> ideal i=x4,x2-y;
//check m-primality
> dim(std(i));
0
//compute k-basis
> kinvSystNC(i);
_[1]=x3+xy
_[2]=x2+y
_[3]=x
_[4]=1
```

A.1.2 Method 2: reduction with respect to a normal form modulo I.

In [23, Section 7.1.8] Elkadi and Mourrain provide a simple algorithm to construct a $\mathbf{k}$-basis of $I^{\perp}$ via the reduction of polynomials with normal forms with respect to I. We reproduce their fundamental results here translated into our setting.

Consider a $\mathbf{k}$-basis $(\overline{x}^{\alpha})_{\alpha\in E}$ of R/I, where E is a finite subset of $\mathbb{N}^n$. Recall that $I^{\perp}$ can be identified with the dual of R/I as a $\mathbf{k}$-vector space, denoted by $(R/I)^*$, see Section 1.4.2. Therefore, we can consider a dual $\mathbf{k}$-basis $(\Lambda_{\alpha})_{\alpha\in E}$ of $I^{\perp}$, in the sense that, for any $\alpha,\beta\in E$,

$$\left(x^{\beta}\circ\Lambda_{\alpha}\right)(0)=\begin{cases}1, & \text{if } \alpha=\beta;\\ 0, & \text{otherwise.}\end{cases} \tag{A.2}$$

Recall that $(f\circ\Lambda)(0)=(g\circ\Lambda)(0)$ for any $f,g\in R$ such that $f-g\in I$.

PROPOSITION A.1.3 [23, Proposition 7.23] Given the $\mathbf{k}$-basis $(\overline{x}^{\alpha})_{\alpha\in E}$ of R/I, the family

$$(y^{\alpha}+\sum_{\beta\in\mathbb{N}^n\setminus E}\lambda_{\alpha,\beta}y^{\beta})_{\alpha\in E},$$

where $\overline{x}^{\beta}=\sum_{\alpha\in E}\lambda_{\alpha,\beta}\overline{x}^{\alpha}$ for any $\beta\notin E$, forms a $\mathbf{k}$-basis of $I^{\perp}$.

Proof: Consider the dual $\mathbf{k}$-basis $(\Lambda_{\alpha})_{\alpha\in E}$ of $I^{\perp}$ with respect to the $\mathbf{k}$-basis $(\overline{x}^{\alpha})_{\alpha\in E}$ of R/I.

For any $\alpha\in E$, we can describe Λ_{α} as $\sum_{\beta\in\mathbb{N}^n}\mu_{\alpha,\beta}y^{\beta}$, for finitely many scalars $\mu_{\alpha,\beta}\neq 0$. Note that $\mu_{\alpha,\beta}=\left(x^{\beta}\circ\Lambda_{\alpha}\right)(0)$. If $\beta\in E$, then $\mu_{\alpha,\beta}=\delta_{\alpha,\beta}$ by A.2, where $\delta_{\alpha,\beta}$ is the Kronecker delta. Otherwise, if $\beta\notin E$, there exist unique scalars $(\lambda_{\alpha,\beta})_{\alpha\in E}$ such that

$$\overline{x}^{\beta}=\sum_{\alpha\in E}\lambda_{\alpha,\beta}\overline{x}^{\alpha}\in R/I,$$

hence

$$\left(x^{\beta}\circ\Lambda_{\alpha}\right)(0)=\left(\left(\sum_{\alpha'\in E}\lambda_{\alpha',\beta}x^{\alpha'}\right)\circ\Lambda_{\alpha}\right)(0).$$

Again by A.2 we get

$$\sum_{\alpha' \in E} \lambda_{\alpha',\beta} \left(x^{\alpha'} \circ \Lambda_\alpha \right)(0) = \lambda_{\alpha,\beta},$$

hence $\mu_{\alpha,\beta} = \lambda_{\alpha,\beta}$. Therefore, $\Lambda_\alpha = y^\alpha + \sum_{\beta \notin E} \lambda_{\alpha,\beta} y^\beta$. $\square$

REMARK A.1.4 Consider a $\mathbf{k}$-basis $(\overline{x}^\alpha)_{\alpha \in E}$ of R/I and let s be its socle degree. Since $\mathfrak{m}^{s+1} \subset I$, then for any β such that $|\beta| \geq s+1$, we have $x^\beta \in I$. Hence $\lambda_{\alpha,\beta} = 0$ for any $\alpha \in E$ and

$$\Lambda_\alpha = y^\alpha + \sum_{\beta \notin E,\, |\beta| \leq s} \lambda_{\alpha,\beta} y^\beta.$$

EXAMPLE A.1.5 Consider $I = (x_1^4, x_1^2 - x_2) \subset \mathbf{k}[\![x_1, x_2]\!]$. The set $(x^\alpha)_{\alpha \in E} := \{x_1^3,$ x_1^2, x_1, $1\}$ is a $\mathbf{k}$-basis of R/I and $\operatorname{socdeg} R/I = 3$. Consider the set of monomials $M = \left(x^\beta\right)_{\beta \leq 3}$. Given a standard basis $S = \{x_2 - x_1^2, x_1^4\}$ of I (with respect to the reverse-degree reverse lexicographical term ordering), we can compute the normal forms $\mathrm{NF}(x^\beta \mid S)$. With local order `ds`, we get

$$\mathrm{NF}(M \mid S) = \{0, 0, 0, 0, x_1^3, x_1^2, x_1^3, x_1^2, x_1, 1\}.$$

We can express these normal forms as a matrix whose entries are the coefficients $\lambda_{\alpha,\beta}$ with $\alpha \in E$ and $\beta \leq 3$ of Proposition A.1.3:

M	x_2^3	$x_1x_2^2$	x_2^2	$x_1^2x_2$	x_1x_2	x_2	x_1^3	x_1^2	x_1	1
$\mathrm{NF}(M \mid S)$	0	0	0	0	x_1^3	x_1^2	x_1^3	x_1^2	x_1	1
x_1^3	0	0	0	0	1	0	1	0	0	0
x_1^2	0	0	0	0	0	1	0	1	0	0
x_1	0	0	0	0	0	0	0	0	1	0
1	0	0	0	0	0	0	0	0	0	1

Hence from the rows of this matrix, we can retrieve the $\mathbf{k}$-basis of $I^\perp$ by considering the entries as coefficients of M: $I^\perp = \langle y_1y_2 + y_1^3, y_2 + y_1^2, y_1, 1\rangle_{\mathbf{k}}$.

Algorithm 6 Computation of $I^\perp$ via reduction with respect to a normal form modulo I

Input: I ideal.
Output: $b_1, \dots, b_t$ **k**-basis of $I^\perp$.
Steps:

(i) Set $s = \mathrm{socdeg}(R/I)$.
(ii) Set $M = (x^\beta)_{\beta \le s}$.
(iii) Compute a standard basis S of I.
(iv) Compute the normal forms $\mathrm{NF}(x^\beta \mid S)$, for any $x^\beta \in M$.
(v) Compute a **k**-basis $F = \{f_1, \dots, f_t\}$ of R/I.
(vi) Compute the matrix A of coefficients of $\mathrm{NF}(x^\beta \mid S)$ over the **k**-basis F of R/I.
(vii) Compute the product of matrix A and column matrix M^t.
(viii) For any $1 \le i \le t$, set b_i as the i-th entry of the column matrix AM^t.

We now provide the implementation of Algorithm 6 in *Singular*, where its default example is precisely Example A.1.5:

```
> LIB "GorensteinCovers.lib";
> example inverseSystem;
// proc inverseSystem from lib GorensteinCovers.lib
EXAMPLE:
ring r=0,(x,y),ds;
ideal i=x^4,x^2-y;
inverseSystem(i);
_[1]=xy+x3
_[2]=y+x2
_[3]=x
_[4]=1
```

Let us note that, if $I \subset R$ is a monomial ideal, the expression of the **k**-basis of $I^\perp$ in Proposition A.1.3 can be simplified:

COROLLARY A.1.6 Let $I \subset R$ be a monomial ideal. Given the **k**-basis $(\overline{x}^\alpha)_{\alpha \in E}$ of R/I, the family $(y^\alpha)_{\alpha \in E}$ forms a **k**-basis of $I^\perp$.

Proof: For each $\beta \notin E$, there exists unique $(\lambda_{\alpha,\beta})_{\alpha \in E}$ such that $x^\beta - \sum_{\alpha \in E} \lambda_{\alpha,\beta} x^\alpha$ in I. Since I is monomial, then $x^\beta \in I$, hence all $\lambda_{\alpha,\beta}$ vanish. By Proposition A.1.3, $(y^\alpha)_{\alpha \in E}$ is a **k**-basis of $I^\perp$. $\square$

```
> LIB "GorensteinCovers.lib";
> ring r=0,(x,y),ds;
//Define a monomial ideal
> ideal i=x3,xy2,y4;
//Computation of a k-basis of
the vector space R/I
> kbase(std(i));
_[1]=y3
_[2]=y2
_[3]=x2y
_[4]=xy
_[5]=y
_[6]=x2
_[7]=x
_[8]=1
//Computation of a k-basis
of the inverse system of I
> inverseSystem(i);
_[1]=y3
_[2]=y2
_[3]=x2y
_[4]=xy
_[5]=y
_[6]=x2
_[7]=x
_[8]=1
```

Algorithm 7 Computation of $I^{\perp}$ for monomial ideals I

Input: I monomial ideal.
Output: $b_1, \ldots, b_t$ **k**-basis of $I^{\perp}$.
Steps:

(i) Compute a **k**-basis $b_1, \ldots, b_t$ of R/I.

