UNIVERSITAT DE BARCELONA

Facultat de Matemàtiques i Informàtica

GRAU DE MATEMÀTIQUES

Treball final de grau

THE GEOMETRISATION CONJECTURE OF 3-MANIFOLDS

Autor: Ángel Prieto de la Cruz

Director: Dr. Javier J. Gutiérrez

Realitzat a: Departament de Matemàtiques i Informàtica

Barcelona, 20 de junio de 2019

Abstract

This thesis aims to be a first approach to Thurston's geometrisation conjecture, which states that each 3-manifold decomposes canonically into pieces admitting geometric structures. Starting from the definition of a model geometry, we will see first that the only three model geometries in dimension 2 are the Euclidean, the elliptic and the hyperbolic. Then we will show how Thurston's theorem asserts that there are a total of eight model geometries in dimension 3, and we will classify six of them as Seifert spaces. We will finish by explaining the geometrisation conjecture through a historical perspective, from the first results on sphere and torus decompositions to Perelman's proof. We will also sketch a proof of the Poincaré conjecture as an immediate corollary.

2010 Mathematics Subject Classification. 57M50, 57M60, 57N16

Acknowledgements

I would like to express my deep gratitude to Dr. Javier J. Gutiérrez, my thesis director, for his valuable technical support on this project. I would also like to thank Dr. Joan Porti, one of the world's most expert researchers in this matter, for receiving me at his office. I also wish to acknowledge the help provided by my girlfriend Sara, whose personal support and advice has been crucial. Finally, I want to dedicate this thesis to my mother. None of this would have been possible without her.

Contents

Introduction

In 1904, Henri Poincaré asked himself the following question:

> *"Consider a compact 3-dimensional manifold V without boundary. It is possible that the fundamental group of V could be trivial, even though V is not homeomorphic to the 3-dimensional sphere?"*

Nowadays this question is known as the Poincaré conjecture, which in its modern formulation says that every simply connected, compact 3-manifold is homeomorphic to the 3-sphere. It was well known by 19th-century mathematicians that the analogous statement held true for the 2-dimensional sphere. However, the Poincaré conjecture could not be proved throughout the 20th century, despite the number of influential mathematicians who tried it, such as J.H.C. Whitehead or E. Moise. The conjecture was gaining reputation as more flawed proofs were published, and in the year 2000 it was classified by the Clay Mathematics Institute as one of the seven Millennium Prize Problems. However, as John Milnor said, the study of the Poincaré conjecture has led not only to many false proofs, but also to a deepening in the understanding of the topology of manifolds. A large number of mathematicians worked on 3-dimensional manifolds in the second half of the 20th century, and the amount of literature on this topic grew exponentially.

One of the most ambitious goals for contemporary topologists was to develop an exhaustive classification of three-dimensional geometries on manifolds, as there was for two-dimensional ones. Indeed, they knew that any compact Riemannian surface could be endowed with a geometric structure modelled only on the Euclidean plane $\mathbb{E}^2$, the sphere S^2 or the hyperbolic plane $\mathbb{H}^2$, as a consequence of the uniformization theorem proved by Poincaré in 1907. The 3-dimensional analogues of such geometries, namely Euclidean, elliptic and hyperbolic, also played an important role in the understanding of the geometric structures on 3-manifolds, since they have constant curvature and therefore a large number of isometries. However, in three dimensions, more complex and non-symmetric geometries emerged.

William Thurston proved that there are a total of 8 geometries where we can model a compact 3-dimensional manifold: The three mentioned above $\mathbb{E}^3$, S^3 and $\mathbb{H}^3$; the two product manifolds $S^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$; the universal covering space of the special linear group $SL_2\mathbb{R}$; and two Lie groups called Nil and Sol. Unfortunately, he realised that each compact 3-manifold did not admit a geometric structure, unlike the two dimensional case. In view of this, he conjectured in 1982 that every compact 3-manifold can be decomposed canonically into smaller manifolds admitting geometric structures.

Thurston's conjecture was known as the geometrisation conjecture of 3-manifolds, and it has been one of the hardest challenges for topology in recent decades. Thurston himself tried to prove it, obtaining some positive partial results. On the other hand, Richard Hamilton proposed a method to prove it, but he was unable to do so. Finally, the mathematician Grigori Perelman, based on Hamilton's method, published in 2003 several articles which included a sketch of the proof. Perelman's work was verified in 2006, leading to his being offered the Fields Medal, which he declined. The Poincaré conjecture was thus automatically proven, since any simply connected, compact 3-manifold can be modelled on S^3. Due to this fact, Perelman was awarded the Millennium Prize in 2010, but he turned down the prize saying that his contribution was no greater than Hamilton's.

Nowadays, the geometrisation conjecture together with Thurston's classification of three-dimensional geometries is called the Thurston-Hamilton-Perelman geometrisation theorem, which states that each compact 3-manifold can be decomposed into smaller manifolds which admit geometric structures modelled on one, and only one, of the 8 Thurston geometries. This theorem, together with all the mathematical tools developed in its study, will be fundamental for the progress of the topology of manifolds in the 21st century.

Chapter 1

Geometric Structures on Manifolds

The aim of this chapter is to recall the basic definitions and properties of differential geometry in order to describe geometric structures on manifolds. First of all, we will explain how to endow a manifold with a Riemannian metric. Then we will see triangulations and we will define the Euler characteristic, which will be useful to classify the 2- and 3-dimensional geometries in the next chapters. Also, we will show the existence of the universal covering space of a manifold, where we will model the geometries, and we will study Lie groups and their actions on manifolds. Finally we will define a model geometry.

1.1. Riemannian Manifolds

A *manifold* is a topological space that locally resembles the Euclidean space near each point. To give a more formal definition, we need to specify what such local maps mean. A *coordinate n-chart* on a topological space X is a pair (U, ϕ), where U is open in X and $\phi : U \to \mathbb{R}^n$ is a homeomorphism onto its image. A collection of coordinate n-charts $\mathcal{A} = \{(U_i, \phi_i)\}_{i \in I}$ on a topological space X whose domains cover X, $\bigcup_{i \in I} U_i = X$, is called an *n-atlas* for X.

Definition 1.1. (Manifold) *An n-dimensional manifold M is a second countable Hausdorff topological space with an n-atlas on it.*

For some purposes, it is useful to define a more general notion, by replacing $\mathbb{R}^n$ in the above definitions by the *upper half space* $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_n \geq 0\}$.

Definition 1.2. (Manifold with boundary) *An n-dimensional manifold with boundary M is a second countable Hausdorff topological space in which every point has a neighbourhood homeomorphic to an open subset of $\mathbb{H}^n$.*

We define the *interior* of a manifold with boundary M as the subspace $\text{Int}(M)$ consisting of points which have some neighbourhood homeomorphic to $\mathbb{R}^n$, and the *boundary* of M as $\partial M = M \setminus \text{Int}(M)$. The interior and the boundary of M are manifolds (without boundary) of dimension n and $n-1$, respectively. Since any open ball in $\mathbb{R}^n$ is homeomorphic to an open subset of $\mathbb{H}^n$, any manifold is a manifold with (empty) boundary. However, the converse is not true. A manifold with boundary is a manifold if and only if $\partial M = \emptyset$. Unless otherwise specified, we will refer to a manifold as a manifold without boundary.

We are interested in adding additional structures on manifolds, such as a differentiable structure. A *$\mathcal{C}^r$-differentiable atlas* for a n-dimensional manifold M is an n-atlas $\mathcal{A} = \{(U_i, \phi_i)\}_{i\in I}$ such that whenever two charts $(U_i, \phi_i), (U_j, \phi_j)$ have a nonempty intersection, $U_i \cap U_j \neq \emptyset$, the transition map

$$\psi_{ij} := \phi_i \circ \phi_j^{-1} : \phi_j \left(U_i \cap U_j\right) \to \phi_i \left(U_i \cap U_j\right)$$

is a $\mathcal{C}^r$-diffeomorphism, for all $i, j \in I$.

One must keep in mind that there is not a unique atlas associated with a given manifold. Two $\mathcal{C}^r$-differentiable n-atlases $\mathcal{A}_1, \mathcal{A}_2$ are said to be *compatible* if their union $\mathcal{A}_1 \cup \mathcal{A}_2$ is a $\mathcal{C}^r$-differentiable n-atlas. It can be shown that compatibility is an equivalence relation. The equivalence class of a differentiable atlas $[\mathcal{A}]$ is called a *differentiable structure* on M.

Definition 1.3. (Differentiable manifold) *An n-dimensional manifold M is said to be $\mathcal{C}^r$-differentiable, or a $\mathcal{C}^r$-manifold, if it has a differentiable structure. $\mathcal{C}^\infty$-manifolds are also called smooth manifolds.*

Another property that we want to define in a manifold is the orientation. We will say that a $\mathcal{C}^r$-differentiable n-atlas $\mathcal{A} = \{(U_i, \phi_i)\}_{i\in I}$ is an *n-oriented atlas* if, for all transition maps $\psi_{ij} := \phi_i \circ \phi_j^{-1}$, the Jacobian matrix $D\psi_{ij}$ has positive determinant.

Definition 1.4. (Orientable manifold) *A n-dimensional $\mathcal{C}^r$-manifold M is orientable if it admits an n-oriented atlas.*

Let M and N be two n-dimensional $\mathcal{C}^r$-manifolds. A map $f : M \to N$ is a *differentiable map at $p \in M$* if there exist coordinate n-charts (U, ϕ) in M and (V, φ) in N, with $p \in U$ and $f(p) \in V$, such that $\varphi \circ f \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is of class $\mathcal{C}^r$ at $\phi(p)$. A map $f : M \to N$ is a *differentiable map* if it is differentiable at each point $p \in M$. We will say that M and N are *diffeomorphic* if there exists a differentiable bijective $\mathcal{C}^r$-map $f : M \to N$ such that the inverse $f^{-1} : N \to M$ is also of class $\mathcal{C}^r$. M. Kervaire and J. Milnor proved in [Ker63] that, if M and N are homeomorphic 3-manifolds, then they are diffeomorphic.

If M is a n-dimensional smooth manifold, we will denote by $\mathcal{C}^\infty(M)$ the set of all the $\mathcal{C}^\infty$-differentiable maps (also called *smooth maps*) $M \to \mathbb{R}$, where we are considering $\mathbb{R}$ as a 1-dimensional manifold with the canonical differentiable structure. The set $\mathcal{C}^\infty(M)$ is a vector space over $\mathbb{R}$, with the sum of smooth maps and multiplication by scalars.

We want to define an analogous notion of the derivative on manifolds. We define a *derivation* D_p in M at $p \in M$ to be a linear map $D_p : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ such that, for every $f, g \in \mathcal{C}^\infty(M)$, $D_p(fg) = D_pf \cdot g(p) + f(p) \cdot D_pg$. With that, we can define the *tangent space* of a manifold M at a point $p \in M$ in a similar way as we do in differential geometry:

Definition 1.5. (Tangent space) *Let M be a n-dimensional smooth manifold. The tangent space T_pM of M at p is the set of all derivations of M at p,*

$$T_pM = \left\{ D_p : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M) \right\}.$$

Note that the tangent space T_pM is also a n-dimensional vector space, with the sum and multiplication by scalars defined as $(\lambda D_{p,1} + \mu D_{p,2})f = \lambda D_{p,1}f + \mu D_{p,2}f$, for $\lambda, \mu \in \mathbb{R}$ and for each pair of derivations $D_{p,1}, D_{p,2}$ in M at p. The disjoint union of all the tangent spaces of M, $\bigsqcup_{p \in M} T_pM$, is called the *tangent bundle* of M. Formally,

Definition 1.6. (Tangent bundle) *Let $(M, [\mathcal{A}])$ be a n-dimensional smooth manifold. Let TM be the set*

$$TM = \left\{ (p, q) \ : \ p \in M, \, q \in T_pM \right\},$$

and let $\pi : TM \to M$ be the projection map satisfying $\pi(p, q) = p$. The triple (TM, M, π) is called the tangent bundle of M.

Note that $\pi^{-1}(\{p\}) = T_pM$, for each $p \in M$. It can be proved that, if M is a n-dimensional smooth manifold, then the tangent bundle TM comes equipped with a *natural topology* and is a $2n$-dimensional smooth manifold. The main idea is that we can define maps $\vartheta_i : \pi^{-1}(U_i) \to \mathbb{R}^n \times \mathbb{R}^n$ for every local chart (U_i, ϕ_i) of M, such that the base of the natural topology on TM are the sets $\vartheta_i^{-1}(B)$, for each $i \in I$ and such each open subset $B \subset \mathbb{R}^n \times \mathbb{R}^n$ and that a smooth atlas for TM is $\left\{ \left(\pi^{-1}(U_i), \vartheta_i\right) \right\}_{i \in I}$.

With the definition of a derivation D_p in a smooth manifold, we can define the *differential* of a smooth map between smooth manifolds:

Definition 1.7. (Differential) *The differential $d\varphi_p$ of a smooth map $\varphi : M \to N$ between smooth manifolds M and N at a point $p \in M$ is a linear map $d\varphi_p : T_pM \to T_{\varphi(p)}N$ satisfying $\left(d\varphi_p\left(D_p\right)\right)(f) = D_p(f \circ \varphi)$, for each smooth map $f \in \mathcal{C}^\infty(N)$. The differential $d\varphi$ of φ is a map $d\varphi : TM \to TN$ such that $d\varphi(p,q) = d\varphi_p(q)$, for each $(p,q) \in TM$.*

Vector fields in the Euclidean space are well known. They can also be defined on manifolds: A *smooth vector field* X on a smooth manifold M is a smooth map $X : M \to TM$ such that $\pi \circ X$ is the identity map on M, where (TM, M, π) is the tangent bundle of M. That is, for each $p \in M$, $X_p := X(p) \in T_pM$, so X_p is a derivation in M at p. The set of all the smooth vector fields on M is usually denoted as $\mathcal{C}^\infty(M, TM)$. It can be shown that it is an $\mathbb{R}$-vector space. If $f \in \mathcal{C}^\infty(M)$ is a smooth map and $X \in \mathcal{C}^\infty(M, TM)$ is a smooth vector field, we can define actions $f \cdot X$ and $X \cdot f$ by setting $(f \cdot X)_p = f(p)X_p$ and $(X \cdot f)(p) = X_p(f)$, respectively, for each $p \in M$. That motivates the following definition:

Definition 1.8. (Affine connection) *An affine connection ∇ on a smooth manifold M is a bilinear map*

$$\begin{array}{rrcl} \nabla : & \mathcal{C}^\infty(M, TM) \times \mathcal{C}^\infty(M, TM) & \to & \mathcal{C}^\infty(M, TM) \\ & (X, Y) & \mapsto & \nabla_X Y \end{array}$$

satisfying:

(I) $\nabla_{f \cdot X} Y = f \cdot \nabla_X Y$,

(II) $\nabla_X (f \cdot Y) = (X \cdot f) \cdot Y + f \cdot \nabla_X Y$,

for each pair of smooth vector fields X, Y and for each smooth map $f : M \to \mathbb{R}$.

Since the tangent space T_pX of a smooth manifold at $p \in M$ is a vector space, we can define a (positive-definite) *inner product* $g_p : T_pM \times T_pM \to \mathbb{R}$ on T_pM. An inner product g_p on T_pM is said to be *smoothly chosen* if for every two vector fields $X, Y : M \to TM$, the maps $p \mapsto g_p(X_p, Y_p)$ are smooth. With all that, we can endow a smooth manifold M with an structure, called a *Riemannian metric*, which will allow us to define several geometric notions on manifolds.

