UNIVERSITAT DE BARCELONA
FACULTAT DE MATEMÀTIQUES

IMPLICATION AND DEDUCTION IN SOME INTUITIONISTIC MODAL LOGICS (1)

by Josep M. Font

PRE–PRINT N.º 10
juny 1982

IMPLICATION AND DEDUCTION IN SOME INTUITIONISTIC MODAL LOGICS ([1])

Josep M. Font

Faculty of Mathematics, University of Barcelona, Spain.

Abstract

We study the deductive properties of the system IM4 of intuitionistic modal logic, paying special attention to the implicative ones. This system is the intuitionistic counterpart of Lewis' S4 and its models are the topological pseudo-Boolean algebras. Its abstract deductive structure is analogous to that of pseudo-Boolean algebras but here with respect to new implicative operations. These satisfy the Deduction Theorem and allow us to find implicative characterizations of several types of deductive systems (such as irreducible, maximal, prime) and some related concepts from universal algebra (simplicity, semisimplicity, radical). Among these operations, $p \Rightarrow q = L(Lp \to Lq)$ is worth noting, because, besides satisfying a lot of classical axioms, it allows us to define L in terms of $\Rightarrow$ and give a formalization of several modal logics of type S4 whose specific non-classical operator is $\Rightarrow$ and not L nor M.

1. The system IM4 and its models

The purpose of making intuitionistic modal logic has benefitted form the modern formulations of modal logic consisting in the addition of an unary operator (L or M) to the language of ordinary propositional calculus and of the corresponding axioms and/or rules of inference. Then, to obtain the intuitionisitc analogue of a given classical modal system, it suffices to weaken the propositional non-modal base, leaving the modal part unaltered. However, this is not free from danger because of the two following remarks.

In the first place, we must remember that in classical modal logic L and M are linked by the two equalities $L = \neg M \neg$ and $M = \neg L \neg$, which are consistent with the boolean principles of duality, and accordingly either L or M can be taken as the only primitive operator. The situation is quite different if we have an intuitionistic base, because of the lack of duality, and so the two previous equalities can not be accepted together. Most of the authors (Prior, Bull, Ono, Sotirov, Monteiro and Varsavski...) take both L and M as primitive, rejecting both equalities, while only a few (Fischer-Servi) prefer to take L as primitive and define M as an abreviation of $\neg L \neg$.

In the second place, the lack of a lot of boolean principles, all concerning negation, destroys the equivalence between several classical axioms; thus, while it is easy to formulate the intuitionistic analogue of S4, we need more additonal criteria to single out an intuitionistic analogue of S5, due to the multiplicity of ways we can take to define it starting from S4. Concerning this problem, the papers of G. Fischer-Servi on this subject are worth reading. In a forthcoming paper we will touch this problem from a different point of view, as well as the study of the properties of the operator $M = \neg L \neg$ in the logic here considered and in some of its extensions.

In the present paper we will concentrate on developping a deductive theory for IM4 (our intuitionistic analogue to S4) in an algebraic-logic setting and paying special attention to the role of the implication.

The language in which we will formulate our intuitionistic modal system is the free algebra P(X) of type (1,1,2,2,2) generated by a set X of propositional letters (which is usually assumed to be denumerable), the operations or connectives being denoted by $L, \neg, \wedge, \vee, \rightarrow$ respectively. From the axioms and inference rules given a syntactical consequence relation $\vdash$ will be defined in the customary finitistic form.

1.1 Definition. We call IM4 the logical system over P(X) which has as axioms schemes, apart form the intuitionistic calculus ones, the following: $Lp \to p$, $L(p\to q)\to(Lp\to Lq)$, $Lp\to LLp$, and which has as deduction rules the Modus Ponens $\{p, p\to q\} \vdash q$ and the Necessity $\{p\} \vdash Lp$.

This system has appeared elsewhere under different names, e.g. in [Bu1] and [O] . As it can easily be seen, it is an intuitionistic analogous of S4 in the sense mentioned in [Bu2]. The analogy can also be applied to its algebraic models, which are a weakening of topological Boolean algebras.

1.2 Definition. A topological pseudo-Boolean algebra (tpBa) is an algebra $\langle A, I, \neg, \wedge, \vee, \cdot \rangle$ of type (1,1,2,2,2) such that $\langle A, \neg, \wedge, \vee, \cdot \rangle$ is a pseudo-Boolean algebra, and I is an interior operator on A, that is it satisfies: (I1) $I(1) = 1$, 1 being the maximum of A;

(I2) $Ia \leqslant a$ for every $a \in A$;

(I3) $I(a\cdot b) \leqslant Ia\cdot Ib$ for all $a,b \in A$;

(I4) $I^2 a = Ia$ for all $a \in A$.

It is well-Known that (I3) can be substituted by (I3') $I(a\wedge b) = Ia \wedge Ib$, and that (I3)+(I4) just as much as (I3')+(I4) are equivalent to the condition (I3") $I(Ia\cdot b) \leqslant Ia\cdot Ib$. The condition (I3) is often called "Gödel's inequality". It can be proved, with apropriate examples, that the three resulting axiom sets are independent. In any tpBa the elements $a \in A$ which satisfy $Ia=a$ are called open and the set of all open elements is denoted by B. The following results can be proved by well-known and standard procedures.