```
> example invSystMon;
// proc invSystMon from lib
//GorensteinCovers.lib
EXAMPLE:
ring r=0,(x,y),ds;
ideal i=x^7,x^6*y^3,
x^2*y^6,y^7;
invSystMon(i);
_[1]=xy6
_[2]=y6
_[3]=x5y5
_[4]=x4y5
_[5]=x3y5
_[6]=x2y5
_[7]=xy5
_[8]=y5
_[9]=x5y4
_[10]=x4y4
_[11]=x3y4
_[12]=x2y4
_[13]=xy4
_[14]=y4
_[15]=x5y3
_[16]=x4y3
_[17]=x3y3
_[18]=x2y3
_[19]=xy3
_[20]=y3
_[21]=x6y2
_[22]=x5y2
_[23]=x4y2
_[24]=x3y2
_[25]=x2y2
_[26]=xy2
_[27]=y2
_[28]=x6y
_[29]=x5y
_[30]=x4y
_[31]=x3y
_[32]=x2y
_[33]=xy
_[34]=y
```

```
_[35]=x6
_[36]=x5
_[37]=x4
_[38]=x3
_[39]=x2
_[40]=x
_[41]=1
```

A.1.3 Method 3: integration.

In [23, Theorem 7.36], Mourrain and Elkadi set the background for an algorithm to compute a $\mathbf{k}$-basis of $I^\perp$ where $I \subset \mathbf{k}[x_1,\dots,x_n]$. We recall here Theorem 3.1.12, where we adapted their results to the local case $I \subset \mathbf{k}[\![x_1,\dots,x_n]\!]$:

THEOREM A.1.7 (Theorem 3.1.12) Given an ideal $I=(f_1,\dots,f_m) \subset R$ and $d>1$. Let $\{b_1,\dots,b_{t_{d-1}}\}$ be a $\mathbf{k}$-basis of $\mathcal{D}_{d-1}$. The polynomials of $\mathcal{D}_d$ with no constant term are of the form

$$\Lambda = \sum_{j=1}^{t_{d-1}} \lambda_j^1 \int_1 b_j|_{y_2=\cdots=y_n=0} + \sum_{j=1}^{t_{d-1}} \lambda_j^2 \int_2 b_j|_{y_3=\cdots=y_n=0} + \cdots + \sum_{j=1}^{t_{d-1}} \lambda_j^n \int_n b_j, \quad \text{(A.3)}$$

where $\lambda_j^k \in \mathbf{k}$, such that

$$\sum_{j=1}^{s} \lambda_j^k (x_l \circ b_j) - \sum_{j=1}^{s} \lambda_j^l (x_k \circ b_j) = 0, 1 \le k < l \le n, \quad \text{(A.4)}$$

and

$$(f_i \circ \Lambda)(0) = 0, \text{ for } 1 \le i \le m. \quad \text{(A.5)}$$

Translating [39, Algorithm 4.3] into the our local setting, we obtain an algorithm to compute the $\mathbf{k}$-basis of $I^\perp$ along with its contraction matrices, see Definition 3.4.1. Observe that the following algorithm consists on the iteration of Algorithm 1 until we reach the socle degree with some extra constrictions derived from the orthogonality condition.

In the *Singular* implementation of the algorithm, given an ideal $I \subset R$, the output is a list of two elements: an ideal whose elements are a $\mathbf{k}$-basis of $I^\perp$ and a list of its contraction matrices $U_1,\dots,U_n$.

```
> LIB "GorensteinCovers.lib";
> ring r=0,(x,y),ds;
> ideal i=x4,x2-y;
> list L=integrate(i);
//First element of the list:
//k-basis of the inverse system of I
```

Algorithm 8 Compute a $\mathbf{k}$-basis of $I^\perp$ and its contraction matrices

Input: $f_1, \dots, f_m$ generators of the ideal I of R.
Output: $D_d = b_1, \dots, b_{s_d}$ $\mathbf{k}$-basis of $I^\perp$;
$U_1, \dots, U_n$ contraction matrices of $I^\perp$ associated to the $\mathbf{k}$-basis D.

(i) Set $d := 0$, $D_d := 1$, $s_d := 1$, $test := true$.
(ii) For $1 \le k \le n$, set an 1×1 matrix $U_k[1] := [0]$ and an $m \times 1$ matrix $A_k[1] := [(f_1 \circ y_k)(0), \dots, (f_m \circ y_k)(0)]$, where $U_k[1]$ and $A_k[1]$ stand for the first column of matrix U_k and A_k, respectively.
(iii) While $test = true$, do
  a) Set $\lambda_i := (\lambda_1^i \cdots \lambda_{s_d}^i)^t$, for any $1 \le i \le n$. Solve the system of equations

$$\begin{aligned} &U_k\,\lambda_l - U_l\,\lambda_k = 0 \text{ for any } 1 \le k < l \le n;\\ &A_k\,\lambda_k = 0 \text{ for any } 1 \le k \le n. \end{aligned} \tag{A.6}$$

  b) Consider a system of generators $\mathbf{H}_1, \dots, \mathbf{H}_m$ of the solutions of Equation (A.6).
  c) For any $\mathbf{H}_i = [\lambda_1, \dots, \lambda_n]$, $1 \le i \le m$, define the associated polynomial

$$\Lambda_{\mathbf{H}_i} = \sum_{k=1}^{n} \left(\sum_{j=1}^{t} \lambda_j^k \int_k b_j|_{y_{k+1}=\cdots=y_n=0} \right).$$

  d) If $\Lambda_{\mathbf{H}_1} \notin \langle D_d \rangle_{\mathbf{k}}$, then $b_{s_d+1} := \Lambda_{\mathbf{H}_1}$ and $D := D, b_{s_d+1}$. Repeat the procedure for $\Lambda_{\mathbf{H}_2}, \dots, \Lambda_{\mathbf{H}_m}$. If no new polynomials appear in this step, set $test := false$.
  e) Set s_{d+1} as the number of elements in D.
  f) For any $1 \le k \le n$, define $s_{d+1} \times s_{d+1}$ matrices U'_k and $m \times s_{d+1}$ matrices A'_k. Set $U'_k[i] := U_k[i]$ and $A'_k[i] := A_k[i]$ for $1 \le i \le s_d$.
  g) For $s_d + 1 \le i \le s_{d+1}$, compute $\mu_j^i \in \mathbf{k}$ such that $x_k \circ b_i = \sum_{j=1}^{s_d} \mu_j^i b_j$ and set

$$U'_k[i] := [\mu_1^i, \cdots, \mu_{s_d}^i], \quad A'_k[i] := [\left(f_1 \circ \int_k b_i\,|_{y_{k+1}=\cdots=y_n=0}\right)(0), \dots, \left(f_m \circ \int_k b_i\,|_{y_{k+1}=\cdots=y_n=0}\right)(0)].$$

  h) Set $d := d + 1$, $U_k := U'_k$ and $A_k := A'_k$.

```
> L[1];
_[1]=1
_[2]=x
_[3]=y+x2
_[4]=xy+x3
//Second element of the list:
//list of contraction matrices.
> print(L[2][1]);
0,1,0,0,
0,0,1,0,
0,0,0,1,
0,0,0,0
> print(L[2][2]);
0,0,1,0,
0,0,0,1,
0,0,0,0,
0,0,0,0
```

A.1.4 Comparison of methods

Let us now compare the computation times of the previous algorithms.

EXAMPLE A.1.8 Let us compute the inverse system of $I = (x^4, x^2y^3 - xy^5, y^5 - xy)$.

```
> LIB "GorensteinCovers.lib";
> ring r=0,(x,y),ds;
> ideal i=x4,x2y3-xy5,y5-xy;
> kinvSystNC(i);
_[1]=1
_[2]=x
_[3]=x2
_[4]=x3
_[5]=y
_[6]=xy+y5
_[7]=y2
_[8]=xy2+y6
_[9]=y3
_[10]=xy3+y7
_[11]=y4
_[12]=xy4+y8
> inverseSystem(i);
_[1]=xy4+y8
_[2]=xy3+y7
_[3]=xy2+y6
_[4]=xy+y5
_[5]=y4
_[6]=y3
_[7]=y2
_[8]=y
_[9]=x3
_[10]=x2
_[11]=x
_[12]=1
> list L=integrate(i);
> L[1];
_[1]=1
_[2]=x
_[3]=y
_[4]=y2
_[5]=x2
_[6]=y3
_[7]=x3
_[8]=y4
_[9]=xy+y5
_[10]=xy2+y6
_[11]=xy3+y7
```

```
_[12]=xy4+y8
> print(L[2][1]);
0,1,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,1,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,1,0,0,0,
0,0,0,0,0,0,0,0,0,1,0,0,
0,0,0,0,0,0,1,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,1,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,1,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0
> print(L[2][2]);
0,0,1,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,1,0,0,0,
0,0,0,1,0,0,0,0,0,0,0,0,
0,0,0,0,0,1,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,1,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,1,0,0,0,
0,0,0,0,0,0,0,0,0,1,0,0,
0,0,0,0,0,0,0,0,0,0,1,0,
0,0,0,0,0,0,0,0,0,0,0,1,
0,0,0,0,0,0,0,0,0,0,0,0
```

TABLE A.1 Computation times of the inverse system of $I = (x^4, x^2y^3 - xy^5, y^5 - xy)$.

Procedure	time (ms)
`invSystNC`	0
`inverseSystem`	0
`integrate`	290

EXAMPLE A.1.9 Let us check the computation times of the inverse system of the monomial ideal $I = (x^7, x^6y^3, x^2y^6, y^7)$.

TABLE A.2 Computation times of the inverse system of $I = (x^7, x^6y^3, x^2y^6, y^7)$.

Procedure	time (ms)
`invSystNC`	20
`inverseSystem`	0
`invSystMon`	0
`integrate`	3810

EXAMPLE A.1.10 Compute the inverse system of

$$I = (x^3y, xz^3t - zt^2, x^2t^2, y^5, z^6, t^3, x^4)$$

and compare its computation times using different methods.

