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A NOTE ON PLANAR REAL CREMONA TRANSFORMATIONS

by Jaume Llibre and Carles Simó

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Jaume Llibre(*) and Carles Simó(**)

(*) Secció de Matemàtiques, Facultat de Ciències, Universitat Autònoma de Barcelona, Bellaterra, Barcelona, Spain.

(**) Facultat de Matemàtiques, Universitat de Barcelona, Barcelona, Spain.

Abstract. *In this note we prove that a polynomial mapping T in two real variables such that its jacobian is constant (the so called planar real Cremona maps) is a bijection between* R^2 *and* R^2. *Let (F,G) be the polynomial components of T. We give a complete global picture of the family of curves F = constant and G = constant.*

1 Introduction

The main purpose of this note is to prove the following theorem.

Theorem A. *Let* $F = F(x_1, x_2)$ *and* $G = G(x_1, x_2)$ *be two real polynomials in the two real variables* x_1, x_2 *such that its jacobian* $J = \det(\partial(F,G)/\partial(x_1,x_2))$ *is a nonzero constant. Then the polynomial map* $(F,G)\colon R^2 \longrightarrow R^2$ *is bijective.*

The key point in the proof of the injectivity of Theorem A is that the algebraic curves $F(x_1,x_2) =$ constant and $G(x_1,x_2) =$ constant are solutions of systems of ordinary differential equations of Hamiltonian type (with Hamiltonians F and G, of course). These systems have only a singularity of index two at the infinity point which consists of two elliptic sectors (see section 2).

For the proof of the onto character we analyze the Newton polygon of F and G. In order to make this analysis easier we note that using triangular maps of the type $(x_1,x_2) \longrightarrow (x_1,x_2+bx_1^p)$, where b is a nonzero real number and p an integer such that $p > 1$, we obtain another Cremona map of the type

$$(x_1,x_2) \longrightarrow (x_1^m + \bar{F}(x_1,x_2), x_1^n + \bar{G}(x_1,x_2)) \quad , \tag{1}$$

where degree $\bar{F} < m$ and degree $\bar{G} < n$. When F and G have the form (1) the only possible diffeomorphic qualitative global pictures of the flow (for the Hamiltonians F or G) are given in Figure 1.a and Figure 1.b (see section 2). In any case the global picture is homeomorphic to one of these two figures (see, again, section 2) where we have used the usual compactification of the plane adding one point p at infinity.

Figure 1 here

As a consequence of the onto character of T the flows with Hamiltonian F and G do not reach the infinity in finite time (see section 3).

Remark 1. The assumption that F and G are polynomials is necessary in Theorem A. The result is not true for analytical maps. For instance, if $F(x_1,x_2) = -\exp(x_2)\cos(x_1\exp(-2x_2))$ and $G(x_1,x_2) = \exp(x_2)\sin(x_1\exp(-2x_2))$ then $J = 1$ but the map $(F,G): R^2 \longrightarrow R^2$ is clearly not injective.

Remark 2. There is no restriction putting $J = 1$ because, if $J = a \neq 0$, we can consider the map $(a^{-1}F,G)$ instead of (F,G).

Remark 3. A theorem similar to Theorem A for complex polynomials in two complex variables would prove the jacobian conjecture for two variables, that is, the inverse map is also polynomial (see [2], [6] and [7, Theorems 38 and 46]). In fact for the validity of the jacobian conjecture it is enough to prove the injectivity.

Remark 4. To prove the injectivity for a mapping $T = (F,G): C^2 \longrightarrow C^2$, F and G complex polynomials with $J = 1$, we claim that it is enough to prove the injectivity for complex valued polynomials maps $T' = (P,Q): R^2 \longrightarrow C^2$ in two real variables with $J = 1$. Suppose T is not injective. Then there exist $z,w \in C^2$ such that $Tz = Tw$. Using translation, scaling and rotation we can assume that z and w are (0,0) and (1,0). Let T" be the map obtained from T by these changes of variables. The restriction of T" to R^2 is of type T' and the claim is proved. Note that T' is of type $T' = (P_1 + iP_2, Q_1 + iQ_2)$ where P_1, P_2, Q_1, Q_2 are real valued polynomials. From Theorem A follows easely that if some of these four polynomials is identically zero T' is injective.

