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ON THE DISTRIBUTION OF A DOUBLE STOCHASTIC INTEGRAL

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Abstract.- Let $\{W(z),\ z\in[0,1]^2\}$ be a Wiener process with a two-dimensional parameter. We evaluate the characteristic function of the stochastic integral $\int_{[0,1]^2} W dW$ and obtain some properties of its moments. Also, a martingale array having this non-symmetric limit distribution is exhibited.

0. Introduction

The law of the stochastic integral $\int_0^1 W_t dW_t$, where W_t is an ordinary Brownian motion can be obtained in an obvious way from Itô's formula: $\int_0^1 W_t dW_t = \frac{1}{2}(W_1^2-1)$. For a two-parameter Wiener process $\{W(s,t), (s,t)\in T\}$, Itô's differentiation formula (see Wong and Zakai [11]) claims that $\frac{1}{2}(W_{11}^2-1)= \int_T WdW + \int_T\int_T 1_D dWdW$, being $D = \{(z,z')\in T\times T: z=(x,y), z'=(x',y'), x\leq x' \text{ and } y\geq y'\}$. In this case we cannot attain from this expression the distribution of the random variables $\int_T WdW$ and $\int_T\int_T 1_D dWdW$. This paper is devoted to discuss the law of these variables. As we shall see they have the same law.

It is known that the law of a double Wiener stochastic integral can be computed in terms of a weighted sum of independent chi-squared random variables. See, for instance, the papers of Varberg [10] and Rosiński- Szulga [9]. In section 1, using a result of this kind for a two-parameter Wiener process, we deduce the characteristic function of $\int_T\int_T 1_D dWdW$.

The distribution of the two-parameter Wiener process W_{st} in the space of continuous functions $C(T)$ is the weak limit of the law of the sequence of processes $n^{-1/2} \Sigma_{i=1}^n X_s^i Y_t^i$, where $\{X^n(t), t\in [0,1], n\geq 1\}$ and $\{Y^n(t), t\in [0,1], n\geq 1\}$ are two independent sequences of infinite dimensional Brownian motions. This result has been proved in [7]. In section 2 we will use this fact to express the indefinite integrals $J_{st}= \int_{R_{st}} 1_D dWdW$ and $K_{st}= \int_{R_{st}} WdW$, where $R_{st}= [0,s]\times[0,t]$, as the weak limit of a sequence of two-parameter continuous processes. This provides a method to compute the moments of the random variable J_{11}.

The sequence of random variables converging to J_{11} can be arranged in order to exhibit an example of a martingale array $\{X_{ni}, n\geq 1, i=1,\dots,k_n\}$, with respect to a family of σ-fields F_{ni}, satisfying the conditional Lindeberg condition

$$\Sigma_{i=1}^{k_n} E(X_{ni}^2 1_{\{|X_{ni}|>\varepsilon\}} / F_{n,i-1}) \xrightarrow{P} 0, \quad \text{for all} \quad \varepsilon>0. \qquad (0.1)$$

The asymptotic behavior of this martingale array is similar to that of the class of degenerate U-statistics discussed by Alvo, Cabilio and Feigin in [1]. Indeed, it is proved that the sequence of conditional variances converges in distribution, as long as $\Sigma_{i=1}^{k_n} X_{ni}$ converges in law to the non-symmetric random variable J_{11}.

1. Let $W = \{W(s,t), (s,t)\in T, T=[0,1]^2\}$ be a two-parameter Wiener process in a probability space (Ω,F,P). For any function $f\in L^2(T\times T)$ the double Itô-Wiener integral $I(f)$ with respect to W can be defined as in Itô [6]. This stochastic integral takes into account just the values of f into the set $\{(z,z')\in T\times T: z\neq z'\}$, and it verifies $I(f)=I(\tilde{f})$, where $\tilde{f}(z,z')=\frac{1}{2}(f(z,z')+f(z',z))$. We are going to recall some known facts about the distribution of I(f).

Consider an orthonormal basis $\{\psi_k\}_{k=1}^{\infty}$ of $L^2(T)$ and form the development $\tilde{f}(z,z') = \Sigma_{j,k=1}^{\infty} a_{jk}\psi_j(z)\psi_k(z')$ of the symmetric function $\tilde{f}$. Then, $X_k = \int_T \psi_k dW$ is a sequence of independent standard Gaussian random variables, and we have

<u>Proposition 1.1</u>.- The sequence $\Sigma_{j,k=1}^{n} a_{jk}X_jX_k - \Sigma_{j=1}^{n} a_{jj}$ converges in quadratic mean to I(f).

Proof: It follows easily from the equalities $I(\psi_j\psi_k)=(\int_T \psi_j dW)(\int_T \psi_k dW)-\delta_{jk}$. □

Now consider the Hilbert-Schmidt operator K on $L^2(T)$ given by the symmetric kernel $\tilde{f}(z,z')$. De note by $\{\mu_k\}_{k=1}^{N}$ ($N<\infty$ or $N=\infty$) the sequence of non zero eigenvalues of K (including multiplicities), and let $\{\phi_k\}_{k=1}^{N}$ be a sequence of orthonormal eigenfunctions of K.