```
LIB "GorensteinCovers.lib";
ring r=0,(x,y,z,t),ds;
ideal i=x3y,xz3t-zt2,x2t2,
y5,z6,t3,x4;
hilb(std(i));
//          1 t^0
//         -2 t^3
//         -2 t^4
//          2 t^5
//         -2 t^7
//          6 t^8
//         -2 t^10
//         -1 t^11
//          1 t^12
//         -4 t^14
//          3 t^15
//          1 t^0
//          4 t^1
//         10 t^2
//         18 t^3
//         25 t^4
//         30 t^5
//         32 t^6
//         28 t^7
//         21 t^8
//         14 t^9
//          8 t^10
//          3 t^11
// dimension (local)   = 0
// multiplicity = 194
```

TABLE A.3 Computation times of the inverse system of $I = (x^3y, xz^3t - zt^2, x^2t^2, y^5, z^6, t^3, x^4)$.

Procedure	time (ms)
`invSystNC`	7530
`inverseSystem`	30
`integrate`	3274940

Let us perform a rough analysis of the arithmetic complexity of these methods to understand better the experimental results we obtain. On one hand, Algorithm 8 has been deeply studied by Mourrain in [39, Proposition 4.1]:

PROPOSITION A.1.11 The total number of arithmetic operations in Algorithm 8 for computing the inverse system $I^\perp$ of an $\mathfrak{m}$-primary ideal I of R is bounded by

$$\mathcal{O}\left((n^2+m)t^3 + n^2 mLt^2\right),$$

where $n = \dim R$, $m = \mu(I)$, $t = \ell(R/I)$ and $L = \binom{n+s}{n}$ is the number of monomials of degree at most s, where $s = \operatorname{socdeg}(R/I)$.

On the other hand, observe that both Algorithm 5 and Algorithm 6 share the first step: the computation of the socle degree s of R/I. In order to do so, it is required, at least, to compute a standard basis of the ideal I. Mayr and Meyer established in [38] that the complexity of the computation of standard basis is doubly exponential in the number of variables in the worst key scenario. However, in this zero-dimensional scenario and the degree-reverse lexicographical ordering, complexity can be reduced, see [25]. Quoting [25]: ”In practice the computations are generally much faster and much feasible that with any other ordering.”

For a thorough analysis of the complexity of Algorithm 5 and Algorithm 6 we should look into their steps in detail. This is out of the scope of this appendix, but let us point out what should be taken into account in each algorithm.

As for Algorithm 5, besides the computation of the socle degree of R/I, there is a kernel computation the arithmetic complexity of which is bounded by $\mathcal{O}\left(mL^3/2\right)$, using the notation from Proposition A.1.11. See [7, Section 2.3.1] for more details on the number of operations.

Regarding Algorithm 6, the complexity is virtually the same complexity as computing the standard basis of the input ideal.

Finally, although the algorithm that appears to be faster in practice is Algorithm 6, for our purposes of finding Gorenstein covers Algorithm 8 is more suitable because it provides an adapted $\mathbf{k}$-basis as outcome.

A.2 Computation of the integral of a module with respect to an ideal

The computation of the integral of the inverse system with respect to a power of the maximal ideal is a key step towards the study of the $MGC(A)$ variety. The available methods to compute inverse systems also provide two essentially different algorithms to compute $\int_{\mathfrak{m}^t} I^\perp$: the 2-duals formula and the integration method.

The 2-duals formula is based in the following result:

PROPOSITION A.2.1 (Proposition 3.1.2) Let M be a finitely generated sub-R-module of S and let K be an ideal of R. Then

$$\int_K M = \left(KM^\perp\right)^\perp. \tag{A.7}$$

Observe that Proposition 3.1.2 gives a much more general formula that applies not only to inverse systems and powers of maximal ideals:

```
> LIB "GorensteinCovers.lib";
> ring r=0,(x(1..3)),ds;
> ideal M=x(1)*x(2),x(3)^3;
> ideal K=x(1),x(2),x(3)^2;
> integral(K,M);
_[1]=x(1)^2
_[2]=x(1)*x(2)
_[3]=x(2)^2
_[4]=x(1)*x(3)^2
_[5]=x(2)*x(3)^2
_[6]=x(3)^5
```

On the other hand, the integration method allows us to avoid the computation of two duals by using Algorithm 1, a generalization of Algorithm 8.

EXAMPLE A.2.2 Consider the sub-R-module $M = \langle y_1y_2, y_3^3\rangle$. To compute the integral $\int_{\mathfrak{m}} M$ in *Singular* using Equation (A.7) we choose the procedure `integral` in **GorensteinCovers.lib**:

```
> LIB "GorensteinCovers.lib";
> ring r=0,(x(1..3)),ds;
> ideal M=x(1)*x(2),x(3)^3;
> ideal K=maxideal(1);
> integral(K,M);
_[1]=x(1)^2
```

```
_[2]=x(1)*x(2)
_[3]=x(2)^2
_[4]=x(1)*x(3)
_[5]=x(2)*x(3)
_[6]=x(3)^4
```

The integration method to compute $\int_{\mathfrak{m}^t} I^\perp$ is naturally used in a setting where we are given the ideal I. Algorithm 1 requires as input both a $\mathbf{k}$-basis of $I^\perp$ and its associated contraction matrices. Therefore, to compute $\int_{\mathfrak{m}} M$ with this algorithm we need some extra steps. First, we need to retrieve $I = \mathrm{Ann}_{\mathrm{R}}(M)$. This computation can be performed using procedure `idealAnnNC` in **InverseSyst.lib**, see [13]. Second, given I, `integrate` provides the desired $\mathbf{k}$-basis and matrices of $I^\perp$. Then the procedure `integrationStep` in **GorensteinCovers.lib** gives a $\mathbf{k}$-basis and contraction matrices of $\int_{\mathfrak{m}} I^\perp$.

```
// Compute the annihilator of M
// in R
> ideal I=idealAnnNC(M);
// Compute a k-basis of the inverse
// system of I and its corresponding
// contraction matrices
> list R=integrate(I);
//Use the previous output to
> list L=integrationStep(R[1],R[2]);
//k-basis of the integral
>L[1];
_[1]=1
_[2]=x(1)
_[3]=x(2)
_[4]=x(3)
_[5]=x(3)^2
_[6]=x(1)*x(2)
_[7]=x(3)^3
_[8]=x(3)^4
_[9]=x(2)*x(3)
_[10]=x(1)*x(3)
_[11]=x(2)^2
_[12]=x(1)^2
//Contraction matrices
print(L[2][1]);
0,1,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,1,
0,0,0,0,0,1,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,1,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0
print(L[2][2]);
0,0,1,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,1,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,1,0,
0,0,0,0,0,0,0,0,1,0,0,0,
```

```
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0
print(L[2][3]);
0,0,0,1,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,1,0,0,
0,0,0,0,0,0,0,0,1,0,0,0,
0,0,0,0,1,0,0,0,0,0,0,0,
0,0,0,0,0,0,1,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,1,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0
//Adapted k-basis of L_{A,1}
>L[3];
_[1]=x(3)^4
_[2]=x(2)*x(3)
_[3]=x(1)*x(3)
_[4]=x(2)^2
_[5]=x(1)^2
```

TABLE A.4 Computation times of $\int_{\mathfrak{m}} M$.

Procedure	time (ms)
`integral`	40
`integrationStep`	180

Besides computational complexity, in our context of computing minimal Gorenstein covers of $A = R/I$, a relevant advantage of Algorithm 1 is that the output provides an adapted $\mathbf{k}$-basis of $\mathcal{L}_{A,t}$, see Definition 3.2.4.

A.3 Computation of minimal covers

The most relevant procedures are `teterVariety` and `MGC2` that implement algorithms Algorithm 2 and Algorithm 3 to provide the variety of minimal Gorenstein covers of rings A with $\mathrm{gcl}(A) = 1$ and $\mathrm{gcl}(A) = 2$, respectively.

Next we will provide some detailed examples on how to study the Gorenstein colength of a given ring $A = R/I$.

EXAMPLE A.3.1 [Teter ring] Let us compute the minimal Gorenstein cover variety of $A = R/I$, where $I = (x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2, x_1^5)$.

