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1.- Introduction. Let C be a plane algebraic curve, defined over the field of complex numbers, and O a singular point of C; we denote P(C) a generic polar of C; it is well known that P(C) passes through O and the intersection multiplicity $(C.P(C))_O$ is given by the term which corresponds to O in the first Plücker's formula.

No much more is known about the local properties of P(C) at O: the geometers of the Italian school claimed to have determined the equisingularity type of P(C) at O from the equisingularity type of C at O, but his statement is false: B. Segre in a memoir of 1952, (S), gave a curve with a multiple infinitely near point that does not belong to the generic polar; this exemple contradicts the Italian statement.

In (M) we find some information about the singularity of the polar curve deduced from the equisingularity type of the original curve. In fact one can expect to determine the equisingularity type of P(C) at O in terms of the equisingularity type of C at O because (as Lê Dũng Tráng showed to me) the equisingularity type of P(C) at O depends on the analytical type of C at O and not only on the equisingularity type.

We consider here the case of irreducible algebroid plane curves (more briefly branches) with only one characteristic exponent. We determine the equisingularity types of the generic polars of these curves and one can observe in particular that the claim of the Italians becomes true for branches with general moduli. I believe that the same holds in general, namely that for any equisingularity class of one--branched algebroid curves there is an open dense subset A in the corresponding moduli space such that for any curve representing a point on A the Italian statement is true.

We fix some terminology. Let $\sum_{\delta,\rho\geq 0} a_{\delta,\rho}x^{\delta}y^{\rho} = 0$ be the equation of a plane algebroid curve. We call Newton-Cramer (briefly N.C.) diagram the set of points on the plane with non-negative integer coordinates (δ,ρ) for which $a_{\delta,\rho} \neq 0$. We consider only N. C. diagrams with some point on each axis. The boundary of the convex hull of the N. C. diagram is a broken line; we call Newton-Cramer polygon the lower-left part of this boundary, i. e., that one whose sides do not contain the origin and leaves it on a halfplane with no points on the N. C. diagram.

<u>2.- The equation of the polar curves</u>. Let Γ be an algebroid one--branched plane curve with one characteristic exponent and defined over the field C of complex numbers. By performing an analitic transformation, if necessary, we may suppose the Puiseux series of Γ written in the form

$$y = S(x) = x^{m/n} + \sum_{i\geq 1} a_i x^{\frac{m+i}{n}}$$

where n is the order and m/n the characteristic exponent of the branch γ. Define

$$S_\nu(x) = \nu^m x^{m/n} + \sum_{i\geq 1} a_i \nu^{m+i} x^{\frac{m+i}{n}} \quad , \quad \nu^n = 1.$$

We can take

$$0 = f(x,y) = \prod_{\nu\in U} (y - S_\nu(x)) \in \mathbb{C}[[x,y]]$$

as implicit equation of γ, where U is the group of n-th roots of unity.

The polar[1] of γ relative to the direction with homogeneous coordinates (λ,μ) is defined by the equation

$$\lambda\frac{\partial f}{\partial x} + \mu\frac{\partial f}{\partial y} = 0$$

or, more explicitly,

$$0 = \lambda\frac{\partial}{\partial x}\left(\prod_{\nu\in U} (y - S_\nu(x))\right) + \mu\frac{\partial}{\partial y}\left(\prod_{\nu\in U} (y - S_\nu(x))\right) =$$

$$= \sum_{\nu\in U}\left(\prod_{\substack{\eta\in U\\ \eta\neq\nu}} (y - S_\eta(x))\right)(\mu - \lambda S'_\nu(x)) =$$

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[1] Or, more clasically, the polar of the improper point $(0,\lambda,\mu)$ with respect to γ. One can consider also polars of proper points, but, as one can easily see, we do not have a more general situation.

$$= \sum_{\nu\in U}\left(\prod_{\substack{\eta\in U\\ \eta\neq\nu}}(y - \eta^m x^{\frac{m}{n}} - \sum_{i\geq 1} a_i \eta^{m+i} x^{\frac{m+i}{n}})\right)\left(-\lambda(\frac{m}{n}\nu^m x^{\frac{m}{n}-1} + \sum_{i\geq 1}\frac{m+i}{n} a_i \nu^{m+i} x^{\frac{m-n+i}{n}}) + \mu\right) \qquad \text{(I)}$$

We determine first the monomials that efectively occur in this equation. Obviously, in despite of its appearance, the equation is rational in x. Take any ρ such that $n - 1 > \rho \geq 0$:

<u>Proposition 1</u>: In the equation (I) we do not have monomials of bidegree (δ,ρ) (on x and y respectively) if $\delta < m(n-1-\rho)/n$. If $\delta \geq m(n-1-\rho)/n$ and if we take $i = i(\delta,\rho) = \delta n - (n-1-\rho)m$, then there occurs in (I) a non--trivial monomial with bidegree (δ,ρ) whose coefficient is a linear function on a_i. Furthermore if $i(\delta,\rho) = i(\delta',\rho')$ we have $(\delta,\rho) = (\delta',\rho')$ and a_j does not occur in the monomials whose bidegree (δ,ρ) verifies $i(\delta,\rho) < j$.

