UNIVERSITAT DE BARCELONA
FACULTAT DE MATEMÀTIQUES

ON FLEXES OF THE KUMMER VARIETY
(Note on a theorem of R. C. Gunning)

by G. E. Welters

PRE–PRINT N.º 14
abril 1983

On flexes of the Kummer variety

(Note on a theorem of R.C. Gunning)

G. E. Welters
Facultad de Matemáticas
Universidad de Barcelona
Barcelona / Spain

In his paper [3], R.C. Gunning has given a new characterization of Jacobi varieties among all principally polarized abelian varieties, by using trisecants of the associated Kummer variety. The present paper is motivated by the link between Gunning's results and the -as yet unanswered- question about the Novikov Hypothesis. Our main statement is Theorem (3.1), which is just a more general version of the key result of [3], allowing also limit cases of the original assumptions. Section 3 is devoted to the proof of this statement. In particular, one obtains similar characterizations of jacobians by means of flexes instead of trisecants (cf Section 1).

After putting Novikov's Hypothesis in geometrical terms (cf (2.18)), its relationship with this version of Gunning's result becomes more apparent. The comparison suggests some intermediate questions which might be useful. We discuss this more closely in Section 2.

In Sections 1 and 2 we assume the groundfield k to be the field $\mathbb{C}$ of complex numbers; in the rest of the paper k is an algebraically closed field of arbitrary characteristic different from 2.

0. Notations and definitions

Let X be a principally polarized abelian variety over k. Let θ be any symmetric theta divisor of X, and call $L = \mathcal{O}_X(\theta)$ the associated line bundle. The linear system $|2\theta|$ is independent from the particular choice of θ, and we write $M = L^{\otimes 2}$ for the corresponding line bundle. Put $g = \dim X$; the global sections of M span a vector space of dimension 2^g, and these correspond classically with the second order theta functions (with zero characteristics).

We shall assume X to be an irreducible principally polarized abelian variety, i.e. that the theta divisor θ is irreducible. In this case, the induced map

$$(0.1) \qquad \phi : X \longrightarrow \mathbb{P}(H^0M) = \mathbb{P}^N, \qquad N = 2^g-1$$

is a (2:1) morphism onto its image. As a matter of fact, ϕ factors through the projection of X onto is Kummer variety $K(X) = X/\{\pm 1\}$, embedding the latter variety into $\mathbb{P}^N$ (cf e.g. [8]). We are interested in trisecants of K(X) and, more particularly, in the limit case of flexes of K(X), that is, lines in $\mathbb{P}^N$ meeting K(X) with multiplicity at least 3 at some smooth point of K(X). (Note that the singular points of K(X) are the images of the points of order two of the abelian variety X, if $g \geq 2$.)

(0.2) DEFINITION. Let $Y \subset X$ be an artinian subscheme of length 3 of a principally polarized abelian variety. The subscheme Y will be called a "secant" subscheme of X if and only if there exists some line $\ell \subset \mathbb{P}^N$ with $Y \subset \phi^{-1}(\ell)$. Equivalently, if and only if the restriction map $H^0M \longrightarrow H^0(M \otimes \mathcal{O}_Y)$ fails to be surjective.

1. Jacobians and flexes

(1.1) Suppose that X is the polarized jacobian of some smooth curve C. Then the Kummer variety K(X) is known to have lots of 3-secants (cf e.g. [6], p.80): Fix three distinct points $a,b,c \in C$. Then, for any

$$\zeta \in \tfrac{1}{2}(C-a-b-c) \subset \mathrm{Pic}^{-1}(C),$$

the points of $\mathbb{P}^N$:

$$\psi(\zeta+a), \quad \psi(\zeta+b), \quad \psi(\zeta+c)$$

are collinear (here the factor ½ denotes counterimage by the multiplication by 2 isogeny). The line ℓ which they determine is a trisecant of K(X) and $\zeta+\{a,b,c\} \subset \psi^{-1}(\ell)$.

By using (0.2), we may rephrase (1.1) as follows: Let $\Gamma \subset X$ be the image of C in X = JC, embedded by translation with an arbitrary element of $\mathrm{Pic}^{-1}(C)$. Then, for any three distinct points $\alpha,\beta,\gamma \in \Gamma$ we obtain a one-dimensional family of secant subschemes $\zeta+Y$ of X, where

$$Y = \{\alpha,\beta,\gamma\} \subset X \quad \text{and} \quad \zeta \in \tfrac{1}{2}(\Gamma-\alpha-\beta-\gamma) \subset X.$$

Moreover, it is known (cf [3]) that, putting:

$$\tilde{V} = \{\zeta \in X \mid \zeta+Y \text{ is a secant subscheme of } X\},$$

one has

$$\tilde{V} = \tfrac{1}{2}(\Gamma-\alpha-\beta-\gamma)$$

and $V = 2\tilde{V}$ is a copy of the curve C embedded in its jacobian. (The factor 2 denotes image by the multiplication by 2 isogeny.)

Conversely, start with a principally polarized abelian variety X, and three distinct points $\alpha,\beta,\gamma \in X$. Define Y and $\tilde{V}$ as above; the set $\tilde{V}$ is an algebraic subvariety of X and one clearly has an inclusion

$$-(\alpha+\beta+\gamma)+Y \subset 2\tilde{V}.$$

In this setting, Gunning proves, among other things:

(1.2) <u>THEOREM</u>. (Gunning, [3]). Assume that X is an irreducible principally polarized abelian variety, and that $2\tilde{V}$ is positive-dimensional at some point of $Y' = -(\alpha+\beta+\gamma)+Y = \{-\alpha-\beta,\ -\alpha-\gamma,\ -\beta-\gamma\}$. Then $2\tilde{V}$ is smooth of dimension one at all three points and there is an irreducible curve $V \subset 2\tilde{V}$ containing them. The endomorphism

$$\alpha_V\colon X \longrightarrow X$$

attached to this 1-cycle of X satisfies:

$$(\alpha_V - I)|Y' = \text{constant}.$$

(We recall that α_V is defined by $\alpha_V(x) = S((\theta_x-\theta)\cdot V)$ for general $x \in X$). In particular, if there are no non-zero complex multiplications of X mapping $\beta-\alpha$ and $\gamma-\alpha$ into zero, it follows that $\alpha_V = I$; hence, by Matsusaka's criterion, X is the jacobian of the (smooth) curve V. Since in the case of a jacobian X = JC one may choose α,β,γ such that the above condition on complex multiplications is satisfied, this yields a characterization of Jacobi varieties among principally polarized abelian varieties (Loc. cit.).