1.3 Proposition. For every $\Sigma \subset P(X)$ the relation $\sim_\Sigma$ defined by $p \sim_\Sigma q$ if and only if $\Sigma \vdash p \leftrightarrow q$ is a congruence relation of the algebra P(X) such that the quotient $P_\Sigma(X)$ is a tpBa where $\tilde{p}=1$ if and only if $\Sigma \vdash p$. □

1.4 Proposition. The Tarski-Lindenbaum algebra $P_\phi(X)$ is the free tpBa generated by $\bar{X} = \{\bar{x} : x \in X\}$. □

The regular matrices for the IM4 calculus are then the tpBa's, $P_\phi(X)$ being characteristic. Using valuations in the usual way, it results that the class of all tpBa's allows us to define a semantical consequence relation $\models$ that satisfies the following

1.5 Theorems. For all $\Sigma \subset P(X)$ and all $p \in P(X)$ the following hold:

a) Completeness: $\Sigma \models p$ if and only if $\Sigma \vdash p$;

b) Compactness: $\Sigma \models p$ if and only if there exists a finite $\Delta \subset \Sigma$ such that $\Delta \models p$; and

c) Finite model property: $\Sigma \models p$ if and only if $\Sigma \models_A p$ for every finite tpBa A if and only if $\Sigma \models_A p$ for every tpBa A with cardinality $\leqslant 2^{2^k}$, k being the number of subformulas of p.

Proof: Part (a) follows from the fact that $P_\phi(X)$ is characteristic, and part (b) directly from (a). Part (c) can be proved by a slight modification of the proof in [Bu1] for a weaker system. □

It is clear that the tpBa's are the algebraic structures which correspond to the IM4 logic. In the remaining sections of the paper we will study the algebraic abstraction of several logical concepts.

2. Deductive systems and abstract logic

In view of the syntactical setting of IM4, the following definition is entirely natural.

2.1 Definition. A deductive system of a tpBa A is a $D \subset A$ such that: (T) $1 \in D$, (MP) If $a \in D$ and $a \cdot b \in D$, then $b \in D$, (N) If $a \in D$ then $Ia \in D$, for all $a, b \in A$. The set of all deductive systems of a given tpBa A will be denoted by $\mathfrak{D}$.

This concept, besides its original logical motivation (a theory) has a doubtless algebraic contents, as shows the following

2.2 Proposition. If for every $D \in \mathfrak{D}$ we define a relation $\equiv_D$ in A by $a \equiv_D b$ if and only if $a \cdot b \in D$ and $b \cdot a \in D$, then $\equiv_D$ is a congruence relation of the algebra A and the correspondence $D \longmapsto \equiv_D$ defines an isomorphism between $\mathfrak{D}$ and the set $\mathfrak{C}(A)$ of all congruence relations of A. Thus it results that $\mathfrak{D}$ is a complete lattice. □

Note that the inverse correspondence associates the deductive system $D_{\equiv} = \{a \in A : a \equiv 1\}$ to every congruence relation $\equiv \in \mathfrak{C}(A)$, and therefore the deductive systems of A are identified with the Shells of all the epimorphisms between tpBa's.

From a logical point of view, the fact that $\mathfrak{D}$ is a closure system is more interesting, and therefore we have the associated consequence operator $\mathbb{D} : \mathbb{D}(X) = \bigcap\{D' \in \mathfrak{D} : D' \supset X\}$ for every $X \subset A$. We can then consider the abstract logic $\mathbb{L} = \langle A, \mathbb{D}\rangle$, which, and because of the considerations we have made until now, will reflect algebraically the deductive properties of the IM4 logic. $\mathfrak{D}$ is obviously algebraic, as it is isomorphic to $\mathfrak{C}(A)$, and so we have:

2.3 Proposition. $\mathbb{L}$ is a finitary logic, that is, for every $X \subset A$, $\mathbb{D}(X) = \bigcup\{\mathbb{D}(F) : F \subset X , F \text{ finite}\}$. □

It would probably not be difficult to obtain directly properties and characterizations of this abstract logic, but the large knowledge we have at present about pseudo-Boolean algebras and their associated abstract logic ([2]) leads us to use the following characterization of $\mathbb{D}$ in terms of $\mathbb{D}'$, the consequence operator associated to $\mathfrak{D}'$, the set of all "deductive systems" of the pseudo-Boolean algebra subjacent to A (that is, the set of all $D \subset A$ that satisfy (T) and (MP) of 2.1, also called the "filters" of the pseudo-Boolean algebra A).

2.4 Theorem. For all $X \subset A$, the identity $\mathbb{D}(X) = \mathbb{D}'(I(X))$ holds.

Proof: It can directly be proved, using known constructive characterizations of $\mathbb{D}'$, the fact that $\mathfrak{D} \subset \mathfrak{D}'$, and specially Gödel's inequality, that $\mathbb{D}'(I(X)) \in \mathfrak{D}$ and that it is the minimum deductive system that contains X. □

A proof that $\mathfrak{D}$ is a distributive lattice can be obtained (using 2.4), by seeing that $\mathfrak{D}$ is a sublattice of $\mathfrak{D}'$. At the end of this section we will refer to another proof of this fact.

