

CHARACTERIZATION OF PRODUCTS OF THETA DIVISORS

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ABSTRACT. We study products of irreducible theta divisors from two points of view. On the one hand, we characterize them as normal subvarieties of abelian varieties such that a desingularization has holomorphic Euler characteristic 1. On the other hand, we identify them up to birational equivalence among all varieties of maximal Albanese dimension. We also describe the structure of varieties X of maximal Albanese dimension, with holomorphic Euler characteristic 1 and irregularity $2 \dim X - 1$.

INTRODUCTION

Given an abelian variety A and an ample line bundle L on A , we say that (A, L) is a principally polarized abelian variety if L has only one section up to constants. The effective divisor associated to L and its translates are called theta divisors.

Irreducible theta divisors of dimension 1 are smooth genus 2 curves. In dimension 2, they are birational to the symmetric product of a curve of genus 3. More precisely, if the genus 3 curve C is non-hyperelliptic they are isomorphic to $C^{(2)}$, while if the curve is hyperelliptic, they have an isolated singularity and $C^{(2)}$ is their natural desingularization. In higher dimensions, they may also be singular, but Ein–Lazarsfeld proved in [EL] that they are always normal and they have only rational singularities. Hence, given a theta divisor $\Theta \subset A$, any desingularization $X \rightarrow \Theta$ has

$$\chi(X, \omega_X) = \chi(\Theta, \omega_\Theta) = h^0(A, \mathcal{O}_A(\Theta)) = 1.$$

Clearly, A is the Albanese variety of X , and the induced morphism $X \rightarrow A$ is its Albanese morphism. Thus, desingularizations of theta divisors are varieties of maximal Albanese dimension, i.e., varieties X such that the Albanese morphism $a_X : X \rightarrow A$ is generically finite onto its image.

The characterization up to birational equivalence of irreducible theta divisors among varieties of maximal Albanese dimension started with the work of Hacon [Ha] and it has been further refined in [HP1, LP, BLNP, Pa] (see the end of Section 3 for more precise references).

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Naturally, if X is a desingularization of a product of irreducible theta divisors, then we also have $\chi(X, \omega_X) = 1$. Indeed, Beauville (in the appendix to [De]) characterized products of two curves of genus 2, as smooth projective surfaces of general type S with irregularity 4 and $\chi(S, \omega_S) = 1$. This characterization of products of curves of genus 2 (irreducible theta divisors of dimension 1) was extended to higher dimensional varieties of maximal Albanese dimension by Hacon–Pardini [HP3].

We are interested here in giving a more general characterization of products of irreducible theta divisors. First we identify them inside arbitrary abelian varieties as normal subvarieties such that a desingularization has holomorphic Euler characteristic 1.

Theorem A. *Let Y be a subvariety of an abelian variety A . Let X be a desingularization of Y . Then, Y is isomorphic to a product of theta divisors if, and only if, Y is normal and $\chi(X, \omega_X) = 1$.*

On the one hand, studying subvarieties of abelian varieties is more general than studying just the images of smooth projective varieties under the Albanese morphism. For example, a singular curve inside an abelian variety is never the image under an Albanese morphism.

On the other hand, given a smooth projective variety, even if it is of maximal Albanese dimension, the Albanese morphism does not need to induce a birational map onto its image.

Indeed, if we consider a subvariety Y of an abelian variety A such that Y is not of general type, then Y is fibred by tori (see [Ue, Theorem 10.9]). Hence, in this case, a desingularization X of Y will have $\chi(X, \omega_X) = 0$. Nevertheless, there exist smooth projective varieties X of maximal Albanese dimension such that $\chi(X, \omega_X) = 0$ and X is of general type. Clearly, in this case the Albanese morphism does not induce a birational map. The first example was constructed by Ein–Lazarsfeld [EL, Example 1.13] and Chen–Debarre–Jiang have studied in detail these varieties in [CDJ]. In particular, it is shown in [CDJ] that the Ein–Lazarsfeld example is essentially the only one in dimension 3.

In any case, if X is a smooth projective variety of maximal Albanese dimension with $\chi(X, \omega_X) = 1$ such that the Albanese image of X is of general type, then the Albanese morphism induces a birational equivalence (see Lemma 1.3). This allows us to characterize products of irreducible theta divisors up to birational equivalence by using a slightly less restrictive version of Theorem A (see Theorem 3.7).

Theorem B. *Let X be a smooth projective variety. Then X is birational to a product of irreducible theta divisors if and only if X satisfies the following conditions:*

- (C.1) X is of maximal Albanese dimension;
- (C.2) $\chi(X, \omega_X) = 1$;
- (C.3) the Albanese image of X is of general type;
- (C.4) the Albanese image of X is smooth in codimension 1.

It is an open question to determine whether condition (C.4) is necessary or not (see Question 3.6).

Let X be a smooth projective variety X of maximal Albanese dimension and $\chi(X, \omega_X) = 1$. Hacon–Pardini show in [HP3] that the irregularity $q(X) = h^0(X, \Omega_X^1)$ sits in the following range:

$$\dim X \leq q(X) \leq 2 \dim X,$$

and, when $q(X)$ is maximal, i.e., $q(X) = 2 \dim X$, X is birational to a product of curves of genus 2 (theta divisors of dimension 1). Hence condition (C.4) is not necessary when $q(X) = 2 \dim X$. We give a complete classification of the submaximal irregularity case, i.e., $q(X) = 2 \dim X - 1$ (see Theorem 3.1), which implies that condition (C.4) is also not necessary when $q(X) = 2 \dim X - 1$.

Theorem C. *Let X be a smooth projective variety of maximal Albanese dimension. Assume that $\chi(X, \omega_X) = 1$ and $q(X) = 2 \dim X - 1$. Then X is birational to one of the following varieties:*

- (i) a product of smooth curves of genus 2 with the 2-dimensional symmetric product of a curve of genus 3;
- (ii) $(C_1 \times \tilde{Z})/\langle \tau \rangle$, where C_1 is a bielliptic curve of genus 2, $\tilde{Z} \rightarrow C_1 \times \cdots \times C_{n-1}$ is an étale double cover of a product of smooth projective curves of genus 2, and τ is an involution acting diagonally on C_1 and \tilde{Z} via the involutions corresponding respectively to the double covers.

This result generalizes to higher dimensions the classification of surfaces S of general type with $p_g(S) = q(S) = 3$ established independently by Hacon–Pardini [HP2] and Pirola [Pi].

The classification of the next step, i.e., $q(X) = 2 \dim X - 2$, seems rather difficult since in this case there does not exist such a nice classification even for surfaces ($\dim X = 2$). Indeed, besides the general example of the double cover of a principally polarized abelian variety (A, Θ) ramified along a smooth divisor in the linear system $|2\Theta|$, there are other more complicated surfaces satisfying the hypothesis (see [CH2] and [PePo1, PePo2]).

Notation. Throughout this paper we work over the complex numbers. More precisely, a variety or a subvariety is a separated scheme of finite type over \mathbb{C} which is integral, i.e., irreducible and

reduced, but not necessarily smooth. In particular, we will say that a variety is a theta divisor, if it is an irreducible divisor that induces a principal polarization on an abelian variety.

We denote by $a_X : X \rightarrow A_X$ the *Albanese morphism* of a smooth projective variety X . We say that a variety X is *irregular* if a_X is non-trivial, i.e., $q(X) := h^0(X, \Omega_X^1) \neq 0$. When a_X is generically finite onto its image, we will say that X is of *maximal Albanese dimension*. Given an abelian variety A , we will call \widehat{A} its dual, usually interpreted as $\text{Pic}^0 A$. For a morphism $a : X \rightarrow A$ to an abelian variety and a coherent sheaf \mathcal{F} on X , we denote by $V^i(\mathcal{F}, a)$ the *i -th cohomological support loci*:

$$V^i(\mathcal{F}, a) := \{P \in \widehat{A} \mid H^i(X, \mathcal{F} \otimes a^*P) \neq 0\}.$$

When $a = a_X$, we will often omit it from the notation, i.e., $V^i(\mathcal{F}) := V^i(\mathcal{F}, a_X)$. We also use $V_0^i(\mathcal{F}, a)$ to denote the components of $V^i(\mathcal{F}, a)$ passing through the identity element of \widehat{A} .

This paper uses the full machinery of generic vanishing theory, applied to irregular varieties (see [Pa] for a nice survey). For example, we will often use that the irreducible components of $V^i(\omega_X, a)$ are abelian subvarieties of $\widehat{B} \subseteq \widehat{A}$ translated by a torsion point $P \in \widehat{A}$, so we will usually denote them by $P + \widehat{B}$ (see [GL1, GL2, Si]). Moreover, if X is of maximal Albanese dimension, then $\text{codim}_{\widehat{A}} V^i(\omega_X, a) \geq i$ for all $i \geq 0$ (see [GL1]). We will also use the *derivative complex* introduced in [GL2], under the form used in [EL]. When \mathcal{G} is a sheaf on an abelian variety A , we will say that \mathcal{G} is a *GV-sheaf*, if $\text{codim}_{\widehat{A}} V^i(\mathcal{G}) \geq i$ for all $i \geq 0$ (see [PaPo3, Proposition/Definition 2.1]) and we will say that \mathcal{G} is *M -regular*, if $\text{codim}_{\widehat{A}} V^i(\mathcal{G}) > i$ for all $i > 0$ (see [PaPo1, Definition 2.1]).

For a smooth projective variety X , we denote by $D^b(X)$ the bounded derived category of coherent sheaves on X . We refer to [Hu] for basic notions on derived categories.

Plan of the paper. The paper is organized as follows. Section 1 collects some basic facts about morphisms $a : X \rightarrow A$ from a smooth projective variety X to an abelian variety A . In particular, we see what happens when $a(X)$ is of general type and $\chi(X, \omega_X) = 1$ (Section 1.1). We also study fibrations where the general fiber is a theta divisor (Section 1.2). Some of these results might be well-known to the experts.

Section 2 concerns the case of subvarieties of abelian varieties $Y \subset A$. We let $X \xrightarrow{f} Y$ be a desingularization of Y . First we review a result of Pareschi (Section 2.1) that allows us to restrict to the case where $V^1(\omega_X, a)$ has positive dimension. In Section 2.2, we consider the case where Y has codimension 1 in A . Finally, in Section 2.3, we consider the general case where Y is smooth in codimension 1. The main result of this section is Theorem 2.8 and we deduce Theorem A from it. Theorem 2.8 is proven by induction on the number of irreducible components of $V_0^1(\omega_X, a_X)$. Hence, the irreducible components of $V_0^1(\omega_X, a_X)$ and their relation with the irreducible components of $V^i(\omega_X, a)$ for $i \geq 1$, play a major role in Section 2.

Section 3 concerns the case where X is a smooth projective variety of maximal Albanese dimension with $\chi(X, \omega_X) = 1$. First, we classify these varieties when they have submaximal

irregularity. We use Fourier-Mukai techniques to prove Proposition 3.2, which is the main technical tool to prove the main Theorem 3.1 of this section (which corresponds to Theorem C). Finally, we recall a question of Pareschi (see Question 3.6) and we deduce Theorem B (see Theorem 3.7) from Theorem 2.8.

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1. PRELIMINARIES

The following list of equivalences is certainly well-known to the experts. We state it for easy reference since it will be needed repeatedly hereafter.

Lemma 1.1. *Let $a : X \rightarrow A$ be a generically finite morphism from X a smooth projective variety to $Y := a(X)$ a subvariety of an abelian variety A . Then the following conditions are equivalent:*

- (i) Y is not fibred by positive dimensional abelian varieties;
- (ii) Y is of general type;
- (iii) $a_*\omega_X$ is a M -regular sheaf;
- (iv) for any $f : V \rightarrow X$ surjective morphism where V is smooth projective, any direct summand of $a_*R^j f_*\omega_V$ is M -regular for any $j \geq 0$;

Moreover, the previous equivalent conditions imply that $\chi(X, \omega_X) > 0$.

Proof. The equivalence of (i) and (ii) is due to Ueno (see [Ue, Theorem 10.9] or [Mo, Theorem 3.7]).

We now show that (i) is equivalent to (iii).

If (i) holds and $a_*\omega_X$ is not M -regular, then there exists an irreducible component T of $V^i(a_*\omega_X)$ of codimension i in \widehat{A} for some $i \geq 1$, by generic vanishing (e.g., [HP3, Theorem 2.2]). Then as in [EL, Proof of Theorem 3, p. 249] and using [GL2, §4], we get that Y is fibred by the fibres of the natural fibration $A \rightarrow \widehat{T}$, which is a contradiction (see also [EL, Remark 1.6]).

Now assume that (iii) holds. Let K be the maximal abelian subvariety of A such that $K + Y = Y$. Let $j = \dim K$. By [EL, Theorem 3], the image Z of the projection $Y \rightarrow A/K$ is of general type and given any desingularization $\mu : Z' \rightarrow Z$, we have $\chi(Z', \omega_{Z'}) > 0$. Consider an appropriate birational model of the Stein factorization of the morphism $X \rightarrow Y \rightarrow Z$, such that there exists Z'' a smooth projective variety that admits a generically finite and surjective

morphism $g : Z'' \rightarrow Z'$ and a fibration $h : X \rightarrow Z''$. Observe that $\chi(Z'', \omega_{Z''}) \geq \chi(Z', \omega_{Z'}) > 0$. We have $\omega_{Z''} \simeq R^j h_* \omega_X$ ([Ko1, Proposition 7.6]). Since Z'' has positive Euler characteristic, we have that

$$\widehat{A/K} = V^0(\omega_{Z''}, \mu \circ g) = V^0(R^j h_* \omega_X, \mu \circ g).$$

Thus, Kollár's theorem [Ko2, Theorem 3.1], or the refined version by Hacon–Pardini [HP3, Theorem 2.1], implies that the codimension- j abelian subvariety $\widehat{A/K} \subset \widehat{A}$ is an irreducible component of $V^j(a_* \omega_X)$. Therefore, $j = 0$ and Y is not fibred by positive dimensional abelian varieties.

On the one hand, condition (iv) clearly implies (iii). On the other hand, if $a_* R^j f_* \omega_V$ is not M-regular, then there exists an irreducible component T of $V^i(a_* R^j f_* \omega_V)$ of codimension i in \widehat{A} for some $i \geq 1$, by [HP3, Theorem 2.2]. As before, the same argument as in [EL, Proof of Theorem 3, p. 249] using [GL2, §4], yields to Y being fibred by the fibres of the natural fibration $A \rightarrow \widehat{T}$, which is a contradiction. Since a direct summand of a M-regular sheaf is M-regular, we are done.