Definition 1.9. (Riemannian manifold) *A Riemannian metric g on a smooth manifold M is a family of smoothly chosen inner products $g_p : T_pM \times T_pM \to \mathbb{R}$ at each point $p \in M$. A Riemannian manifold is a pair (M, g), where M is a smooth manifold and g is a Riemannian metric on M.*

We can think globally in a Riemannian metric g on a smooth manifold M as a positive-definite, symmetric bilinear map $g : \mathcal{C}^\infty(M, TM) \times \mathcal{C}^\infty(M, TM) \to \mathcal{C}^\infty(M)$, satisfying $g(f \cdot X, Y) = g(X, f \cdot Y)$, for each $f \in \mathcal{C}^\infty(M)$. If (M, g) is a Riemannian manifold, then there exists an unique affine connection ∇, called the *Levi-Civita connection*, such that is symmetric and preserves the metric g.

Definition 1.10. (Levi-Civita connection) *The Levi-Civita connection is an affine connection ∇ on a smooth manifold M satisfying that, for every $X, Y, Z \in \mathcal{C}^\infty(M, TM)$, the following properties hold:*

(i) $X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$,

(ii) $\nabla_X Y - \nabla_Y X = [X, Y]$,

where $[X, Y]$ is the Lie bracket defined as $[X, Y](f) = X \cdot (Y \cdot f) - Y \cdot (X \cdot f)$, for each $f \in \mathcal{C}^\infty(M)$.

The Levi-Civita connection is related to the parallel displacement of a vector along a curve. A *parametrized curve* on a n-dimensional smooth manifold M is a smooth map $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an interval. We will say that a map $V : I \to TM$ is a *vector field along a curve* $\gamma : I \to M$ if it is smooth and $V(t) \in T_{\gamma(t)}M$, for every $t \in I$. An important vector field along a curve γ is the *velocity vector* $\dot{\gamma}(t) \in T_{\gamma(t)}M$, whose components are the derivatives of $\gamma_i : I \to \mathbb{R}$ with respect to $t \in I$.

A *geodesic* is a parametrized curve whose tangent vectors remain parallel if they are transported along it. Using the Levi-Civita connection, this is equivalent to:

Definition 1.11. (Geodesic) *Let (M, g) be a Riemannian manifold and let ∇ be the Levi-Civita connection of (M, g). A geodesic is a parametrized curve $\gamma : I \to M$ such that $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$.*

Geodesics allows us to have an intuition about geometric structures on manifolds, since we can define the *length* of a smooth parametrized curve $\gamma : [a, b] \to M$ as

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}\left(\dot{\gamma}(t), \dot{\gamma}(t)\right)}\, dt.$$

We will study this more deeply in Section 1.5.

Affine connections are also useful to describe the notion of curvature. Given an affine connection ∇ on a smooth manifold M, we define the *curvature* of ∇ to be a trilinear map $R_\nabla : \mathcal{C}^\infty(M, TM) \times \mathcal{C}^\infty(M, TM) \times \mathcal{C}^\infty(M, TM) \to \mathcal{C}^\infty(M, TM)$ given by

$$R_\nabla\left((X, Y), Z\right) = \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) Z.$$

When M is a Riemannian manifold, we can give a geometric interpretation of the local curvature of M.

Definition 1.12. (Sectional curvature) *Let (M, g) be a n-dimensional Riemannian manifold and ∇ the Levi-Civita connection of M. If p is a point in M, Π is any 2-dimensional subspace of T_pM and (X_1, X_2) is any basis of Π, we define the sectional curvature $K_p(M)$ of M at p associated with Π to be*

$$K_p(M) = g_p\left(R_\nabla\left((X_1, X_2), X_2\right), X_1\right).$$

The sectional curvature does not depend on the chosen basis of Π. Some of the most interesting Riemannian manifolds are those which have constant sectional curvature. That is, the manifolds M such that $K_p(M)$ has the same value at every point p and for every 2-dimensional vectorial plane $\Pi \subset T_pM$. We will classify such manifolds in Chapter 2 and Chapter 3.

1.2. Triangulations

A basic building block for topological spaces is the simplex. A 0-simplex is a point, a 1-simplex is a closed interval, a 2-simplex is a filled-in triangle and a 3-simplex is a solid tetrahedron. To generalize this to higher dimensions, we will considerer that we are in an affine space $\mathbb{A}^n$.

Definition 1.13. (Euclidean simplex) *An n-dimensional Euclidean simplex σ is the convex hull of $n + 1$ affinely independent points $v_0, \ldots, v_n \in \mathbb{A}^n$.*

Such a Euclidean simplex is denoted by $\sigma = \langle v_0, \ldots, v_n \rangle$. Note that the dimension of the affine space must be at least n, but it could be higher. If $\{e_i\}$ are the points $e_i = (0, \ldots, 1, \ldots, 0)$ of $\mathbb{A}^{n+1}$, for $i \in \{1, \ldots, n+1\}$, we define the *standard n-simplex* as $\triangle^n = \langle e_1, \ldots, e_{n+1} \rangle \subset \mathbb{A}^{n+1}$. However, Euclidean simplices are not general enough for many important applications. We will define simplices in a more abstract way.

Definition 1.14. (Abstract simplicial complex) *An abstract simplicial complex Σ is a pair (V, S), where V is a set whose elements are called vertices, and S is a collection of finite nonempty subsets of V satisfying:*

(I) *For each $v \in V$, $\{v\} \in S$.*

(II) *If $\sigma \in S$ and $\tau \subset \sigma$ is nonempty, then $\tau \in S$.*

An element $\sigma \in S$ consisting of $n+1$ vertices is called an abstract n-dimensional simplex.

Henceforth, we will write $\sigma \in \Sigma$ instead of $\sigma \in S$, and we will assume that a *simplex* is an abstract simplex. If $\sigma = \{v_0, \ldots, v_n\}$ is a n-dimensional simplex, we will continue to denote it as $\sigma = \langle v_0, \ldots, v_n \rangle$. Any nonempty subset of $\sigma \in \Sigma$ is called a *face* of σ. A 0-dimensional face is called a *vertex*, and a 1-dimensional face is an *edge*. A face of dimension $n-1$ is said to be a *facet*. We will say that Σ is a *finite complex* if it has finitely many simplices. The *dimension* of a finite complex Σ is the maximum dimension of any simplex in Σ.

The standard simplex $\triangle^n$ inherits a topology from $\mathbb{A}^{n+1}$. If σ is a k-simplex in a complex Σ, we can take a copy $|\sigma|$ of $\triangle^k$. Note that $|\sigma|$ is a topological space with the topology coinduced by $f : \triangle^k \to |\sigma|$.

Definition 1.15. (Geometric realization) *Let Σ be a n-simplicial complex. For each k-simplex $\sigma \in \Sigma$, we choose pair of facets such that each facet appears in exactly one of the pairs, and identify the facets of each pair. The geometric realization $|\Sigma|$ of Σ is a topological space, obtained as the quotient space of the disjoint union $\coprod_{\sigma \in \Sigma} |\sigma|$ by the equivalence relation generated by these identification maps.*

Definition 1.16. (Triangulation) *Any topological space X homeomorphic to $|\Sigma|$ is called a polyhedron. Such a homeomorphism $h : |\Sigma| \to X$ is called a triangulation of X. Any space that admits a triangulation is said to be triangulable.*

We want to answer whether every smooth manifold admits a triangulation. A manifold is called a *piecewise linear manifold*, or PL-manifold, if the transition maps

$\psi_{ij} = \phi_i \circ \phi_j^{-1}$ are piecewise linear. PL-manifolds are easy to triangulate, since they admit a triangulation in which the manifold structure is evident, called a *combinatorial triangulation*.

Definition 1.17. (Combinatorial triangulation) *If Σ is a simplicial complex, the link of a simplex $\sigma \in \Sigma$ is the union of the simplices $\tau_i \in \Sigma$ such that $\sigma \cap \tau_i = \emptyset$ and σ, τ_i are both faces of a simplex in Σ. A triangulation is called combinatorial if the link of every simplex is homeomorphic to a sphere.*

In 1940, J.H.C. Whitehead proved that every smooth manifold admits an essentially unique compatible PL-structure [Whi40], so any smooth manifold is thus triangulable. Low dimensional manifolds (≤ 3) admit a combinatorial triangulation, since each of these manifolds admits a smooth structure, unique up to isomorphism; this was proved for surfaces by Tibor Radó in the 1920s [Rad25] and for three-manifolds by Edwin Moise in the 1950s [Moi52]. In dimension greater or equal than 4, however, there are some manifolds that do not admit a triangulation.

Since we are interested in triangulating smooth, orientable manifolds, we want to relate the properties of compactness and orientability of polyhedra with some equivalent properties of simplicial complexes. An *edge path* in a simplicial complex Σ is a sequence of vertices such that any two consecutive vertices span an edge. We say that Σ is *edge path-connected* if any two vertices can be joined by a finite edge path.

Proposition 1.18. *Let Σ be a simplicial complex. Then, $|\Sigma|$ is connected if and only if Σ is edge path-connected.*

Proof. See [Lee00, Prop.5.9] □

Furthermore, we are interested in defining an orientation of a simplicial complex.

Definition 1.19. (Oriented simplex) *Let $\sigma = \langle v_0, \dots, v_n \rangle$ be a n-simplex. Given any two orderings $(v_{i_0}, \dots, v_{i_n})$ and $(v_{j_0}, \dots, v_{j_n})$ of the vertices of σ, there is a permutation s such that $s(i_k) = j_k$, for $k \in \{0, \dots, n\}$. Define an equivalence relation $\sim$ on the set of all orderings by saying that two orderings are equivalent if they differ by an even permutation. A choice of an equivalence class of vertex orderings is called an orientation of σ. An oriented simplex $\sigma = [v_0, \dots, v_n]$ is a simplex together with a choice of orientation.*

If $\sigma = [v_0, \dots, v_n]$ is an oriented n-simplex, the orientation of σ determines an orientation on each of its facets.

Definition 1.20. (Induced orientation) *Let $\tau_i = \langle v_0, \ldots, \hat{v}_i, \ldots, v_n \rangle$ be a facet of an oriented n-simplex $\sigma = [v_0, \ldots, v_n]$, where $\hat{v}_i$ denotes that v_i is omitted. The induced orientation on τ_i is defined to be $\tau_i = (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n]$.*

If σ and σ' are two n-simplices such that their intersection is a facet τ, we say that the orientations of σ and σ' are *consistent* if they induce opposite orientations on τ. It can be shown that if M is a triangulated n-manifold with a triangulation $|\Sigma| \to M$, then Σ is a simplicial complex in which every $(n-1)$-simplex is a facet of no more than two n-simplices. This fact motivates the following definition:

Definition 1.21. (Orientable simplicial complex) *Let Σ be an n-dimensional simplicial complex. An orientation of Σ is a choice of orientation of each n-simplex in such a way that any two simplices that intersect in a facet are consistently oriented. If Σ admits an orientation, it is said to be orientable.*

Triangulating a manifold is interesting for several reasons. One of the most important is that we can define a topological invariant: The *Euler characteristic*. It is a well-known result in surface theory that, if $P \subset \mathbb{R}^3$ is a compact polyhedral surface that is the boundary of a convex open set; and F, E, V is the number of faces, edges and vertices of P, respectively, then $V - E + F = 2$. If we define the Euler characteristic as $\chi = V - E + F$, then $\chi = 2$ for all $P \subset \mathbb{R}^3$ as previously defined. We can generalize this for finite simplicial complexes:

Definition 1.22. (Euler characteristic) *Given a finite simplicial complex Σ of dimension n, letting n_k the number of k-simplices in Σ, we define the Euler characteristic $\chi(\Sigma)$ of Σ as*

$$\chi(\Sigma) = \sum_{k=0}^{n} (-1)^k n_k.$$

We would like to prove that, if $|\Sigma| \to X$ and $|\Sigma'| \to X$ are two different triangulations of the space X, then $\chi(\Sigma) = \chi(\Sigma')$. Let's first define a way to subdivide the simplices of a complex into smaller ones.

Definition 1.23. (Subdivision of a complex) *A subdivision of a simplicial complex Σ is any complex Σ' having the same polyhedron as Σ, and such that every simplex in Σ' is contained in some simplex in Σ.*

Definition 1.24. (Elementary subdivision) *Let Σ' be a subdivision of a simplicial complex Σ. We say that is an elementary subdivision if Σ' contains precisely one more vertex than Σ.*

An elementary subdivision can be obtained as follows: If Σ is a n-dimensional simplicial complex and $\sigma = \langle v_0 \dots v_m \rangle \in \Sigma$ is a m-simplex, with $m \leq n$, choose a point $u \in \mathrm{Int}\sigma$, and replace each simplex $\langle v_0, \dots, v_m, w_1, \dots, w_k \rangle$ that has σ as a face (including σ itself) by the set of all simplices of the form $\langle u, v_{i_1}, \dots, v_{i_j}, w_1, \dots, w_k \rangle$, as $\{v_{i_1}, \cdots, v_{i_j}\}$ ranges over proper subsets of $\{v_0, \dots, v_m\}$. We can do iterated elementary subdivisions to obtain any finite subdivision.

Definition 1.25. (Combinatorial equivalence) *Let $\Sigma = (V, S)$ and $\Sigma' = (V', S')$ be two finite simplicial complexes. A map $f : \Sigma \to \Sigma'$ is called an isomorphism if there exists a bijection $f_0 : V \to V'$ such that $\sigma = \{v_0, \cdots, v_n\}$ is a simplex of Σ if and only if $\sigma' = \{f_0(v_0), \cdots, f_0(v_n)\}$ is a simplex of Σ'. We say that Σ and Σ' are combinatorially equivalent if they are isomorphic after finitely many elementary subdivisions.*

Proposition 1.26. *If Σ_1 and Σ_2 are combinatorially equivalent finite simplicial complexes, then $\chi(\Sigma_1) = \chi(\Sigma_2)$.*

Proof. It is enough to prove that if Σ is a finite simplicial complex of dimension n and Σ' is an elementary subdivision of Σ obtained by adding a vertex u in the m-simplex $\sigma = \langle v_0, \dots, v_m \rangle$, then $\chi(\Sigma') - \chi(\Sigma) = 0$.

For every simplex $\tau = \langle v_0, \dots, v_m, w_1, \dots, w_k \rangle$ of Σ that has σ as a face, Σ' has one less $(m+k)$-simplex. In its place, for each j-element proper subset $\{v_{i_1}, \dots, v_{i_j}\}$ of $\{v_0, \dots, v_m\}$, Σ' has a new $(j+k)$-simplex $\langle u, v_{i_1}, \cdots, v_{i_j}, w_1, \dots, w_k \rangle$. There are $\binom{m+1}{j}$ such subsets, for $j \in \{0, \dots, m\}$, so

$$\begin{aligned} \chi(\Sigma') - \chi(\Sigma) = & -(-1)^{m+k} + \sum_{j=0}^{m} \binom{m+1}{j} (-1)^{j+k} = \\ & = \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^{j+k} = (-1)^k (x+1)^{m+1} \big|_{x=-1} = 0. \end{aligned}$$

□

This does not prove yet that the Euler characteristic is a topological invariant, since two triangulations of the same space are not necessarily combinatorially equivalent. However, it is shown for triangulated compact manifolds of dimension 2 and 3, that if their complexes have homeomorphic polyhedra, then they are combinatorially equivalent.

1.3. Covering Spaces

Given a topological space X, we are interested in finding topological spaces which cover X nicely. Roughly speaking, we say that C is a *covering space* of X if C maps onto X in a locally homeomorphic way. Formally,

Definition 1.27. (Covering space) *Let X be a topological space. A covering space of X is a topological space C together with a continuous surjective map $p : C \to X$ such that for every $x \in X$, there exists an open neighbourhood U of x, such that $p^{-1}(U)$ is a union of disjoint open sets in C, each of which is mapped homeomorphically onto U by p. We say that the open neighbourhoods U are evenly covered by p. The map p is called the covering map.*

A topological space X trivially covers itself. Some authors require both spaces C and X to be path-connected and locally path-connected, because many theorems hold only if the spaces have these properties. When the covering space is simply connected, it yields some extremely useful results.