Proposition 1.2.- $I(f)$ has the law of the sum $\Sigma_{k=1}^{N} \mu_k(\xi_k^2-1)$ where $\{\xi_k\}_{k=1}^{N}$ is a sequence of independent standard Gaussian random variables. In particular, the characteristic function of $I(f)$ can be expressed in terms of a modified Fredholm determinant (see Varberg [10]):

$$E(e^{itI(f)}) = (\delta(2it,\tilde{f}))^{-1/2} = \prod_{k=1}^{N} (1-2it\mu_k)^{-1/2}\, e^{-it\mu_k} . \qquad (1.1)$$

Proof: Apply proposition 1.1 to the development $\tilde{f}(z,z')=\Sigma_{k=1}^{N} \mu_k\phi_k(z)\phi_k(z')$. □

In the sequel we will use these results to find the distribution of $I(1_D) = \int_T\int_T 1_D dWdW$, being D the set of points $((x,y),(x',y'))$ in $T\times T$ such that $x \leq x'$ and $y \geq y'$. For any integers j and k set $\alpha_{jk}=(\pi^2(2j-1)(2k-1))^{-1}$.

Proposition 1.3.- There exists a sequence $\{X_{jk},\ j,k\in Z\}$ of independent standard Gaussian random variables such that

$$I(1_D) = \Sigma_{j,k\in Z}\, \alpha_{jk}X_{jk}^2 + 8(\Sigma_{j\geq 1,k\in Z}\, \alpha_{jk}X_{jk})^2 - \frac{1}{4} . \qquad (1.2)$$

Proof: Consider the orthonormal basis of $L^2(T)$ formed by the family of trigonometric functions $\sqrt{2}\sin((2j-1)\pi x+(2k-1)\pi y)$, $\sqrt{2}\cos((2j-1)\pi x+(2k-1)\pi y)$, j,k integers such that $j\geq 1$. Set $G = \{(z,z')\in T\times T\colon (z,z')\in D \text{ or } (z',z)\in D\}$. Then $I(1_D) = \frac{1}{2} I(1_G)$, and the symmetric function $\frac{1}{2} 1_G$ has the following development

$$\frac{1}{2} 1_G((x,y),(x',y')) = 8\, [\Sigma_{j\geq 1,k\in Z}\, \alpha_{jk} \sqrt{2}\cos((2j-1)\pi x+(2k-1)\pi y)]$$
$$[\Sigma_{j\geq 1,k\in Z}\, \alpha_{jk} \sqrt{2}\cos((2j-1)\pi x'+(2k-1)\pi y')] +$$
$$\Sigma_{j\geq 1,k\in Z}\, \alpha_{jk}[2\cos((2j-1)\pi x+(2k-1)\pi y)\cos((2j-1)\pi x'+(2k-1)\pi y') +$$
$$2\sin((2j-1)\pi x+(2k-1)\pi y)\sin((2j-1)\pi x'+(2k-1)\pi y')] . \qquad (1.3)$$

Formula (1.3) can be checked by taking the orthonormal basis $\{e^{i(2k-1)\pi x},\ k\in Z\}$ in $L^2([0,1])$ and computing the coefficients of the Fourier expansion

$$\frac{1}{2}\, 1_G = \Sigma_{j,k,j',k'\in Z}\, \lambda_{jkj'k'}\, e^{-i\pi[(2j-1)x+(2k-1)y+(2j'-1)x'+(2k'-1)y']} .$$

The values of these coefficients are

$$\lambda_{jkj'k'} = 4(\pi^4(2j-1)(2k-1)(2j'-1)(2k'-1))^{-1} \quad \text{if} \quad j+j'\neq 1 \text{ or } k+k'\neq 1,$$

and

$$\lambda_{j,k,1-j,1-k} = 4(\pi^4(2j-1)^2(2k-1)^2)^{-1} + (\pi^2(2j-1)(2k-1))^{-1}.$$

Define

$$X_{jk} = \int_T \sqrt{2}\, \cos((2j-1)\pi x+(2k-1)\pi y)\, dW_{xy} \quad \text{for} \quad j\geq 1,\ k\in Z, \text{ and}$$

$$X_{jk} = \int_T \sqrt{2}\, \sin((2j-1)\pi x+(2k-1)\pi y)\, dW_{xy} \quad \text{for} \quad j\leq 0,\ k\in Z.$$

Then, (1.2) is a consequence of (1.3), using proposition 1.1 and noting that $\Sigma_{j\geq 1,\, k\in Z}\, 8\, \alpha_{jk}^2 = \frac{1}{4}$. □

Proposition 1.4.- The characteristic function of the random variable $I(1_D)$ has the following expression

$$E\,(e^{itI(1_D)}) =$$

$$e^{-it/4}\left[\prod_{k=1}^{\infty} \cos\frac{it}{(2k-1)\pi}\right]^{-1}\left[1 - 4\Sigma_{k=1}^{\infty}\frac{1}{(2k-1)\pi}\tan\frac{it}{(2k-1)\pi}\right]^{-1/2} . \qquad (1.4)$$