If we consider the properties of $\mathbb{D}'$ that are stated in [V1], the previous theorem allows us to prove in an almost mechanical way the following properties of $\mathbb{L}$:

<u>2.5 Theorem</u>. $\mathbb{L}$ satisfies the <u>Adjunction Principle with regard to $\wedge$</u>: $\mathbb{D}(a,b) = \mathbb{D}(a \wedge b)$ for all $a,b \in A$. □

<u>2.6 Theorem</u>. $\mathbb{L}$ satisfies the <u>Strong Disjunction Principle with regard to $\overset{*}{\vee}$</u> (where $a \overset{*}{\vee} b = Ia \vee Ib$) : For all $X \subset A$, $\mathbb{D}(X,a) \cap \mathbb{D}(X,b) = \mathbb{D}(X, a\overset{*}{\vee}b)$, where $a,b \in A$. □

With 2.6 we see that the algebraic function which corresponds to the connective of logical disjunction is assumed here by the operation $\overset{*}{\vee}$; theorem 3.5 will confirm this from another point of view.

<u>2.7 Theorem</u>. $\mathbb{L}$ satisfies the <u>Deduction Principle with regard to $*$</u> : $b \in \mathbb{D}(X,a)$ if and only if $a*b \in \mathbb{D}(X)$ for all $a,b \in A$ and all $X \subset A$, where "$*$" stands for any of the <u>natural implication operations</u>, which are the following: The <u>weak</u> implication $a \nrightarrow b = Ia \cdot b$, the <u>intuitionistic</u> implication $a \Rightarrow b = I(Ia \cdot Ib)$, and the <u>strange</u> implication $a \rightsquigarrow b = Ia \cdot Ib$. □

These three operations ([3]) assume the algebraic role of the connective of logical implication.

The algebra A has a minimum 0 and since the deductive systems are also filters, $\mathbb{D}(0) = A$ and therefore we can consider the (<u>pseudo-</u>) <u>negations</u> $\neg^* a = a*0$ associated to the implications which satisfy:

and

2.8 Theorem. $\mathbb{L}$ satisfies the Pseudo-Reductio ad Absurdum Principle with regard to $\neg^*$: $\neg^* a \in \mathbb{D}(X)$ if and only if $\mathbb{D}(X,a) = A$ for all $a \in A$ and all $X \subset A$. □

Following Verdú's terminology, theorems 2.3,5,6,7 and 8 can be summarized saying that $\mathbb{L}$ is a finitary Heyting logic with an inconsistent element; that is, we see that the abstract logic naturally associated to our logical system IM4 is of an intuitionistic type. The modal character is included in the new connectives ($\overset{*}{\vee}, *, \neg^*$) which, surprisingly, have in their turn an intuitionistic behaviour.

If for every $D \in \mathfrak{D}$ we put $\mathfrak{D}^D = \{D' \in \mathfrak{D} : D' \supset D\}$ we obtain a closure system and with it a consequence operator $\mathbb{D}^D$ which reproduces the properties of $\mathbb{D}$; that is, $\mathbb{L}^D = \langle A, \mathbb{D}^D \rangle$ satisfies 2.3, and 2.5 to 2.8 , $\mathbb{D}^D(\phi) = D$ being its set of "theorems". We can see that the relations " $\mathbb{D}^D(a) = \mathbb{D}^D(b)$" , " $\mathbb{D}(D,a) = \mathbb{D}(D,b)$" and "$a*b \in D$ and $b*a \in D$" are equivalent and define an equivalence relation $\sim_D$ which is compatible with the operations $\wedge$, $\overset{*}{\vee}$, $*$, and $\neg^*$, and satisfies $a \sim_D Ia$ for every $a \in A$, that is, we could say that it is a "logical congruence". Therefore, these operations define a pseudo-Boolean algebra structure (without interior operator) in the quotient $A/\sim_D$. We therefore have two quotients of A : $A/\equiv_D$ which is the properly algebraic quotient, and $A/\sim_D$ which is called "the logical quotient of A by the logic $\mathbb{L}^D$". Certain properties of both quotients will give us information on the deductive system D, as we will see further on.

In the particular case $D = \{1\}$ the relation we obtain is $\sim$, the usual equivalence relation associated to the logic $\mathbb{L}$. Then, according to [V1], there exists a bilogical morphism ([4]) between $\mathbb{L}$ and the logic associated to the pseudo-Boolean algebra $A/\sim$, which obviously is an intuitionistic logic (this allows us to give another proof of the distributivity of $\mathfrak{D}$). In a certain sense, we can say that in this way we obtain a translation of the formulas of the intuitionistic modal logic IM4 into formulas of a certain simply intuitionistic logic, that is, a translation which would be an inverse to Gödel's.