The authors would like to express their gratitude to Prof. P. Menal and Prof. C. Perelló for his interest in the present work.

2 Proof of the injectivity

Let F and G be two polynomials in the hypotheses of Theorem A. This implies that the analytical Hamiltonian system X_F given by

$$\frac{dx_1}{dt} = \frac{\partial F}{\partial x_2} = F_{x_2} ,$$

$$\frac{dx_2}{dt} = - \frac{\partial F}{\partial x_1} = -F_{x_1} ,$$

has no critical points. Of course F is a first integral of this system.

Lemma 1. *For each $y \in R^2$ there is a unique solution $\phi_t(y) = (x_1(t), x_2(t))$ of X_F with $\phi_0(y) = y$ defined on a maximal open interval $(\alpha,\beta) \subset R$ such that $\|\phi_t(y)\| \to \infty$ as $t \to \alpha$ or $t \to \beta$ where $\| \ \|$ denotes the Euclidean norm. It is possible $\alpha = -\infty$, $\beta = +\infty$ or both.*

For a proof of this lemma see [4,p.210].

Next we introduce the Poincaré compactification for polynomial vector fields X(x) in the plane (see [5]). We consider in R^3 the sphere $\bar{S}^2 = \{(x_1,x_2,x_3) \in R^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ and the plane $\bar{P} = \{(x_1,x_2,x_3) \in R^3 : x_3 = 1\}$. For each point x of $\bar{P}$ of type $(x_1,x_2,1)$ we define the map $f_+ : \bar{P} \longrightarrow \bar{S}^2$ given by $f_+(x) = (x_1,x_2,1)/d(x) = (y_1,y_2,y_3) = y$ where $d(x) = (x_1^2 + x_2^2 + 1)^{1/2}$. The image of $\bar{P}$ under f_+ is the upper hemisphere H_+ of $\bar{S}^2$. Then f_+ induces a field on H_+: $\bar{X}(y) = Df_+ \cdot X(x)$.

Let $X = (P,Q)$ a polynomial vector field on $\bar{P}$ of degree $n = \max(\deg P, \deg Q)$. Let $r(y) = y_3^{n-1}$. Then we claim that the field $r\bar{X}$ can be extended analytically to a vector field on $H_+ \cup S^1$ where $S^1 = \bar{S}^2 \cap \{y_3 = 0\}$, the equator of the sphere.

In order to prove the claim (see [5] for details) we use five local carts (U_i,ϕ_i) i=1,2,3, (V_i,ψ_i) i=1,2 where

$U_i = \{y \in \bar{S}^2 : y_i > 0\}$; $V_i = \{y \in \bar{S}^2 : y_i < 0\}$;

$$\phi_i = \frac{1}{y_i}(y_j,y_k) , \quad j < k , \quad i \neq j,k;$$

$$\psi_i = \frac{1}{y_i}(y_j,y_k) , \quad j < k , \quad i \neq j,k.$$

Let $y \in U_1 \cap H_+$ and $z = \phi_1(y)$. Then $(z_1,z_2) = (y_2,y_3)/y_1 = (x_2,1)/x_1$. The vector field $r\bar{X}$ is expressed in the z coordinates as

$$\frac{z_2^n}{d(z)^{n-1}}\left(-z_1 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) + Q\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) , -z_2 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) \right) .$$

In U_2 we get

$$\frac{z_2^n}{d(z)^{n-1}}\left(P\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) -z_1 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) , -z_2 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) \right) .$$

Furthermore the expressions of $r\bar{X}$ on V_1, V_2 are the ones of $r\bar{X}$ on U_1, U_2, respectively, multiplied by $(-1)^{n-1}$. In U_3 the expression obtained is

$$\frac{1}{d(z)^{n-1}}\left(P(z_1,z_2) , Q(z_1,z_2) \right).$$

It is easy to check that the different expressions of $r\bar{X}$ are analytical and compatible. Hence $r\bar{X}$ is extended to $H_+ \cup S^1$ and S^1 is clearly invariant under the flow.