```
> ring r=0,(x(1..3)),ds;
> ideal i=x(1)*x(2),x(1)*x(3),
x(2)^2,x(2)*x(3),x(3)^2,x(1)^5;
> hilb(std(i));
//         1 t^0
//        -5 t^2
//         6 t^3
//        -2 t^4
//        -1 t^5
//         2 t^6
//        -1 t^7
//         1 t^0
//         3 t^1
//         1 t^2
//         1 t^3
//         1 t^4
// dimension (local)   = 0
// multiplicity = 7
> cmType(i);
3
```

Observe that it is a stretched algebra (see Appendix B), hence $\mathrm{gcl}(A) = \mathrm{embd}(A) - \tau(A) + 1 = 3 - 3 + 1 = 1$ by Proposition 5.1.2. We check anyway whether $\mathrm{gcl}(A) = 1$ with procedure `isTeter`:

```
> isTeter(i);
1
```

Since $\mathrm{gcl}(A) = 1$, we can compute the Teter variety of A:

```
ideal a=teterVariety(i);
Dimension of the projective space where the Teter variety lies:
5
Ideal of non-Teter covers:
a(2)^2*a(6)-a(1)*a(4)*a(6)
Polynomial H defining Teter covers:
a(6)*x(1)^5+a(5)*x(1)*x(2)+a(4)*x(2)^2+a(3)*x(1)*x(3)
+a(2)*x(2)*x(3)+a(1)*x(3)^2
```

Therefore, $MGC(A) = \mathbb{P}^5_{\mathbf{k}} \backslash \mathbb{V}_+(a_2^2a_6 - a_1a_4a_6)$ and each point $(a_1 : a_2 : a_3 : a_4 : a_5 : a_6) \in MGC(A)$ is identified with a polynomial

$$H = a_1y_3^5 + a_2y_2y_3 + a_3y_1y_3 + a_4y_2^2 + a_5y_1y_2 + a_6y_1^5.$$

Hence each minimal Gorenstein cover of A is of the form $G = R/\operatorname{Ann}_R H$, where H satisfies $a_2^2 a_6 - a_1 a_4 a_6 \neq 0$.

EXAMPLE A.3.2 [Ring of Gorenstein colength 2] Let us compute the minimal Gorenstein cover variety of $A = R/I$, where $I = (x_1^2, x_1 x_2^3, x_2^5)$.

```
> LIB "GorensteinCovers.lib";
> ring r=0,(x,y),ds;
> ideal i=x2,xy3,y5;
> hilb(std(i));
//          1 t^0
//         -1 t^2
//         -1 t^4
//          1 t^6
```

```
//          1 t^0
//          2 t^1
//          2 t^2
//          2 t^3
//          1 t^4
// dimension (local)   = 0
// multiplicity = 8
> isTeter(i);
2
```

Since $\operatorname{gcl}(R/I) > 1$, now let us study the set MGC_2:

```
> def a,c,D,b=MGC2(i);
```

A first test is whether $\overline{\mathbb{V}_+(\mathfrak{b})\backslash\mathbb{V}_+(\widehat{\mathfrak{d}})}$ is empty or not.

```
> quotient(b,D);
_[1]=b(4)
```

Since $\overline{\mathbb{V}_+(\mathfrak{b})\backslash\mathbb{V}_+(\widehat{\mathfrak{d}})} = \mathbb{V}_+(b_4)$, it is possible that $\mathbb{V}_+(\mathfrak{b})\backslash\mathbb{V}_+(\widehat{\mathfrak{d}}) \neq \emptyset$. Let us do an extra step to understand where $MGC_2(A)$ lies:

```
> ring s=basering; //To be able to retrieve ideals b,D afterwards
> setring r;
> def D,H=candidate(i);
> D*H;
_[1,1]=b(3)*x(2)^6+b(2)*x(1)*x(2)^4+a(1)*x(2)^5+a(2)*x(1)*x(2)^3
+b(4)*x(1)^3+b(1)*x(1)^2*x(2)+a(3)*x(1)^2
```

The points in $MGC_2(A)$ have coordinates $(a_1 : a_2 : a_3 : b_1 : b_2 : b_3 : b_4)$ and are identified with polynomials of the form

$$H = a_1 y_2^5 + a_2 y_1 y_2^3 + a_3 y_1^2 + b_1 y_1^2 y_2 + b_2 y_1 y_2^4 + b_3 y_2^6 + b_4 y_1^3$$

with some restrictions on the coefficients. Hence $MGC_2(A) \subset \mathbb{P}^6_{\mathbf{k}}$. Observe that, since $b_4 = 0$ in $\overline{MGC_2(A)}$, we will be able to reduce by 1 the dimension where $MGC_2(A)$ is embedded.

Next we need to study $\mathbb{V}_+(\mathfrak{b})\backslash\mathbb{V}_+(\widehat{\mathfrak{d}})$:

```
> setring s;
> b;
b[1]=b(3)*b(4)
b[2]=b(2)*b(4)
> D;
D[1]=b(3)*b(4)
D[2]=b(2)*b(4)
D[3]=b(2)^6*b(3)^2
D[4]=b(2)^7*b(3)
D[5]=b(1)*b(2)^6*b(3)
D[6]=b(2)^8
D[7]=b(1)*b(2)^7
D[8]=a(1)*b(2)^7-a(2)*b(2)^6*b(3)
D[9]=b(1)^2*b(2)^6
D[10]=a(2)*b(1)*b(2)^6-a(3)*b(2)^7
D[11]=a(1)*b(1)*b(2)^6
-a(3)*b(2)^6*b(3)
```

```
//Primary decomposition of b
> primdecGTZ(b);
[1]:
   [1]: //primary component
      _[1]=b(4)
   [2]: //radical ideal
      _[1]=b(4)
[2]:
   [1]: //primary component
      _[1]=b(3)
      _[2]=b(2)
   [2]: //radical ideal
      _[1]=b(3)
      _[2]=b(2)
> radical(D);
_[1]=b(2)
_[2]=b(3)*b(4)
```

Since $\mathfrak{b} = (b_4) \cap (b_2, b_3)$, then

$$\mathbb{V}_+(\mathfrak{b}) = \mathbb{V}_+(b_4) \cup \mathbb{V}_+(b_2, b_3),$$

hence the set $MGC_2(A)$ is

$$\mathbb{V}_+(b_4) \cup \mathbb{V}_+(b_2, b_3)\backslash\mathbb{V}_+(b_2, b_3b_4) = \mathbb{V}_+(b_4)\backslash\mathbb{V}_+(b_2, b_3b_4) \simeq \mathbb{P}^5_{\mathbf{k}}\backslash\mathbb{V}_+(b_2).$$

To sum up, $\operatorname{gcl}(A) = 2$ and its minimal Gorenstein covers are rings $G = R/\operatorname{Ann}_R H$, where

$$H = a_1y_2^5 + a_2y_1y_2^3 + a_3y_1^2 + b_1y_1^2y_2 + b_2y_1y_2^4 + b_3y_2^6$$

with $b_2 \neq 0$. H is identified with the point $(a_1 : a_2 : a_3 : b_1 : b_2 : b_3)$ in $\mathbb{P}^5_{\mathbf{k}}\backslash\mathbb{V}_+(b_2)$.

Observe that if $b_3 = 0$, the Hilbert function associated to G is $\{1, 2, 2, 2, 2, 1\}$. Otherwise, if $b_3 \neq 0$, then $\mathrm{HF}_G = \{1, 2, 2, 2, 1, 1, 1\}$.

EXAMPLE A.3.3 [Ring of higher colength] Let us compute the minimal Gorenstein cover variety of $I = (x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^4)$.

```
> ring r=0,(x,y),ds;
> ideal i=x3,x2y,xy2,y4;
> hilb(std(i));
//          1 t^0
//         -3 t^3
//          1 t^4
//          1 t^5

//          1 t^0
//          2 t^1
//          3 t^2
//          1 t^3
// dimension (local)   = 0
// multiplicity = 7
> isTeter(i);
2
> def a,c,D,b=MGC2(i);
> quotient(b,D);
_[1]=1
```

Since $\overline{\mathbb{V}_+(\mathfrak{b})\backslash\mathbb{V}_+(\widehat{\mathfrak{d}})} = \mathbb{V}_+(1) = \emptyset$, then $MGC_2 = \emptyset$. Therefore, $\mathrm{gcl}(A) > 2$. To compute both its Gorenstein colength and its minimal Gorenstein covers we need to another approach. Since $A = R/I$ is a codimension two ideal, we can apply tools from Section 4.2.

A.4 Commands

We end this appendix by providing a list of the main procedures contained in **GorensteinCovers.lib**, together with a brief description of its usage.

```
INVERSE SYSTEMS

kinvSystNC(ideal I)
Computes inverse system of the ideal I. Returns:
[1] k-basis of the inverse system of I (ideal K).

inverseSystem(ideal I)
Computes inverse system of the ideal I. Returns:
[1] k-basis of the inverse system of I (ideal K).
```

```
invSystMon(ideal I)
Computes inverse system of the ideal I. Returns:
[1] k-basis of the inverse system of I (ideal K).
      OR -1 if the ring is not monomial.

integrate(ideal I)
Computes inverse system of the ideal I and its contraction matrices using
the integration method. Returns a list R with:
[1] R[1]=k-basis of the inverse system of I (ideal D),
[2] R[2]=list of contraction matrices of the inverse system (list LU, matrices U).