(1.3) We remark another easy consequence of (1.2) (below we shall prove a similar fact, and the ideas are the same): The presence of an irreducible curve Γ on an irreducible principally polarized abelian variety X, satisfying the property that, for general $\alpha,\beta,\gamma \in \Gamma$ and $\zeta \in \frac{1}{2}(\Gamma-\alpha-\beta-\gamma)$,

$$\phi(\zeta+\alpha), \qquad \phi(\zeta+\beta), \qquad \phi(\zeta+\gamma)$$

are collinear in $\mathbb{P}^N$ is a property that characterizes jacobians. The reader will notice that one may even assume β and γ to be fixed (but otherwise generally chosen) in this condition.

(1.4) We want to infinitesimalize the data in (1.2). To this end, we go back first to (1.1) and let the points a,b,c of C collapse to a single point $x \in C$ or, rather, to the divisor 3x of C. By continuity, we obtain from (1.1): For any

$$\zeta \in \tfrac{1}{2}(C-3x) \subset \mathrm{Pic}^{-1}(C),$$

the subscheme

$$\zeta + \mathrm{Spec}(\mathcal{O}_{C,x}/m^3_{C,x}) \subset X$$

is a secant subscheme of X. Putting it in other words, writing

$$(1.5) \qquad Y_x = -x + \mathrm{Spec}(\mathcal{O}_{C,x}/m^3_{C,x}) \subset X,$$

we have a one-dimensional family of secant subschemes of X:

$\{\zeta + Y_x \mid \zeta \in \tfrac{1}{2}(C-x)\}$.

We aim to reverse things to some extent. In this connection, the following will be proved in Section 3 (cf Theorem (3.1)):

(1.6) VARIATION (of (1.2)). Let X be an irreducible principally polarized abelian variety, and let $Y \subset X$ be a subscheme with $Y \cong \operatorname{Spec} k[\epsilon]/\epsilon^3$ supported, say, at the origin $0 \in X$. Define the algebraic subvariety of X:

(1.7) $\tilde{V} = \{\zeta \in X \mid \zeta + Y \text{ is a secant subscheme of } X\}$.

(Notice that $0 \in 2\tilde{V}$.) Assume that the dimension of $2\tilde{V}$ at the origin is positive. Then $2\tilde{V}$ is smooth one-dimensional at 0. Call V the irreducible component of $2\tilde{V}$ at 0; then $Y \subset V$ and the endomorphism $\alpha_V: X \longrightarrow X$ attached to this 1-cycle of X satisfies $\alpha_V \mid Y = I$.

In analogy with (1.3), we deduce now from (1.6):

(1.8) COROLLARY. Let X be an irreducible principally polarized abelian variety. Then X is a polarized jacobian if and only if there exists an irreducible curve $\Gamma \subset X$ such that, for general $x \in \Gamma$ and $\zeta \in \tfrac{1}{2}(\Gamma - x)$, $\zeta + Y_x$ is a secant subscheme of X. Moreover, in this case Γ is smooth and $X = J\Gamma$.

PROOF. This condition is necessary, by (1.4). Conversely, the assumption implies that for general $x \in \Gamma$ one has: $\tfrac{1}{2}(\Gamma - x) \subset \tilde{V}_x$, where $\tilde{V}_x$ is the variety defined by (1.7) with $Y = Y_x$. Therefore, by (1.6) applied to $V = \Gamma - x$, we infer $\alpha_{\Gamma-x} \mid Y_x = I$ for general $x \in \Gamma$. Since $\alpha_\Gamma = \alpha_{\Gamma-x}$ for all x, we may write finally $d(\alpha_\Gamma - I)(x) = 0$ for general $x \in \Gamma$. Therefore $(\alpha_\Gamma - I)|\Gamma$ is a constant map and, by translating Γ if necessary, we may assume that $0 \in \Gamma$, hence

$$\alpha_\Gamma = I \quad \text{on } \Gamma .$$

Let $A \subset X$ be the abelian subvariety of X generated by Γ. Restricting the polarization of X to A we get an ample divisor class $[D]$ on A. We consider the endomorphism of A attached to Γ and D, defined by

$$\alpha'_\Gamma(a) = S((D_a - D)\cdot\Gamma)$$

for general $a \in A$. Clearly, $\alpha'_\Gamma = \alpha_\Gamma | A = I$, since Γ generates A. Therefore, by the Criterion of Matsusaka ([4]), Γ is smooth and we have an isomorphism of polarized abelian varieties $(A,[D]) \cong (J\Gamma, \theta_\Gamma)$. By the semisimplicity property of the category of principally polarized abelian varieties and the irreducibility of X we conclude that X is the polarized jacobian of Γ, as claimed.

2. Infinitesimalization

We denote again by X an irreducible principally polarized abelian variety of dimension g. Let $Y \hookrightarrow X$ be an artinian subscheme of length 3. We want to sharpen an earlier definition where we considered the reduced subvariety

$$(2.1) \qquad \tilde{V}_Y = \{\zeta \in X \mid \zeta + Y \text{ is a secant subscheme of } X\}$$

(cf (0.2)), and introduce a natural scheme structure on $\tilde{V}_Y$.