Definition 1.28. (Universal cover) *A covering space $\tilde{X}$ of X is a universal covering space if it is simply connected.*

The universal cover owes its name to the following property: If $p : \tilde{X} \to X$ is a covering map, with $\tilde{X}$ simply connected, and $q : C \to X$ is any covering, with C connected, then there exists a covering map $\tilde{p} : \tilde{X} \to C$ such that $q \circ \tilde{p} = p$. In simple words, the universal cover of X covers any connected cover of X. It can be proved that every connected and locally simply connected topological space has a universal covering space (see [Lee00, Th.12.8]). Any manifold is locally simply connected, because it has a basis of Euclidean balls, so every connected manifold M has a universal covering space $\tilde{M}$. If $p : C \to X$ is a covering map, we are interested in the group of automorphisms of C relative to p.

Definition 1.29. (Deck group) *Given a covering map $p : C \to X$, the deck group, denoted $\mathfrak{C}(C, p, X)$, is the group of automorphisms $\varphi : C \to C$ such that the following diagram commutes:*

$$\begin{array}{ccc} C & \xrightarrow{\varphi} & C \\ & \searrow^{p} & \downarrow p \\ & & X. \end{array}$$

It is easy to prove that $\mathfrak{C}(C, p, X)$ is a group with the composition operation.

If X, Y are topological spaces and $h : X \to Y$ is a map with $h(x_0) = y_0$, for some $x_0 \in X$, $y_0 \in Y$, let us define the *induced morphism* $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $h_*([\gamma]) = [h \circ \gamma]$, where $[\gamma] \in \pi_1(X, x_0)$ is the equivalence class of the loop γ in the space X centered at x_0. Note that $H_0 := h_*(\pi_1(X, x_0))$ is a subgroup of $\pi_1(Y, y_0)$, $H_0 \leq \pi_1(Y, y_0)$. Let $p : C \to X$ be a covering map with $p(c_0) = x_0$, for some $c_0 \in C$ and $x_0 \in X$. We say that p is a *regular covering map* if the group $H_0 = p_*(\pi_1(C, c_0)) \leq \pi_1(X, x_0)$ is a normal subgroup, $H_0 \trianglelefteq \pi_1(X, x_0)$.

Proposition 1.30. *Let C and X be topological spaces. If $p : C \to X$ is a regular covering map and $\mathfrak{C} = \mathfrak{C}(C, p, X)$ is its deck group, then X is homeomorphic to $C/\mathfrak{C}$.*

Proof. See [Mun00, Th.81.6]. □

If $\tilde{X}$ is the universal covering space of X and $p : \tilde{X} \to X$ the covering map, the group $H_0 = p_*\left((\pi_1(\tilde{X}, x_0)\right)$ must be trivial, and thus a normal subgroup of $\pi_1(X, x_0)$, making p a regular covering map. Hence, any topological space X is homeomorphic to the quotient $\tilde{X}/\mathfrak{C}$.

1.4. Lie Groups and Actions

A *topological group* is a group with a topology on it such that the group multiplication and the inversion map are continuous. We are especially interested in manifolds carrying a group structure compatible with their differentiable structure. Such manifolds are called *Lie groups*.

Definition 1.31. (Lie group) *A Lie group is a smooth manifold G with a group structure $(G, \cdot)$, such that the multiplication map $m : G \times G \to G$ given by $m(g, h) = gh$, and the inversion map $i : G \to G$ given by $i(g) = g^{-1}$, are both smooth.*

The importance of Lie groups stems primarily from their actions on manifolds. Analogously to the defition of an action of a group on a set, we define a *left action* of a Lie group G on a smooth manifold M as a map $\theta : G \times M \to M$, satisfying $\theta(g_1, \theta(g_2, p)) = \theta(g_1 g_2, p)$ and $\theta(e, p) = p$, where e is the identity element of G. A *right action* can be defined in a similar way, but we will focus our attention on left actions. Furthermore, we will only consider *smooth actions*, that is, when θ is smooth as a map from $G \times M$ into M. In this case, the map $\theta(g, \cdot) : M \to M$ is a diffeomorphism, for each $g \in G$, with inverse $\theta(g^{-1}, \cdot)$.

On the other hand, for any point $p \in M$ we can define the *orbit* as the set of all images of p under elements of G, and the *isotropy group* as the set of elements of G that fix p.

Definition 1.32. (Orbit and isotropy group) *Let p be a fixed point of M. The orbit of p under the action of G is the set $\mathcal{O}(p) = \{\theta(g, p) : g \in G\}$, and the isotropy group of p is the set $G_p = \{g \in G : \theta(g, p) = p\}$.*

The relation $p \sim q$ if $q \in \mathcal{O}(p)$ is an equivalence relation. We denote the set of equivalence classes under the group action G by M/G. With the quotient topology, it is called the *orbit space* of the action.

Definition 1.33. (Transitive action) *Let $\theta : G \times M \to M$ be a left action of a Lie group G on a smooth manifold M. We say that an action $\theta : G \times M \to M$ is transitive (or that the group G acts transitively) if, for every pair of elements $x, y \in M$, there is a group element $g \in G$ such that $gx = y$.*

If G is a Lie group acting transitively on a smooth manifold M, there is a isomorphism $M \cong G/G_p$, for each $p \in M$, and such a manifold is said to be *homogeneus*. We are interested in studying under what conditions the orbit space is a smooth manifold. Before that, we need to define some properties regarding Lie group actions.

Definition 1.34. (Properly discontinuous action) *Let $\theta : G \times M \to M$ be a left action of a Lie group G on a smooth manifold M. We say that the group G acts properly discontinuously on M (or that the action θ is a properly discontinuous action) if, for every compact subset $K \subset M$, $K \cap \theta(g, K) = \emptyset$ for all but finitely many $g \in G$.*

Note that if G acts properly discontinuously on M, then the isotropy group G_p must be finite, for every $p \in M$. Indeed, suppose that G_p is not finite and let K be a compact subset of M containing p. Then, there are infinitely many $g \in G$ such that $\theta(g, p) = p$, and $p \in \theta(g, K)$. Hence $K \cap \theta(g, K)$ is not empty for infinitely many $g \in G$, and G thus do not act properly discontinuously on M.

Definition 1.35. (Free action) *We say that an action $\theta : G \times M \to M$ is free (or that the group G acts freely) if the isotropy group G_p is trivial, for all $p \in M$.*

We already have enough conditions to determine when the orbit space is a smooth manifold:

Theorem 1.36. (Quotient manifold theorem) *Suppose that G is a Lie group acting smoothly, freely and properly discontinuously on a smooth manifold M. Then the orbit space M/G is a manifold of dimension* $\dim M - \dim G$, *and has a unique smooth structure with the property that the quotient map $\pi : M \to M/G$ is a smooth submersion.*

Proof. See [Lee13, Th.7.10]. □

Let M be a smooth manifold and consider its group of diffeomorphisms $\mathrm{Diff}(M)$. This group is a Lie group with the *compact-open topology*, that is, the topology generated by subsets of the form $B(K, U) = \{\varphi \in \mathrm{Diff}(M) : \varphi(K) \subset U\}$, where K is compact and U is open in M. Note that the sets $B(K, U)$ does not form a basis of this topology since they are not closed under intersection. Instead, they form a subbasis, which means that the open sets are arbitrary unions or finite intersections of $B(K, U)$.

A topological group is said to be *discrete* if it is equipped with the discrete topology. If Γ is a subgroup of $\mathrm{Diff}(M)$ acting smoothly and properly discontinuously on M, and we denote the action of Γ on M as $\theta : \Gamma \times M \to M$, then the set $\{\gamma \in \Gamma : K \cap \theta(\gamma, K) \neq \emptyset\}$ is finite for every compact subset $K \subset M$. Hence, Γ is a discrete subgroup of $\mathrm{Diff}(M)$ with the compact-open topology.

1.5. Model Geometries

In this section we are going to study how to endow a manifold with a geometric structure. A first approach to geometric structures comes from the notion of distance. Any Riemannian metric g on a Riemannian manifold M induces a distance d_g on M, since we can define $d_g(a, b) = \inf_\gamma\{L(\gamma)\}$, where $\gamma : [a, b] \to M$ denotes a parametrized curve on M and $L(\gamma)$ its length. Thus, each connected Riemannian manifold (M, g) is a metric space (M, d_g).

It is well-known that two metric spaces are *isometric* if there is a bijection between them that preserves distances. In the context of Riemannian manifolds, isometries also preserve the metric in some way. If X is a smooth vector field on M and $\varphi : M \to M'$ is a diffeomorphism, we define a smooth vector field $d\varphi X$ on M' to be $(d\varphi X)_{\varphi(p)} = d\varphi(X_p)$, for each $p \in M$.

Definition 1.37. (Isometry) *Let $(M, g), (M', g')$ be Riemannian manifolds. A diffeomorphism $\varphi : M \to M'$ is said to be an isometry if $g(X, Y) = g'(d\varphi X, d\varphi Y)$. We say that M and M' are isometric, and we write $M \simeq M'$, if there exist such an isometry between them.*

When the diffeomorphism φ is a local diffeomorphism, we say that φ is a *local isometry*. The set of all isometries $\varphi : M \to M$ of a Riemannian manifold (M, g) onto itself is a group, written as $\mathrm{Isom}(M)$, with the composition operation of isometries. In fact, $\mathrm{Isom}(M)$ is a Lie group (see [Mye39, Sec.8]). Its manifold topology is the *compact-open topology* defined in the previous section.

In order to define a geometric structure on a manifold, the existence of a (Riemannian) metric is insufficient. We need to add additional properties to the metric, such as homogeneity and completeness:

Definition 1.38. (Complete metric) *Let (M, g) be a Riemannian manifold. The Riemannian metric g is said to be complete if each geodesic is isometric to the real line.*

The Hopf-Rinow theorem asserts that a Riemannian manifold (M, g) is complete if, and only if, the induced metric space (M, d_g) is complete, that is, if every Cauchy sequence in (M, d_g) converges. On the other hand, we say that a Riemannian metric g is *homogeneous* if given any points $x, y \in M$, there exist an isometry of M sending x to y. However, it is sufficient for our interests to define a weaker property, the *local homogeneity*:

Definition 1.39. (Locally homogeneous metric) *A Riemannian metric g on a Riemannian manifold M is locally homogeneous if for all points $x, y \in M$ there exists neighbourhoods $U \subset M$ of x and $V \subset M$ of y and an isometry $\varphi : U \to V$.*

Now we have all the elements to define a geometric structure on a manifold:

Definition 1.40. (Geometric structure) *A Riemannian manifold (M, g) admits a geometric structure if g is a complete, locally homogeneous Riemannian metric.*

If (M, g) is a Riemannian manifold which admits a geometric structure and $\tilde{M}$ is the universal covering space of M, then there exists a covering map $p : \tilde{M} \to M$ that is a local isometry, so $\tilde{M}$ inherits a natural metric $\tilde{g}$, called the *pull-back metric*, given by $\tilde{g}(X, Y) = g(dpX, dpY)$, for each $X, Y \in \mathcal{C}^\infty(\tilde{M}, T\tilde{M})$. Hence, $\tilde{g}$ is also complete and locally homogeneous, and $\tilde{M}$ admits a geometric structure. We define a group action $\theta : \text{Isom}(\tilde{M}) \times \tilde{M} \to \tilde{M}$ of the isometry group $\text{Isom}(\tilde{M})$ on $\tilde{M}$ by $\theta(\varphi, x) = \varphi(x)$. I.M. Singer proved in [Sin60] that a locally homogeneous Riemannian metric on a simply-connected Riemannian manifold is homogeneous. As $\tilde{M}$ is simply-connected, $\tilde{g}$ must be homogeneous, so there exist an isometry $\varphi \in \text{Isom}(\tilde{M})$ such that $\varphi(x) = y$, for any points x and y on $\tilde{M}$. Then, the group action of φ on $\tilde{M}$ is transitive, and $\tilde{M}$ is homogeneous. Any element of the deck group $\phi \in \mathfrak{C} = \mathfrak{C}(\tilde{M}, p, M)$ is a diffeomorphism $\phi : \tilde{M} \to \tilde{M}$ such that $p \circ \phi = p$. Since $p : \tilde{M} \to M$ is a local isometry, ϕ is a local isometry, and therefore is an isometry since ϕ is a diffeomorphism. As any element of $\mathfrak{C}$ is an isometry of $\tilde{M}$, then $\mathfrak{C}$ is a subgroup of $\text{Isom}(\tilde{M})$, and by Proposition 1.30 we have that $M \cong \tilde{M}/\mathfrak{C}$. This argument proves the following proposition:

Proposition 1.41. *If M is a Riemannian manifold which admits a geometric structure and $\tilde{M}$ is its universal covering space, then there exist a subgroup Γ of* $\text{Isom}(\tilde{M})$ *such that M is isometric to $\tilde{M}/\Gamma$.* □

This motivates us to define geometries on simply connected Riemannian manifolds, and to model the geometric structure of any Riemannian manifold on its universal cover. As we have seen in the previous section, any subgroup Γ of the group $\text{Diff}(M)$ of a smooth manifold M acting smoothly and properly discontinuously is a discrete subgroup. The converse is false, in general, but is true when M is a complete Riemannian manifold and Γ is a subgroup of $\text{Isom}(M)$. For convenience, we will say that the action of $\Gamma \leq \text{Isom}(M)$ on M is *discrete* (or that Γ acts discretely) if the action is smooth and properly discontinuous.

Definition 1.42. (Model geometry) *A model geometry is a pair $(X, \text{Isom}(X))$, where X is a complete, homogeneous, connected, simply connected Riemannian manifold and* $\text{Isom}(X)$ *is its isometry group. We will say that a Riemannian manifold M has a geometric structure modelled on X if there exist a subgroup* $\Gamma \leq \text{Isom}(X)$ *acting freely and discretely on X, such that M is isometric to X/Γ.*

Our goal in describing a geometry is to identify the model space X, its isometry group $\text{Isom}(X)$, and all the discrete subgroups of $\text{Isom}(X)$ acting freely on X. In the next two chapters we will classify all the 2-dimensional and 3-dimensional geometries.

Chapter 2

Two-Dimensional Geometries

In this chapter we will study the 3 two-dimensional geometries, since they will be necessary to understand the three-dimensional ones. First we will describe the geometries $\mathbb{E}^2, S^2$ and $\mathbb{H}^2$, and then we will prove the uniformization theorem, which says that these are the only 2-dimensional geometries.

2.1. The 3 Model Geometries

There are only 3 geometries where we can modelled a compact surface. From Definition 1.5, we only need to describe the model space X and its isometry group. We will also describe the geodesics and the metric of each of the 3 model spaces. For a complete understanding of the surfaces with a geometric structure modelled on these spaces, one should describe the discrete subgroups of the isometry group. However, this is very laborious and can be found in Section §1 of [Sco83].

2.1.1. $\mathbb{E}^2$

The Euclidean plane $\mathbb{E}^2$ is fairly well-known. It can be equipped with the usual Euclidean metric

$$ds^2 = dx^2 + dy^2,$$

and the geodesics are the straight lines in $\mathbb{E}^2$. We are interested in describing the isometry group $\mathrm{Isom}(\mathbb{E}^2)$, sometimes called the Galilean group. It is easy to show that the isometries of $\mathbb{E}^2$ are translations, rotations, reflections and arbitrary finite combinations of them. Any isometry α can be written as $\alpha(x) = Ax + b$, where A is a 2×2 real orthogonal matrix, $A \in \mathrm{O}(2)$, and b is a vector in $\mathbb{E}^2$.

2.1.2. S^2

The 2-sphere S^2 can be embedded in $\mathbb{E}^3$, and inherits the usual metric from this space,

$$ds^2 = dx^2 + dy^2 + dz^2.$$

The geodesics are the great circles on S^2 and play a similar role for S^2 as straight lines do for E^2. However, there are important differences, since any two great circles must meet, and two diametrically opposite points lie on infinitely many great circles. Note that any isometry of $\mathbb{E}^3$ fixing the origin restricts to an isometry of S^2, and that every isometry of S^2 extends in a natural way to an isometry of $\mathbb{E}^3$ which fixes the origin. The group of distance-preserving transformations of $\mathbb{E}^3$ is $\mathrm{O}(3)$, so we have $\mathrm{Isom}(S^2) \cong \mathrm{O}(3)$. In particular, any element of $\mathrm{SO}(3) \leq \mathrm{O}(3)$ is a rotation of $\mathbb{E}^3$ around some line through the origin. The restriction to the embedded S^2 is a rotation of S^2 and fix exactly two points, those which intersect with the line.