Let $P(x_1,x_2) = \sum_{j=0}^{n} P_j(x_1,x_2)$, $Q(x_1,x_2) = \sum_{j=0}^{n} Q_j(x_1,x_2)$; where P_j , Q_j are homogenous polynomials of degree j. The field on $S^1 \cap U_1$, $S^1 \cap U_2$ is given by

$$\dot{z}_1 = R(z_1) = Q_n(1,z_1) - z_1 P_n(1,z_1) ,$$

$$\dot{z}_1 = S(z_1) = P_n(z_1,1) - z_1 Q_n(z_1,1) ,$$

respectively. In $S^1 \cap V_1$, $S^1 \cap V_2$ we have the same expressions times $(-1)^{n-1}$.

On the other hand, let S^2 be the sphere of R^3 defined by $\{(y_1,y_2,y_3) \in R^3 : y_1^2 + y_2^2 + (y_3 - 1/2)^2 = 1/4 \}$. The plane R^2 may be identified with the sphere S^2 with the "north pole" $p=(0,0,1)$ removed by means of the stereographic projection which assigns to each point $(x_1,x_2) \in R^2$ the point $(y_1,y_2,y_3) \in S^2$ through the relations $x_1 = y_1/(1-y_3)$, $x_2 = y_2/(1-y_3)$.

The differential system X_F on $S^2 - \{p\}$ becomes

$$\frac{dy_1}{dt} = F_1(y_1,y_2,y_3) \ ,$$

$$\frac{dy_2}{dt} = F_2(y_1,y_2,y_3) \ ,$$

where

$$F_1(y_1,y_2,y_3) = 2y_1y_2 \frac{\partial F}{\partial x_1}\left(\frac{y_1}{1-y_3}, \frac{y_2}{1-y_3}\right) - (2y_1^2 + y_3 - 1)\frac{\partial F}{\partial x_2}\left(\frac{y_1}{1-y_3},\frac{y_2}{1-y_3}\right) ,$$

$$F_2(y_1,y_2,y_3) = (2y_2^2 + y_3 - 1)\frac{\partial F}{\partial x_1}\left(\frac{y_1}{1-y_3},\frac{y_2}{1-y_3}\right) - 2y_1y_2\frac{\partial F}{\partial x_2}\left(\frac{y_1}{1-y_3},\frac{y_2}{1-y_3}\right) ,$$

If the degree of F, m, is greater than 2, this system is not defined at p, but it can be extended by a change of the time scale. In any case we introduce a new variable u via

$$\frac{dt}{du} = (1-y_3)^{m-1} \ .$$

Then X_F becomes

$$\begin{aligned} \frac{dy_1}{du} &= (1-y_3)^{m-1}F_1(y_1,y_2,y_3) \ , \\ \frac{dy_2}{du} &= (1-y_3)^{m-1}F_2(y_1,y_2,y_3) \ . \end{aligned} \tag{2}$$

This system extends analytically the flow of X_F from $S^2 - \{p\}$ to S^2 (at least in a neighbohood of p). Note that the point p is the unique critical point of the new flow.

Lemma 2. *In the hypotheses of Theorem A the local phase-protrait of system (2) around the critical point p consists of two elliptic sectors and the rest fans (see [3,p.219] for definitions).*

Proof. By the Poincaré-Hopf Index Theorem (see [3,p.366]) the index of p is equal to two. Now we use the compactification of Poincaré. As we noted in section 1 we can always suppose that $F(x_1,x_2) = x_1^m + \bar{F}(x_1,x_2)$, $\deg \bar{F} < m$. Therefore $P(x_1,x_2) = \bar{F}_{x_2}$,

$Q(x_1,x_2) = -mx_1^{m-1} - \bar{F}_{x_1}$. The vector field (P,Q) on the parts of S^1 which are in the carts U_1, V_1, U_2 and V_2 (see Figure 2) is given by $R(z_1) = -m$, $R(z_1) = (-1)^{m-1}m$, $S(z_1) = mz_1^m$ and $S(z_1) = (-1)^{m-2}mz_1^m$, respectively. If we look at S^1 as $\{(y_1,y_2) \in R^2 : y_1^2 + y_2^2 = 1\}$, then the only critical points are $(0,1)$, $(0,-1)$. We can visualize $H_+ \cup S^1$ as a closed disc bounded by S^1. The flow on it (recall that there are no critical points inside and use Lemma 1) has the two qualitative possibilities given in Figure 3. When we glue the S^1 into a point (and obtain S^2) the respective pictures of Figure 3 become the ones given in Figure 1.a and 1.b. This proves Lemma 2.