Taking for each $x \in X$ the subscheme $x+Y \hookrightarrow X$, one obtains a family

$$\begin{array}{ccc} \mathcal{Y} & \hookrightarrow & X \times X \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & X & \end{array}$$

($p^{-1}(x)$ being embedded as $(x,x+Y)$). Restriction of sections of M to the subschemes $x+Y$ defines a morphism of locally free sheaves on X:

$$(2.2) \qquad (H^oM)\otimes_k \mathcal{O}_X \xrightarrow{\ \varphi\ } R^o_p(\mathcal{O}_{\mathcal{Y}} \otimes pr_2^*M).$$

The set $\tilde{V}_Y$ consists of the points $x \in X$ at which the pointwise fiber of this morphism is of rank ≤ 2. We define a scheme structure on $\tilde{V}_Y$ by taking the scheme of zeros of the morphism

$$(2.3) \qquad \Lambda^3(H^oM)\otimes_k \mathcal{O}_X \xrightarrow{\ \Lambda^3\varphi\ } \Lambda^3 R^o_p(\mathcal{O}_{\mathcal{Y}} \otimes pr_2^*M).$$

Writing $\mathcal{L}$ for the invertible sheaf at the right hand side of (2.3), one has, by definition now, an exact sequence:

$$(2.4) \qquad \Lambda^3(H^oM)\otimes_k \mathcal{L}^{\vee} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{\tilde{V}_Y} \longrightarrow 0.$$

Throughout this section, we shall assume that $k = \mathbb{C}$, and also that $Y \cong \operatorname{Spec} k[\epsilon]/\epsilon^3$, supported at $0 \in X$ (See Remark (2.25)).

Locally, the subscheme $\tilde{V}_Y$ of X can be described formally by means of theta functions. Let B be a period matrix for X, and identify as usual $X = \mathbb{C}^g/(I \mid B)$. Writing $\theta^o, \ldots, \theta^N$, $N=2^g-1$, a basis of the vector space of second order theta functions for B, the mapping ϕ of (0.1) is given by

$$(2.5) \qquad x \longmapsto (\theta^o(x) : \ldots : \theta^N(x))$$

(In the right hand side member, the symbol x is to be understood as a representative in $\mathbb{C}^g$ for $x \in X$. Here and below, this abuse of language will cause no harm, and simplifies the notations). We introduce for convenience the vector notation: $\vec{\theta} = (\theta^0,\dots,\theta^N)$.

To give a subscheme $Y \hookrightarrow X$ as above amounts to give a pair of constant (= translation invariant) differential operators $\Delta_1 \neq 0$ and Δ_2 on X satisfying, together with Δ_o = Identity:

$$(2.6) \qquad \text{for all functions } a,b: \quad \Delta_i(ab) = \sum_{k+l=i} \Delta_k(a)\Delta_l(b).$$

The embedding $\mathrm{Spec}(k[\varepsilon]/\varepsilon^3) \hookrightarrow X$ then corresponds to the ring homomorphism:

$$\mathcal{O}_{X,0} \longrightarrow k[\varepsilon]/\varepsilon^3, \qquad f \longmapsto \sum_{i=o}^{2} \Delta_i(f)(0)\varepsilon^i.$$

The operators Δ_1, Δ_2 are given equivalently by a pair of constant vector fields $D_1 \neq 0$ and D_2 on X, by the formulae:

$$(2.7) \qquad \Delta_1 = D_1, \qquad \Delta_2 = \tfrac{1}{2}D_1^2 + D_2.$$

It is easily seen that a couple (D_1',D_2') defines the same subscheme as (D_1,D_2) if and only if there are constants $a \neq 0, b$ such that

$$(2.8) \qquad D_1' = aD_1, \quad D_2' = a^2D_2 + bD_1.$$

In these terms, a point $x \in X$ belongs to the set $\tilde{V}_Y$ if and only if

$$(2.9) \qquad \mathrm{rk}(\vec{\theta}(x),\ (D_1\vec{\theta})(x),\ ((\tfrac{1}{2}D_1^2+D_2)\vec{\theta})(x)) \leq 2.$$

As for the scheme structure introduced on $\tilde{V}_Y$ by (2.4), the ideal of $\hat{\mathcal{O}}_{X,x}$ defining $\hat{\mathcal{O}}_{\tilde{V}_Y,x}$ is generated by the functions f_{ijk}, $0 \leq i < j < k \leq N$:

$$(2.10)\qquad f_{ijk} = \det\begin{pmatrix} \theta^i(x) & (D_1\theta^i)(x) & ((\tfrac{1}{2}D_1^2+D_2)\theta^i)(x) \\ \theta^j(x) & (D_1\theta^j)(x) & ((\tfrac{1}{2}D_1^2+D_2)\theta^j)(x) \\ \theta^k(x) & (D_1\theta^k)(x) & ((\tfrac{1}{2}D_1^2+D_2)\theta^k)(x) \end{pmatrix}$$

In the rest of the present section, we discuss some elementary facts about the scheme $\tilde{V}_Y$. In the first place, observe that

$$(2.11)\qquad \tilde{V}_Y = \tfrac{1}{2}(2\tilde{V}_Y)$$

(the meaning of the factors 2 and ½ being the same as in Section 1). This is due to the fact that the group ${}_2X$ acts both on X and on $|2\theta|$ (by translations) and that the mapping Φ of (0.1) is equivariant for this action. We define

$$(2.12)\qquad V_Y = 2\tilde{V}_Y.$$

The study of $\tilde{V}_Y$ is equivalent to that of V_Y and, as it seems, the latter scheme is a more natural object to deal with.

We notice that $0 \in V_Y$; this follows by using $(d\Phi)(0) = 0$. We are interested in the study of V_Y at 0. To his end, we introduce a notation: for all $h \geq 1$, put

$$(2.13)\qquad (V_Y)_h = \mathrm{Spec}(\mathcal{O}_{V_Y,0}/m^{h+1}_{V_Y,0}) \hookrightarrow X.$$

Then one has:

(2.14) PROPOSITION. There is an identity of subschemes of X: $(V_Y)_2 = Y$.