3. The natural implication operations

We have affirmed that the operations * have an implicative character since they satisfy the deduction theorem which for us is the most characteristic structural (abstract) property of implication connectives. In the syntactical presentations of the usual logical systems, the implication shows itself through the modus ponens deduction rule, and through the axioms where it appears. In this section we will see how the natural implication operations also reflect the principles expressed by deduction rules and axioms or theorems. First, one can easily prove the

3.1 Proposition. For every $D \subset A$, it holds that $D \in \mathfrak{D}$ if and only if it satisfies the conditons (T) $1 \in D$, and (MP*) If $a \in D$ and $a*b \in D$ then $b \in D$, for all $a,b \in A$. □

This result is quite interesting since it allows us to substitute the two deduction rules (MP) and (N) formulated in terms of $\rightarrow$ and I for one unique rule (MP*) formulated in terms of * , the derived logical theories being preserved.

We will now see the properties of the usual implication which are reproduced, in an algebraic form, by the natural implications. We must bear in mind that $a*b = 1$ is not equivalent to $a \leqslant b$ (which is actually equivalent to $a \dot{\cdot} b = 1$) but is weaker, that is, $a \leqslant b$ implies $a*b = 1$; the reciprocal is not true, and due to this some properties which are classicaly inferred from one to the other are now independent.

3.2 Theorem. In every tpBa A and for every $a,b,c \in A$ the following hold:

(i) $a*a = 1$ (Identity Law);

(ii) $a*(b*a) = 1$ (A fortiori or Absorption Law);

(iii) $(a*(b*c))*((a*b)*(a*c)) = 1$ and $((a*b)*(a*c))*(a*(b*c)) = 1$ (Weak forms of the Autodistributive Law);

(iv) $(a*(b*c))*(b*(a*c)) = 1$ (Weak form of the Permutation of the Premises Law);

(v) $(a*b)*((b*c)*(a*c)) = 1$ and $(a*b)*((c*a)*(c*b)) = 1$ (First weak form of the Hypothetical Syllogism Laws);

(vi) If $a*b = 1$ then $(b*c)*(a*c) = 1$ and $(c*a)*(c*b) = 1$ (Second weak form of the Hypothetical Syllogism Laws); and

(vii) $a*(b*(a \wedge b)) = 1$.

Proof: All these conditions are proved using the properties of the operator $\mathbb{D}$, specially 2.7 and 3.1, and the fact that $\mathbb{D}(\phi) = \{1\}$. For example, since $c \in \mathbb{D}(a,a*b,a*(b*c))$ is clear, we obtain the first part of (iii), and using (ii) we have that $a*b \in \mathbb{D}(b)$ from where $c \in \mathbb{D}(a,b,(a*b)*(a*c))$ and so we obtain the second part of (iii). The remaining are proved in the same way. □

3.3 Theorem. In all tpBa A and for all $a,b,c \in A$ the following hold:

(i) If $a \leqslant b$ then $c*a \leqslant c*b$ (Left isotony);

(ii) If $a \leqslant b$ then $b*c \leqslant a*c$ (Right isotony);

(iii) $a*(a*b) = a*b$ (Elimination of the repeated premises);

(iv) $a*(b \wedge c) = a*b \wedge a*c$; and

(v) $(a \overset{*}{\vee} b)*c = (a*c) \wedge (b*c)$ (Generalized DeMorgan's Law).

Proof: Unlike the previous theorem, these properties cannot be proved for the three operations in a unified way. If we write $\nrightarrow$, $\Rightarrow$, and $\leadsto$ in terms of I and $\cdot$, (i) and (ii) can quickly be proved using the analogous properties of $\cdot$; (iii) is more ardous and we must take into account that $a \Rightarrow b = I(a * b)$, that $1 \Rightarrow a = \qquad = a$, that $1 \Rightarrow a = Ia$ for all $a \in A$, and use 4.1(a). Parts (iv) and (v) are proved patiently but without any special trick. □

A second group of outstanding properties of the natural implication operations is composed by those which characterize some special types of deductive systems and other related algebraic concepts. We will prove them using 2.6, 2.7 and 3.1, which are the most powerful tools we have, in some cases referring to the results of 3.2 and 3.3 and if necessary inspiring ourselves in [D], [M], and [Ra].

<u>3.4 Definitions</u>. We will say that a deductive system D is:

<u>Irreducible</u> when if $D = D' \cap D''$ then $D = D'$ or $D = D''$ for $D', D'' \in \mathfrak{D}$;

<u>Completely irreducible</u> when for every $\mathfrak{E} \subset \mathfrak{D}$, $\mathfrak{E} \neq \emptyset$, if $D = \bigcap \mathfrak{E}$ then $D \in \mathfrak{E}$;

<u>*-prime</u> when if $a \overset{*}{\vee} b \in D$ then $a \in D$ or $b \in D$ for all $a, b \in A$; and

<u>maximal</u> when if $D \subsetneq D' \in \mathfrak{D}$ then $D' = A$.