2.1.3. $\mathbb{H}^2$

The hyperbolic plane $\mathbb{H}^2$ is the most interesting, since the class of surfaces modelled on it is the largest. It can be thought as the upper half plane $\mathbb{R}^2_+$ or, equivalently, as the upper half complex plane $\mathbb{C}^+ = \{x + iy \,:\, y > 0\}$, with the metric

$$ds^2 = \frac{1}{y^2}\left(dx^2 + dy^2\right).$$

The geodesics of the hyperbolic plane are vertical semi-straight lines and semicircles with center on the x-axis. Clearly, for any two points $p, q \in \mathbb{H}^2$, there is a unique geodesic which passes through both of them.

As any isometry $\alpha \in \mathrm{Isom}(\mathbb{H}^2)$ must take a geodesic to a geodesic, we are looking for isometries which take the set of vertical lines and circles to itself. It is a standard result of complex analysis that the transformations which do that are the Möbius transformations $f(z) = (az + b)(cz + d)^{-1}$, for $a, b, c, d \in \mathbb{C}$ satisfying $ad - bc \neq 0$. A Möbius transformation can be obtained as follows: First perform the stereographic projection from the plane to the unit 2-sphere, rotate and move the sphere to a new location and orientation, and then perform the stereographic projection from the sphere to the plane. The set of all these transformations forms a group called the *Möbius group*, where the operation corresponds to composition. We are only considering the upper half plane, so $\mathrm{Isom}(\mathbb{H}^2)$ is a subgroup of the Möbius group. It can be proved that if a, b, c, d are real numbers and $ad - bc < 0$, the corresponding Möbius transformation interchanges the upper half plane and the lower half plane. Hence, the orientation-preserving isometries of $\mathbb{H}^2$ are

$$\left\{ z \mapsto \frac{az+b}{cz+d} \;:\; a,b,c,d \in \mathbb{R},\, ad-bc<0 \right\},$$

and they form a subgroup of the Möbius group. This group can be identified with the group $\mathrm{PSL}_2\mathbb{R}$, which is the quotient $\mathrm{SL}_2\mathbb{R}/\{\pm \mathrm{Id}_2\}$, where Id_2 denotes the 2×2 identity matrix. The orientation-reversing isometries are the composition with the conjugation operation, and we have described all the isometries of $\mathbb{H}^2$.

2.2. Uniformization Theorem

Let $\mathbf{S}, \mathbf{T}, \mathbf{P}, \mathbf{K}$ denote the sphere S^2, the torus $S^1 \times S^1$, the real projective plane $\mathbb{R}\mathrm{P}^2$ and the Klein bottle, respectively. The orientable surfaces are $\mathbf{S}, \mathbf{T}$ and the non-orientable ones are $\mathbf{P}, \mathbf{K}$.

Recall that the connected sum of two surfaces M_1 and M_2 is a surface $M = M_1 \sharp M_2$ formed by deleting an open disk inside each manifold and gluing together the resulting boundaries. If m, n are positive integers, we define $n\mathbf{T}$ as the connected sum of n tori and $m\mathbf{P}$ as the connected sum of m real projective planes. We can enunciate the classification theorem for compact surfaces in this way:

Theorem 2.1. (Classification theorem for compact surfaces) *Every orientable compact surface is homeomorphic either to* $\mathbf{S}$ *or* $n\mathbf{T}$. *Every non-orientable compact surface is homeomorphic to* $m\mathbf{P}$.

Proof. See [Gal13, Chapter 6]. □

The Klein bottle does not appear because $\mathbf{K} \cong 2\mathbf{P}$. Adding some spheres $\mathbf{S}$ does not produce any change, since $\mathbf{S}$ is the identity element for the connected sum. Furthermore, the sum of some projective plane and some tori is a non-orientable surface, $n\mathbf{T} \sharp m\mathbf{P} \cong (2n+m)\mathbf{P}$.

If M_1, M_2 are two compact surfaces, the Euler Characteristic of its disjoint union is $\chi(M_1 \sqcup M_2) = \chi(M_1) + \chi(M_2)$. It is easy to show that the Euler Characteristic of $M_1 \sharp M_2$ is $\chi(M_1 \sharp M_2) = \chi(M_1) + \chi(M_2) - 2$, since $\chi(\mathbf{S}) = 2$. Using this equality and that $\chi(\mathbf{T}) = 0, \chi(\mathbf{P}) = 1$, we have that $\chi(n\mathbf{T}) = 2 - 2n$ and $\chi(m\mathbf{P}) = 2 - m$.

Theorem 2.2. (Uniformization theorem) *Every compact, connected surface admits a geometric structure modelled on one and only one of* $\mathbb{E}^2$, S^2 *or* $\mathbb{H}^2$.

Proof. We have to prove that every compact and connected surface is homeomorphic to the quotient of one of these three spaces by a subgroup of their respective isometry groups. If M is a compact connected surface, it has a well-defined Euler characteristic $\chi(M) \in \mathbb{Z}$, so we can break our argument into three cases, accordingly with the sign of $\chi(M)$. Note that $\chi(M) = 0$ if $M \cong \mathbf{T}$ or $M \cong 2\mathbf{P} \cong \mathbf{K}$; $\chi(M) > 0$ if $M \cong \mathbf{S}$ or $M \cong \mathbf{P}$; and $\chi(M) < 0$ if $M \cong n\mathbf{T}$ or $M \cong m\mathbf{P}$, for $n \geq 2$ and $m \geq 3$.

If $\chi(M) > 0$, then M admits a geometric structure modelled on S^2. Indeed, if $M \cong \mathbf{S}$ it is clear. Otherwise, $M \cong \mathbf{P}$. As the real projective plane is the quotient of the sphere $\mathbf{S}$ with the relation that identifies antipodal points and the antipodal map is an isometry of the sphere, then $\mathbf{P}$ admits a geometric structure on S^2.

If $\chi(M) = 0$, then M admits a geometric structure modelled on $\mathbb{E}^2$. This is because M is either homeomorphic to a torus or a Klein bottle, and these both spaces can be described as the square $[0,1] \times [0,1] \subset \mathbb{E}^2$ with sides identified by the relations $(0,y) \sim (1,y)\,;\ (x,0) \sim (x,1)$ (for the torus) and $(0,y) \sim (1,y)\,;\ (x,0) \sim (1-x,1)$ (for the Klein bottle). As these identifications are isometries of the euclidean plane, $\mathbf{T}$ and $\mathbf{K}$ admit a geometric structure modelled on $\mathbb{E}^2$.

If $\chi(M) < 0$, then M admits a geometric structure modelled on $\mathbb{H}^2$. A proof of this result is complicated, and can be found in [Bon96, pp. 4–8]. The main idea is that, if $\chi(M) < 0$, then $M \cong n\mathbf{T}$ or $M \cong m\mathbf{P}$, and these surfaces can be cut along a curve to obtain "pants". Formally, we can find a 1-dimensional compact submanifold γ of M such that each component of $M \setminus \gamma$ is either a *pair of pants* (namely an open annulus minus a closed disk) or a *pair of Möbius pants* (namely an open Möbius strip minus a closed disk). These pair of pants P are surfaces with boundary, and can be endowed with an Hyperbolic metric such that ∂P are geodesics of $\mathbb{H}^2$. □

A two-dimensional geometry is called *Euclidean*, *elliptic* or *hyperbolic* if the model space is $\mathbb{E}^2, S^2$ or $\mathbb{H}^2$, respectively. It can be shown that these 3 geometries have constant sectional curvature. The curvature of S^2 is $+1$, the curvature of $\mathbb{H}^2$ is -1 and the curvature of $\mathbb{E}^2$ is 0. We will see in the next chapter that this fact can be generalized to higher dimensions.

Chapter 3

Three-Dimensional Geometries

In this chapter we will first describe the 8 three-dimensional geometries, namely $\mathbb{E}^3$, S^3, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}_2\mathbb{R}}$, Nil and Sol. Then we will see the Thurston's classification theorem, which asserts that, under certain conditions, these are the only three-dimensional geometries. Finally we will introduce the Seifert fibre spaces and we will show that 6 of the 8 geometries above are Seifert.

3.1. The 8 Model Geometries

In the previous chapter we have seen that the only 3 two-dimensional geometries are $\mathbb{E}^2, S^2$ and $\mathbb{H}^2$. It is logical to think that their three-dimensional generalizations, $\mathbb{E}^3, S^3$ and $\mathbb{H}^3$, are also three-dimensional geometries. As for the 2-dimensional case, these manifolds have constant sectional curvature, and it can be shown that they are the only simply connected 3-manifolds with this property. Furthermore, any Riemannian 3-manifold which admits a geometric structure and has constant sectional curvature can be modelled in one of these 3 spaces, depending on the sign of the curvature. However, unlike the 2-dimensional case, not every simply-connected space where we can model a geometry has this property. For example, the product spaces $S^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ are simply-connected, but they have much less symmetry. There are 3 more simply-connected manifolds where we can model a geometry, the Lie groups $\widetilde{\mathrm{SL}_2\mathbb{R}}$, Nil and Sol, that we will describe below.

As in the previous chapter, we are going to describe all the pairs $(X, \mathrm{Isom}(X))$ satisfying the conditions of Definition 1.42. We will also describe the metric when possible, and we will avoid describing explicity the geodesics since it is very complicated. Again, for a complete understanding of the 3-manifolds with a geometric structure modelled on X, one should describe the discrete subgroups of $\mathrm{Isom}(X)$. This can be found in Section §4 of [Sco83].

Before explaining the geometries, we are going to define some concepts of group theory in order to understand how to construct some isometry groups. A sequence of groups and group homomorphisms

$$G_0 \xrightarrow{h_1} G_1 \xrightarrow{h_2} G_2 \xrightarrow{h_3} \cdots \xrightarrow{h_n} G_n$$

is said to be an *exact sequence* if $\mathrm{Im}(h_k) = \mathrm{Ker}(h_{k+1})$, for each $k = 1, \ldots, n-1$. In particular, we will say that an exact sequence is a *short exact sequence* if it has the form

$$1 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 1.$$

Note that although the homomorphisms $1 \longrightarrow H$ and $K \longrightarrow 1$ are not mentioned because they are unique, they make α a monomorphism and β an epimorphism. Since α is a monomorphism, it restricts to an isomorphism $H \longrightarrow \alpha(H)$, so we have $H \cong \alpha(H) = \mathrm{Ker}(\beta)$. On the other hand, as β is an epimorphism, the first isomorphism theorem implies that $\mathrm{Im}(\beta) = K \cong G/ker(\beta)$. When $H \lhd G$ is a normal subgroup and $K \cong G/H$, we say that G is an *extension* of K by H. Group extensions are helpful when the groups K and H are known and the properties of G are to be determined. In this case, G is an extension of $G/\mathrm{Ker}\,(\beta)$ by $\mathrm{Ker}(\beta)$. Furthermore, if there exist a homomorphism $\gamma : K \to G$ such that the composition $\beta \circ \gamma$ is the identity map $\mathrm{id} : C \to C$, the short exact sequence defined above is called a *split*, and the group G is isomorphic to the direct sum of H and K, $G \cong H \oplus K$ (see [Con08, Th.3.2]).

3.1.1. $\mathbb{E}^3$

The euclidean space $\mathbb{E}^3$ is very similar to the euclidean plane. We can equip $\mathbb{E}^3$ with the Euclidean metric

$$ds^2 = dx^2 + dy^2 + dz^2.$$

Again, the isometries of $\mathbb{E}^3$ are of the form $\alpha(x) = Ax + b$, where A is a 3×3 real orthogonal matrix, $A \in \mathrm{O}(3)$, and b is a vector in $\mathbb{E}^3$. The homomorphism $\varphi : \mathrm{Isom}(\mathbb{E}^3) \to \mathrm{O}(3)$ sending $Ax + b \mapsto A$ is surjective, and $\mathrm{Ker}\,\varphi$ is the set of all translations of the plane which is isomorphic to $\mathbb{R}^3$. Hence, we have the short exact sequence

$$0 \longrightarrow \mathbb{R}^3 \longrightarrow \mathrm{Isom}(\mathbb{E}^3) \longrightarrow \mathrm{O}(3) \longrightarrow 1.$$

However, the geometric description of isometries of $\mathbb{E}^3$ is slightly different from that in $\mathbb{E}^2$. In the space we have *screw motions*, which consist of the composite of a translation with a rotation around a line left invariant by the translation.

3.1.2. S^3

The 3-sphere S^3 can be understood as the unit sphere in $\mathbb{R}^4$, and it inherits the euclidean metric

$$ds^2 = dx^2 + dy^2 + dz^2 + dt^2.$$

The group of isometries of $\mathbb{E}^4$ which fix the origin is $\mathrm{O}(4)$, so $\mathrm{Isom}(S^3) \cong \mathrm{O}(4)$. Nevertheless, it is more interesting to consider in some situations S^3 as the unit sphere in $\mathbb{C}^2$, consisting of pairs $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ of complex numbers such that $|z_1|^2 + |z_2|^2 = 1$.

3.1.3. $\mathbb{H}^3$

The hyperbolic space $\mathbb{H}^3$ can be constructed in a similar way as we have done with the hyperbolic plane. We can think in the hyperbolic space as the upper half space $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 \,:\, z > 0\}$ endowed with the Riemannian metric

$$ds^2 = \frac{1}{z^2}(dx^2 + dy^2 + dz^2).$$

The isometry group $\mathrm{Isom}(\mathbb{H}^3)$ is generated by reflections across half-planes which are perpendicular to the xy-plane, and inversions in a hemisphere with center on the xy-plane. An isometry of $\mathbb{H}^3$ continuosly extends to its closure in $\mathbb{R}^3 \cup \{\infty\}$. The boundary of $\mathbb{H}^3$ in $\mathbb{R}^3 \cup \{\infty\}$ can be identified with the Riemann Sphere, $\partial\mathbb{H}^3 \cong \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where we identify the xy-plane with $\mathbb{C}$. Hence, any isometry of $\mathbb{H}^3$ is determined by its restriction to $\hat{\mathbb{C}}$. The orientation-preserving isometries are the Möbius transformations defined in Section 2.1.3, with the domain extended to $\hat{\mathbb{C}}$. That is, the transformations $z \mapsto (az + b)(cz + d)^{-1}$ if $z \in \mathbb{C}$ and $\infty \mapsto a/c$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. These isometries form a group that can be identified with the projective linear group $\mathrm{PSL}_2\mathbb{C}$. The orientation-reversing isometries are the composition of the Möbius transformation with the conjugation $z \mapsto \bar{z}$. William P. Thurston did an extensive study of the isometries of $\mathbb{H}^3$. It can be found in Section *2.5 Hyperbolic Isometries* of [Thu97].