PROOF. Assume $Y \hookrightarrow X$ to be given by vector fields D_1, D_2 as in (2.7). In the first place, $T_{V_Y}(0) = \langle D_1 \rangle$ holds. (We identify, as usual, $T_X(0)$ with H^0T_X). To see this, if $D \in H^{O}T_X$ then $D \in T_{V_Y}(0)$ if and only if $(Df_{ijk})(0) = 0$ for all f_{ijk}. Using (2.10), and taking into account that odd derivatives of the functions θ^i vanish at the origin, this is written finally as:

(2.15) $$\mathrm{rk}(\vec{\theta}(0),\ (DD_1\vec{\theta})(0),\ (D_1^2\vec{\theta})(0)) \leq 2.$$

On the other side, it is well known that the irreducibility of X implies that, if $\partial_1, \ldots, \partial_g$ is a basis of $H^{O}T_X$, one has

(2.16) $$\mathrm{rk}(\vec{\theta}(0),\ ((\partial_i \partial_j \vec{\theta})(0))_{i \leq j}) = \tfrac{1}{2}\, g(g+1) + 1$$

(cf Remark (2.25)). In view of this, (2.15) is equivalent with $D \in \langle D_1 \rangle$, as claimed.

To end the proof, it suffices to show that $Y \hookrightarrow V_Y$. This in turn is equivalent with $\tfrac{1}{2}Y \hookrightarrow \tilde{V}_Y$, and it will be enough to check this for the component of $\tfrac{1}{2}Y$ passing through the origin. This component is given by the couple of vector fields $(\tfrac{1}{2}D_1, \tfrac{1}{2}D_2)$ or, equivalently (cf (2.8)) by $(D_1, 2D_2)$. Hence one is finally led to checking that, for all f_{ijk} as in (2.10):

$$(D_1 f_{ijk})(0) = 0, \quad ((\tfrac{1}{2}D_1^2 + 2D_2) f_{ijk})(0) = 0.$$

The first of these conditions has been checked already, and the second one follows in the same way, Q.E.D.

(2.17) So, either V_Y is a smooth curve at the origin, or an infinitesimal piece of such: $V_Y = (V_Y)_h \cong \mathrm{Spec}\ k[\epsilon]/\epsilon^{h+1}$ for some $h \geq 2$. Call this $h = h(Y)$ for a moment, and put $h(Y) = \infty$ if the dimension of V_Y at 0 is positive.

In Theorem (1.6) one assumes that $h(Y) = \infty$. This should be compared with the following

(2.18) FACT. The condition $h(Y) \geq 3$, for some $Y \hookrightarrow X$ as before, is the assumption of the Novikov Hypothesis.

PROOF. Pursuing the formalism used in (2.6),(2.7), an embedding

$$\operatorname{Spec} k[\varepsilon]/\varepsilon^3 \hookrightarrow X$$

supported at the origin is given equivalently by constant vector fields $D_1 \neq 0$, D_2, D_3, by formulae (2.7) together with

$$(2.19) \qquad \Delta_3 = \frac{1}{3!} D_1^3 + D_1 D_2 + D_3.$$

Suppose that $Y \hookrightarrow X$ is given by (D_1, D_2). In view of Proposition (2.14), the assumption $h(Y) \geq 3$ means that there exists a D_3 such that the subscheme $Z \hookrightarrow X$ defined by (D_1, D_2, D_3) is contained in V_Y. As before, this is equivalent with $Z' \hookrightarrow \tilde{V}_Y$, where Z' is the component through the origin, of $\frac{1}{2}Z$. Now, Z' is defined by $(\frac{1}{2}D_1, \frac{1}{2}D_2, \frac{1}{2}D_3)$ or, equivalently, by $(D_1, 2D_2, 4D_3)$. Thus the assumption $h(Y) \geq 3$ is the existence of a D_3 such that, for all f_{ijk} in (2.10):

$$(2.20) \qquad \left(\left(\frac{1}{3!} D_1^3 + 2D_1 D_2 + 4D_3\right) f_{ijk}\right)(0) = 0.$$

Writing this out, this is equivalent to

$$(2.21) \qquad \operatorname{rk}(\vec{\theta}(0), (D_1^2\vec{\theta})(0), ((D_1^4 + 12D_2^2 - 12D_1D_3)\vec{\theta})(0)) \leq 2.$$

In view of (2.16), this reduces finally to the existence of constants c_o and c_1 such that

(2.22) $((D_1^4+12D_2^2-12D_1D_3+c_1D_1^2+c_o)\vec{\theta})(0) = 0,$

which is the assumption of the Novikov Hypothesis, according to Dubrovin ([2], p. 70). To bring it in a more familiar setting, consider the functions (Loc. cit., p. 59)

$$\hat{\theta}[n](z) = \theta[n,0](z \mid 2B)$$

where n runs through the set $(\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g$. The 2^g functions

$$\theta^n(z) = \hat{\theta}[n]\,(2z)$$

are a basis of the vector space of second order theta functions we are considering here ([2], p. 16). Taking $\vec{\theta}$ as made up by this basis and writing furthermore

$$\bar{D}_2 = 2D_2,\ \bar{D}_3 = 3D_3 - \frac{1}{4}\,c_1D_1, \quad d = \frac{1}{16}\,c_o,$$

the equation (2.22) can be rewritten in the standard form ([2], p. 62)

(2.23) $((D_1^4 - D_1\bar{D}_3 + \frac{3}{4}\bar{D}_2^2 + d)\vec{\theta})(0) = 0,$

Q.E.D.

Thus, in this language, the Novikov Hypothesis claims that, if X is an irreducible principally polarized abelian variety containing a subscheme $Y \hookrightarrow X$ as before with $h(Y) \geq 3$, then X is a jacobian.

