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1. *Introduction*

The theory of localization first introduced in topology by D. Sullivan [6] has been exploited by many topologists, e.g. A.K. Bousfield - D.M. Kan [1], P. Hilton - G. Mislin - J. Roitberg [2], and works particularly well in the homotopy category of nilpotent spaces, i.e. spaces whose Postnikov system admits a principal refinement.

Now let $f:E \longrightarrow B$ be a fibre map with E,B and the fibre F connected CW-complexes. Then $\pi_1 E$ acts on the homotopy groups $\pi_n F$, $n \geq 1$, and we say [2] that f is nilpotent if $\pi_1 E$ acts nilpotently on all the homotopy groups of F. This is the case, for example, if E and B are nilpotent. If f is a nilpotent fibre map then the fibre F is nilpotent. The conversely is true only if $\pi_1 B$ operates nilpotently on the homology groups of F [3].

Nilpotent fibre maps turns out to be the right relativization of nilpotent spaces, because their Moore-Postnikov systems admit principal refinements ([2] th. II.2.4).

In [4] P. Hilton has developed a localization theory for the corresponding group-theoretical situation, that of group extensions $N \rightarrowtail G \twoheadrightarrow Q$ such that G acts nilpotently on N.

In this paper we introduce a P-localization theory (where P is a given family of rational primes) for nilpotent fibre maps which generalizes both the existing theory of localization of nilpotent spaces and the localization theory for relative groups of [4]. These ideas had already been treated systematically in [1] with a slight different definition of nilpotent action.

In fact in §2 we prove an universal property which shows in particular the uniqueness of the localization of a nilpotent fibre map and in §3 we construct this licalization in a similar way as is done in [2] for the absolute case.

I wish to thank P. Hilton for suggesting this work to me as well as for some helpful conversations.

2. *An universal property*

We work in the pointed homotopy category of connected CW-complexes. We say that a fibre map $f:E\longrightarrow B$ with connected fibre F is nilpotent if $\pi_1 E$ acts nilpotently on $\pi_n F$, $n\geqslant 1$. Recall that if ω is an action of a group G on N the lower central ω-series of N is defined by

$$\Gamma^0_\omega N = N\,,\qquad \Gamma^i_\omega N = \langle a(xb)a^{-1}b^{-1}; x\in G, a\in N, b\in\Gamma^{i-1}_\omega N\rangle\,,\ i>0\,;$$

and we say that G acts nilpotently on N if $\Gamma^c_\omega N = \{1\}$ for c sufficiently large.

Let now $F\longrightarrow E\xrightarrow{f} B$ be a nilpotent fibre map and let P be an arbitrary collection of rational primes. A P-localization of f is a map $\beta:f\longrightarrow f_{(P)}$ over B, where the map $f_{(P)}:E_{(P)}\longrightarrow B$ is a fibre map with fibre F_P such that the induced map $F\longrightarrow F_P$ is a P-localization:

$$\begin{tikzcd} F \arrow[r] \arrow[d, "e"'] & E \arrow[r, "f"] \arrow[d, "\beta"'] & B \arrow[d] \\ F_P \arrow[r] & E_{(P)} \arrow[r, "f_{(P)}"] & B \end{tikzcd}$$

Theorem 2.1. Let $F\longrightarrow E\xrightarrow{f} B$ be a nilpotent fibre map and let $\beta:f\longrightarrow f_{(P)}$ a P-localization of f. For every nilpotent fibre map $\bar f:\bar E\longrightarrow B$ with P-local fibre $\bar F$,

$$[f_{(P)},\bar f]_B \xrightarrow{\beta^*} [f,\bar f]_B$$

is a bijection, where $[\ ,\ ]_B$ stands for the set of homotopy classes of maps over B.

Proof: Given $\tau \in [f, \bar{f}]_B$ we will show that there is an unique $\delta \in [f_{(P)}, \bar{f}]_B$ such that $\beta^* \delta = \tau$, i.e. rendering commutative the following diagram

$$\begin{tikzcd} E \arrow[r, "\tau"] \arrow[d, "\beta"'] & \bar{E} \arrow[d, "\bar{f}"] \\ E_{(P)} \arrow[r, "f_{(P)}"] \arrow[ur, dashed, "\delta"] & B \end{tikzcd}$$