3.1.4. $S^2 \times \mathbb{R}$

The product space $S^2 \times \mathbb{R}$ is perhaps the simplest of the 8 geometries, since there are only seven manifolds with geometric structure modelled on it. To equip $S^2 \times \mathbb{R}$ with a metric, we need to study how to multiply two Riemannian manifolds. The product of two smooth manifolds $(M_1, [\mathcal{A}_1])$ and $(M_2, [\mathcal{A}_2])$ is a smooth manifold $(M_1 \times M_2, [\mathcal{A}])$ of dimension $\dim(M_1 \times M_2) = \dim M_1 + \dim M_2$, where the space $M_1 \times M_2$ is a topological space with the product topology, and the atlas $[\mathcal{A}]$ can be constructed with the charts of $[\mathcal{A}_1]$ and $[\mathcal{A}_2]$. If (M_1, g^1) and (M_2, g^2) are Riemannian manifolds, we can define a metric on $M_1 \times M_2$:

Definition 3.1. (Product Riemannian metric) *Let $(M_1, g^1), (M_2, g^2)$ be Riemannian manifolds. Using the natural isomorphism $T_{(p_1,p_2)}(M_1 \times M_2) \cong T_{p_1}M_1 \oplus T_{p_2}M_2$ for each $(p_1, p_2) \in (M_1 \times M_2)$, we define a metric g on $T_{(p_1,p_2)}(M_1 \times M_2)$ as*

$$\begin{aligned} g_{(p_1,p_2)} : \quad T_{(p_1,p_2)}(M_1 \times M_2) \times T_{(p_1,p_2)}(M_1 \times M_2) &\to \mathbb{R} \\ ((x_1, x_2), (y_1, y_2)) &\mapsto g^1_{p_1}(x_1, y_1) + g^2_{p_2}(x_2, y_2). \end{aligned}$$

With this definition, $(M_1 \times M_2, g)$ is a Riemannian manifold, and we denote g as $g^1 \oplus g^2$. In our case, $S^2 \times \mathbb{R}$ is a Riemannian manifold endowed with the product Riemannian metric $g^1 \oplus g^2$, where g^1 is the metric of S^2 defined in Section 2.1.2, and g^2 is the Euclidean metric of $\mathbb{R}$.

Since $\text{Isom}(S^2 \times \mathbb{R}) \cong \text{Isom}(S^2) \times \text{Isom}(\mathbb{R})$, the isometries of $S^2 \times \mathbb{R}$ are pairs of isometries of S^2 and $\mathbb{R}$. We have shown in Section 2.1.2 that $\text{Isom}(S^2) \cong \text{O}(3)$. Any isometry $\alpha \in \text{Isom}(\mathbb{R})$ can be written as $\alpha(x) = \epsilon x + b$, where $\epsilon = \pm 1$ (that is, $\epsilon \in \text{O}(1)$) and $b \in \mathbb{R}$, so they are translations and reflections on the line. The group can be made explicit as $\text{Isom}(\mathbb{R}) = \{(u, v) \in \mathbb{R}^2 : v = \pm 1\}$ and a multiplication on the pairs defined by $(u, v)(u', v') = (u + vu', vv')$.

We said that there are only 7 manifolds with geometric structure modelled on $S^2 \times \mathbb{R}$. They are: $S^2 \times \mathbb{R}$, $\mathbb{R}\text{P}^2 \times S^1$, $\mathbb{R}\text{P}^3 \sharp \mathbb{R}\text{P}^3$, the two line-bundles over $\mathbb{R}\text{P}^2$ and the two S^2-bundles over S^1. A proof of that can be found in [Sco83, pp. 457–459].

3.1.5. $\mathbb{H}^2 \times \mathbb{R}$

The product space $\mathbb{H}^2 \times \mathbb{R}$ is similar to $S^2 \times \mathbb{R}$, although there are infinitely many 3-manifolds with geometric structure modelled on $\mathbb{H}^2 \times \mathbb{R}$. It is a Riemannian manifold whose metric is the product Riemannian metric $g = g^1 \oplus g^2$, where g^1 is the Hyperbolic metric defined in Section 2.1.3, and g^2 is again the Euclidean metric of $\mathbb{R}$. We have the natural isomorphism $\text{Isom}(\mathbb{H}^2 \times \mathbb{R}) \cong \text{Isom}(\mathbb{H}^2) \times \text{Isom}(\mathbb{R})$, and these isometry groups are well explained in Sections 2.1.3 and 3.1.4, respectively.

3.1.6. $\widetilde{\text{SL}_2\mathbb{R}}$

It is well-known that the *special linear group* $\text{SL}_2\mathbb{R}$ is the group of all 2×2 real matrices with determinant equal to 1. In fact, it is a 3-dimensional Lie group. The space $\widetilde{\text{SL}_2\mathbb{R}}$ denotes its universal covering space, and it is also a Lie group. We are going to show that a metric on $\widetilde{\text{SL}_2\mathbb{R}}$ can be derived from the metric of $\mathbb{H}^2$.

First, we are going to define some concepts. If (M, g) is a n-dimensional Riemannian manifold, then the *unit tangent bundle* of (M, g) is the set

$$UTM = \{(p, q) \,:\, p \in M,\, q \in T_pM,\, g_p(q, q) = 1\},$$

together with the projection $\pi_1 : UTM \to M$ defined by $\pi_1(p, q) = p$. Clearly, the unit tangent bundle (UTM, M, π_1) is a submanifold of the tangent bundle (TM, M, π) as in Definition 1.6. Note that, since $\pi_1^{-1}(\{p\}) \subset \pi^{-1}(\{p\}) = T_pM$, then $\pi_1^{-1}(\{p\})$ is a $(n-1)$-sphere S^{n-1} on T_pM. It can be proved that the tangent bundle $(TM, \overline{g})$ is a $2n$-dimensional Riemannian manifold, where $\overline{g}$ is the *natural Riemannian metric* on TM (see [Aba05, Def.2.3]). Then, the metric on TM induces a metric on UTM.

If the Riemannian manifold is $\mathbb{H}^2$, with the metric defined in Section 2.1.3, it induces a metric on its unit tangent bundle $UT\mathbb{H}^2$. If $\varphi : \mathbb{H}^2 \to \mathbb{H}^2$ is an isometry of $\mathbb{H}^2$, then the differential $d\varphi : T\mathbb{H}^2 \to T\mathbb{H}^2$ is an isometry of $T\mathbb{H}^2$. In particular, if $\tilde{\varphi}$ is an orientation-preserving isometry of $\mathbb{H}^2$, then $d\tilde{\varphi}$ is an isometry of the unit tangent bundle $UT\mathbb{H}^2$. Hence, there is a natural identification $\text{PSL}_2\mathbb{R} \cong UT\mathbb{H}^2$, and $\text{PSL}_2(\mathbb{R})$ inherits the metric of $\mathbb{H}^2$. We can lift this metric to $\widetilde{\text{SL}_2\mathbb{R}}$, since $\text{PSL}_2\mathbb{R}$ is covered by $\text{SL}_2\mathbb{R}$ which itself is covered by its universal cover $\widetilde{\text{SL}_2\mathbb{R}}$. Summing up, we have translated the metric of $\mathbb{H}^2$ to $\widetilde{\text{SL}_2\mathbb{R}}$ as we can see in the following diagram:

$$\begin{array}{ccccc} & & & & \widetilde{\text{SL}_2\mathbb{R}} \\ & & & & \downarrow \\ \mathbb{H}^2 & \longrightarrow & T\mathbb{H}^2 & & \text{SL}_2\mathbb{R} \\ & & \cup & & \downarrow \\ & & UT\mathbb{H}^2 & \longleftrightarrow & \text{PSL}_2\mathbb{R}. \end{array}$$

The isometries of $\widetilde{\mathrm{SL}_2\mathbb{R}}$ can be also derived from the isometries of $\mathbb{H}^2$. As $\mathrm{PSL}_2\mathbb{R}$ is the group of the orientation-preserving isometries of the hyperbolic plane, $\widetilde{\mathrm{SL}_2\mathbb{R}}$ admits no orientation-reversing isometry. It can be proved that

$$\mathrm{Isom}\left(\widetilde{\mathrm{SL}_2\mathbb{R}}\right) \cong \left(\widetilde{\mathrm{SL}_2\mathbb{R}} \times \widetilde{\mathrm{O}(2)}\right)/\mathbb{Z},$$

since $\mathrm{O}(2) \subset \mathrm{PSL}_2\mathbb{R}$ is the isotropy group of a point and $\mathbb{Z} \cong \pi_1(\mathrm{SO}(2))$.

3.1.7. Nil

The group Nil is a 3-dimensional Lie group, also called the *Heisenberg group*, consisting of all the 3×3 real matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

under multiplication. The name Nil comes from the fact that it is a *nilpotent group*. Remember that a group G is said to be nilpotent if the upper central sequence $1 = Z_0 \leq Z_1 \leq \cdots \leq Z_n$, terminates with $Z_n = G$, for some n, where the sequence is constructed in the following way: Z_1 is the center of G and, for $k > 1$, Z_k is the unique subgroup of G such that Z_k/Z_{k-1} is the center of G/Z_{k-1}. In particular, if we consider the group $(\mathbb{R}^2, +)$ and the homomorphism $\varphi : \mathrm{Nil} \to \mathbb{R}^2$ given by

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x, y),$$

then the kernel of φ is the group consisting of matrices of the form

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This group is isomorphic to $\mathbb{R}$, and one obtains the short exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathrm{Nil} \longrightarrow \mathbb{R}^2 \longrightarrow 0.$$

In order to describe the metric on Nil, we can identify Nil with $\mathbb{R}^3$ sending each 3×3 matrix of Nil to $(x, y, z) \in \mathbb{R}^3$. Then, $\mathbb{R}^3$ is a group with the multiplication

$$(x, y, z) \cdot (x', y', z') = (x + x',\, y + y',\, z + z' + xy').$$

Hence, Nil admits a metric invariant under left multiplication on itself. We can define a metric on $\mathbb{R}^3$ by choosing a formula for ds^2 at some point and determining the corresponding formula at all other points by the group action. If we choose the metric $ds^2 = dx^2 + dy^2 + dz^2$ at the origin, the corresponding invariant metric on $\mathbb{R}^3$ is given by

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2.$$

All isometries of Nil preserve orientation. There exist a natural homomorphism ψ : Isom(Nil) $\to$ Isom($\mathbb{E}^2$) which sends any isometry $\beta(x, y, z) = (x', y', z')$ of Nil to an isometry $\alpha(x, y) = (x', y')$ of $\mathbb{E}^2$, whose kernel Ker ψ can be identified with $\mathbb{R}$. Hence, we have the short exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \text{Isom}\,(\text{Nil}) \longrightarrow \text{Isom}\left(\mathbb{E}^2\right) \longrightarrow 1.$$

3.1.8. Sol

The geometry Sol is the one with least symmetry. The group Sol is a Lie group defined as a split extension of $\mathbb{R}$ by $\mathbb{R}^2$, so we have the short exact sequence

$$0 \longrightarrow \mathbb{R}^2 \longrightarrow \text{Sol} \longrightarrow \mathbb{R} \longrightarrow 0,$$

where $\mathbb{R}$ acts on $\mathbb{R}^2$ by $(t, (x, y)) \mapsto (e^t x, e^{-t} y)$. It owes its name to the fact of being a *solvable group*. Remember that a group G is called solvable if there are subgroups $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$ such that G_{k-1} is normal in G_k, and $G_k \big/ G_{k-1}$ is an abelian group, for $k = 1, \ldots, n$.

We can identify Sol with $\mathbb{R}^3$ so that the xy-plane corresponds to the normal subgroup $\mathbb{R}^2$ and the multiplication is given by

$$(x, y, z) \cdot (x', y', z') = (x + e^{-z} x', y + e^z y', z + z'),$$

The origin is clearly the identity element of the product. If we choose the metric $ds^2 = dx^2 + dy^2 + dz^2$ at the origin, the corresponding invariant metric on $\mathbb{R}^2$ is given by the formula

$$ds^2 = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2.$$

Note that here is nothing special in the election of e as the base of the exponentiation, since it can be replaced by any number greater than 1 by rescaling in the z-direction.

The group Isom(Sol) is generated by all horizontal translations, the reflections across the xz- and yz-planes, the *vertical shifts* $(x,y,z) \mapsto (e^{-\lambda}x, e^{\lambda}y, z+\lambda)$, $\lambda \in \mathbb{R}$, and the flip $(x,y,z) \mapsto (y,x,-z)$. Hence, any isometry of Sol could be

$$(x,y,z) \mapsto \left(\epsilon e^{-\lambda}x + a,\, \epsilon' e^{\lambda}y + b,\, z + x\right)$$

or

$$(x,y,z) \mapsto \left(\epsilon e^{-\lambda}y + a,\, \epsilon' e^{\lambda}x + b,\, -z + x\right),$$

where $a, b, \lambda \in \mathbb{R}$ and $\epsilon, \epsilon' = \pm 1$.

3.2. Thurston's Classification Theorem

William P. Thurston proved that the 8 model spaces explained in the previous section are the only ones where we can model a geometry. He worked in the context of (X,G)-structures, which are a slight generalization of differential structures on manifolds. Let X be a connected manifold and let G be a Lie group of diffeomorphisms acting transitively on X. If M is a manifold of the same dimension as X, we say that a (X,G)-*atlas* on M is a pair $\mathcal{A} = \{\mathcal{U}, \Phi\}$, where $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ is an open covering of M and $\Phi = \{\phi_\alpha : U_\alpha \to X\}_{\alpha\in\Lambda}$ is a collection of maps, called *coordinate charts*, such that for each pair $(U_\alpha, U_\beta) \in \mathcal{U} \times \mathcal{U}$, the transition map $\psi_{\alpha\beta} := \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta\left(U_\alpha \cap U_\beta\right) \to \phi_\alpha\left(U_\alpha \cap U_\beta\right)$ is the restriction of a diffeomorphism in G. Two (X,G)-atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are said to be *compatible* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is a (X,G)-atlas. As compatibility is an equivalence relation, one can define the equivalence class of a (X,G)-atlas $[\mathcal{A}]$, and call a (X,G)-atlas $\hat{\mathcal{A}} \in [\mathcal{A}]$ *maximal* if it is the union of all the (X,G)-atlas of $[\mathcal{A}]$. With this notation:

Definition 3.2. (**(X,G)-manifold**) *A (X,G)-structure on M is a maximal (X,G)-atlas $\hat{\mathcal{A}}$ on M. A (X,G)-manifold is a manifold M together with a (X,G)-structure on it.*

For convenience, Thurston used an alternative (but equivalent) definition of model geometry to the one we used in Definition 1.42.

Definition 3.3. (Thurston model geometry) *A model geometry (X,G) is a connected and simply-connected manifold X together with a Lie group G of diffeomorphisms acting transitively on X such that all the isotropy groups G_x are compact, for all $x \in X$.*

Here the compacity of the isotropy groups means that G_x is compact as a subspace of G equipped with the compact-open topology, for all $x \in X$. We identify two model geometries (X, G) and (X', G') if there is a diffeomorphism from X to X', which sends the action of G to the action of G'. In this context, we will say that a manifold M admits a geometric structure modelled on (X, G) if M is a (X, G)-manifold.

Now we want to prove that Definitions 1.42 and 3.3 we have given of model geometry are, in some way, equivalent. In 1872, Klein defined a *geometry* as follows:

Definition 3.4. (Klein geometry) *A Klein geometry is a pair (G, H), where G is a Lie group and H is a closed Lie subgroup of G such that the space G/H is connected. The group G is called the principal group of the geometry and the space G/H is called the space of the geometry.*

Our objective is to show that both definitions of model geometries are geometries in the sense of Klein. As a consequence of the quotient manifold Theorem 1.36, we have the following proposition:

Proposition 3.5. *If G is a Lie group and H is a closed Lie subgroup of G, the right action of H on G is smooth, free and properly discontinuous. Therefore, the space G/H is a smooth manifold, and the quotient map $\pi : G \to G/H$ is a smooth submersion.*

Proof. See [Lee13, Th.7.15]. □

With the quotient manifold theorem and the proposition above, we deduce that the space $X = G/H$ of a Klein geometry is a smooth manifold of dimension $\dim G - \dim H$, and there is a natural left action of G on $X = G/H$ given by $g \cdot (g'H) = (gg')H$, for every $g, g' \in G$. Taking $g' = 1$, we see that X possesses only a single group orbit, so G acts transitively on X. This proves that a pair $(X, \mathrm{Isom}(X))$ as defined in Definition 1.42 is a Klein geometry (G, H) identifying the group of diffeomorphisms G with $\mathrm{Isom}(X)$ and X with G/H.