A rough but quite natural way of weakening this question consists in building into it a one-dimensional piece somewhere. Following Dubrovin ([1],p.472), one may consider for instance the assumption that there exists a one-dimensional family of subschemes $Y \hookrightarrow X$ as before, with $h(Y) \geq 3$. Let us mention, in this connection, that if $h(Y) \geq 3$ then there is exactly one more Y' with $h(Y') \geq 3$ and having the same tangent direction as Y, namely the image $Y' = -Y$ of Y under the symmetry of X. This follows, as in (2.14), (2.18), by using (2.16).

Finally, a certain strengthening of the latter assumption is obtained by infinitesimalizing the hypotheses in Corollary (1.8). It consists in supposing that X contains a smooth curve C such that, for all $x \in C$, $(V_{Y_x})_3 = -x + \mathrm{Spec}(\mathcal{O}_{C,x}/m^4_{C,x})$ holds (cf (1.5) and (2.13) for notations). In analytical terms, this is essentially equivalent to the existence of a nonconstant holomorphic mapping

$$\Gamma : \Delta \longrightarrow \mathbb{C}^g$$

(Δ=the unit disk) and a holomorphic function $c(t)$ on Δ such that, putting

$$D(t) = \dot{\Gamma}(t) = \sum \frac{d\Gamma_i}{dt} \frac{\partial}{\partial z_i}$$

one has, for all $t \in \Delta$:

$$(2.24) \qquad ((D(t)^4 + 3\dot{D}(t)^2 - 2D(t)\ddot{D}(t) + c(t))\vec{\theta})(0) = 0.$$

(2.25) <u>REMARK</u>. For the time being there seems to be little reason to consider the matters of this section in positive characteristics. However, for later purposes we recall that the most essential fact which has been used here, namely (2.16), is valid in any characteristic $\neq 2$: Let X be a principally polarized abelian variety, and write $\mathbb{P}^{g-1}$ for the projectivized tangent space at $0\in X$. Let $H^0(M-0) = H^0(M-2\cdot 0)$ be the hypersubspace of $H^0(M)$ of those sections vanishing at the origin (hence vanishing doubly there). There is a natural linear mapping

$$H^0(M-0) \longrightarrow H^0\mathcal{O}_{\mathbb{P}^{g-1}}(2)$$

giving equations of the projectivized tangent cones at the origin of the divisors of $|2\theta|$ defined by these sections. Then (2.16) says that this map is surjective. As a matter of fact, this map is surjective if and only if X is irreducible. The "only if" part is quite obvious, and the "if" part follows by considering divisors of $|2\theta|$ of the type $\theta_x+\theta_{-x}$, with $x\in\theta$.

3. An extension of Gunning's results ([3])

The present section is devoted to a proof of the following generalization of [3], Theorem 2, p. 386:

(3.1) THEOREM. Let X be an irreducible principally polarized abelian variety, and let $0\in Y \hookrightarrow X$ be an artinian subscheme of length 3. Assume that there exists a (irreducible, complete) curve $V_1 \hookrightarrow X$ such that, for all $\zeta\in V_1$, $\zeta+Y \hookrightarrow X$ is a secant subscheme (cf (0.2)). Let $V = 2V_1 \hookrightarrow X$, image of V_1 by the multiplication by 2 isogeny of X, and call α_V: $X \longrightarrow X$ the endomorphism attached to the 1-cycle V in the PPAV X. Write Z for the 0-cycle of X defined by Y, and $s=S(Z)\in X$ the abelian sum of its components. Then one has:

(i) If $(-s+Y)\cap V = \emptyset$, then $\alpha_V \mid Y = 0$

(ii) If $(-s+Y)\cap V\neq\emptyset$, then $(-s+Y)\hookrightarrow V$, and V is smooth along this subscheme, and $\alpha_V \mid Y = I$ (identity).

In particular, if there are no complex multiplications $\alpha: X \longrightarrow X$, $\alpha \neq 0$, such that $\alpha | Y = 0$, then V is smooth and $(JV, \theta_V) \cong (X, \theta_X)$.

The last part is clear by Matsusaka's criterion (cf[4]). To begin with the proof of (3.1), let N be the normalization* of the curve V. Then α_V is the following composition:

$$(3.2) \qquad \begin{array}{ccccccccc} X & \xrightarrow{\cong} & \hat{X} & \longrightarrow & \mathrm{Pic}^0 N & \xrightarrow{\cong} & JN & \longrightarrow & X \\ a & \longmapsto & (\theta_a - \theta) & \longmapsto & (\theta_a - \theta) \mid N & & \longmapsto & & S((\theta_a - \theta) \mid N), \end{array}$$

the isomorphism $\mathrm{Pic}^0 N \xrightarrow{\cong} JN$ being the Abel-Jacobi map, and $JN \longrightarrow X$ being the Albanese morphism for the map $N \longrightarrow X$. We keep the notations L,M, etc., introduced in Section 0. Write

$$(3.3) \qquad \delta: X \times X \longrightarrow X, \qquad (x,y) \longmapsto -x+y$$

and let $pr_i: X \times X \longrightarrow X$, i=1,2, be the projections. The isomorphism $X \xrightarrow{\cong} \hat{X} = \mathrm{Pic}^0(X)$ is given by the line bundle $\delta^* L \otimes pr_2^* L^{\vee}$ on $X \times X$. (By this we mean, of course, that this morphism attaches to $a \in X$ the restriction of this line bundle to $\{a\} \times X$). Consequently, the map $X \longrightarrow \mathrm{Pic}^0 N$ in (3.2) is given by the restriction of $\delta^* L \otimes pr_2^* L^{\vee}$ to $X \times N$.