In any case, since X is of maximal Albanese dimension, then $\chi(X, \omega_X) \geq 0$ (see [GL1, Corollary, p. 390]). Moreover, by [EL, Theorem 3], (i) implies $\chi(X, \omega_X) > 0$. \square

Remark 1.2.

- (i) On the one hand, in the notation of Lemma 1.1, if $\chi(X, \omega_X) > 0$, then X is of general type. This is certainly not the converse to the last statement of Lemma 1.1, since X of general type does not imply that Y is of general type (see [EL, Example 1.13]).

The fact that $\chi(X, \omega_X) > 0$ implies that X is of general type follows from [Ca] or [CH1, Corollary 2.4]¹.

- (ii) On the other hand, the fact that the equivalent conditions imply $\chi(A, a_* \omega_X) = \chi(X, \omega_X) > 0$ can be also deduced via a more general argument from the fact that $a_* \omega_X$ is a M-regular sheaf.

Indeed, we recall that if \mathcal{G} is a non-trivial M-regular sheaf on A , then $V^0(\mathcal{G}) = \widehat{A}$ (see for instance [Pa, Lem 1.12]) and, by the invariance of the Euler characteristic, $\chi(A, \mathcal{G}) > 0$.

From now on, we will always have $a : X \rightarrow A$ a generically finite morphism from X a smooth projective variety to $Y := a(X)$ a subvariety of an abelian variety A . For simplicity we will assume that no translate of Y is contained in any abelian subvariety of A , i.e., Y generates A . We will view this setting from two different angles.

- (i) The first point of view considers Y given and let $f : X \rightarrow Y$ be a desingularization. Thus, $f : X \rightarrow Y$ will always be a birational morphism, but $a : X \rightarrow A$ is not

¹There is a small misprint in the statement of this corollary and $q(X)$ should be $\dim A$.

necessarily the Albanese morphism of X .

This will be treated in Section 2.

- (ii) The second point of view considers X a given variety of maximal Albanese dimension and let $a : X \rightarrow A$ be the Albanese morphism, i.e., $a_X : X \rightarrow A_X$. In this case, $f : X \rightarrow Y$ is not necessarily birational, since it may have degree greater than 1.

This will be treated mainly in Section 3.

Note that, in both cases X is of maximal Albanese dimension. Clearly, the study of the two problems is very interrelated.

Recall that by Lemma 1.1 and Remark 1.2(i), we have that $\chi(X, \omega_X) = 0$ implies that Y is not of general type, i.e., Y is fibred by tori. So, from both perspectives, we will be mainly interested in the “next” case, i.e., when $\chi(X, \omega_X) = 1$.

1.1. The hypothesis $\chi(X, \omega_X) = 1$. Let $a : X \rightarrow A$ be a generically finite morphism from X a smooth projective variety to $Y := a(X)$ a subvariety of an abelian variety A . Before trying to study our problem from the two different perspectives, we give some general results.

Lemma 1.3 ([Ti, Theorem 5.2.2]). *Let $a : X \rightarrow A$ be a generically finite morphism from X a smooth projective variety to $Y := a(X)$ a subvariety of an abelian variety A .*

If Y is of general type and $\chi(X, \omega_X) = 1$, then $a : X \rightarrow A$ induces a birational equivalence between X and Y .

Proof. We can consider a birational model of $f : X \rightarrow Y$ such that X and Y are smooth and we denote by $g : Y \rightarrow A$ the morphism that is birational onto its image. Then, since f is generically finite, the trace map $f_*\omega_X \rightarrow \omega_Y$ splits the natural inclusion $\omega_Y \rightarrow f_*\omega_X$. Hence $f_*\omega_X \simeq \omega_Y \oplus \mathcal{Q}$, where \mathcal{Q} is the kernel of the trace map, and $a_*\omega_X \simeq g_*\omega_Y \oplus g_*\mathcal{Q}$. By Lemma 1.1, $g_*\omega_Y$ and $a_*\omega_X$ are M-regular, so, by Remark 1.2(ii), either \mathcal{Q} is trivial and f is birational or $g_*\mathcal{Q}$ is M-regular and $\chi(A, g_*\mathcal{Q}) > 0$.

By [Ko2, Theorem 3.4], $\chi(A, a_*\omega_X) = \chi(X, \omega_X) = 1$ and $\chi(A, g_*\omega_Y) = \chi(Y, \omega_Y)$. Since, by hypothesis Y is of general type, Lemma 1.1 implies that $g_*\omega_Y$ is M-regular, so $\chi(A, g_*\omega_Y) > 0$. Thus, $\chi(A, g_*\mathcal{Q}) = 0$. Hence $\mathcal{Q} = 0$ and f is birational. \square

Combining Lemma 1.1 with [HP3, Lemma 3.2], we have

Lemma 1.4. *Let $a : X \rightarrow A$ be a generically finite morphism from X a smooth projective variety to $Y := a(X)$ a subvariety of an abelian variety A . Suppose Y is of general type and $\chi(X, \omega_X) = 1$.*

If T is any irreducible component of $V^1(\omega_X, a)$ of $\text{codim}_{\widehat{A}} T = i + 1$, then T is an irreducible component of $V^i(\omega_X, a)$.

Proof. By [HP3, Lemma 3.2], we know that $T \subset V^i(\omega_X, a)$. Since $a_*\omega_X$ is M-regular, T is an irreducible component of $V^i(a_*\omega_X) = V^i(\omega_X, a)$. \square

1.2. Fibrations in theta divisors. Under the hypothesis $\chi(X, \omega_X) = 1$, the main aim will be to construct a fibration from Y to some other variety such that the fibres are theta divisors of a fixed abelian subvariety of A . Fibrations by theta divisors are very special. In particular, they are isotrivial, so very close to being a product. Indeed, the following proposition shows that they are products under some mild extra assumptions (compare with [Zh, Theorem 5.2]). The idea of the proof goes back to [Se] and it was already used in [HP3, proof of Theorem 3.1].

Proposition 1.5. *Let $f : X \rightarrow Y$ be a fibration between smooth projective varieties of maximal Albanese dimension. Assume that a general fiber X_y is birational to a theta divisor Θ and $q(X) - q(Y) > \dim X - \dim Y$. Then X is birational to $\Theta \times Y$.*

Proof. We denote by K the kernel of the natural quotient $f_* : A_X \rightarrow A_Y$. Then $a_X|_{X_y} : X_y \rightarrow K_y$ is a generically finite morphism, where K_y is the corresponding translate of K in A_X . From the universal property of Albanese morphisms, we know that the induced morphism $A_{X_y} \rightarrow K_y$ is surjective. By hypothesis, $\dim K = q(Y) - q(X) \geq \dim X_y + 1 = \dim A_{X_y}$. Thus, A_{X_y} is isogeny onto K , so A_{X_y} is fixed, which implies that f is isotrivial.

Note that, since Θ has canonical singularities, the canonical model of X_y is fixed and isomorphic to Θ . Thus, the fibers of the relative canonical model $X' \xrightarrow{f} Y$ are isomorphic to Θ in an open Zariski subset of Y . Since Θ is of general type, $\text{Aut}(\Theta)$ is finite. Therefore, there exists a finite Galois cover $\widehat{Y} \rightarrow Y$ with group G such that the main component of $X' \times_Y \widehat{Y}$ is birational to $\Theta \times \widehat{Y}$. So we can consider the commutative diagram

$$\begin{array}{ccc} \Theta \times \widehat{Y} & \xrightarrow{\quad} & X' \\ \downarrow & \swarrow /G & \downarrow f' \\ \widehat{Y} & \xrightarrow{\quad} & Y, \\ & \searrow /G & \end{array}$$

where G acts on $\Theta \times \widehat{Y}$ diagonally and the horizontal morphisms are quotients by G .

Now, we consider $V := \Theta/G$ and the following commutative diagram

$$\begin{array}{ccccccc} & & W & \xrightarrow{\varphi} & V & \xrightarrow{a_V} & A_V \\ & \nearrow p_1 & & \searrow g & & \nearrow & \\ W \times \widehat{Y} & \longrightarrow & X' & \xrightarrow{a_{X'}} & A_{X'} & & \\ \downarrow p_2 & & \downarrow f' & & \downarrow & & \\ \widehat{Y} & \longrightarrow & Y & \xrightarrow{a_Y} & A_Y & & \end{array}$$

where W is birational to Θ and after birational modifications and abusing notation, we will assume that all varieties are smooth projective. We want to prove that φ is birational.

We denote by K the kernel of $A_{X'} \rightarrow A_Y$. Note that, $h^1(X', \mathcal{O}_{X'}) = (h^1(\widehat{Y}, \mathcal{O}_{\widehat{Y}}) \oplus h^1(W, \mathcal{O}_W))^G$. Hence, since $q(X) - q(Y) > \dim X - \dim Y$, then A_V is isogenous to K . In particular, $\dim K = \dim A_V = \dim \Theta + 1$.

Since W is birational to a theta divisor Θ , for any $P \in \widehat{A}_V$ such that φ^*t^*P is not trivial, we have $h^0(W, \omega_W \otimes \varphi^*t^*P) = 1$ and

$$h^i(W, \omega_W \otimes \varphi^*t^*P) = 0 \quad \text{for all } i \geq 1.$$

By Grauert–Riemenschneider vanishing, $R^i\varphi_*\omega_W = 0$ for all $i > 0$ and

$$\varphi_*\omega_W = \omega_V \oplus \mathcal{Q},$$

where \mathcal{Q} is a torsion-free sheaf.

Since Θ has only rational singularities by [EL, Theorem 1], $\dim V^i(\omega_W, t \circ \varphi) = 0$, so we have also that $\dim V^1(\omega_V, t) = 0$, using that $R^i\varphi_*\omega_W = 0$ for all $i > 0$. Hence, $t_*\omega_V$ is M-regular and then $V^0(\omega_V, t) = \text{Pic}^0(A_{X'}/B_2)$. Thus, for any $P \in \text{Pic}^0(A_{X'}/B_2)$ such that φ^*t^*P is not trivial, we also have $h^0(V, \omega_V \otimes t^*P) = 1$. Then $t_*\mathcal{Q}$ is also M-regular with $\dim V^0(t_*\mathcal{Q}) = 0$, so $t_*\mathcal{Q} = 0$ (recall Remark 1.2(ii)) and φ is birational. Therefore, $(g, f) : X' \rightarrow V \times Y$ is birational. \square

Remark 1.6. The condition $q(X) - q(Y) > \dim X - \dim Y$ is necessary.

Indeed, let $C \rightarrow E$ be a bielliptic curve of genus 2. Consider $\widetilde{C} \rightarrow C$ an étale double cover and $X = (C \times \widetilde{C})/\langle \tau \rangle$, where τ is an involution acting diagonally on C and \widetilde{C} via the involutions corresponding respectively to the double covers.

Then, X has a natural fibration $X \rightarrow C$ such that the general fiber is C . Note that $q(X) = 3$ and $q(C) = 2$ and, indeed, X is not birational to a product of curves of genus 2. Varieties in Theorem C(ii) generalize this construction to higher dimensions.

2. SUBVARIETIES OF ABELIAN VARIETIES WITH HOLOMORPHIC EULER CHARACTERISTIC 1

Let $Y \subseteq A$ be subvariety of an abelian variety A . Without loss of generality, we assume through all Section 2 that Y generates A as a group. We also assume that Y is reduced and irreducible but not necessarily smooth. We consider $X \xrightarrow{f} Y \subseteq A$ a desingularization of Y . We want to study varieties Y such that $\chi(X, \omega_X) = 1$.

2.1. Subvarieties with $\chi(X, \omega_X) = 1$ and small $V^i(\omega_X, a)$. We start by reformulating a theorem of Pareschi (see [Pa, Theorem 5.1]).

Theorem 2.1. *Let $X \xrightarrow{f} Y$ be a desingularization of a subvariety $Y \subseteq A$ of an abelian variety and denote $a : X \xrightarrow{f} Y \hookrightarrow A$ the induced morphism to A . Assume that $\chi(X, \omega_X) = 1$. If $V^i(\omega_X, a)$ has no components of codimension $i + 1$ for all $0 < i < \dim X$, then Y is birational to a theta divisor. Moreover, the induced morphism $t : A_X \rightarrow A$ is an isogeny.*

Pareschi proved the above theorem using Fourier-Mukai techniques and homological algebra. We give an alternative proof, which is closer to the spirit of this paper. The ingredients of our proof are Lemma 1.4 and Hacon's characterization of theta divisors (see [Ha, Section 3]).

Proof. Denote by $n = \dim Y = \dim X$.

We consider an irreducible component T of $V^1(\omega_X, a) = \bigcup_{k \geq 1} V^k(\omega_X, a)$. Denote by $j + 1$ the codimension of T in \widehat{A} . By Lemma 1.4, we know that T is an irreducible component of $V^j(\omega_X, a)$. By assumption, $j = n$ and T contains 0 as an isolated point, since Y generates A , i.e., $\text{Pic}^0 A \rightarrow \text{Pic}^0 X$ is an isogeny onto its image. We then conclude that $\dim A = j + 1 = n + 1$ and $V^1(\omega_X, a)$ contains 0 as an isolated point.

Now we consider the derivative complex over $\mathbb{P}(H^1(A, \mathcal{O}_A)) = \mathbb{P}^n$ as in [EL, Proof of Theorem 3],

$$H^0(X, \omega_X) \otimes \mathcal{O}_{\mathbb{P}^n}(-n) \rightarrow \cdots \rightarrow H^n(X, \omega_X) \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow 0.$$

Since 0 is an isolated point in $V^1(\omega_X, a)$ and $\chi(X, \omega_X) = 1$, the above complex is exact and the kernel on the left is a line bundle $\mathcal{O}_{\mathbb{P}^n}(-t)$. By chasing through the cohomology of the above complex, we see easily $t = n + 1$ and $h^i(X, \omega_X) = \binom{n+1}{i+1}$. Thus, Hacon's criterion [Ha, Corollary 3.4] implies that X is birational to a theta divisor. Hence, so is Y . Since $q(X) = n + 1$, it is clear that $t : A_X \rightarrow A$ is an isogeny. \square

Remark 2.2. Note that A does not need to be principally polarized. Indeed, given $\Theta \subset A_\Theta$ we can consider any isogeny $A_\Theta \rightarrow A$ and let $Y \subset A$ the image of Θ .

The previous theorem exhibits the importance of the codimension- $(i + 1)$ irreducible components of $V^i(\omega_X, a)$. Thus, we place ourselves in the following setting fixing the notation.

Setting 2.3. Let $a : X \rightarrow A$ be a generically finite morphism from X a smooth projective variety to $Y := a(X)$ a subvariety of an abelian variety A . Suppose that there exists a codimension- $(i + 1)$ irreducible component of $V^i(\omega_X, a)$ for some $1 \leq i \leq \dim X - 1$.