<u>3.5 Theorem</u>. For every $D \in \mathfrak{D}$, D is irreducible if and only if D is *-prime. <u>Proof</u>: Assume that D is not *-prime: there are $a, b \notin D$ such that $a \overset{*}{\vee} b \in D$. Then by 2.6 $D = \mathbb{D}(D,a) \cap \mathbb{D}(D,b)$ and this is a non-trivial intersection, so D is not irreducible. Conversely, if we assume that $D = D' \cap D''$ with $D \subsetneq D'$ and $D \subsetneq D''$ we should have $a \in D'$ and $b \in D''$ such that $a, b \notin D$; but $\mathbb{D}(a \overset{*}{\vee} b) = \mathbb{D}(a) \cap \mathbb{D}(b) \subset D' \cap D'' = D$, so $a \overset{*}{\vee} b \in D$ and D is not *-prime. □

With respect to this theorem, see the remark which follows 2.6.

<u>3.6 Theorem</u>. For every $D \in \mathfrak{D}$, the following are equivalent:

(a) D is completely irreducible;

(b) There exists an $a \notin D$ such that $a \in D'$ whenever $D \subsetneq D' \in \mathfrak{D}$; and

(c) There exists an $a \notin D$ such that $b * a \in D$ for all $b \notin D$, $b \in A$.

<u>Proof</u>: Assume that (b) is false: For every $a \notin D$ there is a $D_a \in \mathfrak{D}$

such that $a \notin D_a$ and $D \subsetneq D_a$. Then it is easy to prove that $D = \cap\{D_a : a \notin D\}$ and so D is not completely irreducible. If (b) holds and $b \notin D$, then $D \subsetneq \mathbb{D}(D,b)$ and by hypothesis $a \in \mathbb{D}(D,b)$ which is equivalent to $b*a \in D$ by 2.7. Finally, if we assume (c) and that $D = \cap\{D_i : i \in J\}$ with $D \subsetneq D_i \in \mathfrak{D}$ for all $i \in J$, we have an $a_i \in D_i$ such that $a_i \notin D$; but there is an $a \notin D$ such that $a_i * a \in D$ and so $a_i * a \in D_i$ and by (MP*) is $a \in D_i$ for all $i \in J$; hence $a \in D$ against the assumption. So D is completely irreducible. □

We can add that the family of all completely irreducible deductive systems is the smallest basis of $\mathfrak{D}$ and satisfies certain separation properties.

3.7 Theorem. For every $D \in \mathfrak{D}$ the following conditions are equivalent:

(a) D is maximal;

(b) $D \neq A$ and $a*b \in D$ whenever $a \notin D$ and $b \notin D$ for all $a,b \in A$;

(c) $D \neq A$ and if $(a*b)*b \in D$ then $a \in D$ or $b \in D$ for all $a,b \in A$;

(d) For all $a \in A$, $a \in D$ or $\neg^* a \in D$ but never simultaneously;

(e) $A/{\equiv_D}$ is a simple tpBa (5); and

(f) $A/{\sim_D}$ is a simple pseudo-Boolean algebra, i.e. $A/{\sim_D} = \{\bar{0}, \bar{1}\}$.

Proof: It is easy to prove that (a) implies (b) and that (b) implies (c), using only 2.7 and 3.1. Now assume (c) and suppose that $D \subsetneq D' \in \mathfrak{D}$: there is an $a \in D'$ such that $a \notin D$. From 3.3(iii) it follows that $(a*(a*b))*(a*b) = 1 \in D$ for all $b \in A$, and by (c) and the hypothesis $a*b \in D$ and so $a*b \in D'$ from where we infer that $b \in D'$, that is, $D' = A$: D is maximal. If (b) holds, putting $b = 0$ we have that $a \in D$ or $\neg^* a \in D$ for all $a \in A$; and clearly not simultaneously because $D \neq A$ implies $0 \notin D$; so (b) implies (d); if (d) holds for a given $D \in \mathfrak{D}$, it is easy to see that $0 \in D'$ whenever $D \subsetneq D' \in \mathfrak{D}$, that is, (a) holds. We have thus proved the equivalence between (a), (b), (c) and (d). It is a known result of universal algebra that (a) is equivalent to (e).

If D is maximal, then $\mathfrak{D}^D = \{D,A\}$ and it follows from part (d) and 2.8 that if $a \in D$ then $\bar{a} = \bar{1}$ while if $a \notin D$ then $\bar{a} = \bar{0}$ in $A/\sim_D$, thus establishing that (a) implies (f). If (f) holds, then we have easily that for all $D' \in \mathfrak{D}$, if $D \subset D' \subset A$ then $D'=D$ or $D'=A$, and so D is maximal. □

This theorem is very interesting. For example (d) reminds us that the maximals correspond to the complete consistent theories, while (b) is just a technical tool and (c) relates the maximals with a type of deductive systems known as "strongly prime" ([6]). Conditions (e) and (f) are algebraically canonical.

<u>3.8 Definitions</u>. If A is a tpBa, the <u>radical of A</u>, denoted by <u>R(A)</u> is the intersection of all maximal deductive systems. The <u>$*$-peircean elements</u> are <u>P_*</u> $= \{a \in A :$ There are $b,c \in A$ with $a = ((b*c)*b)*b\}$; the <u>$*$-dense elements</u> are <u>D_*</u>$= \{a \in A : \neg^* a = 0\}$.