On the other hand, since G acts transitively on X, then X is homogeneous, and for each isotropy group G_x, with $x \in X$, we have $X = G/H \cong G/G_x$, making the isotropy groups compact as subgroups of G with the compact-open topology. Therefore, given a connected manifold and a smooth transitive action by a Lie group G on X, we can construct an associated Klein geometry (G, H) by fixing a basepoint $x_0 \in X$ and letting H be the isotropy group G_{x_0} of x_0 in G. This proves that a model geometry (X, G), in the sense of Thurston, is a geometry in the sense of Klein if we associate X to G/H.

However, Definition 3.3 is still imprecise for our objectives, since there are infinitely many three-dimensional model geometries (X, G). Note, for example, that if a geometry (X, G) can be enlarged to a more symmetric geometry (X, G') with $G \subset G'$, then every (X, G)-structure naturally defines an (X, G')-structure. Hence, it makes sense to restrict our attention to geometries (X, G) where G is not contained in any larger group of diffeomorphisms of X. Such geometries are called *maximal*. Furthermore, as we are interested in classify compact 3-dimensional manifolds, we will only consider model geometries (X, G) such that there exists at least one compact manifold M modelled on (X, G). This condition eliminates a whole continuous family of 3-dimensional geometries that do not serve as models for any compact manifolds. With these two conditions, Thurston asserts that there are only 8 three-dimensional model geometries, and classify them according to the dimension of the compact isotropy groups G_x.

Theorem 3.6. (Thurston's classification theorem) *There are eight three-dimensional maximal model geometries (X, G) for which there is at least one compact manifold modelled on (X, G), as follows:*

(I) *If the isotropy groups are three-dimensional, X is $\mathbb{E}^3$, S^3 or $\mathbb{H}^3$.*

(II) *If the isotropy groups are one-dimensional, X is $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}_2\mathbb{R}}$ or* Nil.

(III) *The only geometry with zero-dimensional isotropy groups is* Sol.

Proof. See [Thu97, Th.3.8.4]. □

One might think that this theorem is the analogous version of the uniformization theorem for the three-dimensional case. However, the uniformization theorem classifies all the compact surfaces and the Thurston's classification theorem classifies only the compact 3-manifolds which admit geometric structures. The difference stems mainly in the fact that the classification theorem for compact surfaces asserts that every compact surface is homeomorphic to some surfaces which can be endowed with a well-known metric. The compact 3-dimensional manifolds have much more freedom, and not each of them admits a geometric structure. Thurston conjectured that even though not all of them can be modelled in one of the 8 model geometries, they can be divided into smaller pieces which do it. This was known as the *Thurston's geometrisation conjecture*, which will be explained in the next chapter.

3.3. Seifert Fibre Spaces

Geometric structures on manifolds are intimately connected to the fibrations of topological spaces. Formally, fibred spaces have the structure of a fibre bundle:

Definition 3.7. (Fibre bundle) *Let E, B, F be topological spaces and $\pi : E \to B$ a continuous surjective map satisfying: For every $x \in E$, there is an open neighbourhood $U \subset B$ of $\pi(x)$ such that there is an homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ in such a way that π agrees with the projection $\mathrm{proj}_U : U \times F \to U$ onto the first factor. That is, the following diagram should commute:*

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times F. \\ {\scriptstyle\pi}\downarrow & \swarrow {\scriptstyle \mathrm{proj}_U} & \\ U & & \end{array}$$

Then we say that (E, B, π, F) is a fibre bundle. The space B is called the base space of the bundle, E the total space and F the fibre. The map π is called the bundle projection.

Roughly speaking, a fibre bundle is a space E that is locally a product space $B \times F$, but globally may have a different structure. When E is globally homeomorphic to $B \times F$ we say that E is a trivial bundle over B.

As we have seen in Section 3.1.2, the sphere S^3 can be thought of as the points $(z_1, z_2) \in \mathbb{C}^2$ that verify $|z_1|^2 + |z_2|^2 = 1$. We can think of S^3 locally as $S^2 \times S^1$. In other words, there exists a map $h : S^3 \to S^2$, called the *Hopf map*, such that (S^3, S^2, h, S^1) is a fibre bundle.

Definition 3.8. (Hopf map) *Identifying S^2 with the complex projective line $\mathbb{C}\mathrm{P}^1$, then S^2 can be described by pairs of complex numbers $(z_1, z_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$ under the equivalence relation $[z_1 : z_2] \sim [\lambda z_1 : \lambda z_2]$, for $\lambda \in \mathbb{C} \setminus \{0\}$. The Hopf map $h : S^3 \to S^2$ is defined by $h(z_1, z_2) = [z_1 : z_2]$.*

Proposition 3.9. *The set (S^3, S^2, h, S^1) is a fibre bundle.*

Proof. Clearly the Hopf map h defined as above is continuous. On the other hand, for all $[z_1 : z_2] \in S^2 \cong \mathbb{C}\mathrm{P}^1$, taking $\lambda = \left(|z_1|^2 + |z_2|^2\right)^{-1/2}$ we deduce that $(\lambda z_1, \lambda z_2)$ lies in S^3 and $h(\lambda z_1, \lambda z_2) = [\lambda z_1 : \lambda z_2] = [z_1 : z_2]$ is a surjection.

For every $x \in S^3$, let $\tilde{x} \in S^3$ be a point such that $h(\tilde{x})$ is the antipodal point of $h(x)$ in S^2. Then, $U = S^2 \setminus \{h(\tilde{x})\}$ is an open neighbourhood of $h(x)$ in S^2. Taking $\mu = z_2/z_1$, the map

$$\begin{array}{rccc} \varphi : & h^{-1}(U) & \to & U \times S^1 \\ & (z_1, z_2) & \mapsto & \left([1 : \mu], \dfrac{\mu}{|\mu|} \right) \end{array}$$

is well-defined and $\left(\mathrm{proj}_U \circ \varphi\right)(z_1, z_2) = [1 : \mu] = [z_1 : z_2] = h_{|h^{-1}(U)}(z_1, z_2)$.

□

Note that identifying $S^2 \cong \mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ (the Riemann Sphere), then $U = \mathbb{C}$ and $h(\tilde{x}) = \infty$.

If $I \subseteq \mathbb{R}$ is an interval, we say that a fibre bundle is an *I-bundle* if the fibre is I. I-bundles are useful to describe some topological spaces. For example, if D^2 is the solid disk, the solid cylinder can be fibred as $D^2 \times [0, 1]$, with fibres $p \times [0, 1]$ for each $p \in D^2$. With that, we can define a *trivial fibred solid torus* $\mathbb{T}$ as the space obtained from the solid fibred cylinder by rotating it as follows: We rotate the top $D^2 \times \{1\}$ while holding the bottom $D^2 \times \{0\}$ fixed, and then identify the base $D^2 \times \{0\}$ with $D^2 \times \{1\}$. The fibres of a trivial fibred solid torus are circles. The fibre corresponding to $(0, 0) \times [0, 1]$ is called the *middle fibre* of the torus. We use the adjective trivial because there are other ways to fibre a solid torus. A (non-trivial) *fibred solid torus* $\mathbb{T}(p, q)$ can be easily constructed from a trivial fibred solid torus by rotating the disk D^2 by an angle $2\pi q/p$, where $p > 0, q$ are coprime integers.

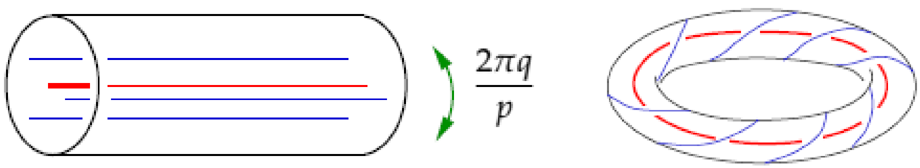

Figure 3.1: Construction of a fibred solid torus $\mathbb{T}(p, q)$ (right) from a fibred cylinder (left). Adapted from [Por08].

Note that the middle fibre, colored red in the figure above, is a fibre whose neighbourhood is not a trivial torus. We will call a fibre *regular* if it has a neighbourhood isomorphic to a trivial fibred solid torus and *critical* otherwise. Any fibred solid torus $\mathbb{T}(p, q)$ has at most one critical fibre: the middle fibre. Hence, a fibred solid torus can be finitely covered by a trivial fibred solid torus.

Analogously, we define a *fibred solid Klein bottle* $\mathrm{K}(p,q)$ to be a solid Klein bottle which is finitely covered by a trivial fibred solid torus. $\mathrm{K}(p,q)$ can be constructed in a similar way as $\mathbb{T}(p,q)$: First by fibrating the solid cylinder and rotating the disk D^2 by an angle $2\pi q/p$, and then glueing the two disks back together by a reflection. Unlike the fibred solid torus, the fibred Klein bottle has a family of critical fibres whose union forms an annulus. With all these definitions, we can define a Seifert fibre space:

Definition 3.10. (Seifert fibre space) *A Seifert fibre space is a* 3*-manifold which admits a decomposition into disjoint circles such that every fibre has a neighbourhood isomorphic to a fibred solid torus or a fibred solid Klein bottle.*

Seifert fiber spaces are also known as Seifert manifolds. The Seifert's original definition did not include the Klein bottle. However, nowadays many authors prefer to include it because the statement "a compact 3-manifold is a Seifert fiber space if, and only if, it is foliated by circles" is only true if we consider the Klein bottle (see [Eps72]).

Seifert manifolds are one of the most important tools for classifying geometries on three-dimensional manifolds, because every Seifert manifold admits a geometric structure. The three-sphere, S^3, is a Seifert manifold with the fiber bundle defined in Proposition 3.9. It can be foliated by circles in the following way: Consider the embedding $S^3 \hookrightarrow \mathbb{C}^2$. Each one-dimensional subespace of $\mathbb{C}^2$ (complex line) intersects S^3 in a great circle, called a *Hopf circle*, which can be represented as $z_1/z_2 = \lambda$, for $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Since exactly one Hopf circle passes through each point of S^3, the family of such circles fills up S^3. For a formal proof of this result, see [Sei80] in Section *Fiberings of* S^3. Furthermore, $S^3 \subset \mathbb{C}^2$ can be described as the *unit quaternion group* $\mathbb{H}^\times$,

$$S^3 \cong \mathbb{H}^\times := \left\{ z_1 + z_2 j \,:\, z_1, z_2 \in \mathbb{C},\, |z_1|^2 + |z_2|^2 = 1,\, i^2 = j^2 = -1,\, ij = -ji \right\},$$

by taking the isomorphism $(z_1, z_2) \mapsto z_1 + z_2 j$. The product in $\mathbb{H}^\times$ is defined as $(z_1 + z_2 j)(w_1 + w_2 j) = (z_1 w_1 - z_2 \overline{w}_2) + (z_1 w_2 + z_2 \overline{w}_1) j$. Hence, S^3 is in fact a three-dimensional Lie group, and we can consider a right action of S^3 on itself by right multiplication of unit quaternions. This action preserve the Hopf fibration of S^3. Indeed, if we consider $(z_1, z_2) \in S^3$ and $w_1 + w_2 j \in \mathbb{H}^\times$, there is a right group action $\theta : S^3 \times \mathbb{H}^\times \to S^3$ defined by

$$((z_1, z_2)\,,\, (w_1 + w_2 j)) \mapsto (z_1 w_1 - z_2 \overline{w}_2\,,\, z_2 \overline{w}_1 + z_1 w_2)$$

which transforms the Hopf circle $z_1/z_2 = \lambda$ to the Hopf circle $z_1/z_2 = \dfrac{\lambda w_1 - \overline{w}_2}{\overline{w}_1 + \lambda w_2}$.

In addition to S^3, six of the eight spaces where we can model a geometry, namely all except $\mathbb{H}^3$ and Sol, are Seifert manifolds:

Theorem 3.11. (Classification of Seifert geometric manifolds) *Let M be a smooth compact 3-manifold. If M is a Seifert manifold, then M possesses a geometric structure modelled on one of* $\mathbb{E}^3$, S^3, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}_2\mathbb{R}}$ *or* Nil.

Proof. See [Sco83, Th.5.3]. □

This theorem gives a nice division of Seifert manifolds into six classes according to which geometric structure they admit. We have seen in Chapter 2 that the surfaces can be classified by the sign of the Euler characteristic. In order to classify the six classes of Seifert manifolds, we need to generalize the notion of manifold considering a topological space which is locally homeomorphic to the quotient space of $\mathbb{R}^n$ by a finite group action. This space is called an *orbifold*.

Definition 3.12. (Orbifold) *A smooth n-dimensional orbifold is a Haussdorf, compact topological space $\mathcal{O}$ endowed with a collection $\{(U_i, \tilde{U}_i, \phi_i, \Gamma_i)\}_{i\in I}$, where for each $i \in I$, U_i is an open subset of $\mathcal{O}$, $\tilde{U}_i$ is an open subset of $\mathbb{R}^n$, $\phi_i : \tilde{U}_i \to U_i$ is a continuous map and Γ_i is a finite group of diffeomorphisms of $\tilde{U}_i$ acting smoothly on $\tilde{U}_i$; satisfying the following properties:*

(I) *$\mathcal{O}$ is covered by the open subsets $\{U_i\}_{i\in I}$, $\mathcal{O} = \bigcup_{i\in I} U_i$.*

(II) *Each ϕ_i factors through a homeomorphism $\overline{\phi}_i$ between $\tilde{U}_i/\Gamma_i$ and U_i. That is, if $\pi_i : \tilde{U}_i \to \tilde{U}_i/\Gamma_i$ is the natural projection, then there is a homeomorphism $\overline{\phi}$ such that $\phi_i = \overline{\phi}_i \circ \pi_i$.*

(III) *For each pair $i, j \in I$ and for every $x \in \tilde{U}_i$ and $y \in \tilde{U}_j$ with $\phi_i(x) = \phi_j(y)$, there exists a diffeomorphism $\psi : V \to W$ between a neighbourhood V of x and a neighbourhood W of y such that $\phi_j(\psi(z)) = \phi_i(z)$, for all $z \in V$.*

This definition extends the classical definition of a manifold in the sense that $\mathcal{O}$ is a manifold if all the groups Γ_i are trivial. Since a Seifert manifold M is foliated by circles, every circle can be shrunk to a point and doing so we usually obtain a surface. In general, the resulting space of this quotient, called the *base space* of M, is a 2-dimensional orbifold.

Definition 3.13. (Seifert bundle) *Let M be a Seifert manifold. The base space X of M is the quotient space of M obtained by identifying each circle to a point. We call the base space X together the projection map $M \to X$ a Seifert bundle η.*

In Section 1.2 we have defined triangulations of manifolds and its Euler characteristic. We can generalise this to orbifolds: A *triangulation* of an orbifold $\mathcal{O} = \{U_i, \tilde{U}_i, \phi_i, \Gamma_i\}_i$ is a decomposition of the underlying topological space into subsets of $\mathcal{O}$, called *orbifold simplices*, such that each point x of this underlying space has a neighbourhood U which is a union of orbifold simplices, which is contained in the image of some map $\phi_i : \tilde{U}_i \to U_i$, and such that the decomposition of U into orbifold simplices lifts to a triangulation of $\phi_i^{-1}(U) \subset \tilde{U}_i \subset \mathbb{R}^n$ which is invariant under the action of Γ_i. The collection of such orbifold simplices is called an *orbifold simplicial complex*. The proof that every smooth manifold admits a triangulation immediately extends to show that every smooth orbifold admits a triangulation. It can be proved that, if σ is an orbifold simplex, the group Γ_i is constant on $\operatorname{Int}(\sigma)$. With all that, we can define the *orbifold Euler characteristic*:

Definition 3.14. (Orbifold Euler characteristic) *Let $\mathcal{O} = \{U_i, \tilde{U}_i, \phi_i, \Gamma_i\}_i$ be an smooth orbifold triangulated by a finite orbifold simplicial complex. For each orbifold simplex σ, let $\Gamma_i(\sigma)$ be the corresponding group on σ. Then, the orbifold Euler characteristic of $\mathcal{O}$ is*

$$\chi(\mathcal{O}) := \sum_{\sigma} (-1)^{\dim(\sigma)} \frac{1}{|\Gamma_i(\sigma)|}.$$

Note that the orbifold Euler characteristic is, in general, a rational number. Again, the proof that the Euler characteristic of a manifold is independent of the triangulation trivially extends to orbifolds.