We can describe this component as $P + \widehat{B}$, where \widehat{B} is an abelian subvariety of \widehat{A} and P is a torsion point (see [GL2, Theorem 0.1] and [Si, §4,5,6]).

Consider the composition of the $a : X \xrightarrow{f} Y \hookrightarrow A$ and the quotient $\pi_B : A \rightarrow B$ given by dualizing the inclusion $\widehat{B} \hookrightarrow \widehat{A}$. Then, denote by $Y_B := \pi_B(Y)$ and let $X \xrightarrow{\beta} X_B \xrightarrow{f_B} Y_B$ be Stein factorization of $X \rightarrow Y_B$. Denote by X_b a general fiber of the fibration β and denote by $Y_b := f(X_b)$ and $f_b := f|_{X_b}$.

Varieties and morphisms described above fit in the following commutative diagram:

$$(2.1.1) \quad \begin{array}{ccccc} X_b & \xrightarrow{f_b} & Y_b & & \\ \downarrow & & \downarrow & & \\ X & \xrightarrow{f} & Y & \hookrightarrow & A \\ \beta \downarrow & & \downarrow & & \downarrow \pi_B \\ X_B & \xrightarrow{f_B} & Y_B & \hookrightarrow & B. \\ & & \searrow a_B & & \end{array}$$

After birational modifications and abusing notation, we may assume that all varieties in the leftmost column are smooth and projective.

We summarize some useful properties in the following lemma.

Lemma 2.4 (Properties of Setting 2.3). *With the same notations as in the previous setting.*

- (i) *The morphism f_B is generically finite and the image $Y_b := f(X_b)$ is an irreducible component of the corresponding fiber of $Y \rightarrow B$.*
- (ii) *If f is birational and Y is of general type, then the morphism $f_b : X_b \rightarrow Y_b$ is birational. Moreover, Y_b is an irreducible ample divisor in a translate of the kernel of π_B .*

Proof. The first assertion is clear from the description. Since f is birational, f_b is also birational. Since $P + \widehat{B}$ is an irreducible component of $V^i(\omega_X, a)$, by [GL2, Theorem 0.2], $\dim X - \dim X_B \geq i$. Moreover, Y is not fibred by abelian subvarieties (see Lemma 1.1), so $\dim X_B = \dim X - i$ and Y_b is an ample divisor in a translate of the kernel of π_B . \square

2.2. Divisors in abelian varieties. If we assume that Y is an ample divisor (in particular, conditions in Lemma 1.1 are satisfied), then we show that $V^i(\omega_X, a)$ are “small” when Y is smooth in codimension 1.

Lemma 2.5. *Let $X \xrightarrow{f} Y$ be a desingularization of an irreducible ample divisor $Y \subseteq A$ of an abelian variety and denote $a : X \xrightarrow{f} Y \hookrightarrow A$. If $V^i(\omega_X, a)$ has a component $T = P + \widehat{B}$ of codimension $i + 1$ in \widehat{A} for some $0 < i < \dim X$, then Y is not smooth in codimension 1.*

The proof is contained in the proof of [DH, Lemma 6]. For reader’s convenience, we present it here.

Proof. By [EL, Proposition 3.1], there exists an adjoint ideal $\mathcal{I}_Z \subset \mathcal{O}_A$, co-supported at the singular locus of Y , which sits in the following exact sequence

$$0 \rightarrow \mathcal{O}_A \rightarrow \mathcal{I}_Z \otimes \mathcal{O}_A(Y) \rightarrow a_*\omega_X \rightarrow 0.$$

Let $p : A \rightarrow B$ be the natural projection. We then have the long exact sequence

$$R^i p_* P \rightarrow R^i p_*(\mathcal{I}_Z \otimes \mathcal{O}_A(Y)) \rightarrow R^i p_*(a_* \omega_X \otimes P) \xrightarrow{\delta} R^{i+1} p_* P$$

Since $P + \widehat{B} \subset V^i(\omega_X, a)$, the coherent sheaf $R^i p_*(a_* \omega_X \otimes P)$ is non-trivial and $V^0(R^i p_*(a_* \omega_X \otimes P)) = \widehat{B}$. Moreover, Kollár's theorem [Ko2, Theorem 3.4] implies that $R^i p_*(a_* \omega_X \otimes P)$ is torsion-free on B . Since $R^{i+1} p_* P$ is a flat line bundle on B , the boundary map δ is not injective. Thus the support of $R^i p_*(\mathcal{I}_Z \otimes \mathcal{O}_A(Y))$ is B .

Now, we consider the short exact sequence

$$0 \rightarrow \mathcal{I}_Z \otimes \mathcal{O}_A(Y) \rightarrow \mathcal{O}_A(Y) \rightarrow \mathcal{O}_Z(Y) \rightarrow 0.$$

Since $R^i p_*(\mathcal{O}_A(Y) \otimes P) = 0$, from the long exact sequence

$$R^{i-1} p_*(\mathcal{O}_Z(Y) \otimes P) \rightarrow R^i p_*(\mathcal{I}_Z \otimes \mathcal{O}_A(Y) \otimes P) \rightarrow R^i p_*(\mathcal{O}_A(Y) \otimes P)$$

and the previous discussion, we get that the support of $R^{i-1} p_*(\mathcal{O}_Z(Y) \otimes P)$ is B . Thus, there exists an irreducible component Z' of Z such that $p|_{Z'} : Z' \rightarrow B$ is surjective and a general fiber of $p|_{Z'}$ is of dimension $\geq i - 1$. As a consequence, $\dim Z \geq \dim B + i - 1 = \dim X - 1$.

Since Z is contained in the singular locus of Y , we conclude that Y is not smooth in codimension 1. \square

As we have seen, the hypothesis that Y is smooth in codimension 1 arises naturally when Y is a divisor and we want to control the size of $V^i(\omega_X, a)$. Moreover, as we will see in the following lemma, it implies that $A_X \rightarrow A$ is a *primitive morphism* of abelian varieties.

Lemma 2.6. *Let $X \xrightarrow{f} Y$ be a desingularization of a subvariety $Y \subseteq A$ of an abelian variety and denote by $a : X \xrightarrow{f} Y \hookrightarrow A$ the induced morphism to A . Assume that Y is a smooth in codimension 1 ample divisor of A . Then, given $P \in \widehat{A}$, $a^* P \cong \mathcal{O}_X$ implies $P = \mathcal{O}_A$.*

Proof. We argue by contradiction. Suppose that there exists a non-trivial $P \in \widehat{A}$ such that $a^* P \cong \mathcal{O}_X$. Then $a : X \rightarrow A$ factors through a non-trivial étale cover $\pi : \widetilde{A} \rightarrow A$ induced by P and f factors through $X \rightarrow \pi^{-1}(Y) \rightarrow Y$. Since Y is smooth in codimension 1 and it is an ample divisor of Y , $\pi^{-1}(Y)$ is irreducible. This is a contradiction, since f is birational, but $\pi^{-1}(Y) \rightarrow Y$ is not. \square

The following corollary will be the “divisor case” in the next section and it will play the role of “building brick”. It is obtained by combining Theorem 2.1 and Lemmas 2.5 and 2.6.

Corollary 2.7. *Let Y be a smooth in codimension 1 divisor of an abelian variety A . Let X be a desingularization of Y . Then, Y is a theta divisor of A if, and only if, $\chi(X, \omega_X) = 1$.*

Proof. Ein–Lazarsfeld prove in [EL, Theorem 1] that irreducible theta divisors are normal and have only rational singularities. Hence, if Y is isomorphic to a theta divisor, in particular, it is smooth in codimension 1.

Now assume that $\chi(X, \omega_X) = 1$ and note that X is of general type by Remark 1.2(i), and so is also Y . By Lemma 2.5, for all $0 < i < \dim X$, $V^i(\omega_X, a)$ has no component of codimension $i + 1$. Therefore, Theorem 2.1 implies that Y is birational to a theta divisor Θ of A_X and $t : A_X \rightarrow A$ is an isogeny. By Lemma 2.6, we have $\deg t = 1$. \square

2.3. Subvarieties smooth in codimension 1. In this section, we identify products of theta divisors among all normal subvarieties of a given abelian variety (see Theorem A). We will deduce this characterization from the following theorem.

Theorem 2.8. *Let Y be a smooth in codimension 1 subvariety of an abelian variety A . Let X be a desingularization of Y . Then, Y is birational to a product of theta divisors if, and only if, $\chi(X, \omega_X) = 1$.*

The rest of this section is mainly devoted to the proof of this theorem.

By Theorem 2.1, we can reduce our study to the case where there exists a codimension- $(i + 1)$ irreducible component of $V^i(\omega_X, a)$ for some $1 \leq i \leq \dim X - 1$. That is, we can assume we are in Setting 2.3 and use the notation therein.

The following technical lemma shows that, under the condition that Y is smooth in codimension 1, we may assume in Setting 2.3 that X_B and Y_B are of general type and, in particular $P = 0$, i.e., the codimension- $(i + 1)$ irreducible component passes through zero.

Lemma 2.9. *Let A be an abelian variety. Let $X \xrightarrow{f} Y$ be a desingularization of a subvariety $Y \subseteq A$ smooth in codimension 1 and denote by $a : X \xrightarrow{f} Y \hookrightarrow A$ the induced morphism to A . Assume that $\chi(X, \omega_X) = 1$.*

For any $i > 0$, all codimension- $(i + 1)$ irreducible components of $V^i(\omega_X, a)$ pass through zero. Moreover, using the notation in Setting 2.3, if \widehat{B} is an irreducible component of $V^i(\omega_X, a)$, then X_B and Y_B are birational and of general type.

Remark 2.10. Since $V^1(\omega_X, a)$ is non-empty, we can suppose that there always exists $P + \widehat{B}$ a codimension- $(i + 1)$ irreducible component of $V^i(\omega_X, a)$ by Lemma 1.4.

In particular, we prove that all irreducible components of $V^1(\omega_X, a)$ pass through zero and the corresponding X_B and Y_B are of general type.

Proof. Let $P + \widehat{B}$ a codimension- $(i + 1)$ irreducible component of $V^i(\omega_X, a)$.

If $i = \dim X$, then $\dim B = 0$ and Y is an ample divisor of A . Then, we conclude by Lemma 2.6. Hence, we assume now that $i < \dim X$ and we are in Setting 2.3 and we use the notation therein.

Step 1. $f_B : X_B \rightarrow Y_B$ is birational.

Assume by contradiction that $\deg f_B > 1$. Then, consider two distinct preimages $b_1, b_2 \in f_B^{-1}(b)$ and their corresponding fibres $X_{b_i} = \beta^{-1}(b_i)$. Recall that $\chi(X, \omega_X) > 0$ implies that X

is of general type (see Remark 1.2(i)), and so is Y . Then, by Lemma 2.4(ii), X_{b_i} are mapped birationally to Y via f . Since $Y_{b_i} := f(X_{b_i})$ are two different ample divisors in a translate of the kernel of π_B (see again Lemma 2.4), their intersection has codimension 1 in each divisor. This is in contradiction with Y being smooth in codimension 1.

Step 2. $P \in \widehat{B}$, so we may assume $P = 0$.

By [HP3, Theorem 2.2], $R^j\beta_*(\omega_X \otimes a^*P)$ are GV -sheaves for all $j \geq 0$. Then by [Ko2, Theorem 3.1], for $P_B \in \widehat{B}$ general $h^i(X, \omega_X \otimes a^*P \otimes a^*P_B) = h^0(Y, R^i\beta_*(\omega_X \otimes a^*P) \otimes a_B^*P_B)$. Since $P + \widehat{B} \subset V^i(\omega_X, a)$, $R^i f_*(\omega_X \otimes a^*P) \neq 0$. This implies that $(a^*P)|_{X_b} \cong \mathcal{O}_{X_b}$.

Hence, if we consider $a_b : X_b \xrightarrow{f_b} Y_b \hookrightarrow K_b$, then $a_b^*(P|_{K_b}) \cong \mathcal{O}_{X_b}$. Since Y_b is smooth in codimension 1 and it is an ample divisor of K_b , then $P|_{K_b} \cong \mathcal{O}_{K_b}$ by Lemma 2.6. Since we have the exact sequence of abelian varieties

$$0 \rightarrow \widehat{B} \rightarrow \widehat{A} \rightarrow \widehat{K} \rightarrow 0,$$

$P|_{K_b} \cong \mathcal{O}_{K_b}$ implies that $P \in \widehat{B}$.

Step 3. X_B and Y_B are of general type.

By Step 1, we only need to prove that X_B is of general type.

For $P_B \in \widehat{B} \setminus \bigcup_j V^1(R^j\beta_*\omega_X, a_B) \neq \emptyset$ (e.g., [HP3, Theorem 2.2]),

$$\begin{aligned} 0 < h^i(X, \omega_X \otimes a^*P_B) &= h^0(Y, R^i\beta_*\omega_X \otimes a_B^*P_B) \\ &= h^0(X_B, \omega_{X_B} \otimes a_B^*P_B) \end{aligned} \quad [\text{Ko1, Proposition 7.6}].$$

By generic vanishing, $\chi(X_B, \omega_{X_B}) = h^0(X_B, \omega_{X_B} \otimes a_B^*P_B)$ for a general $P_B \in \widehat{B}$. Hence $\chi(X_B, \omega_{X_B}) > 0$ and we conclude by Remark 1.2(i). \square

We want to use the diagram (2.1.1) to prove Theorem 2.8 by reducing to the divisor case. In order to do that, we need to prove that the fiber Y_b satisfy the conditions of Corollary 2.7. Before, we need to study the base Y_B and show, in particular, that $\chi(X_B, \omega_{X_B}) = 1$.

Proposition 2.11. *Let $X \xrightarrow{f} Y$ be a desingularization of a subvariety $Y \subseteq A$ of an abelian variety and denote by $a : X \xrightarrow{f} Y \hookrightarrow A$ the induced morphism to A . Assume that $\chi(X, \omega_X) = 1$ and $P + \widehat{B}$ is an irreducible component of $V^1(\omega_X, a)$ and let $i+1$ be the codimension of \widehat{B} in \widehat{A} .*

Then, using notation as in Setting 2.3, we have

$$\chi(X_B, R^i\beta_*(\omega_X \otimes a^*P)) = 1 \quad \text{and} \quad \chi(X_B, \beta_*(\omega_X \otimes a^*P)) = i + 1.$$

In particular, if $P + \widehat{B}$ passes through 0, we may let $P = 0$, and then $\chi(X_B, \omega_{X_B}) = 1$ and $\chi(X_B, \beta_\omega_X) = i + 1$.*

Proof. We use the notation in Setting 2.3, in particular, we will mainly be using the following small part of diagram (2.1.1)

$$\begin{array}{ccc} X & \longrightarrow & A \\ \beta \downarrow & & \downarrow \\ X_B & \xrightarrow{a_B} & B. \end{array}$$

We first apply the argument that Hacon–Pardini use to prove [HP3, Lemma 3.2].