These concepts are common in the implicative studies; later we will point out a logical interpretation of them.

<u>3.9 Lemma</u>. For every $D \in \mathfrak{D}$, D is maximal if and only if D is completely irreducible and $D \supset P_*$.

<u>Proof</u>: Clearly if D is maximal then D is completely irreducible. Let $a,b \in A$. If $a \in D$, as we have that $a \in \mathbb{D}((a*b)*a,a)$ we have also that $a*(((a*b)*a)*a) = 1 \in D$, so $((a*b)*a)*a \in D$. If $a \notin D$ and $b \in D$ then $a*b \in D$ because $b \in \mathbb{D}(D,a,b)$; and if $a \notin D$ and $b \notin D$ then it follows from 3.7 that $a*b \in D$; in these last two cases we must have $(a*b)*a \notin D$ and so $((a*b)*a)*a \in D$. In all cases we have shown that $P_* \subset D$. Conversely suppose that D is completely irreducible and $P_* \subset D$, and let $D' \in \mathfrak{D}$ be such that $D \subsetneq D'$, that is, there is an $a \in D'$ with $a \notin D$. By hypothesis, for all $b \in A$, $((a*b)*a)*a \in D$ and so $(a*b)*a \notin D$; by 3.7 we must have $a*b \in D$ and so $a*b \in D'$ from where it follows that $b \in D'$. We have proved that $D' = A$, i.e. D is maximal. □

From this Lemma we immediately obtain the following

<u>3.10 Proposition</u>. In all tpBa it holds that $R(A) = \mathbb{D}(P_*)$. □

In section 4 we will improve 3.8 and 3.10 for particular operations. For the time being we relate the radical with the semisimplicity through two different results which will offer no surprise to the readers familiar with universal algebra and congruence lattices.

<u>3.11 Theorem</u>. In every tpBa A the following are equivalent:

(a) A is semisimple (7);

(b) $R(A) = \{1\}$ that is, $((b*c)*b)*b = 1$ for all $b,c \in A$; and

(c) $((a*0)*a)*a = 1$ for all $a \in A$. □

<u>3.12 Theorem</u>. For every $D \in \mathfrak{D}$ the following are equivalent:

(a) D is an intersection of maximal deductive systems;

(b) $D \supset R(A)$, that is, $D \supset P_*$;

(c) $A/\equiv_D$ is semisimple; and

(d) $A/\sim_D$ is a semisimple pseudo-Boolean algebra, that is, a Boolean algebra. □

The most outstanding feature of the second part of this section (from 3.4 to the end) is that it deals with results formally analogous to those which characterize the same concepts of pseudo-Boolean algebras in terms of the usual implication (·), fact which confirms the pure intuitionistic character of these structures.

<u>4. The intuitionistic implication</u>

In the first place, we can complete 3.2 and 3.3 with some additional properties of this operation:

<u>4.1 Theorem</u>. In every tpBa A and for every $a,b,c \in A$ the following hold:

(i) $a \Rightarrow (b \Rightarrow c) = (a \Rightarrow b) \Rightarrow (a \Rightarrow c)$ (<u>Autodistributivity</u>);

(ii) $a \Rightarrow (b \Rightarrow c) = b \Rightarrow (a \Rightarrow c)$ (Permutation of the Premises) ;

(iii) $a \Rightarrow b \leq (b \Rightarrow c) \Rightarrow (a \Rightarrow c)$ and $a \Rightarrow b \leq (c \Rightarrow a) \Rightarrow (c \Rightarrow b)$ (Hypothetical Syllogism Laws); and

(iv) $a \Rightarrow (b \Rightarrow c) = (a \wedge b) \Rightarrow c$ (Law of Importation-Exportation of the Implication).

Proof: The proofs are reduced to easy but complicated computations in which the properties of the interior I and of the usual implication · are essentially the only ones used. □

A specially interesting property of $\Rightarrow$ is the following:

4.2 Theorem. In every tpBa A holds that $R(A) = D_{\Rightarrow}$.

Proof: If $a \in D_{\Rightarrow}$, $\neg^{\Rightarrow} a = 0$ and it follows from 3.7 that we have $a \in D$ for every $D \in \mathfrak{D}$ maximal, that is, $a \in R(A)$. Conversely, if we suppose that $a \notin D_{\Rightarrow}$, then $I(\neg^{\Rightarrow} a) = \neg^{\Rightarrow} a \neq 0$ from where we infer that there exists a $D \in \mathfrak{D}$ maximal such that $\neg^{\Rightarrow} a \in D$, and therefore $a \notin D$, from where $a \notin R(A)$ follows. □

The previous property, logically interpreted, affirms the coincidence of two types of "almost certain" elements, i.e. those which belong to every complete consistent theory ($R(A)$) and those whose (intuitionistic) negation is false ($D_{\Rightarrow}$). On the other hand the preceding result gives a simple characterization of the $\Rightarrow$-peircean elements:

4.3 Proposition. For all $a \in A$, $a \in P_{\Rightarrow}$ if and only if $a = ((a \Rightarrow 0) \Rightarrow a) \Rightarrow a$.