<u>Proof</u>: The order of each $S_\eta(x)$ is m/n, so that the coefficient of y^ρ in (I) has order, as a series of x, not less than $(n-1-\rho)m/n$.

Suppose now $\delta \geq (n-1-\rho)m/n$: from g.c.d.$(n,m) = 1$ and $\rho+1 < n$ we have $(n-1-\rho)m \not\equiv 0$ mod. n which gives $\delta > (n-1-\rho)m/n$; hence $i = \delta n - (n-1-\rho)m > 0$ and we can consider in (I) the sum of the products of μ, y^ρ, $n-\rho-2$ terms $\eta^m x^{m/n}$ and one term $\eta^{m+i} a_i x^{(m+i)/n}$ which gives

$$\mu\left(\sum_{\eta\in U}\sum_{\bar\eta\in U-\{\eta\}}\sum_{\{\eta_1,\dots,\eta_{n-\rho-2}\}}\eta_1^m\cdots\eta_{n-\rho-2}^m\bar\eta^{m+i}\right)a_i x^\delta y^\rho \qquad \text{(II)}$$

where the most inner summation runs over the subsets of $n-\rho-2$

elements of $U - \{\eta,\bar{\eta}\}$. No other monomials of bidegree (δ,ρ) in which occurs a_i come from (I). Therefore we only need to prove that the coefficient in (II) is non-zero. Define the polynomials

$$s^p_{j_1,\dots,j_p} = \sum x_{i_1}^{j_1}\dots x_{i_p}^{j_p}$$

where the summation runs over all the p-uples of different indices $i_k \in \{1,\dots,n\}$; classically $s^p_{j_1,\dots,j_p}$ was called a p-uple symmetric polynomial, see (Sev), number 151. As one can easily verify

$$s^p_{j_1,\dots,j_p}.s^1_j = s^p_{j_1,\dots,j_{p-1},j_p+j}+\dots+s^p_{j_1+j,\dots,j_p}+s^{p+1}_{j_1,\dots,j_p,j}\ .$$

Now the coefficient in (II) is

$$\mu\sum_\eta\sum_{\bar{\eta}}\sum_{\{\eta_1,\dots,\eta_{n-\rho-2}\}}\eta_1^m\dots\eta_{n-\rho-2}^m\bar{\eta}^{m+i} =$$

$$= \frac{\mu(\rho+1)}{(n-\rho-2)!}\sum_{\eta_1,\dots,\eta_{n-\rho-1}}\eta_1^m\dots\eta_{n-\rho-2}^m\eta_{n-\rho-1}^{m+i} =$$

$$= \frac{\mu(\rho+1)}{(n-\rho-2)!}.s^{n-\rho-1}_{m,m,\dots,m,m+i}(\eta_1,\dots,\eta_n)$$

if $\rho<n-2$, $m+i \equiv (\rho+2)m \not\equiv 0$ mod. n, therefore

$$0 = \sum_{\eta\in U}\eta^{m+i} = s^1_{m+i}(\eta_1,\dots,\eta_n)$$

hence

$$0 = s^1_{m+i}(\eta_1,\dots,\eta_n)s^{n-\rho-2}_{m,\dots,m}(\eta_1,\dots,\eta_n) =$$

$$= (n-\rho-2)s^{n-\rho-2}_{m,\dots,m,2m+i}(\eta_1,\dots,\eta_n) + s^{n-\rho-1}_{m,\dots,m,m+i}(\eta_1,\dots,\eta_n)$$

that is

$$\frac{\mu(\rho+1)}{(n-\rho-2)!}s^{n-\rho-1}_{m,\dots,m,m+i}(\eta_1,\dots,\eta_n) =$$

$$= -\frac{\mu(\rho+1)}{(n-\rho-3)!}s^{n-\rho-2}_{m,\dots,m,2m+i}(\eta_1,\dots,\eta_n)$$

If $\rho < n-3$ we repeat this procedure and, after $n-\rho-2$ steps, we obtain finally

$$\dots = (-1)^{n-\rho-2}\mu(\rho+1)s^{1}_{m(n-\rho-1)+i}(\eta_1,\dots,\eta_n)$$

Recalling that $m(n-\rho-1)+i = \delta n$, we have

$$s^{1}_{m(n-\rho-1)+i}(\eta_1,\dots,\eta_n) = n$$

and our coefficient is non-zero.