Let

$$\ldots \longrightarrow Y_c \xrightarrow{q_c} Y_{c-1} \longrightarrow \ldots \longrightarrow Y_0 = B$$

be a principal refinement of the Mooré-Postnikov factorisation of $\bar{f}$. Each $q_c : Y_c \longrightarrow Y_{c-1}$ is a principal fibre map with fibre an Eilenberg-MacLane space $K(G_c, n)$ where G_c are (abelian) groups obtained as quotients $\Gamma^i_w \pi_n \bar{F} \,/\, \Gamma^{i+1}_w \pi_n \bar{F}$ (w the action of $\pi_1 \bar{E}$ on $\pi_n \bar{F}$). These groups are trivial $\pi_1 \bar{E}$-modules: for $n \geq 2$ it is clear; for $n=1$ use that w is characterized by $i_*(\xi . \alpha) = (i_* \xi).\alpha$, $\xi \in \pi_1 \bar{F}$, $\alpha \in \pi_1 \bar{E}$, $i : \bar{F} \longrightarrow \bar{E}$ the inclusion, and that $\pi_1 \bar{E}$ acts on itself by conjugation.

The obstruction to the lifting of $f_{(P)}$ to Y_1 lies in $H^2(E_{(P)}, E; G_1)$ and the higher obstructions -to each Y_c- are subsets of $H^n(E_{(P)}, E; G_c)$. Furthermore, at each step c, two liftings maps δ'_c and δ''_c of the same δ_{c-1} are homotopic relative to δ_{c-1} if a certain cocycle of $H^{n-1}(E_{(P)}, E; G_c)$ vanishes. Hence the existence and uniqueness of δ follow from Lemmata 2.2 and 2.3 below, which ensure the vanishing of all groups $H^n(E_{(P)}, E; G_c)$.

Lemma 2.2. Let A be a P-local abelian group. Then, for all $n \geq 0$, $H^n(E_{(P)}, E; A) = 0$.

Proof: We look at the induced homomorphism β^* in the Serre spectral sequences of f and $f_{(P)}$

$$\begin{array}{ccc} H^p(B;\ \mathcal{H}^q(F_P;A)) & \Longrightarrow & H^{p+q}(E_{(P)};A) \\ \big\downarrow e^* & & \big\downarrow \beta^* \\ H^p(B;\ \mathcal{H}^q(F;A)) & \Longrightarrow & H^{p+q}(E;A) \end{array}$$

Since $e:F \longrightarrow F_P$ is a P-localization, by the universal coefficient theorem and [2] I.2.9 and I.1.8

$$H^n(F_P;A) \xrightarrow{e^*} H^n(F;A)$$

is an isomorphism in each dimension. Thus we get an isomorphism at the E_2 level of the Serre spectral sequences and the lemma follows.

<u>*Lemma 2.3.*</u> The groups G_c are P-local.

Proof: Since $\pi_1\bar{F}$ are P-local nilpotent groups $\Gamma^i_\omega\ \pi_n\bar{F}$ are P-local by theorem 2.1 of [5]. Hence G_c are P-local as quotients of P-local nilpotent groups.

<u>*Corollary 2.4.*</u> Let $F \longrightarrow E \xrightarrow{f} B$ be a nilpotent fibre map. If $\beta: f \longrightarrow f_{(P)}$ is a P-localization with $f_{(P)}$ nilpotent, then $f_{(P)}$ and β are uniquely determined up to an homotopy equivalence.

Proof: Let $\bar{\beta}:f \longrightarrow \bar{f}$ be another P-localization of f with $\bar{f}$ nilpotent. By theorem 2.1 there exist $\delta_1:\bar{f} \longrightarrow f_{(P)}$ and $\delta_2:f_{(P)} \longrightarrow \bar{f}$ such that $\bar{\beta}=\delta_1\beta$ and $\beta=\delta_2\bar{\beta}$; thus $\beta=\delta_2\delta_1\beta$ and $\bar{\beta}=\delta_1\delta_2\bar{\beta}$ and therefore $\delta_2\delta_1 \simeq id$ and $\delta_1\delta_2 \simeq id$ by the uniqueness in theorem 2.1.

Corollary 2.5. Let $F \to E \xrightarrow{f} B$ be a nilpotent fibre map and let $\beta : f \to f_{(P)}$ a P-localization. For every nilpotent fibre map $\bar{f} : \bar{E} \to \bar{B}$ with P-local fibre $\bar{F}$ and every map $\rho : B \to \bar{B}$,

$$[f_{(P)}, \bar{f}]_\rho \xrightarrow{\beta^*} [f, \bar{f}]_\rho$$

is a bijection.