INTEGRAL OF A MODULE WITH RESPECT TO AN IDEAL

integral(ideal K,ideal M)
Computes the integral of the module M with respect to the ideal K. Returns:
[1] sub-R-module of S (treated as ideal).

integrationStep(ideal D,list LU)
Computes the integral of the sub-R-module M of S with respect to the maximal
ideal of R. The input is a k-basis of M (treated as ideal b) and the contraction
matrices (list of matrices LU) associated to this k-basis. Returns a list L with:
[1] L[1]=k-basis of the integral (sub-R-module of S, treated as ideal D),
[2] L[2]=list of contraction matrices of the integral (list LU of matrices U)
associated to k-basis L[1].
[3] L[3]=adapted k-basis of the quotient of the integral by the inverse system.

MINIMAL GORENSTEIN COVERS VARIETIES

isTeter(ideal I)
Checks wheter a ring A=R/I is Teter or not. Returns:
[1] integer
 * 0, if gcl(A)=0;
* 1, if gcl(A)=1;
* 2, if gcl(A)>1.

teterVariety(ideal I)
```

```
Given a Teter ring A=R/I, computes its Teter variety. Returns:
[1] an integer h-1
[2] an ideal a such that MGC(A)=P^(h-1)\V_+(a)

MGC2(ideal id)
Given a ring A=R/I with gcl(A)>1, computes MGC_2(A). Returns:
[1] ideal a,
[2] ideal c,
[3] ideal D,
[4] ideal b
such that MGC_2=V_+(b)\V_+(D), pi_1(V_+(c))=V_+(b),
pi_1(V_+(c)\cap V_+(a))=V_+(D).
```

APPENDIX B

Stretched and almost stretched algebras

This appendix intends to provide a summary on structure theorems for stretched and almost stretched Artin $\mathbf{k}$-algebras. Knowing the exact expressions of such rings has been particularly useful in Chapter 5 to determine the Gorenstein colength of some rings, see Section 5.1.1, or to study the unicity of the Hilbert functions of minimal Gorenstein covers of A, see Section 5.2.

Sally already proved in [42] that the analytic type of a stretched algebra $A = R/I$ is determined by its Cohen-Macaulay type. Elias and Valla provide in [21] a complete structure theorem for the generators of the defining ideal I of each analytic type with $\mathrm{HF}_A = \{1, n, 1, \dots, 1\}$.

Regarding almost stretched algebras, Elias and I provide in [15] a complete analytic classification of all Gorenstein such algebras, extending results from [21] and [22].

B.1 Basic notions

Let us start by defining stretched and almost stretched rings.

DEFINITION B.1.1 Let $A = R/I$ be a local Artin ring with maximal ideal $\mathfrak{n}$, $\mathrm{embd}(A) = \dim R = n$ and socle degree $s \geq 2$. We say that A is **stretched** if $\mathfrak{n}^2$ is a principal ideal.

Recall that $\mu\left(\mathfrak{n}^2\right) = \dim_{\mathbf{k}} \mathfrak{n}^2/\mathfrak{n}^3 = \mathrm{HF}_A(2)$, thus the Hilbert function of a stretched ring A is completely determined by its embedding dimension n and its socle degree s:

$$\mathrm{HF}_A(i) = \begin{cases} 1, & \text{if } i = 0; \\ n, & \text{if } i = 1; \\ 1, & \text{if } i = 2, \ldots, s; \\ 0, & \text{if } i \geq s + 1. \end{cases}$$

DEFINITION B.1.2 Let $A = R/I$ be a local Artin ring with maximal ideal $\mathfrak{n}$, $\mathrm{embd}(A) = \dim R = n$ and socle degree $s \geq 2$. We say that A is **almost stretched** if $\mathfrak{n}^2$ is minimally generated by two elements.

Therefore, if A is almost stretched, then $\mathrm{HF}_A(2) = 2$. In addition, if A is Gorenstein, then $s \geq 3$ and $\mathrm{HF}_A(s) = 1$. In this case, the Hilbert function is

$$\mathrm{HF}_A(i) = \begin{cases} 1, & \text{if } i = 0; \\ n, & \text{if } i = 1; \\ 2, & \text{if } i = 2, \ldots, t; \\ 1, & \text{if } i = t+1, \ldots, s; \\ 0, & \text{if } i \geq s + 1; \end{cases}$$

for some $2 \leq t < s$.

DEFINITION B.1.3 If an algebra has this Hilbert function we say that it is **of type** (s, t). We say that a pair (s, t), $3 \leq t + 1 \leq s$, is **regular** if there is not an integer r such $2(r + 1) = s - t + 1$.

Since both structure theorems in [21] and [15] classify rings in terms of their analytic type, let us now provide a precise definition of what it means.

DEFINITION B.1.4 Consider two $\mathbf{k}$-algebras $A_i = \mathbf{k}[\![x_1, \ldots x_n]\!]/I_i$, for $i = 1, 2$. We say that $\varphi : A_1 \longrightarrow A_2$ is an **analytic k-algebra morphism** if

(i) $\varphi|_{\mathbf{k}} = \mathrm{Id}$, and
(ii) φ is a ring morphism.

Note that giving an analytic morphism of Artin $\mathbf{k}$-algebras is equivalent to giving a substitution of variables by polynomials.

DEFINITION B.1.5 Consider a $\mathbf{k}$-algebra morphism $\varphi : A_1 \longrightarrow A_2$. We say that φ is an **analytic k-algebra isomorphism** if exists a morphism $\psi : A_2 \longrightarrow A_1$ such that $\varphi \circ \psi = \mathrm{Id}_{A_2}$ and $\psi \circ \varphi = \mathrm{Id}_{A_1}$. This will be denoted by $A_1 \cong_\varphi A_2$.

Observe that an analytic isomorphism is precisely a change of coordinates.

DEFINITION B.1.6 We say that two Artin $\mathbf{k}$-algebras A_1 and A_2 have the same **analytic type** if there exists an analytic $\mathbf{k}$-algebra isomorphism between A_1 and A_2.

Therefore, A_1 and A_2 have the same analytic type if and only if they only differ by a change of coordinates.

B.2 Structure theorem for Artinian local stretched rings

We reproduce here the structure theorem for generators of the defining ideal I of A depending on their analytic type:

THEOREM B.2.1 [21, 3.1] Let $A = R/I$ be a local Artin of socle degree s, $\mathrm{embd}(A) = \dim R = n$ and $\mathrm{char}(\mathbf{k}) = 0$. Let $\tau := \tau(A)$ be the Cohen-Macaulay type of A.

(i) $1 \leq \tau \leq n$.

(ii) If $\tau < n$, then we can find a basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{m}$ such that I is minimally generated by the elements

$$\{x_i x_j\}_{1 \leq i < j \leq n}, \ \{x_j^2\}_{2 \leq j \leq \tau} \text{ and } \{x_i^2 - u_i x_1^s\}_{\tau+1 \leq i \leq n},$$

where $u_i \in R^*$.

(iii) If $\tau = n$, then we can find a basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{m}$ such that I is minimally generated by the elements

$$\{x_i x_j\}_{1 \leq i < j \leq n}, \ \{x_1 x_j\}_{2 \leq j \leq n} \text{ and } x_1^{s+1}.$$

B.3 Analytic classification of GAAS algebras

In this section we reproduce the key result obtained in [15]. The aim of the paper is to provide a complete analytic classification of Gorenstein Artin almost stretched algebras over a zero characteristic field $\mathbf{k}$, called GAAS for short.