The last two statements are almost trivial: if $i(\delta,\rho) = i(\delta',\rho')$ we have $\delta n-(n-\rho-1)m = \delta' n-(n-\rho'-1)m$ and in particular $(\rho+1)m \equiv (\rho'+1)m$ mod. n; if we recall that $\rho < n-1$, $\rho' < n-1$ and g.c.d.$(n,m) = 1$, we have $\rho = \rho'$ and thus also $\delta = \delta'$.

The monomial of bidegree (δ,ρ) in (I) is obtained by adding products of y^{ρ}, $n-\rho-1$ factors from the $S_{\eta}(x)$'s and one factor from $(\mu-\lambda S'_{\nu}(x))$. If a_j occurs we have

$$\delta \geq (n-\rho-2)\frac{m}{n} + \frac{m+j}{n}$$

from which

$$j \leq \delta n - (n-\rho-1)m = i(\delta,\rho)$$

and the proof is complete.

Remark: The terms μy^{n-1} and $-m\lambda x^{m-1}$ appear also in (I). the last one comes from adding the products of n-1 factors $\eta^m x^{m/n}$ from the $S_\eta(x)$'s and one factor $\frac{m}{n}\nu^m \lambda x^{(m-n)/n}$ from $S'_\nu(x)$. There are no other terms with bidegree (0,n-1), and no other terms with bidegree (m-1,0) involving λ. Therefore, for any value of the a_i there are non-trivial monomials of bidegrees (0,n-1) and (m-1,0) in the equation of a generic polar curve.

3.- Continous fractions. We recall some facts about continous fractions which will be need later on. For more details see (Sev) Chap. VII, §5.20. We shall consider continous fractions of the form

$$h_o + \cfrac{1}{h_1 + \cfrac{1}{h_2 + \ddots + \cfrac{1}{h_s}}}$$

where the h_i are positive integers. We set, if $i > 1$,

$$\frac{p_i}{q_i} = h_o + \cfrac{1}{h_1 + \cfrac{1}{h_2 + \ddots + \cfrac{1}{h_i}}}$$

the so called partial fractions, and we take $p_o = h_o$, $q_o = 1$, $p_{-1} = 1$, $q_{-1} = 0$.

As one can see by using induction, we may take

$$p_{i+1} = h_{i+1}p_i + p_{i-1} \quad , \quad q_{i+1} = h_{i+1}q_i + q_{i-1} \qquad \text{(III)}$$

which gives

$$p_i q_{i-1} - p_{i-1}q_i = (-1)^{i+1} \qquad \text{(IV)}$$

In particular g.c.d.$(p_i, q_i) = 1$ and

$$\frac{p_i}{q_i} - \frac{p_{i-1}}{q_{i-1}} = \frac{(-1)^{i+1}}{q_i q_{i-1}} \qquad \text{(V)}$$

so that we have

$$\frac{p_i}{q_i} = h_o + \frac{1}{q_o q_1} - \frac{1}{q_1 q_2} + \dots + \frac{(-1)^{i+1}}{q_{i-1}q_i} \qquad \text{(VI)}$$

where $1 < q_o q_1 < \dots < q_{i-1}q_i$. In particular if $j < i$

$$0 < \frac{p_i}{q_i} - \frac{p_j}{q_j} \leq \frac{1}{q_j q_{j+1}} \quad \text{if } j \equiv 0 \text{ mod. } 2$$

$$0 < \frac{p_j}{q_j} - \frac{p_i}{q_i} \leq \frac{1}{q_j q_{j+1}} \quad \text{if } j \not\equiv 0 \text{ mod. } 2 \qquad \text{(VII)}$$

<u>4.- Drawing a Newton-Cramer polygon</u>. We shall now analize the singularity of the generic polar curve for a curve with general moduli, i. e., for all the curves γ excepted these ones whose coefficients a_i verify a finite set of algebraic non trivial relations (which will not be explicited). See (Z) for the definition of the moduli space and its topology.

Proposition 1 determines the monomials that occur in the equation of a generic polar of a general γ; in particular if we consider the monomials of degree ρ in y, where $0 \leq \rho < n-1$, their lowest degree in x is the lowest integer not less than $(n-\rho-1)m/n$, i. e., $m - [(\rho+1)m/n]$ because $(\rho+1)m \not\equiv 0$ mod. n. (We denote by square brackets the integral part).