We consider Q to be a general point of \widehat{B} and let $W \subset H^1(A, \mathcal{O}_A)$ be a linear subspace of dimension $i + 1$ transversal to \widehat{B} at Q . Since $P + \widehat{B}$ is an irreducible component of $V^1(\omega_X, a)$, by the same argument as in [EL, Proof of Theorem 3], the derivative complex is exact besides at the leftmost term. Hence, the derivative complex induces the following exact sequence of vector bundles on $\mathbb{P}^i = \mathbb{P}(W)$:

(2.3.1)

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{\mathbb{P}(W)}(-j) \rightarrow H^0(X, \omega_X \otimes a^*P \otimes a^*Q) \otimes \mathcal{O}_{\mathbb{P}(W)}(-i) \rightarrow \cdots \rightarrow \\ \rightarrow H^{i-1}(X, \omega_X \otimes a^*P \otimes a^*Q) \otimes \mathcal{O}_{\mathbb{P}(W)}(-1) \rightarrow H^i(X, \omega_X \otimes a^*P \otimes a^*Q) \otimes \mathcal{O}_{\mathbb{P}(W)} \rightarrow 0, \end{aligned}$$

for some $j \geq i + 1$.

We note that since $Q \in \widehat{B}$ is general, we have the natural isomorphism $H^k(X, \omega_X \otimes a^*P \otimes a^*Q) \cong H^0(X_B, R^k\beta_*(\omega_X \otimes a^*P) \otimes a_B^*Q)$ for any $k \geq 0$. Hence, the exact sequence (2.3.1) becomes

$$\begin{aligned} (2.3.2) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}(W)}(-j) \rightarrow H^0(X_B, \beta_*(\omega_X \otimes a^*P) \otimes a_B^*Q) \otimes \mathcal{O}_{\mathbb{P}(W)}(-i) \rightarrow \cdots \rightarrow \\ \rightarrow H^0(X_B, R^{i-1}\beta_*(\omega_X \otimes a^*P) \otimes a_B^*Q) \otimes \mathcal{O}_{\mathbb{P}(W)}(-1) \\ \rightarrow H^0(X_B, R^i\beta_*(\omega_X \otimes a^*P) \otimes a_B^*Q) \otimes \mathcal{O}_{\mathbb{P}(W)} \rightarrow 0. \end{aligned}$$

Chasing through the above diagram, we get

$$H^i(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(-j)) \cong H^0(X_B, R^i\beta_*(\omega_X \otimes a^*P) \otimes a_B^*Q) \otimes H^0(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}).$$

Hence

$$(2.3.3) \quad \chi(X_B, R^i\beta_*(\omega_X \otimes a^*P)) = h^0(X_B, R^i\beta_*(\omega_X \otimes a^*P) \otimes a_B^*Q) = \binom{j-1}{i}.$$

If we now tensor (2.3.2) by $\mathcal{O}_{\mathbb{P}(W)}(-1)$, chasing through the diagram we get

$$H^i(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(-j-1)) \cong H^0(X_B, \beta_*(\omega_X \otimes a^*P) \otimes a_B^*Q) \otimes H^i(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(-i-1)).$$

Hence

$$(2.3.4) \quad h^0(X, \omega_X \otimes a^*P \otimes a^*Q) = h^0(X_B, \beta_*(\omega_X \otimes a^*P) \otimes a_B^*Q) = \binom{j}{i}.$$

Combining (2.3.3) and (2.3.4), we get

$$\frac{h^0(X, \beta_*(\omega_X \otimes a^*P) \otimes a_B^*Q)}{h^0(X_B, R^i\beta_*(\omega_X \otimes a^*P) \otimes a_B^*Q)} = \frac{j}{j-i} \leq i+1,$$

and the equality holds only if $j = i + 1$.

Let $d > 1$ be the order of P and G the group generated by P . We then consider the étale cover $A' \rightarrow A$ induced by P and the corresponding base change diagram

$$\begin{array}{ccc} & X & \xrightarrow{f} A \\ \eta_X \nearrow & & \nearrow \eta_A \\ X' & \xrightarrow{f'} & A'. \end{array}$$

Combining this diagram with (2.1.1) we obtain,

$$\begin{array}{ccccc} & & X & \longrightarrow & A \\ & \eta_X \nearrow & & & \nearrow \eta_A \\ X' & \longrightarrow & & & A' \\ & \downarrow \beta & & & \downarrow \\ & & X_B & \longrightarrow & B, \\ \beta' \downarrow & \nearrow \nu & & \nearrow \mu & \\ X'_B & & & & \end{array}$$

where $X' \xrightarrow{\beta'} X'_B \xrightarrow{\nu} X_B$ is a birational modification of the Stein factorization of $X' \rightarrow X_B$ such that X'_B is smooth. As in [JLT, Proof of Lemma 3.3], we can arrange the modifications such that $\nu : X'_B \rightarrow X_B$ is still a generically finite G -cover and $\beta' : X \rightarrow X'_B$ is G -equivariant.

Let V be the $(i+1)$ -dimensional quotient vector space $H^0(A, \Omega_A)/H^0(B, \Omega_B)$ and let \mathcal{S} be the saturation of $\beta'^*\Omega_{X'_B} \rightarrow \Omega_{X'}$ and \mathcal{Q} be the quotient of $\Omega_{X'}/\mathcal{S}$. Then, we have on X' the following commutative diagram of coherent sheaves:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \beta'^*H^0(B, \Omega_B) \otimes \mathcal{O}_{X'} & \longrightarrow & H^0(A, \Omega_A) \otimes \mathcal{O}_{X'} & \longrightarrow & V \otimes \mathcal{O}_{X'} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \beta'^*\Omega_{X'_B} & & & & \\ 0 & \longrightarrow & \mathcal{S} & \longrightarrow & \Omega_{X'} & \longrightarrow & \mathcal{Q} \longrightarrow 0. \end{array}$$

Considering i -th wedge of the right column, we have a morphism

$$\bigwedge^i V \otimes \mathcal{O}_{X'} \rightarrow \det \mathcal{Q} \hookrightarrow \omega_{X'/X'_B},$$

and hence a morphism

$$(2.3.5) \quad \bigwedge^i V \otimes \mathcal{O}_{X'_B} \rightarrow \beta'_* \omega_{X'/X'_B}.$$

Recall that, Y_b is an irreducible ample divisor of K_b , by Lemma 2.4(ii). By [Mo, Corollary 3.11], then the natural morphism (2.3.5) is injective. This gives us the following injective morphism between coherent sheaves on X'_B ,

$$\omega_{X'_B}^{\oplus(i+1)} \hookrightarrow \beta'_* \omega_{X'}.$$

Since β' is G -equivariant, the above morphism is also G -equivariant. Hence, if we push-forward to X_B and consider the direct summand corresponding to P , we have the following injective morphism of coherent sheaves on X_B ,

$$(2.3.6) \quad R^i \beta_* (\omega_X \otimes a^* P)^{\oplus(i+1)} \hookrightarrow \beta_* (\omega_X \otimes a^* P).$$

Hence, for any $Q \in \widehat{B}$, we have

$$(i+1)h^0(X_B, R^i \beta_* (\omega_X \otimes a^* P) \otimes Q) \leq h^0(X_B, \beta_* (\omega_X \otimes a^* P) \otimes Q).$$

Therefore, we have $j = i + 1$. So, from (2.3.3), we get $\chi(X_B, R^i \beta_* (\omega_X \otimes a^* P)) = \binom{i+1-1}{i} = 1$ and, from (2.3.4), $\chi(X_B, \beta_* (\omega_X \otimes a^* P)) = \binom{i+1}{i} = i + 1$.

When $P = 0$, we just note that $a^* P = \mathcal{O}_X$ and $R^i \beta_* \omega_X = \omega_{X_B}$ by [Kol, Proposition 7.6]. \square

The following proposition studies the general fibre Y_b . More concretely, we want to prove that $\chi(X_b, \omega_{X_b}) = 1$ and $q(X_b) = i + 1$.

Proposition 2.12. *Under the same assumptions as in Proposition 2.11 and using the notation of Setting 2.3, we consider in addition that Y_B is of general type.*

Then a general fiber X_b of β has $\chi(X_b, \omega_{X_b}) = 1$ and $q(X_b) = i + 1$.

Proof. We use again the notation in Setting 2.3, in particular we recall diagram (2.1.1)

$$\begin{array}{ccccc} X_b & \longrightarrow & Y_b & & \\ \downarrow & & \downarrow & & \\ X & \longrightarrow & Y \subset & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ \beta & & & & \\ X_B & \longrightarrow & Y_B \subset & \longrightarrow & B \\ & & \searrow & \nearrow & \\ & & & a_B & \end{array}$$

First, we claim that the injective morphism (2.3.6) is generically an isomorphism. Otherwise, we have a short exact sequence of coherent sheaves:

$$0 \rightarrow R^i \beta_* (\omega_X \otimes a^* P)^{\oplus(i+1)} \rightarrow \beta_* (\omega_X \otimes a^* P) \rightarrow \mathcal{W} \rightarrow 0,$$

where $\text{rk } \mathscr{W} \geq 1$. Then, we have a short exact sequence of non-trivial coherent sheaves on B ,

$$0 \rightarrow a_{B*} R^i \beta_*(\omega_X \otimes a^* P)^{\oplus(i+1)} \rightarrow a_{B*} \beta_*(\omega_X \otimes a^* P) \rightarrow a_{B*} \mathscr{W} \rightarrow 0.$$

Since Y_B is of general type, $a_{B*} \beta_*(\omega_X \otimes a^* P)$ is M-regular on B (see Lemma 1.1). Therefore, $a_{B*} \mathscr{W}$ is a non-trivial M-regular sheaf on B and $V^0(a_{B*} \mathscr{W}) = \widehat{B}$ (see Remark 1.2(ii)). However, for $Q \in \widehat{B}$ general, $h^0(X_B, \mathscr{W} \otimes a_B^* Q) = h^0(X_B, \beta_*(\omega_X \otimes a^* P) \otimes a_B^* Q) - (i+1)h^0(X_B, R^i \beta_*(\omega_X \otimes a^* P) \otimes a_B^* Q) = 0$, which contradicts $V^0(a_{B*} \mathscr{W}) = \widehat{B}$.

Thus, for a general fiber X_b of β , we have $h^0(X_b, \omega_{X_b}) = \text{rk } \beta_*(\omega_X \otimes a^* P) = i+1$ and then, by [Mo, Corollary 3.11], $\chi(X_b, \omega_{X_b}) = 1$ and $q(X_b) = i+1$. \square

Now we are ready to prove Theorem 2.8 by induction on the number of irreducible components of $V_o^1(\omega_X, a_X)$.

Proof of Theorem 2.8. Note that Y is of general type by Remark 1.2(i). When $\text{codim}_A Y = 1$, we are in the divisor case and we conclude by Corollary 2.7. In particular $V^1(\omega_X, a) = \{0\}$.

So we may assume $\text{codim}_A Y \geq 2$. We consider the Albanese morphism of $a_X : X \rightarrow A_X$, and we denote by $M = a_X(X)$ and $t : A_X \rightarrow A$ the induced morphism between abelian varieties. Then we have the commutative diagram

$$(2.3.7) \quad \begin{array}{ccccc} X & \xrightarrow{f} & Y & \hookrightarrow & A \\ & \searrow g & & & \nearrow t \\ & & M & \xrightarrow{t_Y} & A_X \end{array}$$

where $a : X \rightarrow Y \hookrightarrow A$ and $a_X : X \rightarrow M \hookrightarrow A_X$. Since f is birational, g and t_Y are also birational. As we already mentioned, we assume that Y generates A , so $\widehat{t} : \widehat{A} \rightarrow \widehat{A}_X$ is an isogeny onto its image

By Lemma 1.4, any irreducible component of codimension- $(i+1)$ of $V^1(\omega_X, a)$ (respectively of $V^1(\omega_X, a_X)$) is an irreducible component of $V^i(\omega_X, a)$ (resp. $V^i(\omega_X, a_X)$). Thus, all irreducible components of $V^1(\omega_X, a)$ and $V^1(\omega_X, a_X)$ are positive dimensional, since $\text{codim}_{A_X} M \geq \text{codim}_A Y \geq 2$. Hence, the conditions in Setting 2.3 are fulfilled and we use the notation therein.

By Lemma 2.9 (see also Remark 2.10), each codimension- $(i+1)$ component \widehat{B} of $V^i(\omega_X, a)$ passes through 0 and X_B and Y_B are of general type.

We denote by $V_o^1(\omega_X, a_X)$ the union of irreducible components of $V^1(\omega_X, a_X)$ passing through 0. We denote by k the number of positive dimensional irreducible components of $V_o^1(\omega_X, a_X)$,

$$V_o^1(\omega_X, a_X) = \bigcup_{1 \leq j \leq k} \widehat{B}_j,$$

and $\text{codim}_{\widehat{A}_X} \widehat{B}_j = i_j + 1$.

We consider $\widehat{B}_j = \widehat{t}^{-1}(\widehat{B}_j)$. Then, $\text{codim}_{\widehat{A}} \widehat{B}_j \leq i_j + 1$. Moreover, since M is of general type, then \widehat{B}_j is an irreducible component of $V^{i_j}(\omega_X, a_X)$ by Lemma 1.4. Thus, $\widehat{B}_j \subset V^{i_j}(\omega_X, a)$.

Since Y is also of general type, then $\text{codim}_{\widehat{A}} \widehat{B}_j \geq i_j + 1$ by Lemma 1.1. Thus, \widehat{B}_j is a component of $V^{i_j}(\omega_X, a)$ of codimension $i_j + 1$ in \widehat{A} .