We shall denote by

$$(3.4) \qquad \delta_N: Y \times N \longrightarrow X$$

the restriction of δ to $Y \times N$. Then the composition

$$(3.5) \qquad Y \hookrightarrow X \longrightarrow \mathrm{Pic}^0 N$$

is given by the bundle

$$(3.6) \qquad \delta_N^* L \cong (L^{\vee} \mid N)$$

on $Y \times N$. For the sake of symmetry, it will be convenient to introduce also the composite map

$$(3.7) \qquad Y \times Y \longrightarrow X \longrightarrow \mathrm{Pic}^0 N,$$

where the first arrow is the difference map, restriction to $Y \times Y$ of $X \times X \longrightarrow X$, $(x,y) \longmapsto x-y$. Notice that the data (3.5) and (3.7) are mutually equivalent. Denoting by p_i: $Y \times Y \times N \longrightarrow Y \times N$, $i=1,2$ the projection maps, the composition (3.7) is given by the line bundle

$$(3.8) \qquad p_1^*(\delta_N^* L) \cong (p_2^*(\delta_N^* L))^{\vee}.$$

(3.9) Next we construct a natural projective line bundle on N. Introduce first $\tilde{V} = \frac{1}{2}V$. (The curve V_1 is an irreducible component of $\tilde{V}$.) We define $\tilde{N}$ by the left hand side pullback square in:

$$\begin{array}{ccccc}
\tilde{N} & \longrightarrow & \tilde{V} & \hookrightarrow & X \\
\downarrow & & \downarrow & & \downarrow 2 \\
N & \longrightarrow & V & \hookrightarrow & X
\end{array}$$

The curve $\tilde{N}$ is smooth and complete. The finite group ${}_2X$ acts freely on $\tilde{N}$, and $N = \tilde{N}/{}_2X$.

The natural map $\tilde{V} \longrightarrow \mathrm{Grass}(\mathbb{P}^1, \mathbb{P}^N)$ $(\mathbb{P}^N = |M|^{\vee})$, attaching to a general point $\zeta \in \tilde{V}$ the unique line $\ell_\zeta \subset \mathbb{P}^N$ such that $\zeta + Y \hookrightarrow \psi^{-1}(\ell_\zeta)$, induces a well-defined morphism

(3.10) $\tilde{N} \longrightarrow \mathrm{Grass}(\mathbb{P}^1, \mathbb{P}^N), \qquad \zeta \longrightarrow \ell_\zeta.$

Equivalently, this is a $\mathbb{P}^1$-bundle

(3.11) $\tilde{\pi} : \tilde{P} \longrightarrow \tilde{N}, \qquad \tilde{\pi}^{-1}(\zeta) = \ell_\zeta.$

The map ψ of (0.1) being equivariant for the ${}_2X$-action on both sides, we deduce an action of ${}_2X$ on $\tilde{P}$, compatible with the action of ${}_2X$ on $\tilde{N}$. Taking quotients, we get the claimed $\mathbb{P}^1$-bundle

(3.12) $\pi : P \longrightarrow N.$

(3.13) One defines a section $\tilde{\sigma}$ of the bundle $\tilde{P}$ by putting, for a general point of $\tilde{N}$ (identified with its image in $\tilde{V}$):

$$\tilde{\sigma}(\zeta) = \psi(\zeta) \in \ell_\zeta .$$

The action of ${}_2X$ leaves this section invariant, hence $\tilde{\sigma}$ drops to a section σ of the bundle P.

More generally, $\tilde{\sigma}$ is the restriction to $\{0\} \times \tilde{N}$ of a well-defined morphism of $\tilde{N}$-schemes:

(3.14) $\tilde{p}: Y \times \tilde{N} \longrightarrow \tilde{P},$

which above a general point $\zeta \in \tilde{N}$ is the composition $Y \xrightarrow{+\zeta} \zeta + Y \xrightarrow{\psi} \ell_\zeta$. The map $\tilde{p}$ being ${}_2X$-equivariant, one defines on this way a morphism of N-schemes

(3.15) $p: Y \times N \longrightarrow P$

which restricts to the section σ on $\{0\}\times N$.

Theorem (3.1) will be a corollary of the two propositions below.

(3.16) PROPOSITION. With the notations above, we have on $Y\times Y\times N$:

$$p_1^*(\delta_N^*L)\otimes(p_2^*(\delta_N^*L))^{\vee}\cong p_2^*(p^*\mathcal{O}_P(\sigma))\otimes(p_1^*(p^*\mathcal{O}_P(\sigma)))^{\vee}.$$

(Notice that the sheaf on the right hand side remains unchanged, if we replace $\mathcal{O}_P(\sigma)$ by $\mathcal{O}_P(\sigma)\otimes F$, $F\in\mathrm{Pic}(N)$ being arbitrary.)

(3.17) PROPOSITION. i) Assume $(s+Y)\cap V=\emptyset$. Then

$$p_2^*(p^*\mathcal{O}_P(\sigma))\otimes(p_1^*(p^*\mathcal{O}_P(\sigma)))^{\vee}\cong\mathcal{O}_{Y\times Y\times N}.$$

ii) If $(-s+Y)\cap V\neq\emptyset$, then $(-s+Y)\hookrightarrow V$, and V is smooth along this subscheme. Putting $\Gamma\hookrightarrow Y\times N$, Γ = graph of the morphism $Y\xhookrightarrow{-s}N$, and $\Gamma'=p_1^{-1}(\Gamma)$, $\Gamma''=p_2^{-1}(\Gamma)$, one has:

$$p_2^*(p^*\mathcal{O}_P(\sigma))\otimes(p_1^*(p^*\mathcal{O}_P(\sigma)))^{\vee}\cong\mathcal{O}_{Y\times Y\times N}(\Gamma'-\Gamma'').$$

For the way in which (3.1) is deduced from these two propositions, we remark that, in Case (ii), the morphism $Y\times Y\longrightarrow\mathrm{Pic}^0N$ of (3.7) is defined by $\mathcal{O}_{Y\times Y\times N}(\Gamma'-\Gamma'')$; therefore, by the definition of the Abel-Jacobi isomorphism $\mathrm{Pic}^0N\xrightarrow{\cong}JN$, the composition of (3.7) with this isomorphism equals

$$\begin{array}{ccccc} Y\times Y & \xrightarrow{(-s,-s)} & N\times N & \longrightarrow & JN \\ & & (x,y) & \longmapsto & x-y \end{array}.$$

Composing this with $JN\longrightarrow X$ we find that $\alpha_V\mid Y=I$, as claimed. In Case (i), the morphism (3.7) is zero, hence $\alpha_V\mid Y=0$.