Proof: From proposition 1.3 we obtain the decomposition $I(1_D)=J_1+J_2$, where $J_1 = \Sigma_{j\leq 0,k\in Z}\, \alpha_{jk}X_{jk}^2 - \frac{1}{4}$ and $J_2 = \Sigma_{j\geq 1,k\in Z}\, \alpha_{jk}X_{jk}^2 + 8(\Sigma_{j\geq 1,k\in Z}\, \alpha_{jk}X_{jk})^2$ are independent random variables. Then

$$E(e^{itJ_1}) = e^{-it/4} \prod_{j\leq 0}\prod_{k\in Z} (1-2it\alpha_{jk})^{-1/2} =$$

$$e^{-it/4}\left[\prod_{k=1}^{\infty} \cos\frac{it}{(2k-1)\pi}\right]^{-1/2} . \qquad (1.5)$$

In order to compute the characteristic function of J_2, we put

$$J_2 = \Sigma_{j,k=1}^{\infty}\, \alpha_{jk}(X_{jk}^2 - X_{j,-k}^2) + 8(\Sigma_{j,k=1}^{\infty}\, \alpha_{jk}(X_{jk}-X_{j,-k}))^2 =$$

$$\Sigma_{j,k=1}^{\infty} M_{jk}N_{jk} + 8\ (\Sigma_{j,k=1}^{\infty} N_{jk})^2,$$

where M_{jk} and N_{jk} are independent random variables with distribution $N(0,2)$ and $N(0,2\alpha_{jk}^2)$, respectively. Thus,

$$E(e^{itJ_2}) = \lim_{N\to\infty} E\,[\exp(8it(\Sigma_{j,k=1}^{N} N_{jk})^2 - t^2\Sigma_{j,k=1}^{N} N_{jk}^2)] =$$

$$\lim_{N\to\infty} \int_{R^{2N}} \exp(8it(\Sigma_{j,k=1}^{N} x_{jk})^2 - \frac{1}{2}\Sigma_{j,k=1}^{N} x_{jk}^2(2t^2 + \frac{1}{2}\alpha_{jk}^{-2})) \prod_{j,k=1}^{N} \frac{dx_{jk}}{\sqrt{4\pi\alpha_{jk}^2}} =$$

$$(\prod_{k=1}^{\infty} \cos\frac{it}{(2k-1)\pi})^{-1/2}(1 - 4\,\Sigma_{k=1}^{\infty} \frac{1}{(2k-1)\pi} \tan\frac{it}{(2k-1)\pi})^{-1/2}. \qquad (1.6)$$

Finally, (1.4) follows from (1.5) and (1.6). □

Note that if we write $g(t) = (\prod_{k=1}^{\infty} \cos\frac{it}{(2k-1)\pi})^{-2}$, then $E(e^{itI(1_D)}) = e^{-it/4}(g(t)-2ig'(t))^{-1/2}$. Unfortunately, as far as we know, there is not a simpler or more reduced expression for the function g.

Although we have already obtained an infinite product expansion for the characteristic function of $I(1_D)$, it may be interesting to exhibit the eigenvalues and the eigenfunctions of the integral operator K on $L^2(T)$ with kernel $\frac{1}{2} 1_G$. Observe that they are given by the partial differential equation

$$\lambda \frac{\partial^2\psi}{\partial x\partial y} = -\psi(x,y), \quad \psi(0,0)=\psi(1,1)=0, \quad \psi(0,y)+\psi(1,y)=\psi(x,0)+\psi(x,1)=1.$$

If a function $\psi \in L^2(T)$ has the Fourier development

$$\psi(x,y) = \Sigma_{j\geq 1, k\in Z}[x_{jk}\sqrt{2}\cos((2j-1)\pi x+(2k-1)\pi y) + x_{-j,k}\sqrt{2}\sin((2j-1)\pi x+(2k-1)\pi y)],$$

then, using the expansion (1.3), we see that the equation $K\psi = \lambda\psi$ is equivalent to the system of equations

$$\begin{cases} 8\,\alpha_{jk}\left(\Sigma_{j\geq 1,k\in Z}\,\alpha_{jk}x_{jk}\right) = (\lambda - \alpha_{jk})x_{jk} \\ \alpha_{jk}x_{-j,k} = \lambda x_{-j,k}\,, \end{cases}$$

for all $j\geq 1$, $k\in Z$.