To draw the N.C. polygon we have to consider only the points on the plane

$$P_\rho = \Big(m - [(\rho+1)m/n], \rho\Big) \quad , \ \rho = 0,\ldots,n-1.$$

Note that we include the point $(0,n-1)$ which corresponds to the only monomial of degree $n-1$ in y; thus we have the points $(m - [m/n], 0)$ and $(0,n-1)$ as first and last vertices of the N.C. polygon.

We perform the Euclidian algorithm to compute 1 = = g.c.d.(m.n):

$$m = hn + \sigma_1$$
$$n = h_1\sigma_1 + \sigma_2$$
$$\dots$$
$$\sigma_{s-2} = h_{s-1}\sigma_{s-1} + \sigma_s$$
$$\sigma_{s-1} = h_s\sigma_s$$

and we write m/n as continous fraction:

$$\frac{m}{n} = h + \cfrac{1}{h_1 + \cfrac{1}{h_2 + \ddots + \cfrac{1}{h_s}}}$$

Suppose, as induction hypotesis, that we know that the (j+1)-th vertex of the N.C. polygon is the point

$$P_{q_{2j}-1} = \left(m - \left[\frac{q_{2j}m}{n}\right], q_{2j}-1 \right)$$

Note that if $j = 0$, we have the point $(m-h,0) = (m-[m/n],0)$ and if $2j = s$ we stop because $q_s-1 = n-1$. We suppose $j < s/2$.

To draw the next side of the N.C. polygon we select among the lines $P_\rho P_{q_{2j}-1}$, $q_{2j}-1 < \rho \leq n-1$, the one of maximal slope. We must determine

$$\text{Max}\left(\frac{\left[\frac{(\rho+1)m}{n}\right]-\left[\frac{q_{2j}m}{n}\right]}{\rho-q_{2j}+1}\right), \quad q_{2j}-1 < \rho \leq n-1$$

where we take, instead of the slope, the opposite of his inverse (note that the slopes are all negative). Write

$$t = \rho - q_{2j} + 1$$

thus, we have

$$0 < t \leq n - q_{2j}$$

and we must determine the maximal value of

$$\Delta = \frac{1}{t}\left(\left[(t+q_{2j})\frac{m}{n}\right] - \left[\frac{q_{2j}m}{n}\right]\right).$$

We have

$$\left[q_{2j}\frac{m}{n}\right] = \left[q_{2j}\frac{p_{2j}}{q_{2j}} + q_{2j}\left(\frac{p_s}{q_s} - \frac{p_{2j}}{q_{2j}}\right)\right] =$$

$$= \left[p_{2j} + q_{2j}\left(\frac{p_s}{q_s} - \frac{p_{2j}}{q_{2j}}\right)\right] = p_{2j}$$

where the last equality comes from (VII) §3. Hence

$$\Delta = \frac{1}{t}\left(\left[(t+q_{2j})\frac{m}{n}\right] - p_{2j}\right).$$

Suppose now $2j+1 < s$; we shall consider the case $j = \frac{s-1}{2}$ later on.

$$\frac{m}{n} = \frac{p_s}{q_s} = \frac{p_{2j}}{q_{2j}} + \left(\frac{p_s}{q_s} - \frac{p_{2j}}{q_{2j}}\right) =$$

$$= \frac{p_{2j}}{q_{2j}} + \left(\frac{1}{q_{2j}q_{2j+1}} - \ldots + (-1)^{s+1}\frac{1}{q_{s-1}q_s}\right)$$

by using (VI) §3. Therefore

$$\Delta = \frac{1}{t}\left[(t+q_{2j})\frac{p_{2j}}{q_{2j}} + (t+q_{2j})\left(\frac{1}{q_{2j}q_{2j+1}} - \ldots + (-1)^{s+1}\frac{1}{q_{s-1}q_s}\right) - p_{2j}\right]$$

$$= \frac{1}{t}\left[t\frac{p_{2j}}{q_{2j}} + \frac{t}{q_{2j}q_{2j+1}} + \frac{1}{q_{2j+1}} - (t+q_{2j})\left(\frac{1}{q_{2j+1}q_{2j+2}} - \ldots + (-1)^{s}\frac{1}{q_{s-1}q_s}\right)\right]$$

$$= \frac{1}{t}\left[t\frac{p_{2j+1}}{q_{2j+1}} + \frac{1}{q_{2j+1}} - (t+q_{2j})\left(\frac{1}{q_{2j+1}q_{2j+2}} - \ldots + (-1)^{s}\frac{1}{q_{s-1}q_s}\right)\right].$$