In [22] Elias and Valla give a complete characterization of the analytic types of Gorenstein Artin almost stretched algebras under the assumption $s \geq 2t$. For a general pair (t, s), $s \geq 2t$, there are finitely many analytic types and for some special pairs (s, t) there are finitely many analytic types plus finitely many of one-dimensional families of analytic types.

To remove the restriction on the type of A, the techniques considered are Grauert's division theorem (see Theorem 1.5.12), the multivariate Hensel's lemma and the resolution process of a zero-dimensional scheme.

DEFINITION B.3.1 [15, Definition 2.2] For all $s \geq t+1$ we denote by $J_{s,t}$ the ideal generated by

$$x_i x_j, 1 \leq i < j \leq n, (i,j) \neq (1,2); \quad x_j^2 - x_1^s, 3 \leq j \leq n; \quad x_1^t x_2.$$

Given $w \in R$, for all integer $0 \leq q \leq t-1$ we denote by $I_{q,w}$ the ideal of R generated by $J_{s,t}$ and $x_2^2 - x_1^{q+1}x_2 - w x_1^{s-t+1}$; I_∞ is the ideal generated by $J_{s,t}$ and $x_2^2 - x_1^{s-t+1}$.

THEOREM B.3.2 [15, Theorem 4.9] Let $A = R/I$ be a Gorenstein Artin almost stretched algebra of type (s,t) with $3 \leq t+1 \leq s$.
If (s,t) is regular or $s \geq 3t-1$, then I is isomorphic to one and only one of the following ideals:

$$I_{0,1}; \ I_{1,1}; \dots; \ I_{\min\{t-1,s-t-1\},1} = I_\infty.$$

Assume that (s,t) is non-regular and $s \leq 3t-2$. Let r be the integer such that $s-t+1 = 2(r+1)$.
If $r = 0$, then I is isomorphic to one and only one of the following t ideals:

$$I_{0,1}; \quad I_{0,-1/4}; \quad I_{0,-1/4+x_1^d}, d = 1, \cdots, t-2.$$

If $1 \leq r \leq (t-2)/2$, then the different isomorphism classes of I are

$$I_{0,1};\ \cdots;\ I_{r-1,1};\ I_{r+1,1};\ \ldots;\ I_{\min\{t-1,s-t\},1} = I_\infty, \text{ and}$$

(i) $I_{r,a};\ I_{r,a+x_1^d}, d = 1, \cdots, r$, if $a \neq 0, \frac{-1}{4}, \frac{-(r+1)}{2(t+r+1)}$,
(ii) $I_{r,\frac{-1}{4}};\ I_{r,\frac{-1}{4}+x_1^d}, d = 1, \cdots, t-r-2$,
(iii) $I_{r,\frac{-(r+1)}{2(t+r+1)}};\ I_{r,\frac{-(r+1)}{2(t+r+1)}+x_1^d}, d = 1, \cdots, r-1$.

If $r \geq (t-1)/2$, then I is isomorphic to one and only one of the following ideals:

$$I_{0,1};\ \ldots;\ I_{r-1,1};\ \{I_{r,a}\}_{a\in\mathbf{k}^*};\ \{I_{r,a+x_1}\}_{a\in\mathbf{k}^*};\ \ldots;\ \{I_{r,a+x_1^{t-r-2}}\}_{a\in\mathbf{k}^*};$$

$$I_{r+1,1};\ \ldots;\ I_{\min\{t-1,s-t\},1} = I_\infty.$$

REMARK B.3.3 The algebraically closed condition on $\mathbf{k}$ is used in [15, Proposition 2.3] because to prove that certain ideals have the same analytic type we need to ensure the existence of square roots.

Notice that the first case for which there is a continuous moduli is $(s,t) = (7,4)$, i.e. a non-regular case with $r = 1$ and $s \leq 2t-1$:

EXAMPLE B.3.4 [15, Example 4.10] Let us consider the non-regular case $(s,t) = (7,4)$, i.e. $r = 1$ and $s \leq 2t-1$. Notice that this case is not covered by [22] because $s \leq 2t-1$. The analytic types are are defined by the ideals

$$I_{0,1};\ I_{2,1};\ I_{3,1};\ I_{1,a}, a \neq 0;\ I_{1,a+x_1}, a \neq 0, \frac{-1}{6}.$$

Notice that the continuous moduli are parametrized by $\mathbf{k} - \{0\}$ and $\mathbf{k} - \{0, -1/6\}$.

EXAMPLE B.3.5 Let us consider the regular case $(s,t) = (5,3)$. This case is not covered by [22] because $s \leq 2t-1$. The analytic types are are defined by two ideals: $I_{0,1}, I_{1,1}$.

If $n = 2$, which corresponds to the Hilbert function $\{1,2,2,2,1,1\}$, then there are the following two possible ideals:

$$I_{0,1} = (x^3y, y^2 - xy - x^3),\ I_{1,1} = (x^3y, y^2 - x^2y - x^3).$$

APPENDIX C

Varieties of minimal Gorenstein covers

In this appendix we list the varieties of minimal Gorenstein covers of all analytic types of $\mathbf{k}$-algebras of low Gorenstein colength that appear in Poonen's classification, see [40]. This means that we provide an explicit description for any $MGC(A)$ for $\operatorname{gcl}(A) = 1, 2$ and $\ell(A) \leq 6$. We assume $\operatorname{char}(\mathbf{k}) = 0$ for the sake of simplicity.

For every representative $A = R/I$ of an analytic type, we give the general form of a polynomial H in $\mathcal{L}_{A,t}$, with $t = 1, 2$, and the expression of $MGC(A)$. By Theorem 3.3.2, $G = R/\operatorname{Ann}_{\mathrm{R}} H$ is a minimal Gorenstein cover of A if and only if $[H]$ in $MGC(A)$, by taking the coefficients of H as coordinates in $MGC(A)$.

All the computations are done using our implementation of Algorithm 2 and Algorithm 3 in *Singular*, using library **GorensteinCovers.lib**, see Appendix A.

C.1 Teter varieties

Let us describe the variety of minimal Gorenstein cover for any Teter ring A with $\ell(A) \leq 6$.

Case 2 of $\ell(A) = 3$: $H = a_3 y_1^2 + a_2 y_1 y_2 + a_1 y_2^2 \in \mathcal{L}_{A,1}$,

$$(a_1 : a_2 : a_3) \in MGC(A) = \mathbb{P}^2_{\mathbf{k}} \backslash \mathbb{V}_+(a_2^2 - a_1 a_3).$$

Case 3 of $\ell(A) = 4$: $H = a_3 y_1^3 + a_2 y_1 y_2 + a_1 y_2^2 \in \mathcal{L}_{A,1}$,

$$(a_1 : a_2 : a_3) \in MGC(A) = \mathbb{P}^2_{\mathbf{k}} \backslash \mathbb{V}_+(a_1 a_3).$$

Case 4 of $\ell(A) = 4$: $H = a_6y_1^2 + a_5y_1y_2 + a_4y_2^2 + a_3y_1y_3 + a_2y_2y_3 + a_1y_3^2 \in \mathcal{L}_{A,1}$,

$$(a_1 : \cdots : a_6) \in MGC(A) = \mathbb{P}^5_{\mathbf{k}} \backslash \mathbb{V}_+(a_3^2a_4 - 2a_2a_3a_5 + a_1a_5^2 + a_2^2a_6 - a_1a_4a_6).$$

Case 3 of $\ell(A) = 5$: $H = a_1y_2^4 + a_3y_1^2 + a_2y_1y_2 \in \mathcal{L}_{A,1}$,

$$(a_1 : a_2 : a_3) \in MGC(A) = \mathbb{P}^2_{\mathbf{k}} \backslash \mathbb{V}_+(a_1a_3).$$

Case 4 of $\ell(A) = 5$: $H = a_3y_1^3 + a_1y_2^3 + a_2y_1y_2 \in \mathcal{L}_{A,1}$,

$$(a_1 : a_2 : a_3) \in MGC(A) = \mathbb{P}^2_{\mathbf{k}} \backslash \mathbb{V}_+(a_1a_3).$$

Case 5 of $\ell(A) = 5$: $H = a_1y_1y_2^2 + a_2y_2^3 + a_3y_1^2 \in \mathcal{L}_{A,1}$,

$$(a_1 : a_2 : a_3) \in MGC(A) = \mathbb{P}^2_{\mathbf{k}} \backslash \mathbb{V}_+(a_1).$$

Case 8 of $\ell(A) = 5$: $H = a_6y_1^3 + a_5y_1y_2 + a_4y_2^2 + a_3y_1y_3 + a_2y_2y_3 + a_1y_3^2 \in \mathcal{L}_{A,1}$,

$$(a_1 : a_2 : a_3 : a_4 : a_5 : a_6) \in MGC(A) = \mathbb{P}^5_{\mathbf{k}} \backslash \mathbb{V}_+(a_2^2a_6 - a_1a_4a_6).$$

Case 9 of $\ell(A) = 5$: $H = a_{10}y_1^2 + a_9y_1y_2 + a_8y_2^2 + a_7y_1y_3 + a_6y_2y_3 + a_5y_3^2 + a_4y_1y_4 + a_3y_2y_4 + a_2y_3y_4 + a_1y_4^2 \in \mathcal{L}_{A,1}$,

$$(a_1 : \cdots : a_{10}) \in MGC(A) = \mathbb{P}^9_{\mathbf{k}} \backslash \mathbb{V}_+(\mathfrak{a}),$$

where the ideal $\mathfrak{a}$ is generated by