Proof: In one direction there is nothing to prove. If we suppose that $a \in P_{\Rightarrow}$, since $P_{\Rightarrow} \subset \mathfrak{D}(P_{\Rightarrow}) = R(A) = D_{\Rightarrow}$, it follows that $a \Rightarrow 0 = \neg^{\Rightarrow} a = 0$ and therefore $((a \Rightarrow 0) \Rightarrow a) \Rightarrow a = (0 \Rightarrow a) \Rightarrow a = 1 \Rightarrow a = Ia = a$ since by its own definition every $\Rightarrow$-peircean element is open. □

A more complete and detailed study of the three sets D_{*} and P_{*} and of the relations between them and of them with the radical can be seen in [F]; the results can be summarized in the two chains of inclusions and equalities:

$$P_{\Rightarrow} \subset P_{\rightsquigarrow} \subset P_{\leftrightarrow} = R(A) = D_{\Rightarrow} \subset D_{\rightsquigarrow} = D_{\leftrightarrow}$$

$$D_{\leftrightarrow} \cap B \subset P_{\Rightarrow} = I(P_{\rightsquigarrow}) = P_{\rightsquigarrow} \cap B = P_{\leftrightarrow} \cap B = R(A) \cap B = D_{\Rightarrow} \cap B \ .$$

Moreover, $\leftrightarrow$ also satisfies 4.3 and parts (i), (ii) and (iv) of 4.1 ([8]). $\rightsquigarrow$ does not satisfy any of the previous results.

Although the preceding results are interesting from an algebraic point of view, its importance depends on the result we state next and which says that the operation $\Rightarrow$ completely characterizes the structure of the topological pseudo-Boolean algebras.

4.4 Theorem. Let A be a pseudo-Boolean algebra and $*$ a binary operation on A which satisfies the following properties:

(1) $a*a = 1$; (2) $a*(b*c) = (a*b)*(a*c)$; (3) $a*(b\cdot c) \leqslant a*(b*c)$; and (4) $(a*b)*c \leqslant (a*b)\cdot c$ for all $a,b,c \in A$. Then the unary operator on A defined by $Ia = 1*a$ is an interior operator on A (that is, A is a tpBa) such that $a \Rightarrow b = a*b$ for every $a,b \in A$.

Proof: Using (1) to (4) we can step by step prove (5) $a*1 = 1$, (6) if $a \leqslant b$ then $c*a \leqslant c*b$, (7) $1*a \leqslant a$, and (8) $1*(a*b) = a*b$. So, if we put $Ia = 1*a$, then we have seen that $I1 = 1$, $Ia \leqslant a$, $I^2a = Ia$; and then using (2) to (4) we also obtain $I(a\cdot b) \leqslant Ia\cdot Ib$, that is, I is an interior operator. Finally if $a \Rightarrow b = I(Ia\cdot Ib) = 1*((1*a)\cdot(1*b))$, using (2),(3), (4), (6) and (8) we obtain $a \Rightarrow b = a*b$. □

A half of the conclusion of this theorem can be stated by saying that $\Rightarrow$ is the unique binary operation on a tpBa which satisfies (1) to (4) and $Ia = 1 \Rightarrow a$.

In the first place, an algebraic consequence of this result is that we can define the topological pseudo-Boolean algebras as an equational subclass of the algebras of type (1,2,2,2,2). In the second place, the logical importance 3.1 and 4.4 have together is that they allow us to give a formalization of the intuitionistic modal logic IM4 ([9]) in which the concept of necessity does not intervene at all, being substituted by axioms and rules for a new connective, the "intuitionistic implication", which at the present situation deserves with no doubt a more proper name, and which we would maybe classify as "super-modal" sentence since it seems to combine "de dicto" and "de re" modalities on all its terms.

Specifically, having 1.3 in mind, the previous theorem suggests to us the following set of axiom schemes:

$$p \Rightarrow p$$

$$(p \Rightarrow (q \Rightarrow r)) \rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$$

$$((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) \rightarrow (p \Rightarrow (q \Rightarrow r))$$

$$(p \Rightarrow (q \rightarrow r)) \rightarrow (p \Rightarrow (q \Rightarrow r))$$

$$((p \Rightarrow q) \Rightarrow r) \rightarrow ((p \Rightarrow q) \rightarrow r)$$

that together with the intuitionistic ones and with the two deduction rules

$$\text{(MP)} \quad \{p, p \rightarrow q\} \vdash q$$

$$(\text{MP}\Rightarrow) \quad \{p, p \Rightarrow q\} \vdash q$$

will develop the same logic IM4. In this situation the necessity operator is a derived operator which could be defined, for example, by

$$\text{(Def)} \quad Lp = (p \Rightarrow p) \Rightarrow p \ .$$

We don't known if this is the more adequate definition of L in terms of $\Rightarrow$; this would also depend on the adoption of a philosophical interpretation of the "intuitionistic implication" $\Rightarrow$ and its relation to necessity. We have made no attempts to examine this matter although it seems a quite interesting one, specially after the results we have shown.