We first consider Setting 2.3 with respect to \widetilde{B}_1 and B_1 , irreducible components of $V^{i_1}(\omega_X, a_X)$ and $V^{i_1}(\omega_X, a)$ of codimension $i_1 + 1$ in \widehat{A}_X and \widehat{A} . Using the perspective of (2.3.7) and (2.1.1), we have

$$\begin{array}{ccccc}
 X_{b_1} & \xrightarrow{f_{b_1}} & Y_{b_1} & & \\
 \downarrow & & \downarrow & & \\
 X & \xrightarrow{f} & Y \subset & & A \\
 \downarrow \beta_1 & \searrow g & \downarrow t_Y & \searrow t & \downarrow \\
 & M & & A_X & \\
 & \downarrow & & \downarrow & \\
 X_{\widetilde{B}_1} & \xrightarrow{f_{B_1}} & Y_{B_1} \subset & & B_1 \\
 \downarrow g_1 & \searrow & \downarrow & \searrow & \\
 & M_{\widetilde{B}_1} & & \widetilde{B}_1 & \\
 & \downarrow & & \downarrow & \\
 & & & & B_1
 \end{array}$$

where $X_{\widetilde{B}_1}$ is smooth projective and $\beta_1 : X \rightarrow X_{\widetilde{B}_1}$ is a birational model of the Stein factorization of $X \rightarrow M \rightarrow M_{\widetilde{B}_1}$. Then β_1 is also a model of the Stein factorization of $X \rightarrow Y \rightarrow Y_{B_1}$, so in the notation of Setting 2.3, $X_{\widetilde{B}_1} = X_{B_1}$. Note that, since Y is smooth in codimension 1, by Lemma 2.9 we have that Y_{B_1} is of general type and $f_{B_1} : X_{\widetilde{B}_1} \rightarrow Y_{B_1}$ is a birational morphism. Thus, $g_1 : X_{\widetilde{B}_1} \rightarrow M_{\widetilde{B}_1}$ is also a birational morphism.

Since $M_{\widetilde{B}_1}$ is of general type, we can apply Proposition 2.12 to $X \rightarrow M \rightarrow M_{\widetilde{B}_1}$. Hence, for a general point $b_1 \in X_{\widetilde{B}_1}$, and X_{b_1} the fiber of β_1 over b_1 , we have $\chi(X_{b_1}, \omega_{X_{b_1}}) = 1$.

Since f is birational, $f_{b_1} = f|_{X_{b_1}}$ is a birational morphism onto the corresponding fiber Y_{b_1} of $Y \rightarrow Y_{B_1}$. Moreover, since Y is smooth in codimension 1, so is Y_{b_1} . Let K_1 be the kernel of $A \rightarrow B_1$. Then, by Lemma 2.4(ii), Y_{b_1} is an irreducible smooth-in-codimension-1 ample divisor of a translate of K_1 . We conclude by Corollary 2.7 that Y_{b_1} is a theta divisor Θ_1 of $K_1 \subset A$. In particular, X_{b_1} is birational to Θ_1 . By Proposition 1.5, we can assume that $X = X_{b_1} \times X_{\widetilde{B}_1}$. Thus, we have

$$(2.3.8) \quad A_X = K_1 \times A_{X_{\widetilde{B}_1}} \quad \text{and} \quad M = \Theta_1 \times M_1,$$

where we define $M_1 := M_{X_{\widetilde{B}_1}} = a_{X_{\widetilde{B}_1}}(X_{\widetilde{B}_1})$. Hence, we also have that $\widetilde{B}_1 = \{0\} \times \widehat{A}_{X_{\widetilde{B}_1}}$ and

$$V_0^1(\omega_X, a_X) = \left(\{0\} \times \widehat{A}_{X_{\widetilde{B}_1}} \right) \cup \left(\widehat{K}_1 \times V_0^1(\omega_{X_{\widetilde{B}_1}}, a_{X_{\widetilde{B}_1}}) \right).$$

Therefore $\widetilde{B}_2 = \widehat{K}_1 \times \widehat{B}'_2$ for some abelian subvariety $\widehat{B}'_2 \subseteq V_0^1(\omega_{X_{\widetilde{B}_1}}, a_{X_{\widetilde{B}_1}}) \subset \widehat{A}_{X_{\widetilde{B}_1}}$.

We now consider Setting 2.3 with respect to \widetilde{B}_2 and B_2 , irreducible components of codimension $i_2 + 1$. Using the perspective of (2.3.7), (2.1.1), as before, and adding the decomposition

(2.3.8), we have

$$\begin{array}{ccccc}
X_{b_2} & \xrightarrow{f_{b_2}} & Y_{b_2} & & \\
\downarrow & & \downarrow & & \\
X = X_{b_1} \times X_{\tilde{B}_1} & \xrightarrow{f} & Y \subset A & \xrightarrow{t} & A \\
\downarrow \beta_2 & \searrow g & \uparrow t_Y & \searrow t & \downarrow \\
X_{\tilde{B}_2} & \xrightarrow{f_{B_2}} & Y_{B_2} \subset & \xrightarrow{} & B_2 \\
\downarrow g_2 & \searrow & \downarrow & \searrow & \downarrow \\
\Theta_1 \times M_1 & \xrightarrow{} & K_1 \times A_{X_{\tilde{B}_1}} & \xrightarrow{} & \\
\downarrow & \searrow & \downarrow & \searrow & \downarrow \\
\Theta_1 \times M'_2 & \xrightarrow{} & \tilde{B}_2 = K_1 \times B'_2 & \xrightarrow{} &
\end{array}$$

where we have denoted by M'_2 the image of the composition of morphism $M_1 \rightarrow A_{X_{\tilde{B}_1}} \rightarrow B'_2$. Note that f_{B_2} and g_2 are birational and Y_{B_2} is of general type by Lemma 2.9.

Let K_2 be the kernel of $A \rightarrow B_2$. As before, Lemma 2.4, Proposition 2.12, and Corollary 2.7 imply that a general fiber X_{b_2} of β_2 is birational to a theta divisor Θ_2 of $K_2 \subset A$. Then, by Proposition 1.5, $X_{b_1} \times X_{\tilde{B}_1}$ is birational to $X_{b_2} \times X_{\tilde{B}_2}$. Moreover, the morphism $g_2 : X_{\tilde{B}_2} \rightarrow \Theta_1 \times M'_2$ induces a birational equivalence between $X_{\tilde{B}_2}$ and the product of X_{b_1} with another variety that we call X_2 . That is, X is birational to $X_{b_1} \times X_{b_2} \times X_2$ and we have

$$A_X = K_1 \times K_2 \times A_{X_2} \quad \text{and} \quad M = \Theta_1 \times \Theta_2 \times M_2,$$

where $M_2 = a_{X_2}(X_2)$. Hence, we also have that $\widehat{B}_2 = \widehat{K}_1 \times \{0\} \times \widehat{A}_{X_2}$ and

$$V_o^1(\omega_X, a_X) = \left(\{0\} \times \widehat{A}_{X_{\tilde{B}_1}} \right) \cup \left(\widehat{K}_1 \times \{0\} \times \widehat{A}_{X_2} \right) \cup \left(\widehat{K}_1 \times \widehat{K}_2 \times V_o^1(\omega_{X_2}, a_{X_2}) \right).$$

Therefore, $\widehat{B}_3 = \widehat{K}_1 \times \widehat{K}_2 \times \widehat{B}'_3$ for some abelian subvariety $\widehat{B}'_3 \subseteq V_o^1(\omega_{X_2}, a_{X_2}) \subset \widehat{A}_{X_2}$.

We can iterate this process using all \widehat{B}_j for $1 \leq j \leq k$. For each fibration $\beta_j : X \rightarrow X_{\tilde{B}_j}$, we get a further decomposition of X . Finally, we conclude that X is birational to $X_{b_1} \times \dots \times X_{b_k} \times X_k$,

$$A_X = K_1 \times \dots \times K_k \times A_{X_k} \quad \text{and} \quad M = \Theta_1 \times \dots \times \Theta_k \times M_k,$$

where $M_k = a_{X_k}(X_k)$. Hence, we have that

$$\begin{aligned}
V_o^1(\omega_X, a_X) = & \left(\{0\} \times \widehat{A}_{X_{\tilde{B}_1}} \right) \cup \left(\widehat{K}_1 \times \{0\} \times \widehat{A}_{X_2} \right) \cup \left(\widehat{K}_1 \times \widehat{K}_2 \times \{0\} \times \widehat{A}_{X_3} \right) \cup \\
& \cup \dots \cup \left(\widehat{K}_1 \times \dots \times \widehat{K}_{k-1} \times \{0\} \times \widehat{A}_{X_k} \right) \cup \\
& \cup \left(\widehat{K}_1 \times \dots \times \widehat{K}_k \times V_o^1(\omega_{X_k}, a_{X_k}) \right).
\end{aligned}$$

To conclude, we only need to show that M_k is a point. Suppose by contradiction that $\dim M_k \geq 1$. Then $\dim X_k = \dim M_k \geq 1$ and hence $V^1(\omega_{X_k}, a_{X_k})$ contains 0. Then, $\widehat{K}_1 \times \cdots \times \widehat{K}_k \times V^1(\omega_{X_k}, a_{X_k})$ exhibits a new component of $V_0^1(\omega_X, a_X)$, which is a contradiction.

Hence, X is birational to a product of k theta divisors and $A_X = K_1 \times \cdots \times K_k$. \square

Once we have proven Theorem 2.8, we can deduce Theorem A from it.

Proof of Theorem A. Only one direction needs to be proved.

We fix a desingularization $f : X \rightarrow Y$. Then by assumption, $\chi(X, \omega_X) = 1$. As we already said, we assume that Y generates A . By Theorem 2.8, X is birational to a product of theta divisors and the induced morphism $t : A_X \rightarrow A$ is surjective.

The image $a_X(X)$ is a product of theta divisors $\Theta_1 \times \cdots \times \Theta_m$. We denote by K_j the Albanese variety A_{Θ_j} of Θ_j . In order to show that the induced morphism $\tau : \Theta_1 \times \cdots \times \Theta_m \rightarrow Y$ is an isomorphism, we just need to prove that $t : A_X = K_1 \times \cdots \times K_m \rightarrow A$ is an isomorphism. We will show that if t is not an isomorphism, Y can not be normal.

First of all, we assume that t is an isogeny of degree $d > 1$. Then $t^{-1}(Y)$ has d irreducible components, each of which is a translate of the product of theta divisors $\Theta_1 \times \cdots \times \Theta_m$ in A_X . Theta divisors are ample divisors, so the d irreducible components meet. Thus, $t^{-1}(Y)$ is connected but not irreducible, hence not normal. This gives a contradiction, since Y is normal and t is étale.

Now we assume that $\dim \ker t > 0$. The strategy is to prove that there always exist non-connected fibers of τ to get a contradiction with Zariski's Main Theorem.

Let K be the neutral component of $\ker t$. We denote by $k = \dim K$ and $\varphi_j : K \rightarrow K_j$ the natural projection. By induction on the number of theta divisors, we can assume that $\varphi_j \neq 0$ for all j . Let $t_1 : A_X \rightarrow A_X/K$ be the quotient and \bar{Y} the image $t_1(a_X(X))$. Then we have the following commutative diagram

$$\begin{array}{ccc} \Theta_1 \times \cdots \times \Theta_m & \hookrightarrow & K_1 \times \cdots \times K_m \\ \downarrow \tau_1 & & \downarrow t_1 \\ \bar{Y} & \hookrightarrow & A_X/K \\ \downarrow \tau_2 & & \downarrow t_2 \\ Y & \hookrightarrow & A \end{array} \begin{array}{l} \tau \\ t \end{array}$$

Note that, since $Y \subsetneq A$, we have $k = \dim K < m$.

For $c = (c_1, \dots, c_m) \in \Theta_1 \times \cdots \times \Theta_m$, we have $\tau_1^{-1}\tau_1(c) = (K + c) \cap (\Theta_1 \times \cdots \times \Theta_m)$. Thus,

$$\tau_1^{-1}\tau_1(c) \cong \varphi_1^{-1}(\Theta_1 - c_1) \cap \cdots \cap \varphi_m^{-1}(\Theta_m - c_m).$$

We observe the following:

- 1) $\bigcup_{c_j \in \Theta_j} (\Theta_j - c_j) = (\Theta_j - \Theta_j) = K_j$, since $\dim K_j > 1$;
- 2) Since τ is birational, for $c \in \Theta_1 \times \cdots \times \Theta_m$ general, we have $\bigcap_{j=1}^m \varphi_j^{-1}(\Theta_j - c_j) = \{0_K\}$ the identity element of K .

Thanks to the second observation, there exists $1 \leq t \leq m - 1$ and $d_j \in \Theta_j$ for $1 \leq j \leq t$ such that,

$$\begin{cases} \dim \left(\bigcap_{i=1}^t \varphi_i^{-1}(\Theta_i - d_i) \right) = 1 & \text{and} \\ \dim \left(\bigcap_{i=1}^{t+1} \varphi_i^{-1}(\Theta_i - d_i) \right) = 0 & \text{for a general } d_{t+1} \in \Theta_{t+1}. \end{cases}$$

We first claim that $\bigcap_{i=1}^t \varphi_i^{-1}(\Theta_i - d_i)$ is connected. Otherwise, let $\nu \in \bigcap_{i=1}^t \varphi_i^{-1}(\Theta_i - d_i)$ be a point not in the connected component containing 0_K . We then choose $d'_j \in \Theta_j \cap (\Theta_j - \varphi_j(\nu))$ for $t+1 \leq j \leq m$. Then

$$\{0_K, \nu\} \subset \left(\bigcap_{i=1}^t \varphi_i^{-1}(\Theta_i - d_i) \right) \cap \left(\bigcap_{j=t+1}^m \varphi_j^{-1}(\Theta_j - d'_j) \right) \subset \bigcap_{i=1}^t \varphi_i^{-1}(\Theta_i - d_i).$$

Thus, $\left(\bigcap_{i=1}^t \varphi_i^{-1}(\Theta_i - d_i) \right) \cap \left(\bigcap_{j=t+1}^m \varphi_j^{-1}(\Theta_j - d'_j) \right)$ is also not connected. We denote

$$d' = (d_1, \dots, d_t, d'_{t+1}, \dots, d'_m).$$

Then $\tau_1^{-1}\tau(d')$ is not connected. Since t_2 is an isogeny, τ_2 is finite. Hence, $\tau_2^{-1}(\tau(d'))$ is a set of finite points containing $\tau_1(d')$. Thus, $\tau^{-1}\tau(d')$ is also not connected. This contradicts Zariski's Main Theorem, since τ is birational and Y is normal.