Perform the Euclidean division

$$tp_{2j+1} = cq_{2j+1} + r \ , \ 0 \leq r < q_{2j+1} \ .$$

Now

$$\Delta = \frac{1}{t}\left[c + \frac{r+1}{q_{2j+1}} - (t+q_{2j})\left(\frac{1}{q_{2j+1}q_{2j+2}} - \ldots + (-1)^{s}\frac{1}{q_{s-1}q_s}\right)\right] \leq \frac{c}{t}$$

Because

$$\frac{r+1}{q_{2j+1}} \leq 1$$

and

$$(t+q_{2j})\left(\frac{1}{q_{2j+1}q_{2j+2}} - \ldots + (-1)^{s}\frac{1}{q_{s-1}q_s}\right) > 0$$

(remember that $2j + 2 \leq s$). Also

$$\frac{c}{t} = \frac{1}{t}\left(\frac{tp_{2j+1} - r}{q_{2j+1}}\right) \leq \frac{p_{2j+1}}{q_{2j+1}}$$

so that we have determined a upper bound for Δ. We shall prove now that this upper bound is in fact the wanted maximum. We shall prove also that the maximum is reached if and only if $t = \alpha q_{2j+1}$, $\alpha = 1,\ldots,h_{2j+2}$.

From the last computation it is obvious that the upper bound is not reached if $r > 0$. Suppose $r = 0$: q_{2j+1} divides tp_{2j+1}, hence also t because g.c.d.$(p_{2j+1},q_{2j+1}) = 1$. Thus we take $t = \alpha q_{2j+1}$ and we suppose first $0 < \alpha \leq h_{2j+2}$.

$$\Delta = \frac{1}{\alpha q_{2j+1}}\left[\alpha p_{2j+1} + \frac{1}{q_{2j+1}} - (\alpha q_{2j+1} + q_{2j})\left(\frac{1}{q_{2j+1}q_{2j+2}} - \ldots + \frac{(-1)^s}{q_{s-1}q_s}\right)\right]$$

$$\geq \frac{1}{\alpha q_{2j+1}}\left[\alpha p_{2j+1} + \frac{1}{q_{2j+1}} - (h_{2j+2}q_{2j+1} + q_{2j})\left(\frac{1}{q_{2j+1}q_{2j+2}} - \ldots + \frac{(-1)^s}{q_{s-1}q_s}\right)\right]$$

$$= \frac{1}{\alpha q_{2j+1}}\left[\alpha p_{2j+1} + \frac{1}{q_{2j+1}} - q_{2j+2}\left(\frac{1}{q_{2j+1}q_{2j+2}} - \ldots + \frac{(-1)^s}{q_{s-1}q_s}\right)\right]$$

$$= \frac{1}{\alpha q_{2j+1}}\left[\alpha p_{2j+1} + q_{2j+2}\left(\frac{1}{q_{2j+2}q_{2j+3}} - \ldots + \frac{(-1)^{s+1}}{q_{s-1}q_s}\right)\right]$$

$$= \frac{p_{2j+1}}{q_{2j+1}}$$

because either

$$0 < q_{2j+2}\left(\frac{1}{q_{2j+2}q_{2j+3}} - \ldots + \frac{(-1)^{s+1}}{q_{s-1}q_s}\right) \leq \frac{q_{2j+2}}{q_{2j+2}q_{2j+3}} < 1,$$

if 2j+2 < s, or this term does not exist if 2j+2 = s.

Suppose now $\alpha \geq h_{2j+2}+1$: we prove first

$$(t+q_{2j})\left(\frac{1}{q_{2j+1}q_{2j+2}} - \ldots + \frac{(-1)^s}{q_{s-1}q_s}\right) > \frac{1}{q_{2j+1}} \qquad \text{(VIII)}.$$