NOTES

(1) This material constitutes a portion of the author's Ph.D. Dissertation.

(2) With respect to the pseudo-Boolean algebras, also called Heyting algebras or in their dual form Brouwerian algebras, [McKT] and [Ra] can be read; some special techniques are inspired in [D] and [M]. The theory of abstract logics which is used derives from the one formulated in [BB] and [BS] and has been developped by V. Verdú in [V1]and[V2]. Note that we use some terms of the former in a slightly different sense.

(3) The weak implication is mentioned in [Po] and the intuitionistic one in [M], although we ignore if for the first time.

(4) If $\mathbb{L} = \langle A , \mathbb{C} \rangle$ and $\mathbb{L}' = \langle A' , \mathbb{C}' \rangle$ are two abstract logics, a <u>bilogical morphism</u> of $\mathbb{L}$ in $\mathbb{L}'$ is a mapping h of A onto A' such that $\mathbb{C} = h^{-1} \bullet \mathbb{C}' \circ h$. In these conditions there exists a lattice isomorphism between the closure systems of the closed sets of both logics such that the properties of one of them (as e.g. 2.3, 2.5 to 2.8) are transferred from one to the other.

(5) For us a simple algebra is the one which only has two different congruence relations. It is obvious that a tpBa is simple if and only if $B = \{0,1\}$ and $0 \neq 1$.

(6) For a very wide class of implicative algebras the deductive systems which satisfy the condition of being prime with respect to the operation $a \circ b = (a \cdot b) \cdot b$ are called "strongly prime". In the implication algebras of [Ra] and the Sales algebras and Wajsberg algebras of [Ro] this operation is effectively a supremum. In the case of positive implication algebras (alias Hilbert algebras), in [Pl] is proved that that they coincide with maximal deductive systems.

(7) We understand by a <u>semisimple</u> algebra the one which is a subdirect product of simple algebras.

(8) We want to emphasize $P_{++} = R(A)$ for its analogy with a property of pseudo-Boolean algebras we have not yet seen anywhere, namely $P_{\bullet} = R(A)$. We observe this is a purely implicative result which for its proof uses the negation. We ignore if it remains valid in implicative structures without a negation, e.g. Hilbert algebras.

(9) Observe that 4.4 remains valid if we change "pseudo-Boolean" for "Boolean". Therefore this remark applies in the same way to the classical modal logic S4, fact which has some interest, too.

REFERENCES

[BB] <u>Bloom</u>,S.D.-<u>Brown</u>,D.J. "Classical Abstract Logics" Dissertationes Mathematicae CII (1973) pp 43-51 .

[BS] <u>Brown</u>,D.J.-<u>Suszko</u>,R. "Abstract Logics" Dissertationes Mathematicae CII (1973) pp 9-40 .

[Bu1] <u>Bull</u>,R.A. "Some modal calculi based on IC" in "Formal Systems and Recursive Relations" North-Holland, Amsterdam, 1965, pp 3-7.

[Bu2] <u>Bull</u>, R.A. "A modal extension of intuitionistic logic" Notre Dame Journal of Formal Logic 6 (1965) pp 142-145 .

[D] <u>Diego</u>,A. "Sur les algèbres de Hilbert" Gauthier-Villars, Paris,1966.

[F] Font,J.M. "Estudi algebraic de certes lògiques intuicionistes modals" Publ. of the Univ. of Barcelona, 1982 .

[McKT] <u>McKinsey</u>,J.C.C.-<u>Tarski</u>,A. "On closed elements on closure algebras" Annals of Mathematics 47 (1946) pp 122-162 .

[M] Monteiro,A. "La semisimplicité des algèbres de Boole topologiques et les systèmes déductifs" Revista de la UMA XXV (1971) pp 417-448.

[Pl] Pla,J. "Sobre l'estructura reticular de les àlgebres de Hilbert" Actas del V Congreso de la Agrupación de Matemáticos de Expresión Latina (Madrid, 1978) pp 209-210 .

[Po] Porte,J. "Quelques extensions du théorème de la déduction" Revista de la UMA XX (1962) pp 259-266 .

[Ra] Rasiowa,H. "An algebraic approach to non-classical logics" North--Holland, Amsterdam, 1974 .

[Ro] Rodríguez,A.J. "Un estudio algebraico de los cálculos proposicionales de Lukasiewicz" Ph.D. Thesis, Univ. of Barcelona, 1980.

[V1] Verdú,V. "Contribució a l'estudi de certs tipus de lògiques abstractes" Publ. of the Univ. of Barcelona, 1980 .

[V2] Verdú,V. "Lògiques distributives i Booleanes" Stochastica 3 (1979) pp 97-108 .

[O] Ono, H. "On some intuitionistic modal logics" Publ. of the RIMS, Kyoto Univ., 13 (1977) pp 687-722 .

publicaciones
ediciones
universidad
de barcelona

Dipòsit Legal B.: 25.143-1982
BARCELONA–1982