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Figure 2 here

Figure 3 here

Remark 1. The statement of Lemma 2 is obviously true for the analytical Hamiltonian system X_G.

Remark 2. In the proof of Lemma 2 we strongly rely on the existence of the polynomial G such that the jacobian of (F,G) with respect to (x_1,x_2) is equal 1. In fact for a general analytic Hamiltonian system X_F without proper critical points the statement is false. A counterexample is displayed by $F(x_1,x_2)=x_2(x_1x_2-1)$ which has two hyperbolic sectors and four elliptic sectors at the infinity.

Lemma 3. *(i) The global flow of X_F on R^2 is diffeomorphic to the flow given in Figure 4.*

(ii) If the algebraic curve $F(x_1,x_2)$ = constant on R^2 is nonempty, then it has only one connected component. This component is diffeomorphic to R.

(iii) For all $\bar{x} \in R^2$ the algebraic curve $F(x) = F(\bar{x})$ and the trajectory $\phi_t(\bar{x})$ of X_F represent the same curve in R^2.

Proof. (i) follows from Figure 3.

(ii) The curves $F(x)$ = constant on R^2 have only one connected component. Otherwise, between two curves with the same value of F there would be a curve where F is a extremum and hence it would be composed of critical points.

(iii) follows immediately from the Hamiltonian character and (ii).

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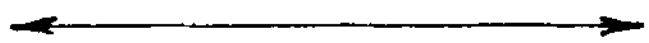
Figure 4 here



Proof of the injectivity of T. Let ϕ_t and ψ_t be the flows of X_F and X_G, respectively. Since the jacobian is 1 we have that

$$\frac{d}{dt}(G\circ\phi_t) = \{G\circ\phi_t\ ,\ F\} = -1\ ,$$

$$\frac{d}{dt}(F\circ\psi_t) = \{F\circ\psi_t\ ,\ G\} = 1\ ,$$

where { , } denotes the Poisson bracket (for more details see [1,p.193]). Then for each point $\bar{x} \in R^2$ we have that

$$G(\phi_t(\bar{x})) = -t + G(\bar{x})\ ,$$
$$F(\psi_t(\bar{x})) = t + F(\bar{x})\ . \tag{3}$$

From (2) and Lemma 3 it follows that the curves $F(x)$ = constant and $G(\bar{x})$ = constant have at most one point in common. So the polynomial map $(F,G) : R^2 \longrightarrow R^2$ is injective.

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3 Proof of the exhaustivity

It is not excluded up to now that for ϕ_t or ψ_t some point of R^2 reaches the infinity in finite time. By Lemma 3 (ii) the image of R^2 under F is an interval I of R. There are three possibilities: I is bounded, I is bounded from one side or $I = R$. If I is bounded from above, for each point $\bar{x} \in R^2$ it follows from (2) that there is some time $t_\infty(\bar{x}) < +\infty$ such that $\psi_t(\bar{x})$ goes to infinity when $t \uparrow t_\infty(\bar{x})$. In a similar way if I is bounded from below there is some time $t_{-\infty}(\bar{x}) > -\infty$ such that $\psi_t(\bar{x})$ goes to infinity when $t \downarrow t_{-\infty}(\bar{x})$. These values $t_\infty(\bar{x})$, $t_{-\infty}(\bar{x})$ are the values α and β of Lemma 1. Let us show that F is exhaustive, that is, $I = R$ (and therefore if follows that the orbits of X_G reach the infinity only for unbounded time). The same will be true for G.