From these equations we can deduce the next results, which could also have been derived in a direct form from (1.4) and (1.1).

a) For any integer h put $A_h = \{(j,k)\in Z^2:\ \alpha_{jk}=(\pi^2(2h-1))^{-1}\}$ and denote by m_h the cardinal of A_h. Then, the numbers $(\pi^2(2h-1))^{-1}$, $h\in Z$, are eigenvalues of K, each one with multiplicity m_h-1. The invariant subspace associated to $\lambda=(\pi^2(2h-1))^{-1}$ is $\{\psi\in L^2(T):\ x_{jk}=0$ for all $(j,k)\notin A_h$, and $\Sigma_{(j,k)\in A_h, j\geq 1}\, x_{jk}=0\}$.

b) The rest of eigenvalues have multiplicity one and are the solutions of the equation $\frac{1}{8} = \Sigma_{j\geq 1,k\in Z}\,\alpha_{jk}^2/(\lambda-\alpha_{jk})$, which equivalent to

$$\Sigma_{k=1}^{\infty}\ \frac{1}{(2k-1)\pi}\tan\frac{1}{2\lambda(2k-1)\pi} = \frac{1}{4}\,. \tag{1.7}$$

There is exactly one solution of this equation in every one of the open intervals (π^{-2},∞) and $(\pi^{-2}(2k+1)^{-1},\ \pi^{-2}(2k-1)^{-1})$, $(-\pi^{-2}(2k-1)^{-1},\ -\pi^{-2}(2k+1)^{-1})$, $k=1,2,\ldots$ Unlike the first ones, these eigenvalues are not symmetrically placed about the origin, and can only be evaluated aproximately. For instance, using the development of the function $f(\lambda) = \Sigma_{k=1}^{\infty}\ \frac{1}{(2k-1)\pi}\tan\frac{1}{2\lambda(2k-1)\pi}$ in the interval (π^{-2},∞), we can find the aproximate value 0.276203 for the maximum eigenvalue.

2. Consider the two-parameter continuous processes defined by $K_{st} = \int_{R_{st}} WdW$ and $J_{st} = \int_{R_{st}} \int_{R_{st}} 1_D dWdW$, $(s,t) \in T$, where $R_{st} = [0,s] \times [0,t]$. It is known that $W_{st}^2 = 2K_{st} + 2J_{st} + st$. The processes J_{st} and K_{st} are two-parameter martingales with respect to the natural filtration of W, and K_{st} is a strong martingale (cf. [3]).

Let $\{X^n(t),\ t \in [0,1],\ n \geq 1\}$ and $\{Y^n(t),\ t \in [0,1],\ n \geq 1\}$ be two independent infinite dimensional Brownian motions. We know (cf. [7]) that $n^{-1/2} \Sigma_{i=1}^n X_s^i Y_t^i$ converges weakly to W.

<u>Lemma 2.1</u>.- The sequences of two-parameter continuous processes

$$S_{st}^n = n^{-1} \Sigma_{i,j=1}^n (\int_0^s X^i dX^j)(\int_0^t Y^i dY^j), \quad \text{and} \tag{2.1}$$

$$T_{st}^n = n^{-1} \Sigma_{i,j=1}^n (\int_0^s X^i dX^j)(\int_0^t Y^j dY^i) \tag{2.2}$$

converge weakly to the processes K_{st} and J_{st}, respectively.

Proof: We shall only prove the convergence of T_{st}^n to J_{st}, and the other statement has a similar demonstration. Put $t_k^n = k2^{-n}$ for $k=0,1,\dots,2^n$ and $n \geq 1$. Define

$$J_{st}^n = \Sigma_{h,k=0}^{2^n-1} [W(t_{h+1}^n \wedge s, t_k^n \wedge t) - W(t_h^n \wedge s, t_k^n \wedge t)] \cdot$$
$$[W(t_h^n \wedge s, t_{k+1}^n \wedge t) - W(t_h^n \wedge s, t_k^n \wedge t)] \quad , \quad \text{and}$$

$$T_{st}^{nm} = n^{-1} \Sigma_{i,j=1}^n \Sigma_{h,k=0}^{2^m-1} X^i(t_h^m \wedge s)[X^j(t_{h+1}^m \wedge s) - X^j(t_h^m \wedge s)] Y^j(t_k^m \wedge t) \cdot$$
$$[Y^j(t_{k+1}^m \wedge t) - Y^j(t_k^m \wedge t)] \ .$$

For any $m \geq 1$, the sequence T^{nm} converges weakly to J^n as n tends to infinity. Also, the following convergence holds

$$\sup_n E(|T_{st}^{nm} - T_{st}^n|^2) \xrightarrow[m \to \infty]{} 0, \tag{2.3}$$

for all $(s,t) \in T$. In fact,

$$E(|T^{nm}_{st} - T^{n}_{st}|^2) = n^{-2} \Sigma^{n}_{i,j=1} E(|\Sigma^{2^m-1}_{h,k=0} X^i(t^m_h \wedge s)[X^j(t^m_{h+1}\wedge s)-X^j(t^m_h\wedge s)]$$

$$Y^j(t^m_k \wedge t)[Y^i(t^m_{k+1}\wedge s)-Y^i(t^m_k\wedge t)] - (\int_0^s X^i dX^j)(\int_0^t Y^j dY^i)|^2) \leq$$

$$2n^{-2} \Sigma^{n}_{i,j=1}[E((\Sigma^{2^m-1}_{h=0} X^i(t^m_h\wedge s)[X^j(t^m_{h+1}\wedge s)-X^j(t^m_h\wedge s)] - \int_0^s X^i dX^j)^2 \cdot$$