We now consider $\bigcap_{i=1}^t \varphi_i^{-1}(\Theta_i - d_i)$ with the reduced scheme structure and we denote it by C . Hence, C is a connected curve containing 0_K . Then $\varphi_{t+1}(C)$ is a curve in K_{t+1} containing the identity element of K_{t+1} . We have the following Zariski open dense subset of Θ_{t+1} :

$$U := \{u \in \Theta_{t+1} \mid \dim(C \cap \varphi_{t+1}^{-1}(\Theta_{t+1} - u)) = 0\}.$$

Considering the following set

$$V := \bigcup_{\substack{c \in \varphi_{t+1}(C) \\ c \neq 0_{t+1}}} (\Theta_{t+1} - c) = \Theta_{t+1} - (\varphi_{t+1}(C) \setminus \{0_{t+1}\}),$$

we have

$$V \cap \Theta_{t+1} = \bigcup_{\substack{c \in \varphi_{t+1}(C) \\ c \neq 0_{t+1}}} (\Theta_{t+1} \cap (\Theta_{t+1} - c)).$$

Since Θ_{t+1} is a theta divisor of K_{t+1} , the induced morphism

$$\begin{aligned} K_{t+1} &\rightarrow \text{Pic}^0 K_{t+1} \\ c &\rightarrow t_c^* \mathcal{O}_{K_{t+1}}(\Theta_{t+1}) \otimes \mathcal{O}_{K_{t+1}}(-\Theta_{t+1}) \end{aligned}$$

is an isomorphism. Moreover, since Θ_{t+1} has rational singularities, $\text{Pic}^0 K_{t+1} \rightarrow \text{Pic}^0 \Theta_{t+1}$ is an isomorphism. Thus, $(\Theta_{t+1} - c)|_{\Theta_{t+1}}$ are pairwise different effective divisors of Θ_{t+1} for different $c \in \varphi_{t+1}(C) \setminus \{0_{t+1}\}$. Therefore $V \cap \Theta_{t+1}$ is a Zariski dense subset of Θ_{t+1} .

Hence $U \cap V$ is also a Zariski dense subset of Θ_{t+1} . Given any $d_{t+1} \in U \cap V \subset \Theta_{t+1}$, the set

$$(\Theta_{t+1} - d_{t+1}) \cap (\varphi_{t+1}(C))$$

is finite with at least 2 distinct points. Thus,

$$\bigcap_{j=1}^{t+1} \varphi_j^{-1}(\Theta_j - d_j) \supset (\varphi_{t+1}^{-1}(\Theta_{t+1} - d_{t+1}) \cap C)$$

is also a finite set containing at least 2 distinct points 0_K and κ .

By observation 1), we can choose $d_j \in \Theta_j$ for $t+1 \leq j \leq m$ such that $\kappa \in \varphi_j^{-1}(\Theta_j - d_j)$ for any $t+1 \leq j \leq m$. Let $d = (d_1, \dots, d_m) \in \Theta_1 \times \dots \times \Theta_m$. Then

$$\tau_1^{-1}\tau_1(d) \cong \varphi_1^{-1}(\Theta_1 - d_1) \cap \dots \cap \varphi_m^{-1}(\Theta_m - d_m)$$

is a finite set with at least two distinct points. In particular, $\tau_1^{-1}\tau_1(d)$ is not connected and hence $\tau^{-1}\tau(d)$ is also not connected, which is a contradiction to Zariski's Main Theorem. \square

As we have seen in the previous proof the difference between Theorem A and Theorem 2.8 is quite subtle. The following examples may help to distinguish the two situations.

Example 2.13. Let X be a desingularization of $\Theta_1 \times \Theta_2 \subset A_1 \times A_2 = A$. Then, the Albanese variety of X is $A = A_1 \times A_2$ and the image $a_X(X) = Y$ is $\Theta_1 \times \Theta_2$, which is normal.

Now, let $P_1 \in A_1$ and $P_2 \in A_2$, be two 2-torsion points. Consider $t : A_1 \times A_2 \rightarrow A'$ the isogeny induced by (P_1, P_2) (just the quotient by the finite group generated by (P_1, P_2)). Indeed, $t : A_1 \times A_2 \rightarrow A'$ is an étale covering of degree 2. Let $Y' \subset A'$ be $t(\Theta_1 \times \Theta_2)$. Note that Y' is smooth in codimension 1. But Y' is not normal, since $t^{-1}(Y') = (\Theta_1 \times \Theta_2) \cup ((\Theta_1 + P_1) \times (\Theta_2 + P_2))$ is connected but not irreducible, hence not normal.

Thus, $Y' \subset A'$ exhibits a case Y' is birational to a product of theta divisors, but not isomorphic. Moreover, A' is not necessarily isomorphic to the Albanese variety of its desingularization.

Summing up, in the following diagram, the first row is a paradigmatic example of Theorem A, while the second row is a paradigmatic example of Theorem 2.8 when t is an isogeny:

$$\begin{array}{ccccc}
 & & a_X & & \\
 & & \curvearrowright & & \\
 X & \xrightarrow{f} & Y = \Theta_1 \times \Theta_2 & \hookrightarrow & A = A_X \\
 \parallel & & \downarrow \text{bir} & & \downarrow t \text{ 2:1} \\
 X & \xrightarrow{f'} & Y' = t(\Theta_1 \times \Theta_2) & \hookrightarrow & A' \\
 & & \curvearrowleft & & \\
 & & a' & &
 \end{array}$$

Example 2.14. Let $C \rightarrow E$ a bielliptic curve, such that $g(C) = 2$ and let K be the kernel of $JC = A_C \rightarrow E$. Then consider $X = C \times C \times C$ and let

$$\begin{array}{ccccc}
 X & \xrightarrow{a_X} & Y & \hookrightarrow & A_X \\
 \parallel & \xrightarrow{f} & \downarrow \text{bir} & & \downarrow t \\
 X & \xrightarrow{f'} & Y' = t(Y) & \hookrightarrow & A' \\
 & \xrightarrow{a'} & & &
 \end{array}$$

where $t : A_X = JC \times JC \times JC \rightarrow A$ is the quotient by the diagonal of K , i.e., $\{(x, x, x) \in A_X \mid x \in K\}$. Note that $t|_Y$ is a birational morphism and Y' is smooth in codimension 1.

This example shows that, in Theorem 2.8, t does not need to be an isogeny.

Finally, we want to mention the following difference between Theorems A and Theorem 2.8.

Remark 2.15. On the one hand, as we have already seen in the previous examples, the abelian variety in Theorem 2.8 does not need to be principally polarized (see also Remark 2.2).

On the other hand, we emphasize that in Theorem A the abelian variety is always principally polarized. Indeed, Y induces a principal polarization on A and $A = A_X$.

3. VARIETIES OF MAXIMAL ALBANESE DIMENSION & HOLOMORPHIC EULER CHARACTERISTIC 1

In the previous section, we have considered $a : X \rightarrow A$ generically finite morphisms from a smooth projective variety X to a subvariety of an abelian variety $Y := a(X) \subset A$, from the point of view of Y , letting X be its desingularization. Now we will consider that X is given and a will be its Albanese morphism.

If we consider $a_X : X \xrightarrow{f} Y := a_X(X) \hookrightarrow A_X$, we note, even if X is of general type, $f : X \rightarrow Y$ might not be birational. We have two paradigmatic examples of smooth complex projective varieties X of maximal Albanese dimension and general type, such that f is not birational. The first one is when X is of general type and $\chi(X, \omega_X) = 0$. The second one is when $\chi(X, \omega_X) = 1$ and $q(X) = \dim X$. As we have mentioned in the introduction, the complete classification of these examples seems to be a rather difficult problem and clearly the knot comes from studying $f : X \rightarrow Y = a_X(X)$ rather than just $a_X(X)$ as we have done in the previous section.

Now, X will be always a smooth complex projective variety such that

$$(C.1) \quad X \text{ is of maximal Albanese dimension;}$$

$$(C.2) \quad \chi(X, \omega_X) = 1.$$

Then Hacon–Pardini show in [HP3] that

$$\dim X \leq q(X) \leq 2 \dim X,$$

and, when $q(X)$ is maximal, i.e., $q(X) = 2 \dim X$, then Hacon–Pardini prove that X is birational to a product of curves of genus 2 (theta divisors of dimension 1). In particular, f is birational.

We want to extend the classification Theorem of Hacon–Pardini to lower values of $q(X)$. First, we will give a complete classification of the submaximal case, i.e., varieties satisfying (C.1–2) and $q(X) = 2 \dim X - 1$. Already in this case the examples where f is not birational will step into our classification problem. More precisely, we will have two cases depending whether f is birational or it has degree 2.

Second, we will study how to characterize birationally products of higher dimensional theta divisors. For this, we will need to impose the following extra conditions

$$(C.3) \quad a_X(X) \text{ is of general type;}$$

$$(C.4) \quad a_X(X) \text{ is smooth in codimension 1.}$$

where (C.3) prevents f from being non birational and it is an open problem to determine if condition (C.4) is necessary (see Question 3.6).

Submaximal irregularity. The case when $q(X) = 2 \dim X - 1$, was already studied for surfaces, i.e., $\dim X = 2$ and $q(X) = p_g(X) = 3$, by Hacon–Pardini [HP2] and Pirola [Pi]. This section will be devoted to prove the following theorem, which can be seen as a generalization of both [HP2, Pi] and [HP3].

Theorem 3.1. *Let X be a smooth projective variety of maximal Albanese dimension. Assume that $\chi(X, \omega_X) = 1$ and $q(X) = 2 \dim X - 1$. Then X is birational to one of the following varieties:*

- (i) *a product of smooth curves of genus 2 with the 2-dimensional symmetric product of a curve of genus 3 (i.e., X is a product of theta divisors);*
- (ii) *$(C_1 \times \tilde{Z})/\langle \tau \rangle$, where C_1 is a bielliptic curve of genus 2, $\tilde{Z} \rightarrow C_1 \times \cdots \times C_{n-1}$ is an étale double cover of a product of smooth projective curves of genus 2, and τ is an involution acting diagonally on C_1 and \tilde{Z} via the involutions corresponding respectively to the double covers.*

The next natural step would be to classify varieties satisfying (C.1–2) and $q(X) = 2 \dim X - 2$, but we note that this classification would include the classification of surfaces S with $q(S) = p_g(S) = 2$, which is not complete.

To prove the previous theorem, we need the following general proposition, which can be seen as a variant of the main Theorem in [HP3, Theorem 3.1], but it is also the main technical tool to prove Theorem 3.1.

Proposition 3.2. *Let $f : V \rightarrow Z$ be a surjective morphism between smooth projective varieties, where Z is of maximal Albanese dimension. Let \mathcal{F} be a direct summand of $f_*\omega_V$, such that $\chi(Z, \mathcal{F}) = 1$. Then,*

$$q(Z) \leq 2 \dim Z.$$

Moreover, if equality holds, then

- (i) Z is birational to a product of smooth curves of genus 2;
- (ii) the Albanese morphism factors through $a_Z : Z \xrightarrow{g} C_1 \times \cdots \times C_m \hookrightarrow A_Z$, where $m = \dim Z$ and

$$g_*\mathcal{F} = (\omega_{C_1} \boxtimes \cdots \boxtimes \omega_{C_m}) \otimes Q,$$

for some $Q \in \widehat{A}_Z$.

Proof. We argue by induction on $\dim Z$.

If $\dim Z = 1$, then Z is a smooth projective curve of genus ≥ 1 . Since \mathcal{F} is a direct summand of $f_*\omega_V$, we know that $\mathcal{F} \otimes \omega_Z^{-1}$ is a nef vector bundle on Z (see for instance [V1, Corollary 3.6]). From $\chi(Z, \mathcal{F}) = 1$, we deduce by Riemann-Roch that $g(Z) \leq 2$ and, if $g(Z) = 2$, then $\mathcal{F} = \omega_Z \otimes Q$ for some $Q \in \widehat{A}_Z$.

By induction, we may assume the proposition holds for $\dim Z \leq m - 1$.

Step 1. Suppose that $a_Z(Z)$ is fibred by positive dimensional abelian varieties.

Then, let K be the maximal abelian subvariety of A_Z , such that $a_Z(Z) + K = a_Z(Z)$. We consider the quotient $A_Z \rightarrow A_Z/K$ and $Z \xrightarrow{\kappa} Z' \rightarrow A_Z/K$ the Stein factorization of $Z \rightarrow A_Z/K$. After birational modifications and abusing notation, we have the following commutative diagram of smooth projective varieties

$$\begin{array}{ccc} Z & \xrightarrow{a_Z} & A_Z \\ \kappa \downarrow & & \downarrow \\ Z' & \xrightarrow{a_{Z'}} & A_Z/K, \end{array}$$

where $A_{Z'} = A_Z/K$ and $a_{Z'}$ is indeed the Albanese morphism of Z' .

Since Z is of maximal Albanese dimension, so is the general fiber of κ . Hence, for $Q \in \widehat{A}_Z$ a general torsion point and $i \geq 1$, $\mathrm{rk} R^i \kappa_*(f_*\omega_V \otimes Q) = 0$, by generic vanishing on the fibres of κ . By Kollár's Theorem [Kol, Theorem 2.1], the sheaves $R^i \kappa_*(f_*\omega_V \otimes Q)$ are torsion-free, hence

zero for all $i > 0$. We denote by $\mathcal{F}' = \kappa_*(\mathcal{F} \otimes Q)$. Since $R^i \kappa_*(\mathcal{F} \otimes Q) = 0$ for all $i \geq 1$, then $\chi(Z', \mathcal{F}') = \chi(Z, \mathcal{F} \otimes Q) = \chi(Z, \mathcal{F}) = 1$.

Let V' be a connected component of the étale cover of V induced by f^*Q . Let $f' : V' \rightarrow Z'$ be the natural surjective morphism. Then, \mathcal{F}' is clearly a direct summand of $f'_*\omega_{V'}$. Hence by induction on the dimension of Z , we have $q(Z') \leq 2 \dim Z'$ and we get $q(Z) \leq 2 \dim Z - \dim K < 2 \dim Z$.

Step 2. Suppose that $a_Z(Z)$ is not fibred by positive dimensional abelian varieties.

We consider the Fourier-Mukai functors

$$\begin{aligned} \mathbf{R}\Phi_{\mathcal{P}} : D^b(A_Z) &\rightarrow D^b(\widehat{A}_Z), & \mathbf{R}\Phi_{\mathcal{P}}(\cdot) &:= \mathbf{R}p_{\widehat{A}_Z*}(p_{A_Z}^*(\cdot) \otimes \mathcal{P}), \\ \mathbf{R}\Psi_{\mathcal{P}} : D^b(\widehat{A}_Z) &\rightarrow D^b(A_Z), & \mathbf{R}\Psi_{\mathcal{P}}(\cdot) &:= \mathbf{R}p_{A_Z*}(p_{\widehat{A}_Z}^*(\cdot) \otimes \mathcal{P}), \end{aligned}$$

where $\widehat{\mathcal{P}}$ is the Poincaré line bundle on $A_Z \times \widehat{A}_Z$.

Since $a_Z(Z)$ is not fibred by positive dimensional abelian varieties, by Lemma 1.1 (iv), $a_{Z*}\mathcal{F}$ is M-regular. Then, by [PaPo2, Thm 2.2 and Corollary 3.2], the following element in $D^b(\widehat{A}_Z)$,

$$\widehat{\mathbf{R}\Delta\mathcal{F}} := \mathbf{R}\Phi_{\mathcal{P}}(\mathbf{R}\mathcal{H}om_{A_Z}(a_{Z*}\mathcal{F}, \mathcal{O}_{A_Z}))[q(Z)]$$

is indeed a torsion-free coherent sheaf in cohomological degree 0 and of rank $\chi(Z, \mathcal{F}) = 1$. Hence, $\widehat{\mathbf{R}\Delta\mathcal{F}} = M \otimes \mathcal{I}_W$, where M is a line bundle on \widehat{A}_Z and \mathcal{I}_W is the ideal sheaf of $W \subset \widehat{A}_Z$.