We suppose again $F(x_1,x_2) = x_1^m + \sum_{j<m} F_j(x_1,x_2)$. We consider the Newton polygon of F for the neighborhood of the infinity, i.e., the outer part of the boundary of the convex closure of the set $\{(r,s) \in N \times N : F(x_1,x_2) = \sum a_{rs} x_1^r x_2^s$ with $a_{rs} \neq 0\}$, see Figure 5. We claim that if there is some vertex (i,j) in the Newton polygon with one odd coordinate then F is exhaustive. We set $x_2 = x_1^{p/q}$, q odd, where $-q/p$ belongs to the interval J whose extrema are the slopes of the sides of the polygon which meet in (i,j) (eventually p is negative if these two slopes are positive or one is positive and the other zero). Suppose i,j odd. Then we select p even and for $|x_1|$ large we have $F(x_1, x_2 = x_1^{p/q}) = a_{ij} x_1^{(qi+pj)/q}(1 + o(1))$ with $qi + pj$ odd. Therefore F is exhaustive. If i odd, j even, take any p such

that $p/q \varepsilon J$ and again F is exhaustive. Finally, for i even, j odd it is enough to take p odd. Hence we can suppose that all the verteces have even coordinates. If the coefficients associated to the verteces change the sign we have exhaustivity. We suppose for definiteness that all these coefficients are positive.

Figure 5 here

Let us take a side of the Newton polygon. The involved terms are of the type $F_s = \sum_{k=0}^{r} a_k x_1^{i+kp} x_2^{j-kq}$ where r is even and g.c.d. $(p,q) = 1$. These terms are dominant when $x_2 = ax_1^{p/q}$ (or when $x_1 = c = \text{constant}$, if $q = 0$). On this curve $F_s(x_1,x_2) = x_1^{(qi+pj)/q} a^j f_r(b)$ where $f_r(b) = \sum_0^r a_k b^k$, $b = a^{-q}$ (or $F_s(x_1,x_2) = x_2^j c^j f_r(b)$, $b = c^p$ if $q = 0$) and $F(x_1,x_2) = F_s(x_1,x_2)(1 + o(1))$ when $(x_1,x_2) \to \infty$ if $F_s(x_1,x_2)$ is unbounded on this curve. If $f_r(b)$ has some real zero $\bar{b}$ of odd multiplicity, then in any neighborhood of $\bar{b}$ there are points b_-, b_+ such that $f_r(b_-) < 0$, $f_r(b_+) > 0$ and hence F is exhaustive.

Let us suppose that all the real zeros of $f_r(b)$ are of even (greater than zero) multiplicity. Then we consider the terms in the highest line parallel to the side through one of the points in the Newton diagram and let $f_r^{(1)}$ be the related polynomial in b. If for one of the zeros b^* of f_r we have $f_r^{(1)}(b^*) < 0$, we have done. For the case $f_r^{(1)}(b^*) > 0$ see later. If $f_r^{(1)}(b^*) = 0$ we continue the process with other lines parallel to the selected side, obtaining $f_r^{(2)}$, $f_r^{(3)}$, etc. Let t be the first index such that $f_r^{(t)}(b^*) \neq 0$. If $f_r^{(t)}(b^*) < 0$ the exhaustivity is clear. If for all $t, f_r^{(t)}(b^*) = 0$, then we have that $x_2^q - ax_1^p$ is a factor of F (or $x_2^q x_1^{-p} - a$ if $p = 0$). Therefore $F(x_1,x_2) = (x_2^q - ax_1^p)\bar{F}(x_1,x_2) + C$, where C is a constant. Then for the Hamiltonian field we get $F_{x_1} = apx_1^{p-1} - (x_2^q - ax_1^p)\bar{F}_{x_1}$, $F_{x_2} = qx_2^{q-1}\bar{F} + (x_2^q - ax_1^p)\bar{F}_{x_2}$. First we suppose $p > 0$, $q \neq 0$. Then $p > 1$ and we have a critical point at the origin if $q > 1$, which is an absurdity. Therefore $F(x_1,x_2) = (x_2 - ax_1^p)\bar{F} + C$. Then the algebraic curve $F = C$ has the real components $x_2 - ax_1^p = 0$, $\bar{F} = 0$. If $\bar{F} = 0$ has real points in the curve $x_2 - ax_1^p = 0$, then these points will be critical points and this is impossible. If $\bar{F} = 0$ has real points outside $x_2 - ax_1^p = 0$, we have a contradiction with Lemma 3 (ii). If $\bar{F} = 0$ has no real components then $\bar{F}(0,x_2)$ is a polynomial of even degree and therefore $F(0,x_2)$ is of odd degree, showing exhaustivity. A similar reasoning applies to the cases $p < 0$ and $q = 0$.