If M is a seifert manifold with base orbifold X, we have a short exact sequence

$$1 \longrightarrow K \longrightarrow \pi_1(M) \longrightarrow \pi_1(X) \longrightarrow 1,$$

where K is a cyclic group generated by a regular fibre. Furthermore, K is only finite when $X \cong S^2$.

Let M be a a Seifert manifold whose base space X is a compact orientable surface not homeomorphic to S^2, and let $K = \langle k \rangle$ be the infinite cyclic group defined in the short exact sequence above. It is well-known that the fundamental group of a compact surface X admits a presentation

$$\pi_1(X) = \left\langle \bar{a}_1\bar{b}_1, \cdots, \bar{a}_g\bar{b}_g \;\middle|\; \prod_{i=1}^{g}[\bar{a}_1, \bar{b}_1] \right\rangle,$$

where $g := \frac{1}{2}(2-\chi(X))$ is the *genus* of X and $[a,b] := a^{-1}b^{-1}ab$ denotes the *commutator* of the elements a and b. Let $\varphi : \pi_1(M) \to \pi_1(X)$ be the epimorphism sending the elements $a_i, b_i \in \pi_1(M)$ to the elements $\overline{a}_i, \overline{b}_i \in \pi_1(X)$, for every $i = 1, \cdots, g$. The element $\prod_{i=1}^{g}[a_1, b_1]$ lies on $\mathrm{Ker}(\varphi)$, and thus in K. Since K is cyclic, there must exist a (minimal) positive integer r such that

$$\prod_{i=1}^{g}[a_1, b_1] = k^r.$$

Definition 3.15. (Euler number) *If η is a Seifert bundle with base space a compact orientable surface not homeomorphic to the sphere, then the integer $e(\eta) = -r$ is called the Euler number of η.*

We have only defined the Euler number for some Seifert bundles. However, it can be defined for any Seifert bundle with base space a 2-dimensional smooth orbifold X. The Euler number is a topological invariant of the Seifert bundle. For a detailed description of this, see [Sco83] in Section §3.

The orbifold Euler characteristic and the Euler number are two topological invariants which allow us to classify the six classes of Seifert manifolds claimed in Theorem 3.11. If η is the Seifert bundle of a Seifert manifold M with base space X and we denote $\chi = \chi(X)$ the orbifold Euler characteristic and $e = e(\eta)$ the Euler number, then the geometry of M is determined as shown in the following table:

	$\chi > 1$	$\chi = 0$	$\chi < 0$
$e = 0$	$S^2 \times \mathbb{R}$	$\mathbb{E}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$e \neq 0$	S^3	Nil	$\widetilde{\mathrm{SL}_2\mathbb{R}}$

The two non-Seifert geometries are Sol and $\mathbb{H}^3$. However, a torus bundle over S^1 can be modelled on Sol. In particular, we say that an *Anosov diffeomorphism* is a diffeomorphism of the 2-torus T^2 whose action on $\mathrm{SL}_2\mathbb{Z}$ has two distinct real eigenvalues. If $\varphi : T^2 \to T^2$ is an Anosov diffeomorphism and M_φ is the torus bundle over S^1 with bundle projection φ, then M_φ is modelled in Sol. On the other hand, the vast range of manifolds modelled on $\mathbb{H}^3$ makes them the most mysterious, rich and difficult to classify. Thurston's research focused mainly on the study of hyperbolic manifolds.

Chapter 4

The Geometrisation Conjecture of 3-Manifolds

This final chapter will be devoted to introduce the Thurston's geometrisation conjecture of 3-manifolds. Firstly, we will describe how to decompose a manifold into canonical pieces, in order to understand the statement of the conjecture, and then we will introduce it through a historical perspective. Finally we will show its main applications and corollaries, such as the Poincaré conjecture.

4.1. Canonical Decomposition

In this section we will see that any compact 3-manifold can be split in a reasonably canonical way into smaller pieces. First we are going to cut M along spheres, which is equivalent to decompose M into connected summands.

Definition 4.1. (Connected sum) *Let M_1, M_2 be two n-dimensional manifolds. The connected sum $M_1 \sharp M_2$ of M_1 and M_2 is formed by deleting the interiors of n-dimensional balls $B_1 \subseteq M_1$ and $B_2 \subseteq M_2$, and attaching the resulting manifolds $M_1 \setminus \mathring{B}_1$ and $M_2 \setminus \mathring{B}_2$ to each other by a homeomorphism $h : \partial B_1 \to \partial B_2$. We write*

$$M_1 \sharp M_2 = \left(M_1 \setminus \mathring{B}_1\right) \cup_h \left(M_2 \setminus \mathring{B}_2\right).$$

The choice of the identification h only matters up to isotopy. Note however that if M_1 and M_2 are oriented manifolds, then there is a unique isotopy class of identifications such that $M_1 \sharp M_2$ admits an orientation compatible with those of M_1 and M_2.

Henceforth, we will suppose M a connected, oriented manifold unless otherwise specified. It is clear that S^n is an identity element for the connected sum, meaning that if M is a n-dimensional manifold, then $M = M\sharp S^n$. Therefore, if M_1 and M_2 are two n-dimensional manifolds, we will say that the connected sum $M_1\sharp M_2$ is *trivial* if either M_1 or M_2 is the n-sphere. This motivates the following definition:

Definition 4.2. (Prime manifold) *A prime manifold P is a n-dimensional manifold that cannot be expressed as a non trivial connected sum of two n-manifolds.*

That is, P is a prime manifold if $P = M_1\sharp M_2$ implies that either M_1 or M_2 is S^n. On the other hand, we will say that a manifold is *irreducible* if every embedded sphere bounds an embedded ball. Formally,

Definition 4.3. (Irreducible manifold) *An irreducible n-dimensional manifold N is a manifold in which for every submanifold $S \subset N$ homeomorphic to the $(n-1)$-sphere S^{n-1}, there exist a subset $D \subset N$ homeomorphic to the n-ball B^n such that $\partial D = S$.*

In general, it is not clear that these two definitions have anything in common. However, they are in fact very nearly equivalent, especially for 3-manifolds. We want to prove that irreducibility is stronger than primeness:

Proposition 4.4. *Every irreducible 3-manifold N is prime.*

Proof. If N is an irreducible manifold and we express $N = N_1\sharp N_2$, then N is obtained by removing a 3-ball each from N_1 and N_2 and gluing the resulting 2-spheres together. Let $B_k \subset N_k$ be the ball that was removed from N_k, for $k \in \{1,2\}$, and ∂B_k the corresponding spheres. These united spheres form a 2-sphere in N, and the fact that N is irreducible means that this sphere must bound a ball $B \subset N$. Then, B is equal to one of the two pieces obtained from N by cutting along a sphere, and those two pieces are $N_1 \setminus B_1$ and $N_2 \setminus B_2$. It follows that for either $k = 1$ or $k = 2$, the manifold N_k is reconstructed by gluing B and B_k. That gluing identifies ∂B and ∂B_k. But any manifold that is obtained by gluing two 3-balls, identifying their 2-sphere boundaries, is homeomorphic to S^3, and N is thus prime. □

We have seen that if M is a 3-manifold in which any embedded 2-sphere bounds and embedded 3-ball, then M is prime. In particular this is true for any *separating sphere*, that is, an sphere S^2 such that $M \setminus S^2$ is not connected. The converse result is also true: If M is a prime manifold, any embedded separating sphere bounds a ball. Indeed, if S^2 separates M into two components M_1 and M_2 such that neither M_1 nor M_2 is a ball, then M would not be prime.

The three sphere S^3 is an irreducible manifold, and hence prime, as a consequence of Alexander's theorem:

Theorem 4.5. (Alexander's theorem) *Every embedded* 2*-sphere in* $\mathbb{R}^3$ *bounds an embedded* 3*-ball.*

Proof. See [Hat80, Th.1.1]. □

We know that irreducible 3-manifolds are prime, but we would like to know which prime 3-manifolds are irreducible. Surprisingly, all the prime orientable 3-manifolds are irreducible, except one: $S^1 \times S^2$.

Lemma 4.6. $S^1 \times S^2$ *is prime.*

Proof. Let $S \subset S^1 \times S^2$ be a separating 2-sphere, so $\left(S^1 \times S^2\right) \setminus S$ consists of two compact 3-manifolds M and M', each with boundary a 2-sphere. By the Seifert-van Kampen theorem, we have $\mathbb{Z} = \pi_1(S^1 \times S^2) \cong \pi_1(M) * \pi_1(M')$, so either M or M' must be simply-connected, say M is simply-connected. This is because, if G is a subgroup of $\mathbb{Z}$, then either $G \cong \mathbb{Z}$ or $G \cong *$. The universal cover $\tilde{X}$ of $S^1 \times S^2$ can be identified with $\mathbb{R}^3 \setminus \{0\}$. Indeed, the maps $\nu : \mathbb{R} \to S^1$, $\nu(t) = e^{it}$ and $\mathrm{id} : S^2 \to S^2$ are covering maps of S^1 and S^2, respectively. As $\mathbb{R}$ and S^2 are simply connected, they are the respective universal covers of S^1 and S^2. Consider the map $f : \mathbb{R}^3 \setminus \{0\} \to S^2 \times \mathbb{R}^+ \setminus \{0\}$ defined by $f(x) = \left(x\|x\|^{-1}, \|x\|\right)$. Since f is a diffeomorphism and $\mathbb{R}^+ \setminus \{0\}$ is diffeomorphic to $\mathbb{R}$, the universal cover of $S^1 \times S^2$ is diffeomorphic to $\mathbb{R}^3 \setminus \{0\}$. If $p : \mathbb{R}^3 \setminus \{0\} \to S^1 \times S^2$ is the cover map, $M \subset S^1 \times S^2$ lifts to a diffeomorphic copy $\tilde{M}$ of itself in $\mathbb{R}^3 \setminus \{0\}$. As ∂M is a 2-sphere, then $\partial \tilde{M}$ must bound a ball in $\mathbb{R}^3 \setminus \{0\}$ by Alexander's theorem. We conclude that $\tilde{M}$ is a ball, hence also M. Thus every separating sphere in $S^1 \times S^2$ bounds a ball, so $S^1 \times S^2$ is prime. □

Proposition 4.7. *The only orientable prime* 3*-manifolds which are not irreducible are those isomorphic to* $S^1 \times S^2$*.*

Proof. By Lemma 4.6, $S^1 \times S^2$ is prime. If M is a prime 3-manifold, every embedded 2-sphere in M which separates M into two components bounds a ball. So if M is prime but not irreducible there must exists a non-separating sphere S in M. We can embed the product $S^2 \times [0,1]$ in M, $h : S \times [0,1] \to M$ with $h\left(S^2 \times \{1/2\}\right) = S$, call $h\left(S^2 \times [0,1]\right) = S \times I$. In $M \setminus (S \times I)$, let α be an arc joining $S \times \{0\}$ to $S \times \{1\}$, and let $N(\alpha)$ be a tubular neighbourhood of α. In an orientable manifold M, the union $M' = (S \times I) \cup N(\alpha)$ is a manifold diffeomorphic to $S^1 \times S^2$ minus a ball. As the boundary $\partial M'$ of M' separates M, it has $S^1 \times S^2$ as a connected summand. Assuming M is prime, then $M = S^1 \times S^2$. □

With all this, we can already face the main theorem of this chapter, and one of the most important theorems of the 3-manifold topology:

Theorem 4.8. (Kneser's-Milnor theorem) *Every oriented compact 3-manifold M is a finite connected sum of manifolds*

$$M \cong P_1 \sharp P_2 \sharp \cdots \sharp P_n$$

where each P_i is prime. Such a decomposition is unique up to ordering of the factors and homeomorphism of the factors.

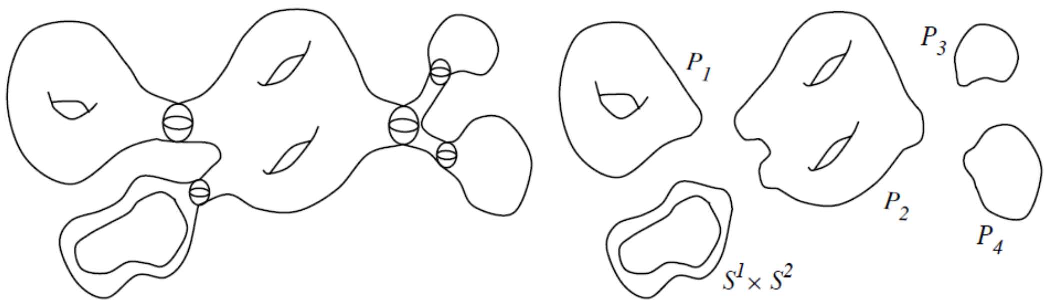

Figure 4.1: Kneser-Milnor decomposition. Taken from [Est06].

This theorem owes its name to both Kneser, who proved the existence in 1929 [Kne29], and John Milnor, who proved the uniqueness in 1962 [Mil62]. Even though the proof exceeds the level of this thesis, we can take an heuristic approach to understand it: If P is not itself prime, then it admits a decomposition $P \simeq P_1 \sharp P_2$, with P_1 and P_2 not S^3, and we can repeat this process on P_1 and P_2 if they are non-prime. On the other hand, if P has a non-separating S^2, there exists a decomposition $P = N \sharp \left(S^1 \times S^2\right)$. This process must end after finitely many steps because each $S^1 \times S^2$ summand gives a summand of $\mathbb{Z}$ for $\pi_1(M)$.

In fact, Proposition 4.7 gives a stronger version of the Kneser-Milnor Theorem: Every oriented compact 3-manifold M is a finite connected sum of manifolds

$$M \cong N_1 \sharp N_2 \sharp \cdots \sharp N_s \sharp S^1 \times S^2 \sharp \cdots \sharp S^1 \times S^2$$

where each N_i is irreducible. Again, this decomposition is unique up to ordering and homeomorphism of the factors.

Beyond the prime decomposition, there is a further decomposition of irreducible compact orientable 3-manifolds, splitting along tori rather than spheres.

Firstly, we need to define some properties of embedded surfaces on 3-manifolds. Let M be a 3-manifold and let $\mathcal{S} \hookrightarrow M$ be an embedded connected surface. An embedded disk $D \hookrightarrow M$ with $D \cap \mathcal{S} = \partial D$ is called a *compressing disk* for $\mathcal{S}$.

Definition 4.9. (Incompressible surface) *A connected surface $\mathcal{S} \subset M$ other than the 2-sphere S^2 or the 2-disk D^2 is called incompressible if for each compressing disk $D \subset M$ for $\mathcal{S}$ there is a disk $D' \subset \mathcal{S}$ such that $\partial D = \partial D'$.*

In other words, an incompressible surface on a 3-manifold is a surface such that, for each disk whose boundary lies on the surface, it bounds a disk on it. A simpler but less intuitive way to define it is as follows: A surface $\mathcal{S}$ is said to be incompressible if the map $\iota : \pi_1(\mathcal{S}) \hookrightarrow \pi_1(M)$ is injective. We will not prove that these two definitions are equivalent; but it comes from the fact that if $D \subset M$ is a compressing disk, then ∂D is nullhomotopic (i.e. homotopic to a constant map) in M, hence also in $\mathcal{S}$ if the map ι is injective. It remains to show that a nullhomotopic embedded circle in a surface must bound a disk.

Incompressible surfaces are interesting because, if they are removed from an irreducible manifold, it remains irreducible:

Proposition 4.10. *If $\mathfrak{S} \subset N$ is a finite collection of disjoint incompressible surfaces and N is irreducible, then $N \setminus \mathfrak{S}$ is irreducible.*

Proof. If N is an irreducible manifold, then every 2-sphere must bound a ball in N. In particular, there exist a 2-sphere $S^2 \subset N \setminus \mathfrak{S}$ such that it bounds a ball B^3 in N. Since $\mathbb{R}^3$ is simply connected, $\pi_1(\mathbb{R}^3) \simeq *$, and it is immediate from the previous definition that there are not incompressible surfaces in it. Indeed, the only connected surfaces $\mathcal{S}$ that make the map $\iota : \pi_1(\mathcal{S}) \to *$ injective are those homeomorphic to the 2-sphere, and they are not incompressible by Definition 4.9.