The rest of this section is devoted to the proofs of (3.16) and (3.17).

PROOF OF (3.16). Pulling the two bundles back to $Y \times Y \times \tilde{N}$ we get line bundles with a ${}_2X$-linearization. To prove the proposition, it suffices to exhibit an isomorphism between these linearized bundles. The inverse image of $\mathcal{O}_P(\sigma)$ in $\tilde{P}$ is $\mathcal{O}_{\tilde{P}}(\tilde{\sigma})$, the linearization being defined by keeping fixed an equation for the divisor $\tilde{\sigma}$.

On the other side, the inverse image of $p_1^*(\delta_N^*L) \otimes (p_2^*(\delta_N^*L))^\vee$ in $Y \times Y \times \tilde{N}$ yields $p_1^*(\delta_{\tilde{N}}^*M) \otimes (\tilde{p}_2^*(\delta_{\tilde{N}}^*M))^\vee$. Here we have written $\tilde{p}_i\colon Y \times Y \times \tilde{N} \longrightarrow Y \times \tilde{N}$, $i=1,2$ for the projections and $\delta_{\tilde{N}}\colon Y \times \tilde{N} \longrightarrow X$ for the restriction of δ to $Y \times \tilde{N}$. The linearization is defined as follows: for $\epsilon \in {}_2X$, choose a relative isomorphism $\lambda\colon M \xrightarrow{\cong} M$ over the translation with ϵ, $T_\epsilon\colon X \longrightarrow X$. Then $\tilde{p}_1^*(\delta_{\tilde{N}}^*\lambda) \otimes (\tilde{p}_2^*(\delta_{\tilde{N}}^*\lambda^{-1}))^\vee$ gives the action of ϵ on the bundle $\tilde{p}_1^*(\delta_{\tilde{N}}^*M) \otimes (\tilde{p}_2^*(\delta_{\tilde{N}}^*M))^\vee$. These facts are easily deduced from the following ones:

On $X \times X \times X$, write $s_i\colon X \times X \times X \longrightarrow X \times X$, $i=1,2$, $s_1(x,y,z) = (x,z)$, $s_2(x,y,z) = (y,z)$; put also $r_i\colon X \times X \times X \longrightarrow X$, $i=1,2$, the first two projections. Finally, let $q\colon X \times X \times X \longrightarrow X \times X \times X$ be the isogeny $q(x,y,z) = (x,y,2z)$. Then, by using the symmetry property of L, the Theorem of the Square and the See-Saw Principle, it is easily seen that

$$q^*((s_1^*\delta^*L) \otimes (s_2^*\delta^*L)^\vee) \cong ((s_1^*\delta^*M) \otimes (s_2^*\delta^*M)^\vee) \otimes ((r_1^*L)^\vee \otimes (r_2^*L))$$

(cf e.g. [7], p.320, for a similar reasonning). Moreover, this is an isomorphism of ${}_2X$-linearized bundles, if one takes the obvious linearization on the left-hand side, and, on the right-hand side, the linearization of $(s_1^*\delta^*M) \otimes (s_2^*\delta^*M)^\vee$ as described above, times the identity on the factor $(r_1^*L)^\vee \otimes (r_2^*L)$.

Next, we produce an isomorphism of line bundles

$$\tilde{p}_1^*(\delta_{\tilde{N}}^*M) \otimes (\tilde{p}_2^*(\delta_{\tilde{N}}^*M))^\vee \cong \tilde{p}_2^*(\tilde{p}^*\mathcal{O}_{\tilde{P}}(\tilde{\sigma})) \otimes (\tilde{p}_1^*(\tilde{p}^*\mathcal{O}_{\tilde{P}}(\tilde{\sigma})))^\vee. \tag{3.18}$$

The verification of its compatibility with the above described linearizations is rather boring and straightforward, so we shall omit this, leaving it to the reader. The datum of (3.18) is equivalent with an isomorphism

$$(3.19)\qquad \tilde{p}_1^*(\delta_{\tilde{N}}^*M \otimes \tilde{p}^*\mathcal{O}_{\tilde{P}}(\tilde{\sigma})) \cong \tilde{p}_2^*(\delta_{\tilde{N}}^*M \otimes \tilde{p}^*\mathcal{O}_{\tilde{P}}(\tilde{\sigma})).$$

Since $\tilde{p}_1^*\tilde{p}^*\tilde{\pi}^*F \cong \tilde{p}_2^*\tilde{p}^*\tilde{\pi}^*F$ for all $F \in \mathrm{Pic}(\tilde{N})$, it will suffice to exhibit an isomorphism

$$(3.20)\qquad \tilde{p}_1^*(\delta_{\tilde{N}}^*M \otimes \tilde{p}^*\mathcal{O}_{\tilde{P}}(1)) \cong \tilde{p}_2^*(\delta_{\tilde{N}}^*M \otimes \tilde{p}^*\mathcal{O}_{\tilde{P}}(1))$$

(here $\mathcal{O}_{\tilde{P}}(1)$ denotes the pullback of $\mathcal{O}_{\mathbb{P}^N}(1)$ by the obvious map $\tilde{P} \longrightarrow \mathbb{P}^N$).