If for all the sides of the Newton polygon we have no zeros of the related $f_r(b)$ polynomials or if the zeros being even there is some t such that $f_r^{(t)}(b^*) > 0$ for a zero b^* of f_r then F is positive definite for large values of $\|(x_1,x_2)\|$. Therefore the curve $F(x_1,x_2) = K$ is closed for K large enough. This implies that this curve is a periodic orbit for X_F showing the existence of a critical point inside (see [4,p.254]) and therefore leading to an absurdity. This ends the proof that F is exhaustive. The same is true for G. Figure 5 displays the dominant terms in the Newton polygon near infinity. If there are no terms with exponents (i,j), $i < r$ but there is some point (r,s) with nonzero coefficient the Newton polygon ends on the point (r,s) with the maximum value of s.

Proof of the exhaustivity of T. We know that the flows ϕ_t , ψ_t exist for all t without going to infinity in finite time. Let $(x_0,y_0) \in R^2$ be any point and $(t_0,s_0) = T(x_0,y_0)$. Then select any point (t,s) in R^2. From (2) it follows that $\phi_{s_0-s} \cdot \psi_{t-t_0}(x_0,y_0) = (\bar{x},\bar{y})$ with $T(\bar{x},\bar{y}) = (t,s)$. Hence T is exhaustive.

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Captions for the figures

Fig. 1. Qualitative picture of the flow of X_F on S^2.

Fig. 2. The four carts used for S^1.

Fig. 3. Qualitative picture of the flow of (1) in $H_+ \cup S^1 \subset \tilde{S}^2$. Case (a) m odd; case (b) m even.

Fig. 4. Qualitative picture of the flow of X_F.

Fig. 5. The Newton polygon of dominant terms near infinity.

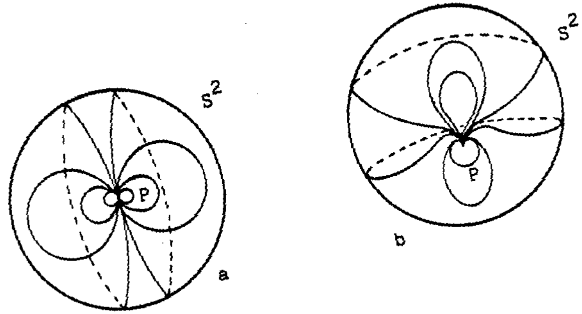

Fig. 1.

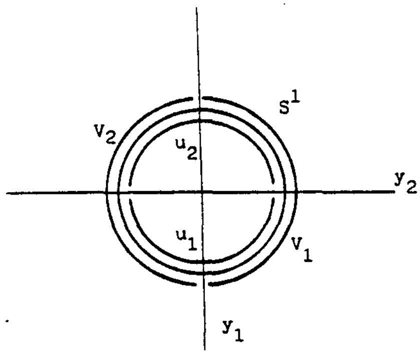

Fig. 2.

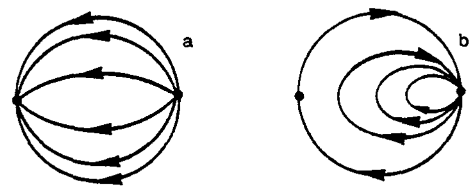

Fig. 3.

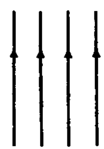

Fig. 4.

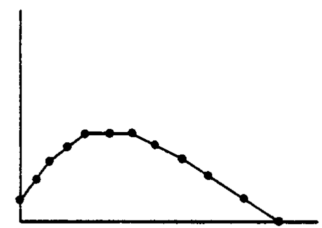

Fig. 5.

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