$$(\Sigma^{2^m-1}_{k=0} Y^j(t^m_k\wedge t)[Y^i(t^m_{k+1}\wedge t)-Y^i(t^m_k\wedge t)])^2 + E((\Sigma^{2^m-1}_{k=0} Y^j(t^m_k\wedge t)\cdot$$

$$[Y^i(t^m_{k+1}\wedge t)-Y^i(t^m_k\wedge t)] - \int_0^t Y^j dY^i)^2 (\int_0^s X^i dX^j)^2)] \leq$$

$$2Kn^{-2}[(n^2-n)(E[|\Sigma^{2^m-1}_{h=0} B^1(t^m_h)[B^2(t^m_{h+1})-B^2(t^m_h)] - \int_0^1 B^1 dB^1|^4])^{1/2} +$$

$$n(E[|\Sigma^{2^m-1}_{h=0} B^1(t^m_h)[B^1(t^m_{h+1})-B^1(t^m_h)] - \int_0^1 B^1 dB^1|^4])^{1/2}] \quad ,$$

where

$$K^2 = \sup_{m\geq 1} \{E(|\Sigma^{2^m-1}_{h=0} B^1(t^m_h)[B^2(t^m_{h+1})-B^2(t^m_h)]|^4),$$

$$E(|\Sigma^{2^m-1}_{h=0} B^1(t^m_h)[B^1(t^m_{h+1})-B^1(t^m_h)]|^4), E(|\int_0^1 B^1 dB^2|^4),\ E(|\int_0^1 B^1 dB^1|^4)\} \ ,$$

and B^1, B^2 are two independent standard Brownian motions. Therefore, (2.3) is proved. In addition, we have

$$E(|J^n_{st} - J_{st}|^2) \xrightarrow[n\to\infty]{} 0, \qquad (2.4)$$

for all $(s,t)\in T$.

Using Cairoli-Doob's maximal inequalities for two-parameter martingales, the convergences (2.3) and (2.4) can be transformed into

$$\sup_n E(\sup_{s,t} |T^{nm}_{st} - T^n_{st}|^2) \xrightarrow[m\to\infty]{} 0, \qquad (2.5)$$

and

$$E(\sup_{s,t} |J^n_{st} - J_{st}|^2) \xrightarrow[n\to\infty]{} 0. \qquad (2.6)$$

Let d be a metric on the set of all probabilities on C(T) which induces the weak convergence, and such that $d(\mathcal{L}(X),\mathcal{L}(Y) \leq E(\sup_{s,t} |X_{st} - Y_{st}|)$ for any C(T)-valued random variables X and Y. Here $L(X)$ stands for the distribution of X. Then

$$d(L(T^n),L(J)) \leq \sup_n E(\sup_{s,t}|T^n_{st}-T^{nm}_{st}|) + d(L(T^{nm}),L(J^m)) + $$
$$+ E(\sup_{s,t}|J^n_{st}-J_{st}|)$$

converges to zero as $n\to\infty$, and the lemma is proved. □

Lemma 2.2.- For all (s,t) in T, the random variables K_{st}, stK_{11}, J_{st} and stJ_{11} are identically distributed.

Proof: An invariance property for the Wiener process states that for any $a>0$ and $b>0$, $\{\sqrt{ab}\, W(s/a,t/b),\ (s,t)\in T\}$ has the law of a two-parameter Wiener process. Therefore, it suffices to show that K_{11} and J_{11} have the same distribution. To do this, set $\Delta_{ij}=(i2^{-n},(i+1)2^{-n}]\times(j2^{-n},(j+1)2^{-n}]$ for $0\leq i,j\leq 2^n-1$, $n\geq 1$, and $W(\Delta_{ij}) = W((i+1)2^{-n},(j+1)2^{-n})-W(i2^{-n},(j+1)2^{-n})-W((i+1)2^{-n},j2^{-n})+W(i2^{-n},j2^{-n})$. Then, we have in the L^2 sense $K_{11}=\lim_n \sum_{\substack{i'<i\\ j'<j}} W(\Delta_{ij})W(\Delta_{i'j'})$ and $J_{11}=\lim_n \sum_{\substack{i'<i\\ j'<j}} W(\Delta_{ij})W(\Delta_{i'j'})$, which implies the assertion of the lemma. □

Next we will use lemma 2.1 to obtain some information about the moments $m_p=E(K^p_{11}) = E(J^p_{11})$, $p\geq 1$. For $i,j=1,\dots,p$ define

$$\xi_{ij}=(\int_0^1 X^i dX^j)(\int_0^1 Y^i dY^j) + (\int_0^1 X^j dX^i)(\int_0^1 Y^j dY^i), \qquad (2.7)$$

and

$$\mu_p=E(\xi_{12}\xi_{23}\dots\xi_{p-1,p}\,\xi_{p,1}). \qquad (2.8)$$

Proposition 2.1.- For all $p \geq 1$, we have

$$m_p = \Sigma_{k=1}^{p} \frac{(p-1)!}{2(p-k)!} \mu_k \, m_{p-k} \ . \tag{2.9}$$

Proof: Set

$$E((S_{11}^n)^p) = n^{-p} \Sigma_{(i,j)\in\{1,\dots,n\}^p\times\{1,\dots,n\}^p} [E(\prod_{k=1}^{p} \int_0^1 X^{i_k} dX^{j_k})]^2 . \tag{2.10}$$