In fact

$$(t+q_{2j})\left(\frac{1}{q_{2j+1}q_{2j+2}} - \ldots + \frac{(-1)^s}{q_{s-1}q_s}\right) \geq$$

$$\geq (h_{2j+2}q_{2j+1} + q_{2j+1} + q_{2j})\left(\frac{1}{q_{2j+1}q_{2j+2}} - \ldots + \frac{(-1)^s}{q_{s-1}q_s}\right) =$$

$$= (q_{2j+2} + q_{2j+1})\left(\frac{1}{q_{2j+1}q_{2j+2}} - \ldots + \frac{(-1)^s}{q_{s-1}q_s}\right) \quad ;$$

we distinguish now three cases:

a) 2j + 3 < s:

$$(q_{2j+2} + q_{2j+1})\left(\frac{1}{q_{2j+1}q_{2j+2}} - \ldots + \frac{(-1)^s}{q_{s-1}q_s}\right) >$$

$$> (q_{2j+2} + q_{2j+1})\left(\frac{1}{q_{2j+1}q_{2j+2}} - \frac{1}{q_{2j+2}q_{2j+3}}\right) =$$

$$= \frac{1}{q_{2j+1}} + \frac{q_{2j+3} - q_{2j+1} - q_{2j+2}}{q_{2j+2}q_{2j+3}} \geq$$

$$\geq \frac{1}{q_{2j+1}} + \frac{q_{2j+3} - q_{2j+1} - h_{2j+3}q_{2j+2}}{q_{2j+2}q_{2j+3}} = \frac{1}{q_{2j+1}}$$

b) $2j + 3 = s$: we proceed as in the case a): the first inequality is an equality here, but $h_{2j+3} = h_s > 1$ and hence the second inequality is a strict one.

c) $2j + 2 = s$:

$$(q_{2j+2} + q_{2j+1})\left(\frac{1}{q_{2j+1}q_{2j+2}} - \dots + \frac{(-1)^s}{q_{s-1}q_s}\right) =$$

$$= (q_s + q_{s-1})\frac{1}{q_{s-1}q_s} = \frac{1}{q_{s-1}} + \frac{1}{q_s} > \frac{1}{q_{s-1}}$$

and we have proved (VIII). We have now, by (VIII)

$$\Delta = \frac{1}{\alpha q_{2j+1}}\left[\alpha p_{2j+1} + \frac{1}{q_{2j+1}} - (t+q_{2j})\left(\frac{1}{q_{2j+1}q_{2j+2}} - \dots + \frac{(-1)^s}{q_{s-1}q_s}\right)\right]$$

$$< \frac{1}{\alpha q_{2j+1}}\left[\alpha p_{2j+1}\right] = \frac{p_{2j+1}}{q_{2j+1}}$$

and, as expected, the maximum is not reached.

Thus we know that, if $2j+1 < s$, the greatest value of Δ is p_{2j+1}/q_{2j+1} and it is reached for $\rho = \alpha q_{2j+1}+q_{2j}-1$, $\alpha = 1,\dots,h_{2j+2}$. Consequently the side of the N.C. polygon which starts at $P_{q_{2j}-1}$ has slope $-q_{2j+1}/p_{2j+1}$, passes through the points

$$P_{\alpha q_{2j+1}+q_{2j}-1}\ , \quad \alpha = 1,\dots,h_{2j+2}-1$$

and ends at the point

$$P_{h_{2j+2}q_{2j+1}+q_{2j}-1} = P_{q_{2j+2}-1}$$

according the induction hypotesis. If s is even, this holds for $j = s/2 - 1$ and we reach the last vertex $P_{q_s-1} = P_{n-1}$; the N.C. polygon is completely drawn in this case.

If s is odd, we must consider separately the case $2j = = s-1$, which is not included in the former computation. We will determine the side starting at $P_{q_{s-1}-1}$: now

$$\Delta = \frac{1}{t}\left[(t+q_{s-1})\frac{p_s}{q_s} - p_{s-1}\right]$$

and from

$$1 = p_s q_{s-1} - p_{s-1} q_s \qquad ((IV)\ \S 3)$$

we have

$$\Delta = \frac{1}{t}\left[t\frac{p_s}{q_s} + \frac{p_{s-1}q_s+1}{q_s} - p_{s-1}\right] =$$

$$= \frac{1}{t}\left[t\frac{p_s}{q_s} + \frac{1}{q_s}\right] = \frac{1}{t}\left[\frac{tp_s+1}{q_s}\right]$$

First we note that

$$tp_s + 1 \equiv 0 \ \mathrm{mod.}\ q_s \qquad \text{(IX)}$$

if and only if

$$t \equiv -q_{s-1} \text{ mod. } q_s$$

where we used again (IV), §3. As $0 < t \leq n-q_{2j} = q_s-q_{s-1}$, we have (IX) if and only if $t = q_s - q_{s-1}$.