Hence, there are not incompressible surfaces in $B^3 \cong \mathbb{R}^3$, and $B^3 \cap \mathfrak{S} = \emptyset$. So the sphere S^2 bounds a ball in $N \setminus \mathfrak{S}$, and $N \setminus \mathfrak{S}$ is thus irreducible. □

The incompressible surfaces we are interested in are incompressible tori. A compact 3-manifold N is said to be *atoroidal* if we cannot embed any incompressible tori in it. The following theorem, known as the *JSJ splitting theorem*, was discovered in the 1970s by W. Jaco and P. Shalen from one side, and K. Johannson independently [Jac76].

Theorem 4.11. (JSJ splitting theorem) *Let N be an irreducible, compact and orientable 3-manifold. There exists a finite collection $\mathcal{T} = \{T_1, \cdots, T_k\}$ of disjoint incompressible tori such that each component of $N \setminus \mathcal{T}$ is either atoroidal or a Seifert manifold, and a minimal such collection $\mathcal{T}$ is unique up to isomorphism.*

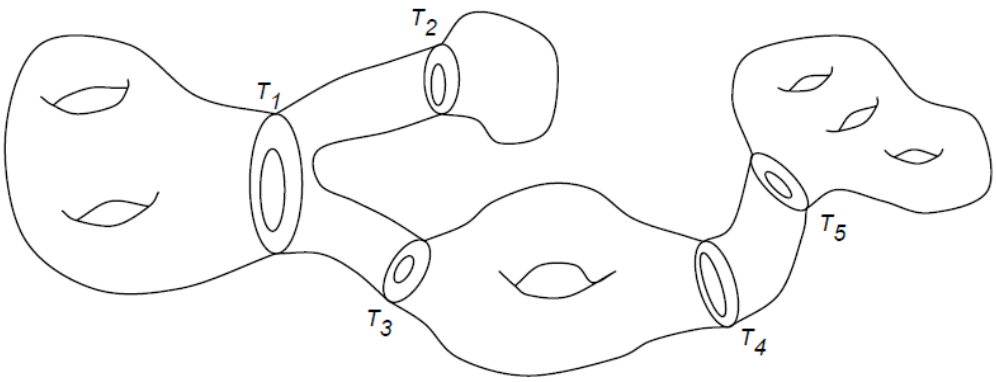

Figure 4.2: Jaco-Shalen, Johannson splitting. Taken from [Est06].

Here minimal means that, if $\mathcal{T}'$ is another collection, then $\mathcal{T} \subseteq \mathcal{T}'$. Hence, $\mathcal{T} = \emptyset$ if N is either atoroidal or a Seifert manifold. Note that each component of $N \setminus \mathcal{T}$ is a manifold N_i with boundary $\partial N_i = T_i$, for some $T_i \in \mathcal{T}$.

Summing up, the *canonical decomposition* of a compact, orientable 3-manifold M can be obtained as follows: First, we decompose the manifold in its prime components, $M = P_1 \sharp \cdots \sharp P_n$, known as the Kneser-Milnor decomposition. As every prime component P_i is either $S^1 \times S^2$ or an irreducible manifold N_i, we can write $M = N_1 \sharp N_2 \sharp \cdots \sharp N_s \sharp S^1 \times S^2 \sharp \cdots \sharp S^1 \times S^2$. Note that $S^1 \times S^2$ is a Seifert manifold. On the other hand, the irreducible components N_i can be split along tori, by a JSJ

Splitting, so that each subcomponent (which is a manifold with boundary) is either atoroidal or a Seifert manifold (or both). The interior of such subcomponents is a manifold. In this way, we have obtained the canonical pieces into which a 3-manifold decomposes, that can be Seifert manifolds or atoroidal manifolds. These pieces will be necessary to understand the geometrisation conjecture that we will study in the next section.

4.2. The Geometrisation Conjecture

As we have seen in Section 3.2, William P. Thurston proved that each manifold which admits a geometric structure can be modelled on one and only one of eight model geometries. However, unlike the 2-dimensional case, not every 3-dimensional manifolds admits a geometric structure. For example, if we take two hyperbolic manifolds with toral boundary and we glue them along their boundary tori, the resulting space is a manifold (without boundary) that does not admit a geometric structure. In 1982, Thurston gave a first version of the Geometrisation conjecture:

Conjecture 4.12. (Thurston's Geometrisation Conjecture) *Every compact, orientable* 3*-manifold decomposes canonically into pieces whose interior is either Seifert fibred or hyperbolic.*

The canonical decomposition he refers is the explained in the previous section. However, this statement does not mention geometric structures. Nowadays, the geometrisation conjecture is enunciated as follows:

Conjecture 4.13. (Geometrisation Conjecture) *The interior of any compact, orientable* 3*-manifold M can be split along a finite collection of disjoint embedded spheres and incompressible tori into a canonical collection of* 3*-submanifolds $M_1, \ldots, M_n$ such that, for each i, the manifold obtained from M_i by capping off all sphere components by balls admits a geometric structure.*

The equivalence between these both formulations of the Geometrisation conjecture comes from the fact that a 3-manifold admits a locally homogeneous metric if, and only if, it is either hyperbolic, Seifert fibred or a torus bundle over S^1.

The geometrisation conjecture was motivated because some partial results were known before its formulation. Waldhausen noticed that some manifolds, called *graph manifolds*, decomposes into geometric pieces.

Definition 4.14. (Graph manifold) *A 3-dimensional compact, orientable manifold is called a graph manifold if it is a union of Seifert manifolds along toral boundaries.*

More specifically, he proved in 1967 that any any graph manifold is the connected sum of manifolds whose JSJ split contains only Seifert manifolds. Note that even though this result seems trivial, it is not since the toral boundaries do not have to be incompressible tori after the union. On the other hand, Thurston proved also a partial result for *Haken manifolds*.

Definition 4.15. (Haken manifold) *A compact, orientable, irreducible 3-dimensional manifold is called a Haken manifold (or a sufficiently large manifold) if it contains an irreducible surface.*

In particular, he proved in 1981 that any Haken manifold is hyperbolic if, and only if, it is atoroidal. Clearly, any graph manifold and any Haken manifold satisfies the geometrisation conjecture.

The JSJ splitting Theorem 4.11 asserts that the interior of the pieces obtained in the canonical decomposition are either Seifert manifolds or atoroidal. In Section 3.3 we have seen that any Seifert manifold admits a geometric structure, so in order to prove the geometrisation conjecture one only needs to prove that atoroidal manifolds also admit geometric structures. It was conjectured that atoroidal manifolds have either an elliptic or hyperbolic geometry, according to whether the fundamental group is finite or not. Hence, the geometrisation conjecture splits into two simpler conjectures:

Conjecture 4.16. (Elliptization Conjecture) *A closed, orientable 3-dimensional manifold is elliptic if and only if its fundamental group is finite.*

Conjecture 4.17. (Hyperbolization Conjecture) *A compact, orientable 3-dimensional manifold is hyperbolic if and only if it is atoroidal and has infinite fundamental group.*

Recall that a 3-manifold is called elliptic or hyperbolic if it admits a geometric structure modelled on S^3 or $\mathbb{H}^3$, respectively. Hence, the geometrisation conjecture is true if both elliptization and hyperbolization conjectures are true.

In 1982, Richard Hamilton introduced a method to evolve a Riemannian metric in any Riemannian manifold, called *the Ricci flow*. He tried to show that manifolds endowed with complicated metrics also admit more symmetric metrics (for example, with constant sectional curvature or homogeneous). The Ricci flow is a one-parametric family of Riemannian metrics $\{g_t\}_{t\in I}$ on a smooth manifold M satisfying the differential equation

$$\frac{\partial g_t}{\partial t} = -2\mathrm{Ric}_{g_t}.$$

The term Ric_{g_t} denotes the *Ricci curvature* of the metric g_t, which is defined to be

$$\mathrm{Ric}_{g_t}(X,Y) := \mathrm{Trace}\,(Z \mapsto R_{\nabla_t}(X,Z)Y)$$

for every smooth vector fields $X, Y, Z \in \mathcal{C}^\infty(M, TM)$, where R_{∇_t} is the curvature of the Levi-Civita connection of g_t defined in Section 1.1. Hamilton proved that for any metric g_0 on M, there exist $\epsilon > 0$ such that the differential equation above has a unique solution defined on the interval $[0, \epsilon)$ with initial condition $g = g_0$. He believed that the Ricci flow could be a powerful tool to prove the geometrisation conjecture.

And Hamilton was right. Finally, the mathematician Grigori Perelman published in 2003 a proof of the geometrisation conjecture, based on the notes of Hamilton about the Ricci flow. More specifically, Perelman showed that any compact orientable manifold M decomposes as

$$M \cong M_1 \sharp \cdots \sharp M_r \sharp E_1 \sharp \cdots \sharp E_k \sharp S^1 \times S^2 \sharp \cdots \sharp S^1 \times S^2$$

where E_i are elliptic manifolds and each M_i admits a torus decomposition as $H_i \sqcup G_i$, with H_i an hyperbolic manifold and G_i a graph manifold.

Perelman's proof was written in several publications that far exceed the level of this thesis. However, the reader can consult [Est06] and [Por08] for a sketch of the proof, or [Bes10] for a more exhaustive explanation. In 2006, the proof was verified, so the geometrisation conjecture should be called today the *Thurston-Hamilton-Perelman geometrisation theorem of* 3*-manifolds*.

4.3. Applications and Consequences

The geometrisation theorem, together with the Thurston's classification theorem, gives a complete understanding of the geometric structures on compact, orientable 3-dimensional manifolds, such as the uniformization theorem for 2-dimensional ones. We can say that such a manifold can be decomposed into canonical pieces whose interior can be modelled in one and only one of the 8 Thurston's geometries. One might note that the canonical decomposition described is sufficient but not necessary, since Sol geometry does not appear. This is because the canonical pieces are Seifert manifolds or atoroidal, and as a consequence of the hyperbolization conjecture, atoroidal manifolds are elliptic or hyperbolic. But any manifold modelled on Sol is neither Seifert nor hyperbolic. If φ is an Anosov diffeomorphism and M_φ is a torus bundle over S^1 modelled on Sol, it admits a geometric structure although its JSJ splitting is not trivial.

Note that the Poincaré conjecture is a particular case of the elliptization conjecture. It is easy to prove it as a corollary of the geometrisation theorem:

Theorem 4.18. (Poincaré Conjecture) *Every simply connected, compact 3-manifold is homeomorphic to S^3.*

Proof. Let Σ be a compact, simply connected manifold. As the fundamental group of Σ is trivial, it has no incompressible tori since the fundamental group of the torus is $\mathbb{Z} \oplus \mathbb{Z}$. Hence, the canonical decomposition of Σ must be trivial, and thus by the geometrisation theorem Σ admits a geometric structure. Again, since Σ is simply connected, it is its own universal cover and the model space of Σ is Σ. Then Σ must be one of the 8 model geometries, but S^3 is the only compact one. □

Another consequence of the geometrisation theorem is the *spherical space form conjecture*, which asserts that a finite group acting on S^3 is conjugate to a group of isometries of S^3. However, the applications do not depend only on the geometrisation theorem, but on the tools developed to prove it. In the last decade, many works have been published applying the Ricci flow with cosmological motivations.

The generalized Poincaré conjecture for n-dimensional manifolds is already proven. Surprisingly, the case $n = 3$ was proved the last. Higher dimensional manifolds are often easier than lower, because there is more space to carry out geometric constructions. However, 4-dimensional spaces are a jungle, because of the difficulty of classifying all finitely presented groups. Whether it is possible to generalise the geometrisation theorem to higher dimensions or not is a question that topology will answer in this century or in the coming ones.

Bibliography

[Aba05] M.T.H.K. Abbassi, M. Sarih, *On Natural Metrics on Tangent Bundles of Riemannian Manifolds*, Archivum Mathematicum, Brno, **41** (1996), 71–92.

[Bes10] L. Bessières, G. Besson, M. Boileau, S. Maillot, J. Porti, *Geometrisation of* 3*-manifolds*, EMS Tracts in Mathematics, **13**, Zürich (2010).

[Bon96] F. Bonahon, *Geometric Structures on 3-Manifolds*, Handbook of Geometric Topology, edited by F.D. Mesman and W.J. Maas (1996).

[Con08] K. Conrad, *Splitting of short exact sequences for groups,* (2008). Available on: `https://kconrad.math.uconn.edu/blurbs/grouptheory/splittinggp.pdf`

[Eps72] D.B.A. Epstein, *Periodic flows on three-manidolds*, Ann. of Math. **95** (1972), 66–82.

[Est06] E.C. Rivas, V.M. Molina, *Demostración de Hamilton-Perelman de las Conjeturas de Poincaré y Thurston*, La Gaceta de la RSME, Vol. 9.1 (2006), 15–42.

[Gal13] J. Gallier, D. Xu, *A guide to the classification theorem for compact surfaces*, Geometry and Computing **9.** Berlin: Springer Verlag (2013).

[Hat80] A. Hatcher, *Notes on Basic 3-manifold topology*, (1980s). Avaliable on: `http://www.math.cornell.edu/~hatcher`

[Jac76] W. Jaco, P.B. Shalen, *Seifert fibered spaces in irreducible, sufficiently large 3-manifolds* Bull. Amer. Math. Soc. **82** (1976), 765–767.

[Ker63] M.A. Kervaire, J. Milnor, *Groups of Homotopy Spheres*, Ann, of Math **77** (1963), 504–537 .

[Kne29] H. Kneser, *Geschlossen Flächen in dreidimensionalen Mannigfaltigkeiten*, Jahresbericht der Deutschen Mathematiker Vereinigung **38** (1929), 248–260.

[Lee00] J.M. LEE, *Introduction to Topological Manifolds*, Graduate Texts in Mathematics **202.** New York: Springer Verlag (2000).

[Lee13] J.M. LEE, *Introduction to Smooth Manifolds*, Graduate Texts in Mathematics **218.** New York: Springer Verlag (2013).

[Mil62] J. MILNOR, *A unique decomposition theorem for 3-manifolds*, Amer. J. Math **84** (1962), 1–7.

[Moi52] E.E. MOISE, *Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung*, Ann. of Math. (2) **56** (1952), 96–114.

[Mun00] J.R. MUNKRES, *Topology*, Prentice -Hall Inc, New Delhi, (2000).

[Mye39] S.B. MYERS, N.E. STEENROD, *The group of isometries of a Riemannian Manifold*, Ann. of Math. (2) **40** (1939), 400–416.

[Por08] J. PORTI, *Geometrization of three manifolds and Perelman's proof*, Rev. R. Acad. Cien. Serie A. Mat. (1) **102** (2008), 101–125.

[Rad25] T. RADÓ, *Über den Begriff der Riemannschen Fläche*, Acta Sci. Math. (Szeged) (2) **2**, (1925), 101–121.

[Sco83] P. SCOTT, *The Geometries of 3-Manifolds*, Bull. London Math. Soc., **15**, (1983), 401–487.

[Sei80] H. SEIFERT, W. THRELFALL, *Topology of 3-Dimensional Fibered Spaces*, Academic Press Inc. New York, (1980).

[Sin60] I.M. SINGER, *Infinitesimally homogeneous spaces*, Comm. Pure Appl. Math., **13**, (1960), 685–697.

[Thu97] W.P. THURSTON, *Three-Dimensional Geometry and Topology*, Volume 1, Princenton University Press, (1997).

[Whi40] J.H.C. WHITEHEAD, *On C^1-complexes*, Ann. of Math. (2) **41**, no. 4, (1940), 809–824.