In this sum all terms vanish except those corresponding to multi-indexes (i,j) such that for any $m \in \{1,\dots,n\}$ there is an even number of indexes equal to m. Denote by ν_{ij} the number of different integers appearing in the multi-index (i,j). Then, for all $k=1,\dots,p$, the sum of the terms with multi-indexes verifying $\nu_{ij} = k$ is of order n^k. Therefore, if G_p denotes the set of permutations of the numbers $1,1,2,2,\dots,p,p$, we obtain

$$\lim_n E((S_{11}^n)^p) = \frac{1}{p!} \Sigma_{(i,j)\in G_p} [E(\prod_{k=1}^{p} \int_0^1 X^{i_k} dX^{j_k})]^2 , \tag{2.11}$$

where (i,j) represents the permutation $(i_1,j_1,i_2,j_2,\dots,i_p,j_p)$. In view of lemma 2.1, (2.11) is the value of m_p. Two permutations (i,j) and (i',j') of G_p such that $i_k = j_h \Leftrightarrow i'_k = j'_h$ for any $k,h=1,\dots,p$, will be called equivalent and they give rise to identical terms in the sum (2.11). If Q_p stands for the quotient set, we have

$$m_p = \Sigma_{(i,j)\in Q_p} [E(\prod_{k=1}^{p} \int_0^1 X^{i_k} dX^{j_k})]^2 . \tag{2.12}$$

Observe that the cardinal of Q_p is $(2p)!/p!2^p$. A permutation of G_p will be called irreductible if it cannot be a product of cycles. All permutations equivalent to an irreductible one are also irreductibles. Denote by $I_p \subset Q_p$ the set of equivalence classes of irreductible permutations and define

$$n_p = \Sigma_{(i,j)\in I_p} [E(\prod_{k=1}^{p} \int_0^1 X^{i_k} dX^{j_k})]^2 . \tag{2.13}$$

Then,

$$n_p = \Sigma_{k=1}^{p} \binom{p-1}{k-1} n_k m_{p-k}, \quad \text{for all } p \geqslant 1, \tag{2.14}$$

with the convention $n_0 = m_0 = 1$.

Finally, if we set $\int_0^1 X^i dX^{i+1} = \xi_i^+$ and $\int_0^1 X^{i+1} dX^i = \xi_i^-$, for $i=1,\dots,p$, with the assumption $p+1=1$, it can be shown that

$$n_p = \frac{1}{2} (p-1)! \ \Sigma_{\varepsilon \in \{+,-\}^p} \left[E(\xi_1^{\varepsilon_1} \dots \xi_p^{\varepsilon_p}) \right]^2 = \frac{1}{2} (p-1)! \, \mu_p, \tag{2.15}$$

which completes the proof of the proposition. □

Remarks:

1. Suppose that for any $\varepsilon \in \{+,-\}^p$ we define the set A_ε of points x in $[0,1]^p$ such that $x_i \varepsilon_i x_{i+1}$ for all $i=1,\dots,p$ (with the convention $p+1=1$), where the symbols + and - mean $\leqslant$ and $\geqslant$, respectively. Then, using the formal rules $E(X_u^i X_v^i) = u \wedge v$, $E(X_u^i dX_v^i) = 1_{\{u>v\}} dv$ and $E(dX_u^i dX_v^i) = 1_{\{u=v\}} du$, it can be seen that $\mu_p = \sum_\varepsilon |A_\varepsilon|^2$, where $|A_\varepsilon|$ is the Lebesgue measure of A_ε.

2. The expectations $E(\xi_1^{\varepsilon_1} \dots \xi_p^{\varepsilon_p})$ can be cumputed recursively by means of Itô's formula. Indeed, if we define $J_\varepsilon = \{i: \varepsilon_i = + \text{ and } \varepsilon_{i+1} = -\}$, then

$$E(\xi_1^{\varepsilon_1} \dots \xi_p^{\varepsilon_p}) =$$

$$\frac{1}{p} \Sigma_{i \in J_\varepsilon} \left[E(\xi_1^{\varepsilon_1} \dots \xi_{i-1}^{\varepsilon_{i-1}} \xi_i^+ \xi_{i+2}^{\varepsilon_{i+2}} \dots \xi_p^{\varepsilon_p}) + E(\xi_1^{\varepsilon_1} \dots \xi_{i-1}^{\varepsilon_{i-1}} \xi_i^- \xi_{i+2}^{\varepsilon_{i+2}} \dots \xi_p^{\varepsilon_p}) \right].$$

Using this algorithm it is nor hard to evaluate the first moments of J_{11}. For instance, $\mu_1 = m_1 = 0$; $\mu_2 = \frac{1}{2}$, $m_2 = \frac{1}{4}$; $\mu_3 = \frac{1}{6}$, $m_3 = \frac{1}{6}$; $\mu_4 = \frac{7}{96}$, $m_4 = \frac{23}{48}$; $\mu_5 = \frac{37}{720}$, $m_5 = \frac{31}{30}$.