Proof: Consider the diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "e"'] & E \arrow[r, "f"] \arrow[d, "\beta"] \arrow[dd, bend right, "\eta"'] \arrow[ddd, bend right=60, "\tau"'] & B \arrow[d, equal] \\
F_P \arrow[r] & E_{(P)} \arrow[r, "f_{(P)}"] \arrow[d, dashed, "\delta"'] & B \arrow[d, equal] \\
\bar{F} \arrow[r] \arrow[d, equal] & \tilde{E} \arrow[r, "\tilde{f}"] \arrow[d, "\sigma"'] & B \arrow[d, "\rho"] \\
\bar{F} \arrow[r] & \bar{E} \arrow[r, "\bar{f}"] & \bar{B}
\end{tikzcd}$$

where $\tilde{f}$ is induced by ρ. Every $\tau \in [f, \bar{f}]_\rho$ induces a map $\eta \in [f, \tilde{f}]_B$ and we can apply theorem 2.1 to ensure the existence of $\delta \in [f_{(P)}, \tilde{f}]_B$ such that $\delta\beta = \eta$. Hence $\beta^*(\sigma\delta) = \tau$. Observe that the nilpotency of $\tilde{f}$ follows from that of $\bar{f}$.

A standard argument shows the uniqueness of δ.

3. *Existence of a P-localization*

Let $F \to E \xrightarrow{f} B$ be a nilpotent fibre map and let

$$\cdots \longrightarrow Y_e \xrightarrow{q_e} Y_{e-1} \longrightarrow \cdots \longrightarrow Y_1 \xrightarrow{q_1} Y_0 = B$$

be a principal refinement of its Moore-Postnikov system. We will construct a P-localization of f by induction on the height of the system.

Suppose first we have a principal fibre map

$$K(G,1) \longrightarrow Y \xrightarrow{q} B$$

induced by a map $a : B \longrightarrow K(G,2)$. We define $q_{(P)} : Y_{(P)} \longrightarrow B$

as the principal fibre map induced by $e_{\bullet}g : B \longrightarrow K(G_p, 2)$ where e is a P-localization

$$
\begin{tikzcd}
& K(G,1) \arrow[rr, equal] \arrow[dl, "e"'] \arrow[dd] & & K(G,1) \arrow[dl, "e"'] \arrow[dd] \\
K(G_p,1) \arrow[rr, equal, crossing over] \arrow[dd] & & K(G_p,1) & \\
& Y \arrow[rr] \arrow[dl, dashed, "\beta"'] \arrow[dd, "q"] & & PK(G,2) = * \arrow[dl] \arrow[dd] \\
Y_{(P)} \arrow[rr, crossing over] \arrow[dd, "q_{(P)}"'] & & PK(G_p,2) = * \arrow[from=uu, crossing over] & \\
& B \arrow[rr, "g"] \arrow[dl, equal] & & K(G,2) \arrow[dl, "e"] \\
B \arrow[rr] & & K(G_p,2) \arrow[from=uu, crossing over] &
\end{tikzcd}
$$

Let $\beta : Y \longrightarrow Y_{(P)}$ be induced by $q : Y \longrightarrow B$ and $Y \longrightarrow PK(G_p, 2)$. From the commutativity of the above diagram it follows that the map $K(G,1) \longrightarrow K(G_p, 1)$ induced by β on the fibres is a P-localization.

Let $s_{c-1} = q_1 \dots q_{c-1} : Y_{c-1} \longrightarrow B$ and assume now that we have already defined $s_{c-1(P)} : Y_{c-1(P)} \longrightarrow B$ and also $\beta : Y_{c-1} \longrightarrow Y_{c-1(P)}$ over B inducing a P-localization on the fibres.

If $q_c : Y_c \longrightarrow Y_{c-1}$ is induced by $g : Y_{c-1} \longrightarrow K(G, n+1)$ we define $q_{c(P)} : Y_{c(P)} \longrightarrow Y_{c-1(P)}$ as the principal fibre map induced by a $\delta : Y_{c-1(P)} \longrightarrow K(G_p, n+1)$ such that $\delta\beta = e\, g$ where e is a P-localization. See diagram below.