If $W = \emptyset$, i.e., $\widehat{\mathbf{R}\Delta\mathcal{F}} = M$ is a line bundle on \widehat{A}_Z , then $\mathbf{R}\Psi_{\mathcal{P}}(M)$ is supported on a translate of an abelian subvariety of A_Z . By Mukai's duality ([Mu, Theorem 2.2]),

$$\mathbf{R}\Psi_{\mathcal{P}}(M) = (-1)_{A_Z}^* \mathbf{R}\mathcal{H}om_{A_Z}(a_{Z*}\mathcal{F}, \mathcal{O}_{A_Z}).$$

Thus, $a_{Z*}\mathcal{F}$ is also supported on a translate of an abelian subvariety of A_Z . Since $a_{Z*}\mathcal{F}$ is supported on $a_Z(Z)$ and $a_Z(Z)$ is not fibred by positive dimensional abelian varieties by hypothesis, we get a contradiction.

We can suppose then that W is not empty. That is, $\widehat{\mathbf{R}\Delta\mathcal{F}}$ is not reflexive so, by [PaPo2, Corollary 3.2], there exists $j > 0$ such that $V^j(\mathcal{F})$ has a component $P + \widehat{B}$ of codimension $j + 1$ in \widehat{A}_Z .

As usual, we consider $Z \xrightarrow{\beta} Z_B \rightarrow B$, the Stein factorization of $Z \rightarrow B$. After birational modifications and abusing notation, we have the following commutative diagram of smooth projective varieties

$$\begin{array}{ccc} Z & \xrightarrow{a_Z} & A_Z \\ \beta \downarrow & & \downarrow \\ Z_B & \xrightarrow{a_{Z_B}} & B, \end{array}$$

where $a_{Z_B} : Z_B \rightarrow B = A_{Z_B}$ is indeed the Albanese morphism of Z_B .

Note that $\dim Z_B = \dim Z - j$ and $q(Z_B) = \dim B = q(Z) - j - 1$. As before, for $Q \in \widehat{A_Z}$ a general torsion point, we have $\chi(Z_B, \beta_*(\mathcal{F} \otimes Q)) = 1$. Then, considering $\mathcal{F}_B := \beta_*(\mathcal{F} \otimes Q)$, by induction on dimensions, we have $q(Z_B) \leq 2 \dim Z_B$. Hence, $q(Z) \leq 2 \dim Z + 1 - j \leq 2 \dim Z$.

We have already proved that $q(Z) \leq 2 \dim Z$ in any case, and now we want to study what happens when equality holds.

Step 3. The variety Z when equality $q(Z) = 2 \dim Z$ holds.

We have seen that in Step 1, we always have an strict inequality. In Step 2, equality can hold only when $j = 1$ and by induction on dimensions, we may assume Z_B is isomorphic to a product of smooth curves of genus 2 and $\mathcal{F}_Q = \beta_*(\mathcal{F} \otimes Q) = \omega_{Z_B} \otimes Q_B$ for some $Q_B \in \widehat{B}$.

Since \mathcal{F} is a direct summand of $f_*\omega_V$, $\mathcal{F} \otimes \omega_Z^{-1}$ is weakly positive (see [V2, Theorem III]). Thus, $(\mathcal{F} \otimes \omega_Z^{-1} \otimes Q)|_C$ is nef, where C is a general fiber of β . Since $\text{rk } \mathcal{F}_Q = 1$, we have $\chi(C, (\mathcal{F} \otimes \omega_Z^{-1} \otimes Q)|_C) = 1$, and we conclude that $g(C) = 2$ and $\text{rk } \mathcal{F} = 1$. Thus, by Proposition 1.5, Z is birational to $C \times Z_B$ and, hence, it is birational to a product of genus 2 curves.

Step 4. The sheaf \mathcal{F} when equality $q(Z) = 2 \dim Z$ holds.

Without loss of generality we may assume that $Z = C_1 \times \cdots \times C_m$. Denote the natural projections by

$$\begin{aligned} p_1 : Z &\rightarrow C_2 \times \cdots \times C_m, \\ p_2 : Z &\rightarrow C_1 \times C_3 \times \cdots \times C_m, \text{ and} \\ p_{12} : Z &\rightarrow C_3 \times \cdots \times C_m. \end{aligned}$$

Fix a general $Q \in \widehat{A_Z}$ and consider the natural sequence

$$(3.1.1) \quad 0 \rightarrow p_{12}^* p_{12*}(\mathcal{F} \otimes Q) \rightarrow \begin{array}{c} p_1^* p_{1*}(\mathcal{F} \otimes Q) \\ \oplus \\ p_2^* p_{2*}(\mathcal{F} \otimes Q) \end{array} \xrightarrow{\varphi} \mathcal{F} \otimes Q.$$

Note that $p_{12}^* p_{12*}(\mathcal{F} \otimes Q)$ lies naturally in the kernel of φ .

By induction hypothesis and comparing the elements in $\text{Pic}^0(C_k)$ on further push-forwards, we have

$$\begin{aligned} p_{1*}(\mathcal{F} \otimes Q) &= (\omega_{C_2} \otimes Q_2) \boxtimes ((\omega_{C_3} \boxtimes \cdots \boxtimes \omega_{C_m}) \otimes Q_{12}), \\ p_{2*}(\mathcal{F} \otimes Q) &= (\omega_{C_1} \otimes Q_1) \boxtimes ((\omega_{C_3} \boxtimes \cdots \boxtimes \omega_{C_m}) \otimes Q_{12}), \text{ and} \\ p_{12*}(\mathcal{F} \otimes Q) &= (\omega_{C_3} \boxtimes \cdots \boxtimes \omega_{C_m}) \otimes Q_{12}, \end{aligned}$$

for some $Q_i \in \text{Pic}^0(C_i)$ and $Q_{12} \in \text{Pic}^0(\prod_{k \neq 1,2} C_k)$ that depend on Q . Now, since $\ker \varphi$ has rank 1, it is easy to see that the sequence (3.1.1) is exact. Moreover, if we choose Q appropriately, then

$$0 \rightarrow \mathcal{O}_{C_1 \times C_2} \rightarrow \begin{array}{c} \mathcal{O}_{C_1} \boxtimes (\omega_{C_2} \otimes Q_2) \\ \oplus \\ (\omega_{C_1} \otimes Q_1) \boxtimes \mathcal{O}_{C_2} \end{array} \rightarrow (\omega_{C_1} \otimes Q_1) \boxtimes (\omega_{C_2} \otimes Q_2) \rightarrow \mathcal{O}_x \rightarrow 0$$

is exact, where x is a point in $C_1 \times C_2$ that depends on Q_1 and Q_2 . Thus, we have a non-trivial morphism

$$(\omega_{C_1} \otimes Q_1) \boxtimes (\omega_{C_2} \otimes Q_2) \boxtimes ((\omega_{C_3} \boxtimes \cdots \boxtimes \omega_{C_m}) \otimes Q_{12}) \longrightarrow \mathcal{F} \otimes Q.$$

Both sheaves are torsion-free of $\text{rk} = 1$ and $\chi = 1$, so they are isomorphic (e.g., [Pa, Lemma 1.12]). \square

Our aim is to apply induction on the dimension to prove Theorem 3.1. By Theorem 2.1, we will be able to reduce to Setting 2.3. But now we are assuming that $a : X \rightarrow A$ is the Albanese morphism and we get some extra properties.

Lemma 3.3 (Extra properties of Setting 2.3). *With the same notations as in Setting 2.3. Assume that $a : X \rightarrow A$ is the Albanese morphism a_X .*

If $Y = a_X(X)$ is of general type, then the morphism $a_B : X_B \xrightarrow{f_B} Y_B \hookrightarrow B$ is the Albanese morphism a_{X_B} of X_B . Moreover, if $q(X) \geq \dim X + 2$, then Y_B is not an abelian subvariety.

Proof. Note that both β and π_B are fibrations and Y_b is an ample divisor in a translate of the kernel of π_B . Hence, a_B is the Albanese morphism of X_B and $Y_B \hookrightarrow B$ is the Albanese image of X_B . If $q(X) \geq \dim X + 2$, then $\dim B \geq \dim Y_B + 1$, so Y_B is not an abelian subvariety of B . \square

Now, we have all the tools to attack Theorem 3.1.

First, we add Condition (C.3), and we get that only case (i) is possible.

Proposition 3.4. *Let X be a smooth projective variety satisfying (C.1–3) and $q(X) = 2 \dim X - 1$. Then X is birational to a product of theta divisors.*

If X is a product of theta divisors and $q(X) = 2 \dim X - 1$, we clearly need to have $\dim X - 1$ theta divisors of dimension 1 (i.e., smooth curves of genus 2) and a theta divisor of dimension 2 (i.e., the 2-dimensional symmetric product of a curve of genus 3).

Proof of Proposition 3.4. The surface case was solved independently by Hacon–Pardini [HP2] and Pirola [Pi]. So, we may assume that $n = \dim X \geq 3$. Since $\mathcal{O}_X \in V^1(\omega_X)$, by Theorem 2.1, there exists \widehat{B} a non-trivial irreducible component of $V^1(\omega_X)$ passing through 0. Hence, we are in Setting 2.3 and we use the notation therein, in particular we recall diagram (2.1.1)

$$\begin{array}{ccccc} X_b & \longrightarrow & Y_b & & \\ \downarrow & & \downarrow & & \\ X & \longrightarrow & Y & \hookrightarrow & A \\ \beta \downarrow & & \downarrow & & \downarrow \\ X_B & \longrightarrow & Y_B & \hookrightarrow & B. \end{array}$$

Now, as in the proof of Theorem 2.8, we would like to apply induction by using Proposition 2.12. The main difficulty is that Y_B may not be of general type.

Consider $Q \in \widehat{A}_X$ a general point such that $R^i \beta_*(\omega_X \otimes Q) = 0$ for all $i > 0$ and we denote $\mathcal{F} = \beta_*(\omega_X \otimes Q)$. Since $\chi(X, \omega_X) = 1$, then $\chi(X_B, \mathcal{F}) = 1$. Denote by $i + 1 = \text{codim}_{\widehat{A}_X} \widehat{B}$, then $\dim X_B = n - i$ and $q(X_B) = 2n - 2 - i$. By Proposition 3.2 applied to β and \mathcal{F} , we have $i \leq 2$.

If $i = 2$, then equality holds in Proposition 3.2 and X_B is birational to a product of smooth projective curves of genus 2. Hence, we can indeed use Proposition 2.12 and we have $\chi(X_b, \omega_{X_b}) = 1$ and $q(X_b) = \dim X_b + 1$. Hence, A_{X_b} is isogenous to the kernel of $\pi_B : A_X \rightarrow B$ and $Y_b = f_b(X_b)$ is not fibred by positive dimensional abelian subvarieties. That is, X_b satisfies the hypothesis of the proposition in dimension 2. Hence by [HP2, Theorem 2.2] or [Pi, Theorem 1.1.1], X_b is birational to an irreducible theta divisor of a principally polarized abelian threefold. Thus, by Proposition 1.5, we have X is birational to $X_B \times X_b$ and we are done.

If $i = 1$, we see that $\dim X_B = n - 1$ and $q(X_B) = 2n - 3$.

If Y_B is of general type, we can apply again Proposition 2.12 to show that the general fiber X_b of β is a genus 2 curve, hence we conclude by induction on dimensions and Proposition 1.5.

If Y_B is not of general type, then we need to go to a further quotient. Let K be the maximal abelian subvariety of B such that $Y_B + K = Y_B$. Denote by $i' = i + \dim K$ and denote by B' the quotient B/K . Let $\pi_{B'} : A_X \rightarrow B'$ be the natural projection. Then, denote by $Y_{B'} := \pi_{B'}(Y)$ and let $X \xrightarrow{\beta'} X_{B'} \xrightarrow{f_{B'}} Y_{B'}$ be Stein factorization of $X \rightarrow Y_{B'}$. Varieties and morphisms described above are compatible with diagram (2.1.1) and fit in the following extended commutative diagram:

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & Y & \hookrightarrow & A_X \\
 \downarrow \beta & & \downarrow & & \downarrow \pi_B \\
 X_B & \xrightarrow{f_B} & Y_B & \hookrightarrow & B \\
 \downarrow & & \downarrow & & \downarrow \\
 X_{B'} & \xrightarrow{f_{B'}} & Y_{B'} & \hookrightarrow & B' \\
 & & \searrow & & \nearrow \\
 & & & & a_{X_{B'}}
 \end{array}$$

After birational modifications and abusing notation, we assume that all varieties in the leftmost column are smooth.

Note that, $Y_{B'}$ is of general type, so also $X_{B'}$ is also of general type.

Note that $\dim X_{B'} = n - 1 - \dim K$ and $q(Z) = 2n - 3 - \dim K$. Consider $Q \in \widehat{A}_X$ a general point such that $R^i \beta'_*(\omega_X \otimes Q) = 0$ for all $i > 0$ and we denote $\mathcal{F}' = \beta'_*(\omega_X \otimes Q)$. Since $\chi(X, \omega_X) = 1$, then $\chi(X_{B'}, \mathcal{F}') = 1$.

By Proposition 3.2 applied to β' and \mathcal{F}' , we have $\dim K = 1$ (i.e., $i' = 2$) and $X_{B'}$ is birational to a product of $n - 2$ curves of genus 2. Without loss of generality, we can assume that $X_{B'}$ is isomorphic to a product of $n - 2$ curves of genus 2.

Let $S_{b'}$ be a general fiber of β' . By Proposition 3.2, we also get that \mathcal{F}' has rank 1, so we have $\chi(S_{b'}, \omega_{S_{b'}}) = 1$. Since $S_{b'}$ is a surface of maximal Albanese dimension and $a_{S_{b'}}(S_{b'})$ is of general type, then either $S_{b'}$ is birational to a symmetric product of curves of genus 3 (when $q(S_{b'}) = 3$) or it is birational to a product of curves of genus 2 (when $q(S_{b'}) = 4$).