Take $t = q_s - q_{s-1}$:

$$\Delta = \frac{1}{q_s-q_{s-1}}\left[\frac{q_s p_s - q_{s-1}p_s + 1}{q_s}\right] = \frac{p_s - p_{s-1}}{q_s - q_{s-1}}$$

We claim that this is the maximal value for Δ. If $t < q_s-q_{s-1}$, $(tp_s+1)/q_s$ is not an integer, thus

$$\left[\frac{tp_s + 1}{q_s}\right] \leq \frac{tp_s}{q_s}$$

and hence

$$\frac{1}{t}\left[\frac{tp_s + 1}{q_s}\right] \leq \frac{p_s}{q_s} \tag{X}$$

If we recall (IV) §3 again,

$$-p_s q_{s-1} < -p_{s-1} q_s$$

i. e.

$$p_s q_s - p_s q_{s-1} < p_s q_s - p_{s-1} q_s$$

this is

$$\frac{p_s}{q_s} < \frac{p_s - p_{s-1}}{q_s - q_{s-1}} ,$$

from which, using (X) we conclude

$$\Delta < \frac{p_s - p_{s-1}}{q_s - q_{s-1}}$$

for $t < q_s - q_{s-1}$. Thus we have a new side of the N.C. polygon which starts at $P_{q_{s-1}-1}$. Note that if $t = q_s - q_{s-1}$, $\rho = n - 1$ so that the new side ends at P_{n-1} and is in fact the last one.

We summarize:

<u>Proposition 2</u>. The Newton-Cramer polygon of a generic polar of our generic curve γ has $\left[\frac{s+1}{2}\right]$ sides, r_j, $j = 0,\ldots,\left[\frac{s-1}{2}\right]$. The side r_j has slope either $-q_{2j+1}/p_{2j+1}$ if $j < (s-1)/2$, or $-(q_s-q_{s-1})/(p_s-p_{s-1})$ if $j = (s-1)/2$.Furthermore the points of the N.C. diagram on the side r_j are, for $j < (s-1)/2$

$$P_{q_{2j}+q_{2j+1}-1} = (m-p_{2j},q_{2j}-1) + \alpha(-p_{2j+1},q_{2j+1}),$$

$\alpha = o,\ldots,h_{2j+2}$. If s is odd, the last side $r_{(s-1)/2}$, contains only the points $(m-p_{s-1},q_{s-1}-1)$ and $(0,n-1)$.

As proposition 1 shows, one can order the points on the sides of the N.C. polygon, excepted $(0,n-1)$, by the values of $i(\delta,\rho)$ in such a way that, for each point, there

there occurs in the corresponding monomial one a_j which does not appears in the monomials corresponding to former points. Hence the coefficients of the forms which approximate the polar equation in correspondence to the sides of the N.C. polygon are general for a general γ. In particular each side yelds an equation with no multiple roots for the determination of the first coefficients of the Puiseux series of the polar curve. From the N.C. polygon we determine easily the equisingularity type of the general polar curve and we obtain:

Theorem 1. Let m/n be be a irreducible fraction which we write as continous fraction in the form

$$\frac{m}{n} = h + \cfrac{1}{h_1 + \cfrac{1}{h_2 + \ddots + \cfrac{1}{h_s}}}$$

with $h_s > 1$. Let p_i/q_i, $i = 0,\dots,s$, be, in irreducible form, the partial fractions

$$\frac{p_i}{q_i} = h + \cfrac{1}{h_1 + \cfrac{1}{h_2 + \ddots + \cfrac{1}{h_s}}} .$$

Let M be the moduli space of the one-branched algebroid plane curves with characteristic exponent m/n. Then there exists in M a dense open subset A such that if γ represents a point on A, the singularity of the generic polar curve of γ can be described in the following way:

Let $O_1, \dots, O_h$ the free infinitely near points which follow O in γ. The generic polar of γ is composed by

h_2 branches with characteristic exponent p_1/q_1

h_4 branches with characteristic exponent p_3/q_3

. . .

h_{2j} branches with characteristic exponent p_{2j-1}/q_{2j-1}

. . .

and at last, either

a) h_s branches with characteristic exponent p_{s-1}/q_{s-1} if s is even;

or

b) h_{s-1} branches with characteristic exponent p_{s-2}/q_{s-2}; g.c.d.$(p_s - p_{s-1}, q_s - q_{s-1})$ branches with characteristic exponent $(p_s - p_{s-1})/(q_s - q_{s-1})$ if s is odd.

All these branches pass through the points $O, O_1, \dots, O_h$ and no two of them share the first free infinitely near point which follows O_h.

We can reformulate the last statement by saying that in the Puiseux series of γ and in those of the branches of the generic polar the h terms before the characteristic term are the same ones and the characteristic terms are all different (modulo the choice of determination on the fractionary powers of x).