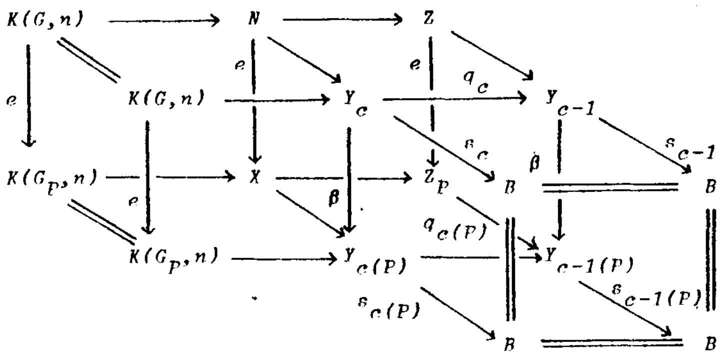
$$\begin{tikzcd}
& K(G,n) \arrow[rr, equal] \arrow[dl, "e"'] \arrow[dd] & & K(G,n) \arrow[dl, "e"'] \arrow[dd] \\
K(G_P,n) \arrow[rr, equal, crossing over] \arrow[dd] & & K(G_P,n) & \\
& Y_c \arrow[rr] \arrow[dl, "\beta"'] \arrow[dd, "q_c"] & & PK(G,n+1)=* \arrow[dl] \arrow[dd] \\
Y_{c(P)} \arrow[rr, crossing over] \arrow[dd, "q_{c(P)}"'] & & PK(G_P,n+1)=* \arrow[from=uu, crossing over] & \\
& Y_{c-1} \arrow[rr, "g"] \arrow[dl, "\beta"'] & & K(G,n+1) \arrow[dl, "e"] \\
Y_{c-1(P)} \arrow[rr, dashed, "\delta"] & & K(G_P,n+1) \arrow[from=uu, crossing over] &
\end{tikzcd}$$

We can argue as in the first step in order to obtain a map $\beta : Y_c \longrightarrow Y_{c(P)}$ inducing a P-localization on the fibres.

The existence and uniqueness of δ follow then applying Corollary 2.5 to

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "e"'] & Y_{c-1} \arrow[r, "s_{c-1}"] \arrow[d, "\beta"] \arrow[dd, bend right, "e\,g"'] & B \arrow[d, equal] \\
Z_P \arrow[r] & Y_{c-1(P)} \arrow[r, "s_{c-1(P)}"] \arrow[d, dashed, "\delta"] & B \arrow[d] \\
K(G_P,n+1) \arrow[r] & K(G_P,n+1) \arrow[r] & *
\end{tikzcd}$$

It remains only to see that $\beta : s_c \longrightarrow s_{c(P)}$ induces a P-localization on the fibres. Consider the diagram of fibre maps



We may apply now the homotopy exact sequence to the back squares to conclude that $\pi_i N \longrightarrow \pi_i X$ is a P-localization for all i. Hence $N \longrightarrow X$ is a P-localization.

If the refined principal Moore-Postnikov system is finite the above procedure yields the desired P-localization. In the general case there is a weak homotopy equivalence $E \longrightarrow \varprojlim Y_c$ over B. Let $E_{(P)}$ be the geometric realization of the singular complex of $\varprojlim Y_{c(P)}$. Then there is a map $\beta : E \longrightarrow E_{(P)}$ such that the diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\beta"'] & \varprojlim Y_c \arrow[d, "\varprojlim \beta"] \\ E_{(P)} \arrow[r] & \varprojlim Y_{c(P)} \end{tikzcd}$$

is homotopy-commutative over B.

Let F_c be the fibre of $s_c : Y_c \longrightarrow B$. The fibre maps $\varprojlim Y_c \longrightarrow B$ and $E \longrightarrow B$ have weak homotopy equivalent fibres $\varprojlim F_c$ and F respectively. Analogously $\varprojlim F_{c(P)}$ and

F_P are weak homotopy equivalent. Moreover the induced map $\varprojlim F_c \longrightarrow \varprojlim F_{c(P)}$ is a P-localization. Thus

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "e"'] & E \arrow[r, "f"] \arrow[d, "\beta"'] & B \arrow[d, equal] \\
F_P \arrow[r] & E_{(P)} \arrow[r, "f_{(P)}"] & B
\end{tikzcd}$$

is a P-localization.

Note that $\ldots \longrightarrow Y_{c(P)} \longrightarrow Y_{c-1(P)} \longrightarrow \ldots \longrightarrow B$ is a principal refinement of the Moore-Postnikov system of $f_{(P)}$; the fibre map $f_{(P)}$ is therefore nilpotent.

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