In the first case, we conclude by Proposition 1.5. In the last case, let $C_{b'}$ be the fiber of $X_B \rightarrow X_{B'}$ corresponding to $S_{b'}$. Note that $C_{b'}$ is of general type, since X_B is of general type. Then, the morphism $S_{b'} \rightarrow C_{b'}$ needs to be the projection to either C_1 or C_2 . In any case, the general fiber of β is a curve C of genus 2. By Proposition 1.5, we have that X is birational to $C \times X_B$ and we conclude by induction on the dimension of X . \square

Remark 3.5. Note that, a posteriori, we know that, in the previous proof, X_B and Y_B are always birational, so Y_B is of general type. In particular, the second case for $i = 1$ does not appear.

Proof of Theorem 3.1. By Proposition 3.4, we just need to consider varieties not satisfying condition (C.3), i.e., such that $Y = a_X(X)$ is fibred by positive dimensional abelian subvarieties (see Lemma 1.1).

Consider K to be the largest abelian subvariety of A_X such that $Y + K = Y$ and consider the quotient $A_X \rightarrow A_X/K$. Let $X \xrightarrow{\kappa} Z \rightarrow Y/K$ be the Stein factorization of $Z \rightarrow Y/K$. After birational modifications and abusing notation, we have the following commutative diagram of smooth projective varieties

$$(3.1.2) \quad \begin{array}{ccccc} X & \longrightarrow & Y \subset & \longrightarrow & A_X \\ \kappa \downarrow & & \downarrow & & \downarrow \\ Z & \xrightarrow{f_K} & Y/K \subset & \longrightarrow & A_X/K = A_Z, \\ & & \searrow & \nearrow & \\ & & & a_Z & \end{array}$$

where both Z and Y/K are of general type.

Note that $\dim Z = n - \dim K$ and $q(Z) = 2n - 1 - \dim K$. Consider $Q \in \widehat{A}_X$ a general point such that $R^i \kappa_*(\omega_X \otimes Q) = 0$ for all $i > 0$ and we denote $\mathcal{F} = \kappa_*(\omega_X \otimes Q)$. Since $\chi(X, \omega_X) = 1$, then $\chi(Z, \mathcal{F}) = 1$.

By Proposition 3.2 applied to κ and \mathcal{F} , we have $\dim K = 1$ and Z is birational to a product of $n - 1$ curves of genus 2. Without loss of generality, we can assume that Z is isomorphic to a product of $n - 1$ curves of genus 2.

The natural morphism $t : X \rightarrow Z \times_{A_Z} A_X$ is a generically finite and surjective morphism. Thus, $t_* \omega_X = \omega_{Z \times_{A_Z} A_X} \oplus \mathcal{Q}$ for some torsion-free coherent sheaf \mathcal{Q} on $Z \times_{A_Z} A_X$. We then

have $\kappa_*\omega_X = \omega_Z \oplus \mathcal{G}$, for some torsion-free coherent sheaf \mathcal{G} . Note that

$$\begin{aligned}\chi(Z, \mathcal{G}) &= \chi(Z, \kappa_*\omega_X) - \chi(Z, \omega_Z) \\ &= \chi(Z, \kappa_*\omega_X) - \chi(Z, R^1\kappa_*\omega_X) \\ &= \chi(X, \omega_X) = 1.\end{aligned}$$

By Proposition 3.2 applied to κ and \mathcal{G} , the sheaf $\mathcal{G} = \omega_Z \otimes Q$ for some $Q \in \widehat{A_Z}$. In particular, a general fiber of κ is a curve C_1 of genus 2. We also observe that Q is torsion line bundle by Simpson's theorem (see [Si]) and we denote by d the order of Q .

Let $\pi : \tilde{Z} \rightarrow Z$ be the étale cover induced by the cyclic group generated by Q . We denote the Galois group by $G = \mathbb{Z}/d\mathbb{Z}$. We consider the base change of the commutative diagram (3.1.2):

$$\begin{array}{ccccc}\tilde{X} & \longrightarrow & X & \longrightarrow & A_X \\ \downarrow \tilde{\kappa} & & \downarrow \kappa & & \downarrow \\ \tilde{Z} & \xrightarrow{\pi} & Z & \xrightarrow{a_Z} & A_X/K.\end{array}$$

Then

$$\begin{aligned}q(\tilde{X}) &= h^{n-1}(\tilde{X}, \omega_{\tilde{X}}) = h^{n-2}(\tilde{Z}, R^1\tilde{\kappa}_*\omega_{\tilde{X}}) + h^{n-1}(\tilde{Z}, \tilde{\kappa}_*\omega_{\tilde{X}}) \\ &= q(\tilde{Z}) + h^{n-1}(\tilde{Z}, \pi^*\kappa_*\omega_{\tilde{X}}) \\ &= q(\tilde{Z}) + \sum_{i=0}^{d-1} h^{n-1}(Z, \kappa_*\omega_X \otimes Q^i) \\ &= q(\tilde{Z}) + \sum_{i=0}^{d-1} h^{n-1}(Z, (\omega_Z \oplus (\omega_Z \otimes Q)) \otimes Q^i) \\ &= q(\tilde{Z}) + 2.\end{aligned}$$

Since a general fiber of $\tilde{\kappa}$ is still the genus 2 curve C_1 , by Proposition 1.5, we conclude that \tilde{X} is birational to a product $C_1 \times \tilde{Z}$. Thus, there exists a faithful G -action on C_1 and X is birational to the quotient by the diagonal action $(C_1 \times \tilde{Z})/G$ (see, e.g., [Be, Lemma VI.10]). Since $q(X) = q(Z) + 1$, we conclude that C_1/G is an elliptic curve. Since $g(C_1) = 2$ and G is cyclic, we have $d = 2$ and $G = \mathbb{Z}/2\mathbb{Z}$. \square

Birational characterization of products of theta divisors. It is clear from the previous results that we need the condition (C.3) to characterize birationally the products of theta divisors and force f to be birational. Hence, it is natural ask if this extra condition is enough (the question due to Pareschi, see [Ti, Chapter 5]).

Question 3.6 (Pareschi). Let X be a smooth projective variety. If X satisfies the 3 conditions (C.1–3), is X birational to a product of theta divisors?

If X is birational to a product of irreducible theta divisors, then X clearly satisfies all conditions (C.1–3). Hence, if the answer is positive this will provide a birational characterization of products of theta divisors.

To our knowledge, until now, the evidences for a positive answer were:

- (i) the surface case $\dim X = 2$, starting from the work of Beauville (in the appendix of [De]) and finishing with Hacon–Pardini [HP2] and Pirola [Pi];
- (ii) the maximal irregularity case, i.e., when $q(X) = 2 \dim X$, by Hacon–Pardini [HP3];
- (iii) the characterization of theta divisors under some extra assumptions, which started with the work of Hacon [Ha, Section 3]. This initial characterization has been refined by Hacon–Pardini [HP1, Corollary 4.3], Lazarsfeld–Popa [LP, Proposition 3.8(ii)], Barja–Lahoz–Naranjo–Pareschi [BLNP, Proposition 3.1], and Pareschi [Pa, Theorem 5.1].

We have given some new evidences, namely

- (iv) the submaximal irregularity case, i.e., when $q(X) = 2 \dim X - 1$ (see Proposition 3.4);
- (v) a further characterization of theta divisors (see Corollary 2.7).

We also push forward the characterization of theta divisors of Corollary 2.7 to the case of arbitrary codimension of $a_X(X)$ inside A_X . That is, we consider the already mentioned condition (C.4):

$$(C.4) \quad a_X(X) \text{ is smooth in codimension 1.}$$

Then, as a direct corollary of Lemma 1.3 and Theorem 2.8, we obtain the following birational characterization of products of theta divisors.

Theorem 3.7. *Let X be a smooth projective variety. Then X is birational to a product of theta divisors if, and only if, X satisfies the 4 conditions (C.1–4).*

We have already mentioned that Ein–Lazarsfeld prove in [EL] that an irreducible theta divisor is normal and has only rational singularities. Hence, if X is birational to a product of irreducible theta divisors, then $a_X(X)$ is a product of irreducible theta divisors, so it is normal and, in particular, smooth in codimension 1.

It is also possible to give an easier proof of the previous theorem by using induction on the dimension of X , instead of induction on the number of components of $V_0^1(\omega_X, a_X)$. Indeed, since in this case we work directly with the Albanese variety of X , we can avoid using the full machinery of Theorem 2.8 and we can directly use Propositions 1.5, 2.12, and Corollary 2.7.

REFERENCES

- [BLNP] Barja, M.A., Lahoz, M., Naranjo, J.C., and Pareschi, G., *On the bicanonical map of irregular varieties*, J. Algebraic Geom. **21** (2012), pp. 445–471.
- [Be] Beauville, A., *Surfaces algébriques complexes*. Astérisque **54** (1978).
- [Ca] Campana, F., *Remarques sur le revêtement universel des variétés kählériennes compactes*, Bull. Soc. Math. France **122** (1994), no. 2, pp. 255–284.
- [CDJ] Chen, J.A., Debarre, O., Jiang, Z., *Varieties with vanishing holomorphic Euler characteristic*, Published online, J. reine angew. Math. doi: 10.1515/crelle-2012-0073
- [CH1] Chen, J.A., and Hacon, C.D., *Pluricanonical maps of varieties of maximal Albanese dimension*, Math. Ann. **320** (2001), pp. 367–380.
- [CH2] Chen, J.A., and Hacon, C.D., *A surface of general type with $p_g = q = 2$ and $K_X^2 = 5$* , Pacific J. Math. **223** (2006), no. 2, pp. 219–228.
- [De] Debarre, O. *Inégalités numériques pour les surfaces de type général*, Bull. Soc. math. France **110** (1982), pp. 319–346.
- [DH] Debarre, O., and Hacon, C.D., *Singularities of divisors on abelian varieties*, Manuscripta Math. **122** (2007), pp. 217–228.
- [EL] Ein, L., and Lazarsfeld, R., *Singularities of theta divisors and the birational geometry of irregular varieties*, J. Amer. Math. Soc. **10** (1997), pp. 243–258.
- [GL1] Green, M., Lazarsfeld, R., *Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville*, Invent. Math. **90** (1987), pp. 389–407.
- [GL2] Green, M., Lazarsfeld, R., *Higher obstructions to deforming cohomology groups of line bundles*, J. Amer. Math. Soc. **4** (1991), pp. 87–103.
- [Ha] Hacon, C.D., *Fourier transforms, generic vanishing theorems and polarizations of abelian varieties*, Math. Z. **235** (2000), no. 4, pp. 717–726.
- [HP1] Hacon, C.D. and Pardini, R., *On the birational geometry of varieties of maximal Albanese dimension*, J. reine angew. Math. **546** (2002), pp. 177–199.
- [HP2] Hacon, C.D. and Pardini, R., *Surfaces with $p_g = q = 3$* , Trans. Amer. Math. Soc. **354** (2002), no. 7, pp. 2631–2638.
- [HP3] Hacon, C.D. and Pardini, R., *Birational characterization of products of curves of genus 2*, Math. Res. Lett. **12** (2005), no. 1, pp. 129–140.
- [Hu] D. Huybrechts, *Fourier–Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs (2006).
- [JLT] Jiang, Z., Lahoz, M., and Tirabassi, S., *On the Itaka Fibration of Varieties of Maximal Albanese Dimension*, Int. Math. Res. Not. **2013** (2013), no. 13, 2984–3005.
- [Ko1] Kollár, J., *Higher Direct Images of Dualizing Sheaves. I*, Ann. of Math. **123** (1986), pp. 11–42.
- [Ko2] Kollár, J., *Higher Direct Images of Dualizing Sheaves. II*, Ann. of Math. **124** (1986), no. 1, pp. 171–202.
- [LP] Lazarsfeld, R., and Popa, M., *Derivative complex, BGG correspondence, and numerical inequalities for compact Kähler manifolds*, Invent. Math. **182** (2010), pp. 605–633.
- [Mo] Mori, S., *Classification of higher-dimensional varieties*, in Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), pp. 269–331, Proc. Sympos. Pure Math. **46**, Part 1, Amer. Math. Soc., Providence, RI, 1987.
- [Mu] S. Mukai, *Duality between $D(X)$ and $D(\widehat{X})$ with its application to Picard sheaves*, Nagoya math. J. **81** (1981), pp. 153–175.

- [Pa] Pareschi, G., *Basic results on irregular varieties via Fourier-Mukai methods*, in Current developments in algebraic geometry, pp. 379–403, Math. Sci. Res. Inst. Publ., **59**, Cambridge Univ. Press, Cambridge, 2012.
- [PaPo1] Pareschi, G., and Popa, M., *Regularity on abelian varieties I*, J. Amer. Math. Soc. **16** (2003), pp. 285–302.
- [PaPo2] Pareschi, G., and Popa, M., *Strong generic vanishing and a higher-dimensional Castelnuovo-de Franchis inequality*, Duke Math. J. **150** (2009), no. 2, pp. 269–285.
- [PaPo3] Pareschi, G., Popa, M., *Regularity on abelian varieties III: relationship with generic vanishing and applications*, Grassmannians, moduli spaces and vector bundles, 141–167, Clay Math. Proc., **14**, Amer. Math. Soc., Providence, RI, 2011.
- [PePo1] Penegini, M. and Polizzi, F., *Surfaces with $p_g = q = 2$, $K^2 = 6$, and Albanese Map of Degree 2*, Canad. J. Math. **65** (2013), pp. 195–221.
- [PePo2] Penegini, M. and Polizzi, F., *On surfaces with $p_g = q = 2$, $K^2 = 5$ and Albanese map of degree 3*, to appear in Osaka J. Math.
- [Pi] Pirola, G.P., *Surfaces with $p_g = q = 3$* , Manuscripta math. **108** (2002), pp. 163–170.
- [Se] Serrano, F., *Isotrivial fibered surfaces*, Annali di Matematica Pura ed Applicata (IV), **171** (1996), pp. 63–81.
- [Si] Simpson, C., *Subspaces of moduli spaces of rank one local systems*, Ann. Sci. École Norm. Sup. **26** (1993), pp. 361–401.
- [Ti] Tirabassi, S., *Szygies, Pluricanonical Maps and the Birational Geometry of Irregular Varieties*, Università degli studi Roma TRE, 2012.
- [Ue] Ueno, K., *Classification Theory of algebraic Varieties and Compact Complex Spaces*, Lecture Notes in Math., no. 439, Springer-Verlag (1975).
- [V1] Viehweg, E., *Positivity of direct image sheaves and applications to families of higher dimensional manifolds*, ICTP Lecture Notes **6** (2001), pp. 249–284.
- [V2] Viehweg, E., *Weak positivity and the additivity of the Kodaira dimension for certain fiber spaces*, Algebraic varieties and analytic varieties (Tokyo, 1981), 329–353, Adv. Stud. Pure Math. 1 (1983), North-Holland, Amsterdam.
- [Zh] Zhang, L., *On the bicanonical maps of primitive varieties with $q(X) = \dim(X)$: the degree and the Euler number*. arXiv:1210.2060.

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