3. The following expression for the characteristic function of J_{11} can be deduced from proposition 2.1,

$$E(e^{itJ_{11}}) = \exp(\Sigma_{k=1}^{\infty} \frac{\mu_k i^k t^k}{2k}) .$$

Set $\phi = \frac{1}{2}(\xi^2-1)$, where ξ is a random variable with law $N(0,1)$. An argument similar to that used in the proof of proposition 2.1 shows that $E(\phi^p) = \Sigma_{(i,j)\in Q_p} E(\prod_{k=1}^{p} \int_0^1 X^{i_k} dX^{j_k})$. Then, the following inequalities hold

$$E((\phi/2)^p) \leq n_p \leq E(\phi^p). \qquad (2.16)$$

In fact, to verify the second inequality observe that the terms in the sum (2.12) are less or equal than one. For the first inequality note that $J_{11}+K_{11}$ has the same law as ϕ . As a consequence of (2.16), J_{11} has finite exponential moments $E(e^{tJ_{11}})$ for $t<1$. The next corollary shows that really $E(e^{tJ_{11}})<\infty$ for $t<\sqrt{2}$.

Corollary 2.1.- $\mu_p \leq 2^{-p/2}$ if p is even, and $\mu_p \leq C2^{-p/2}$ if p is odd, being $C=\frac{2}{3}\sqrt{11}$. Moreover,

$$E(e^{tJ_{11}}) \leq 2^{1/4}(\sqrt{2}+t)^{(C-1)/4}(\sqrt{2}-t)^{-(C+1)/4} . \qquad (2.17)$$

Proof: If p is even, the first statement is an immediate consequence of Schwarz inequality,

$$\mu_p = E[(\xi_{12}\xi_{34} \cdots)(\xi_{23}\xi_{45} \cdots)] \leq$$

$$\leq [E[(\xi_{12}\xi_{34} \cdots)^2]E[(\xi_{23}\xi_{45} \cdots)^2]]^{1/2} = 2^{-p/2} .$$

For $p=2q+1$, $q\geq 0$, we apply Itô's formula and Schwarz inequality,

$$\mu_p = \Sigma_{i=1}^{p} \int_0^1 E[\xi_{12}(t)\ldots\xi_{i-1,i}(t)\xi_{i+2,i+3}(t)\ldots\xi_{p1}(t)(X_t^i X_t^{i+2}\int_0^1 Y_s^i Y_s^{i+2} ds +$$

$$+ Y_t^i Y_t^{i+2} \int_0^1 X_s^i X_s^{i+2} ds)]\, dt =$$

$$= E[\xi_{12} \cdots \xi_{i-1,i}\xi_{i+2,i+3} \cdots \xi_{p1}(X_1^i X_1^{i+2}\int_0^1 Y_s^i Y_s^{i+2} ds + Y_1^i Y_1^{i+2}\int_0^1 X_s^i X_s^{i+2} ds)] \leqslant$$

$$\leqslant [\,(E(\xi_{12}^2))^{2q-1}\; E\,[\,(X_1^i X_1^{i+2}\int_0^1 Y_s^i Y_s^{i+2} ds + Y_1^i Y_1^{i+2}\int_0^1 X_s^i X_s^{i+2} ds)^2]]^{1/2} =$$

$$= 2^{-p/2}\,\frac{2}{3}\sqrt{11}.$$

Finally, (2.17) follows immediately from the preceeding inequalities. □

3. Consider the sequence of random variables $U_n = n^{-1} \Sigma_{i,j=1}^{n} \int_0^1 X^i dX^j \int_0^1 Y^i dY^j$. It is clear that U_n converges in distribution to J_{11}. Indeed, $U_n = S_{11}^n - n^{-1} \Sigma_{i=1}^{n} \int_0^1 X^i dX^i \int_0^1 Y^i dY^i$. For each $n \geqslant 2$ and $j=2,\ldots,n$ define

$$X_{nj} = n^{-1} \Sigma_{i=1}^{j-1} \xi_{ij}, \tag{3.1}$$

and let F_{nj}, $j=1,\ldots,n$, be the σ-field generated by the processes $X_1,\ldots,X_j$, $Y_1,\ldots,Y_j$. Then X_{nj} is a martingale array, that means, $X_{n2},\ldots,X_{nj}$ are F_{nj}-measurable and $E(X_{nj}/F_{n,j-1})=0$, for $j=2,\ldots,n$. Furthermore $\Sigma_{j=2}^{n} X_{nj} = U_n$. Set $V_{nj}^2 = \Sigma_{i=2}^{j} E(X_{ni}^2 / F_{n,i-1})$, and $V_n^2 = V_{nn}^2$.