```
a(4)^2*a(6)^2-2*a(3)*a(4)*a(6)*a(7)+a(3)^2*a(7)^2-a(4)^2*a(5)*a(8)
+2*a(2)*a(4)*a(7)*a(8)-a(1)*a(7)^2*a(8)+2*a(3)*a(4)*a(5)*a(9)-2*a(2)*a(4)*a(6)*a(9)
-2*a(2)*a(3)*a(7)*a(9)+2*a(1)*a(6)*a(7)*a(9)+a(2)^2*a(9)^2-a(1)*a(5)*a(9)^2
-a(3)^2*a(5)*a(10)+2*a(2)*a(3)*a(6)*a(10)-a(1)*a(6)^2*a(10)
-a(2)^2*a(8)*a(10)+a(1)*a(5)*a(8)*a(10)
```

Case 3 of $\ell(A) = 6$: $H = a_3y_1^5 + a_2y_1y_2 + a_1y_2^2 \in \mathcal{L}_{A,1}$,

$$(a_1 : a_2 : a_3) \in MGC(A) = \mathbb{P}^2_{\mathbf{k}} \backslash \mathbb{V}_+(a_1a_3).$$

Case 6 of $\ell(A)=6$: $H = a_1y_2^4 + a_3y_1^3 + a_2y_1y_2 \in \mathcal{L}_{A,1}$,

$$(a_1 : a_2 : a_3) \in MGC(A) = \mathbb{P}^2_{\mathbf{k}} \backslash \mathbb{V}_+(a_1a_3).$$

Case 8 of $\ell(A)=6$: $H = a_1y_2^4 - a_1y_1^2y_2 + a_2y_1y_2^2 + a_3y_2^3 \in \mathcal{L}_{A,1}$,

$$(a_1 : a_2 : a_3) \in MGC(A) = \mathbb{P}^2_{\mathbf{k}} \backslash \mathbb{V}_+(a_1).$$

Case 12 of $\ell(A)=6$: $H = a_6y_1^4 + a_5y_1y_2 + a_4y_2^2 + a_3y_1y_3 + a_2y_2y_3 + a_1y_3^2 \in \mathcal{L}_{A,1}$,

$$(a_1 : a_2 : a_3 : a_4 : a_5 : a_6) \in MGC(A) = \mathbb{P}^5_{\mathbf{k}} \backslash \mathbb{V}_+(a_2^2a_6 - a_1a_4a_6).$$

Case 19 of $\ell(A)=6$: $H = a_6y_1^2 + a_5y_1y_2 + a_4y_2^2 + a_3y_1y_3 + a_2y_3^3 + a_1y_2y_3^2 \in \mathcal{L}_{A,1}$,

$$(a_1 : a_2 : a_3 : a_4 : a_5 : a_6) \in MGC(A) = \mathbb{P}^5_{\mathbf{k}} \backslash \mathbb{V}_+(a_1a_6).$$

Case 20 of $\ell(A)=6$: $H = a_4y_2^3 + a_1y_3^3 + a_6y_1^2 + a_5y_1y_2 + a_3y_1y_3 + a_2y_2y_3 \in \mathcal{L}_{A,1}$,

$$(a_1 : a_2 : a_3 : a_4 : a_5 : a_6) \in MGC(A) = \mathbb{P}^5_{\mathbf{k}} \backslash \mathbb{V}_+(a_1a_4a_6).$$

See Example 3.5.1.

Case 24 of $\ell(A)=6$: $H = a_1y_4^2 + a_2y_3y_4 + a_3y_2y_4 + a_4y_1y_4 + a_5y_3^2 + a_6y_2y_3 + a_7y_1y_3 + a_8y_2^2 + a_9y_1y_2 + a_{10}y_1^3 \in \mathcal{L}_{A,1}$,

$$(a_1 : \cdots : a_{10}) \in MGC(A) = \mathbb{P}^9_{\mathbf{k}} \backslash \mathbb{V}_+(\mathfrak{a}),$$

where $\mathfrak{a} = (a_3^2a_5a_{10} - 2a_2a_3a_6a_{10} + a_1a_6^2a_{10} + a_2^2a_8a_{10} - a_1a_5a_8a_{10})$.

Case 25 of $\ell(A)=6$: $H = a_1y_5^2 + a_2y_4y_5 + a_3y_3y_5 + a_4y_2y_5 + a_5y_1y_5 + a_6y_4^2 + a_7y_3y_4 + a_8y_2y_4 + a_9y_1y_4 + a_{10}y_3^2 + a_{11}y_2y_3 + a_{12}y_1y_3 + a_{13}y_2^2 + a_{14}y_1y_2 + a_{15}y_1^2 \in \mathcal{L}_{A,1}$,

$$(a_1 : \cdots : a_{15}) \in MGC(A) = \mathbb{P}^{14}_{\mathbf{k}} \backslash \mathbb{V}_+(\mathfrak{a}),$$

where the ideal $\mathfrak{a}$ is generated by:

```
a(5)^2*a(8)^2*a(10)-2*a(4)*a(5)*a(8)*a(9)*a(10)+a(4)^2*a(9)^2*a(10)
-2*a(5)^2*a(7)*a(8)*a(11)+2*a(4)*a(5)*a(7)*a(9)*a(11)+2*a(3)*a(5)*a(8)*a(9)*a(11)
```

```
-2*a(3)*a(4)*a(9)^2*a(11+a(5)^2*a(6)*a(11)^2-2*a(2)*a(5)*a(9)*a(11)^2
+a(1)*a(9)^2*a(11)^2+2*a(4)*a(5)*a(7)*a(8)*a(12)-2*a(3)*a(5)*a(8)^2*a(12)
-2*a(4)^2*a(7)*a(9)*a(12)+2*a(3)*a(4)*a(8)*a(9)*a(12)-2*a(4)*a(5)*a(6)*a(11)*a(12)
+2*a(2)*a(5)*a(8)*a(11)*a(12)+2*a(2)*a(4)*a(9)*a(11)*a(12)-2*a(1)*a(8)*a(9)*a(11)*a(12
+a(4)^2*a(6)*a(12)^2-2*a(2)*a(4)*a(8)*a(12)^2+a(1)*a(8)^2*a(12)^2+a(5)^2*a(7)^2*a(13)
-2*a(3)*a(5)*a(7)*a(9)*a(13)+a(3)^2*a(9)^2*a(13)-a(5)^2*a(6)*a(10)*a(13)
+2*a(2)*a(5)*a(9)*a(10)*a(13)-a(1)*a(9)^2*a(10)*a(13)+2*a(3)*a(5)*a(6)*a(12)*a(13)
-2*a(2)*a(5)*a(7)*a(12)*a(13)-2*a(2)*a(3)*a(9)*a(12)*a(13)+2*a(1)*a(7)*a(9)*a(12)*a(13)
+a(2)^2*a(12)^2*a(13)-a(1)*a(6)*a(12)^2*a(13)-2*a(4)*a(5)*a(7)^2*a(14)
+2*a(3)*a(5)*a(7)*a(8)*a(14)+2*a(3)*a(4)*a(7)*a(9)*a(14)-2*a(3)^2*a(8)*a(9)*a(14)
+2*a(4)*a(5)*a(6)*a(10)*a(14)-2*a(2)*a(5)*a(8)*a(10)*a(14)-2*a(2)*a(4)*a(9)*a(10)*a(14)
+2*a(1)*a(8)*a(9)*a(10)*a(14)-2*a(3)*a(5)*a(6)*a(11)*a(14)+2*a(2)*a(5)*a(7)*a(11)*a(14)
+2*a(2)*a(3)*a(9)*a(11)*a(14)-2*a(1)*a(7)*a(9)*a(11)*a(14)-2*a(3)*a(4)*a(6)*a(12)*a(14)
+2*a(2)*a(4)*a(7)*a(12)*a(14)+2*a(2)*a(3)*a(8)*a(12)*a(14)-2*a(1)*a(7)*a(8)*a(12)*a(14)
-2*a(2)^2*a(11)*a(12)*a(14)+2*a(1)*a(6)*a(11)*a(12)*a(14)+a(3)^2*a(6)*a(14)^2
-2*a(2)*a(3)*a(7)*a(14)^2+a(1)*a(7)^2*a(14)^2+a(2)^2*a(10)*a(14)^2-a(1)*a(6)*a(10)*a(14)^2
+a(4)^2*a(7)^2*a(15)-2*a(3)*a(4)*a(7)*a(8)*a(15)+a(3)^2*a(8)^2*a(15)-a(4)^2*a(6)*a(10)*a(15)
+2*a(2)*a(4)*a(8)*a(10)*a(15)-a(1)*a(8)^2*a(10)*a(15)+2*a(3)*a(4)*a(6)*a(11)*a(15)
-2*a(2)*a(4)*a(7)*a(11)*a(15)-2*a(2)*a(3)*a(8)*a(11)*a(15)+2*a(1)*a(7)*a(8)*a(11)*a(15)
+a(2)^2*a(11)^2*a(15)-a(1)*a(6)*a(11)^2*a(15)-a(3)^2*a(6)*a(13)*a(15)
+2*a(2)*a(3)*a(7)*a(13)*a(15)-a(1)*a(7)^2*a(13)*a(15)-a(2)^2*a(10)*a(13)*a(15)
+a(1)*a(6)*a(10)*a(13)*a(15)
```

C.2 Gorenstein colength 2

In this section, we study the variety of minimal Gorenstein cover for any ring A of Gorenstein colength 2 with $\ell(A) \leq 6$.

Recall that minimal Gorenstein covers of Teter rings have a unique Hilbert function, see Theorem 2.0.4, but in Gorenstein colength 2 we cannot deduce unicity from Theorem 2.2.5. As a side effect of the computation of $\mathcal{L}_{A,2}$ and $MGC(A)$, we obtain information on the possible Hilbert functions of any minimal Gorenstein cover $G = R/\operatorname{Ann}_R H$ of A. The socle degree of any minimal Gorenstein cover G cannot be higher than the degree of any polynomial in $\mathcal{L}_{A,2}$. In particular, for any $H \in MGC(A)$, $\operatorname{socdeg} R/\operatorname{Ann}_R H = \deg H$.

Along with $MGC(A)$, we also provide all possible Hilbert functions of any minimal Gorenstein cover. Note that in some cases, $MGC(A)$ has a too long description to be included here, hence we only give the generic form of a polynomial H in $\mathcal{L}_{A,2}$.

Case 7 of $\ell(A) = 5$: $H = a_1 y_3^2 + a_2 y_1 y_3 + a_3 y_2^2 + a_4 y_1 y_2 + a_5 y_1^2 + b_3 y_2^2 y_3 + b_5 y_3^3 +$

$b_6y_2y_3^2+b_9y_2^3$, $(a_1:a_2:a_3:a_4:a_5:b_3:b_5:b_6:b_9)\in MGC(A)$,

$$MGC(A)=\mathbb{V}_+(b_3b_6-b_5b_9)\setminus(\mathbb{V}_+(a_5)\cup\mathbb{V}_+(\mathfrak{d}))\subset\mathbb{P}^8,$$

where $\mathfrak{d}=(b_5^2b_9-b_6^3,b_3b_5-b_6^2,b_3^2-b_6b_9)$.
If $[H]\in MGC(A)$, then b_3,b_5,b_6,b_9 do not vanish simultaneously, hence $\operatorname{socdeg} G=3$. Unique Hilbert function for any minimal Gorenstein cover G: $\mathrm{HF}_G=\{1,3,2,1\}$. See Example 3.5.2.

Case 7 of $\ell(A)=6$: $H=b_3y_2^5+b_2y_1y_2^3+a_1y_2^4+b_1y_1^2y_2+a_2y_1y_2^2+a_3y_1^2$,

$$(a_1:a_2:a_3:b_1:b_2:b_3)\in MGA(A)=\mathbb{P}^5_{\mathbf{k}}\setminus\mathbb{V}_+(b_2^2-b_1b_3).$$

If $b_3\neq 0$, then $\deg H=5$. Otherwise, if $b_3=0$, then $b_2\neq 0$ and $\deg H=4$. Therefore, minimal Gorenstein covers G can have one of the following two Hilbert functions: $\mathrm{HF}_G=\{1,2,2,2,1\}$ and $\mathrm{HF}_G=\{1,2,2,1,1\}$. See Example 2.1.9.

Case 11 of $\ell(A)=6$: $H=a_1y_3^2+a_2y_2y_3+a_3y_1y_3+a_4y_2^2+a_5y_1y_2+b_3y_1y_3^2+b_5y_3^3+b_6y_1^2y_3-b_{10}y_1^4$, $(a_1:a_2:a_3:a_4:a_5:b_3:b_5:b_6:b_{10})\in MGC(A)$,

$$MGC(A)=\mathbb{V}_+(b_3^2-b_5b_6+b_3b_{10})\setminus(\mathbb{V}_+(a_4)\cup\mathbb{V}_+(b_{10})\cup\mathbb{V}_+(b_3,b_5))\subset\mathbb{P}^8.$$

If $[H]\in MGC(A)$, then $b_{10}\neq 0$, hence $\operatorname{socdeg} R/\operatorname{Ann_R} H=4$. Unique Hilbert function for any minimal Gorenstein cover G: $\mathrm{HF}_G=\{1,3,2,1,1\}$. See Example 3.5.3.

Case 13 of $\ell(A)=6$: $H=a_5y_1^3+b_9y_1^2y_2+b_5y_1y_2^2+b_7y_1y_2^2-b_8y_2^3+b_5y_1^2y_3+a_4y_1y_2+a_3y_1y_3+a_2y_2y_3+a_1y_3^2$, $(a_1:a_2:a_3:a_4:a_5:b_5:b_7:b_8:b_9)\in MGC(A)$,

$$MGC(A)=\mathbb{V}_+(b_5b_7+b_7^2+b_8b_9)\setminus\mathbb{V}_+(\mathfrak{d})\subset\mathbb{P}^8,$$

where $\mathfrak{d}=(b_5b_8,b_7^4b_8+3b_7^2b_8^2b_9+2b_8^3b_9^2,b_5^6-b_7^6+7b_7^2b_8^2b_9^2+6b_8^3b_9^3)$.
If $[H]\in MGC(A)$, then b_5,b_7,b_8,b_9 do not vanish simultaneously, hence $\operatorname{socdeg} G=3$. Unique Hilbert function for any minimal Gorenstein cover G: $\mathrm{HF}_G=\{1,3,3,1\}$.

Case 14 of $\ell(A)=6$: $H=a_1y_3^2+a_2y_2y_3+a_3y_1y_3+a_4y_1y_2+b_1y_1y_2y_3-b_2(y_2^2y_3+y_2^3)+b_3y_2y_3^2+b_4y_3^3+b_5(y_2^2y_3-y_1y_2^2-y_1^2y_3+y_2^3)+b_6(y_2^2y_3+y_2y_3^2+y_1y_3^2+y_2^3)+b_7(y_1y_2^2-y_1^3)-b_8y_2^3+b_9(y_2^3-y_1^2y_2)-b_{10}y_1^3\in\mathcal{L}_{A,2}$. Since $\deg H\leq 3$ for any $H\in\mathcal{L}_{A,2}$, then $\operatorname{socdeg} G\leq 3$ and there is a unique Hilbert function for any minimal cover: $\mathrm{HF}=\{1,3,3,1\}$.

Case 15 of $\ell(A)=6$: $H = a_5y_1^3 + b_8y_2^3 + b_2y_2^2y_3 + b_6y_2y_3^2 + b_4y_3^3 + a_4y_1y_2 + a_3y_2^2 + a_2y_1y_3 + a_1y_3^2$, $(a_1 : a_2 : a_3 : a_4 : a_5 : b_2 : b_4 : b_6 : b_8) \in MGC(A)$,

$$MGC(A) = \mathbb{V}_+(b_2b_6 - b_4b_8)\setminus(\mathbb{V}_+(a_5)\cup\mathbb{V}_+(\mathfrak{d})) \subset \mathbb{P}^8,$$

where $\mathfrak{d} = (-b_6^3 + b_4^2b_8, b_2b_6 - b_4b_8, b_2b_4 - b_6^2, b_2^2 - b_6b_8)$.
If $[H] \in MGC(A)$, then $a_5 \neq 0$ and hence $\operatorname{socdeg} G = 3$. Unique Hilbert function for any minimal Gorenstein cover G: $\mathrm{HF}_G = \{1,3,3,1\}$.

Case 16 of $\ell(A)=6$: $H = a_1y_3^2 + a_2y_2y_3 + a_3y_1y_3 + a_4y_1y_2 + b_1y_1y_2y_3 + b_2(y_1y_2^2 + y_1y_3^2) + b_3y_2y_3^2 + b_4y_3^3 - b_5(y_3^3 + y_1^2y_3) + b_6(y_2^2y_3 + y_3^3) + b_7(y_1y_2^2 - y_1^3) - b_8y_2^3 + b_9(y_2^3 - y_1^2y_2) - b_{10}y_1^3 \in \mathcal{L}_{A,2}$. Since $\deg H \leq 3$ for any $H \in \mathcal{L}_{A,2}$, then $\operatorname{socdeg} G \leq 3$ and there is a unique Hilbert function for any minimal cover: $\mathrm{HF} = \{1,3,3,1\}$.

Case 17 of $\ell(A)=6$: $H = a_1y_3^2 + a_2y_2y_3 + a_3y_1y_2 + a_4y_1^2 + b_1(y_1^2y_3 + y_1y_2y_3 - y_1y_2^2) - b_2y_2^3 + b_3y_2y_3^2 + b_4y_3^3 + b_5(y_3^3 + y_1y_2^2 + y_1y_3^2) + b_6(y_2^2y_3 + y_3^3) + +b_7y_1^2y_2 + b_8y_1y_2^2 - b_9y_2^3 + b_{10}y_1^3 \in \mathcal{L}_{A,2}$ Since $\deg H \leq 3$ for any $H \in \mathcal{L}_{A,2}$, then $\operatorname{socdeg} G \leq 3$ and there is a unique Hilbert function for any minimal cover: $\mathrm{HF} = \{1,3,3,1\}$.

Case 18 of $\ell(A)=6$: $H = b_1y_1^2y_3 + b_2y_1y_2y_3 + b_3y_2^2y_3 + b_4y_3^3 + a_1y_3^2 + a_2y_2^2 + a_3y_1y_2 + a_4y_1^2$,

$$(a_1 : a_2 : a_3 : a_4 : b_1 : b_2 : b_3 : b_4) \in MGC(A) = \mathbb{P}^7\setminus\mathbb{V}_+(b_2^2 - b_1b_3).$$

If $[H] \in MGC(A)$, then b_1, b_2, b_3 do not vanish simultaneously, hence $\operatorname{socdeg} G = 3$. Unique Hilbert function for any minimal Gorenstein cover G: $\mathrm{HF}_G = \{1,3,3,1\}$. See Example 3.5.4.

Case 23 of $\ell(A)=6$: $H = b_{16}y_3^3 + b_6y_3^2y_4 + b_{10}y_3y_4^2 + b_9y_4^3 + a_9y_1^2 + a_8y_1y_2 + a_7y_2^2 + a_6y_1y_3 + a_5y_2y_3 + a_4y_3^2 + a_3y_1y_4 + a_2y_2y_4 + a_1y_4^2$, $(a_1 : \cdots : a_9 : b_6 : b_9 : b_{10} : b_{16}) \in MGC(A)$,

$$MGC(A) = \mathbb{V}_+(b_6b_{10} - b_9b_{16})\setminus(\mathbb{V}_+(d_1)\cup\mathbb{V}_+(d_2)) \subset \mathbb{P}^{12},$$

where $d_1 = (a_7a_9 - a_8^2)$ and $d_2 = (b_9^2b_{16} - b_{10}^3, b_6b_9 - b_{10}^2, b_6^2 - b_{10}b_{16})$. If $[H]$ is in $MGC(A)$, then b_6, b_9, b_{10}, b_{16} do not vanish simultaneously, hence $\operatorname{socdeg} G = 3$. Unique Hilbert function for any minimal Gorenstein cover G: $\mathrm{HF}_G = \{1,4,3,1\}$. See Example 3.5.5.

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