3.- The non generic branches. It is known (see (Z)) that in the moduli space of one-branched plane algebroid curves with characteristic exponent m/n there is a "most particular point" which is adherent to every point on M; this is the analytical type of γ_o: $x^m = y^n$. We call elementary branches the branches with the analytical type of γ_o. It is very easy to determine the equisingularity type of the generic polar of an elementary branch: we calculate from γ_o and we obtain a N.C. polygon with only two vertices, (m-1,0) and (0,n-1), which gives g.c.d.(m-1,n-1) branches with characteristic exponent (m-1)/(n-1). Any two of these branches have infinitely near points in common up to the last satellite point.

As one can expect, the singularity of the generic polar of any one-branched plane curve with characteristic exponent m/n is, in certain sense, "intermediate" between the singularities of the generic polars of the general and most particular curves, γ and γ_o, respectively. We prove also the converse, that is, any such intermediate singularity corresponds to the generic polar of a one-branched curve with characteristic exponent m/n.

Let Θ be the closed domain on the plane bounded by the N.C. polygon described in Proposition 2, the line segment from (0,n-1) to (m-1,0) and the δ-axis, i. e., the part of the upper-half plane limited by the N.C. polygons of γ and γ_0.

Theorem 2. If $\bar{\gamma}$ is a one branched algebroid plane curve with characteristic exponent m/n, upon an analitic transformation of the coordinates, the generic polar of $\bar{\gamma}$ has its N.C. polygon contained in Θ. Conversely, any N.C. drawn in Θ, from (0,n-1) to the δ-axis, is the N.C. polygon of the generic polar of a one branched algebroid plane curve with characteristic exponent m/n.

Proof: We may suppose that $\bar{\gamma}$ was given, as in §2, by the Puiseux series

$$y = x^{m/n} + \sum_{i>0} a_i x^{\frac{m+i}{n}} .$$

The first statement is obvious: from the note which follows Proposition 1, we have the points (0,n-1) and (m-1,0) in the N.C. polygon and we know (from the definition of the N.C. polygon) that the points below the lower boundary of Θ correspond to monomials that do not occur in the polar equations for general values, hence for any values, of the a_i

For the second statement we note first that all the points on Θ are in the half-plane $\delta \geq (n-1-\rho)m/n$ and thus, by proposition 1, correspond to non trivial monomials on the polar equation for general values of the a_i;we select from Θ the points with integral coordinates excluding those in the upper boundary line, getting

$$\Xi = \left\{(\delta,\rho)\varepsilon\Theta \;\middle|\; \delta \text{ and } \rho \text{ are integer and } \frac{\delta}{m-1} + \frac{\rho}{n-1} \neq 1\right\}.$$

We order the points of Ξ by the values of $i = i(\delta,\rho)$ so that (Proposition 1) in the monomial which corresponds to a point there appears a coefficient a_i that does not occur in the monomials corresponding to former points.

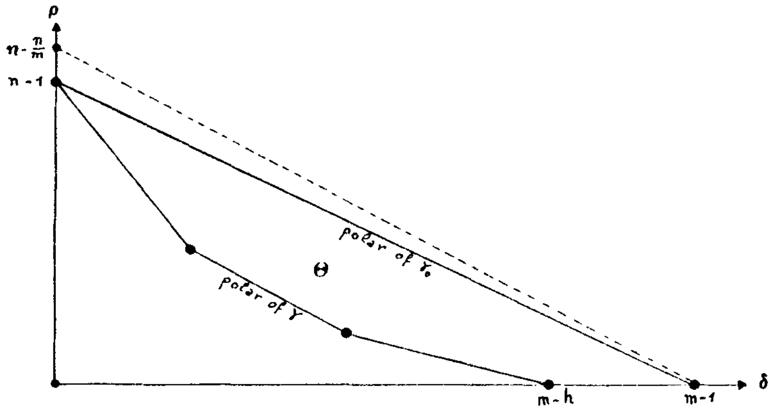
On the other hand we note that any term in the equation (I), §2, in which occurs the parameter λ has bidegree (δ,ρ) with

$$\delta \geq (n-\rho-1)\frac{m}{n} + \frac{m}{n} - 1$$

i. e.,

$$\frac{\delta}{m-1} + \frac{\rho}{n - n/m} \geq 1$$

thus, see figure below, λ does not occur on the monomials which correspond to points of Ξ. Hence if we select any subset $T \subset \Xi$, by the first remark we can select the values



figure

of the a_i for a polar equation with zero coefficients corresponding to the points on T and non-zero coefficients corresponding to the points of Ξ not in T. By the second remark this is done independently of λ, i. e., for a generic polar, and the proof is complete.

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