<u>Lemma 3.1</u>.- The martingale array X_{nj} satisfies the conditional Lindeberg condition (0.1).

Proof: Put $Z_j = \Sigma_{i=1}^{j-1} \xi_{ij}$. Then,

$$E(Z_j^4) \leqslant \text{const.}\{\; E\,[\,(\Sigma_{i=1}^{j-1}(\int_0^1 X^j dX^i)^2(\int_0^1 Y^j dY^i)^2)^2] +$$

$$+ E[(\Sigma_{i=1}^{j-1}(\int_0^1 X^i dX^j)^2(\int_0^1 Y^i dY^j)^2)^2] = \text{const. } j^2 + o(j).$$

Therefore,

$$n^{-2}\, \Sigma_{j=2}^n\, E(Z_j^2\, 1_{\{|Z_j|>n\varepsilon\}}) \leq n^{-4}\varepsilon^{-2}\, \Sigma_{j=2}^n\, E(Z_j^4) \longrightarrow 0,$$

as $n \to \infty$, for all $\varepsilon > 0$.

If this Lindeberg condition holds, Hall [5] and Rootžen [8] have shwon that $V_n^2 \xrightarrow{P} \eta$ with $P\{\eta > 0\} = 1$ implies that U_n converges in distribution to a mixture of normal distributions with characteristic function $E(\exp(-\frac{1}{2}t^2\eta))$. If there is only convergence in distribution of the sequence V_n^2, this result may fail as it has been proved by a counterexample of Dvoretsky [4] . Also, Alvo, Cabilio and Feigin [1] exhibit a class of martingales, which are degenerate U-statistics, and such that the sequence U_n of row sums converge in distribution to a weighted sum of chi-squared independent random variables as long as the sequence of conditional variances converges in law. The next result shows that the martingale array (3.1) satisfies these same properties.

<u>Proposition 3.1</u>.- The sequence V_n^2 converges in law to the random variable

$$\int_{[0,1]^3} (W_{stu} + \tilde{W}_{stu})^2 ds dt du, \tag{3.2}$$

where $\tilde{W}_{stu} = W_{11u} - W_{1tu} - W_{s1u} + W_{stu}$, and $\{W_{stu},\ (s,t,u) \in [0,1]^3\}$ is a zero mean continuous Gaussian process with covariance function $E(W(s_1,t_1,u_1) \cdot W(s_2,t_2,u_2)) = (s_1 \wedge s_2)(t_1 \wedge t_2)(u_1 \wedge u_2)$.

Proof: Compute

$$E(X_{nj}^2/F_{n,j-1}) =$$

$$\Sigma_{i=1}^{j-1}(\int_0^1(X_1^i-X_u^i)^2du)(\int_0^1(Y_1^i-Y_u^i)^2du + \Sigma_{i=1}^{j-1}(\int_0^1(X_u^i)^2du)(\int_0^1(Y_u^i)^2du) +$$

$$2\ \Sigma_{i,i'=1}^{j-1}(\int_0^1(X_1^i-X_u^i)(X_1^{i'}-X_u^{i'})du)(\int_0^1(Y_1^i-Y_u^i)(Y_1^{i'}-Y_u^{i'})du) +$$

$$2\ \Sigma_{i,i'=1}^{j-1}(\int_0^1 X_u^iX_u^{i'}du)(\int_0^1 Y_u^iY_u^{i'}du) +$$

$$2\ \Sigma_{i,i'=1}^{j-1}(\int_0^1(X_1^i-X_u^i)X_u^{i'}du)(\int_0^1(Y_1^i-Y_u^{i'})Y_u^{i'}du =$$

$$\int_0^1\int_0^1 (\Sigma_{i=1}^{j-1}\ (X_1^i-X_s^i)(Y_1^i-Y_t^i) + X_s^iY_t^i)^2\ dsdt.$$

Then,

$$V_n^2 = n^{-2}\int_0^1\int_0^1 \Sigma_{j=1}^{n-1}\ (\Sigma_{i=1}^{j}(X_1^i-X_s^i)(Y_1^i-Y_t^i) + X_s^iY_t^i)^2\ dsdt. \qquad (3.3)$$

Denote by D_3 the set of functions from $[0,1]^3$ to R which are continuous from above, with limits from below, and define the D_3-valued processes

$$Z_n(s,t,u) = n^{-1/2}\Sigma_{i=1}^{[nu]}\ X^i(s)Y^i(t). \qquad (3.4)$$

Using theorem 6 of Bickel and Wichura [2], we obtain the weak convergence of the sequence $Z_n(s,t,u)$ to $W(s,t,u)$. Further, the mapping $x(s,t,u) \longrightarrow \int_{[0,1]^3}[x(1,1,u)-x(1,t,u)-x(s,1,u)+x(s,t,u)]^2\ dudsdt$ from D_3 to R is continuous. Therefore, noting that $V_n^2 = \int_{[0,1]^3}[Z_n(1,1,u)-Z_n(1,t,u)-Z_n(s,1,u)+2Z_n(s,t,u)]\ dsdtdu$, the proof of the proposition is complete. □

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