Write

$$\mu_{\tilde{N}}: Y \times \tilde{N} \longrightarrow X$$

the restriction to $Y \times \tilde{N}$ of the addition map $\mu : X \times X \longrightarrow X$. Clearly

$$\tilde{p}^*\,\mathcal{O}_{\tilde{P}}(1) \cong \mu_{\tilde{N}}^*M.$$

Thus (3.20) is equivalent with

$$(3.21)\qquad \tilde{p}_1^*(\delta_{\tilde{N}}^*M \otimes \mu_{\tilde{N}}^*M) \cong \tilde{p}_2^*(\delta_{\tilde{N}}^*M \otimes \mu_{\tilde{N}}^*M).$$

On the other side, if

$$\Phi : X \times X \longrightarrow X \times X$$

denotes the isogeny sending (x,y) to (-x+y,x+y), one has (cf [7], p.320):

$$\Phi^*(pr_1^*H \otimes pr_2^*H) \cong pr_1^*H^{\otimes 2} \otimes pr_2^*H^{\otimes 2},$$

for any symmetric line bundle H on X. Thus, applying this to H=M we obtain

$$(3.22) \qquad \delta_N^*M \otimes \mu_N^*M \cong pr_Y^*(M^{\otimes 2} \mid Y) \otimes pr_N^*(M^{\otimes 2} \mid \tilde{N}).$$

Since Y is a sum of local schemes, Pic(Y) = 0. Thus $M^{\otimes 2} \mid Y \cong \mathcal{O}_Y$ and, by (3.22), both members of (3.21) become identified with the stheaf $\mathcal{O}_{Y\times Y} \otimes (M^{\otimes 2} \mid N)$, Q.E.D.

PROOF OF (3.17). (i) Three possible types are allowed for Y:

$$(3.23) \qquad \begin{array}{l} \text{a) } Y \cong \sum_{i=1}^{3} \operatorname{Spec} k \\ \text{b) } Y \cong \operatorname{Spec} k[\varepsilon]/\varepsilon^3 \\ \text{c) } Y \cong \operatorname{Spec} k[\varepsilon]/\varepsilon^2 + \operatorname{Spec} k. \end{array}$$

An easy case-by-case inspection shows that, if $\zeta \in X$, then

$$\psi : \zeta + Y \longrightarrow \mathbb{P}^N$$

(cf (0.1)) is an immersion if and only if ζ does not belong to $\frac{1}{2}(-s+Y)$. Therefore the morphism p: $Y \times N \longrightarrow P$ of (3.15) is an immersion above points of N not mapping into $-s+Y \subset X$. Consequently, if $(-s+Y) \cap V = \emptyset$, the map p is an immersion. Taking any embedding $Y \hookrightarrow \mathbb{P}^1$ we get a commutative diagram of N-schemes

$$\begin{array}{ccc} \mathbb{P}^1 \times N & \xrightarrow{\ \cong\ } & P \\ \big\uparrow & \nearrow_{p} & \\ Y \times N & & \end{array}$$

From this we derive $p^*\mathcal{O}_P(\sigma) \cong (\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_N) \otimes \mathcal{O}_{Y\times N} \cong \mathcal{O}_Y(1) \boxtimes \mathcal{O}_N \cong \mathcal{O}_{Y\times N}$ (recall that $\mathrm{Pic}(Y) = 0$), and Part (i) follows.

(ii) we shall deal with the three cases of (3.23) separatedly.

Case (a). This is the original one, from Gunning's paper [3]. Put $Y = \{x_1=0, x_2, x_3\}$, three distinct points in X. Here $s = \Sigma x_i$, and $-s+Y = \{-x_1-x_2, -x_1-x_3, -x_2-x_3\}$. The map p of (3.15) is described equivalently as the datum of three sections $\sigma=\sigma_1$, σ_2 and σ_3 of $\Pi: P \to N$. Two sections σ_i and σ_j, $i \neq j$, meet above $\xi \in N$ if and only if ξ is mapped to $-x_i-x_j \in X$ by $N \to X$.

Write, in $\mathrm{Pic}(P) = \mathrm{Pic}\, N \oplus \mathbb{Z}\sigma$:

$$\sigma_1 = \sigma, \qquad \sigma_2 = \sigma+\lambda_2, \qquad \sigma_3 = \sigma+\lambda_3$$

with $\lambda_2, \lambda_3 \in \mathrm{Pic}(N)$. Proposition (3.16) together with (3.7), (3.8) implies that $\lambda_2, \lambda_3 \in \mathrm{Pic}^0(N)$. Namely, restricting the second member of the isomorphism formula in (3.16) to $\{(x_1, x_i)\} \times N$ $(i=2,3)$, we get: $\mathcal{O}_N(\lambda_i) = R^0_\Pi \mathcal{O}_{\sigma_i}(\sigma) \otimes R^0_\Pi \mathcal{O}_\sigma(-\sigma) \in \mathrm{Pic}^0(N)$.

Thus the intersection numbers $(\sigma_i \cdot \sigma_j)$ are independent from $i,j \in \{1,2,3\}$.

By assumption, $(-s+Y) \cap V \neq \emptyset$. Therefore, by the foregoing, at least two sections σ_i, σ_j, $i \neq j$, hence all of them, meet each other, and $-s+Y$ is contained in V.

Next we use

(3.24) LEMMA. ([3], Lemma 2, p. 382). The curve V is smooth at the points of $-s+Y \subset V$, and the sections σ_i, $i=1,2,3$ meet transversally above these points.

PROOF. Consider the point $-x_i-x_j \in V$, and let $\zeta \in \tilde{V}$ with $2\zeta = -x_i-x_j$. We show that, equivalently, $\tilde{V} = \frac{1}{2}V$ is smooth at ζ and that the sections $\tilde{\sigma}_i$ and $\tilde{\sigma}_j$ meet transversally at $\tilde{\sigma}_i(\zeta) = \psi(\zeta+x_i) = \psi(\zeta+x_j) = \tilde{\sigma}_j(\zeta)$. Choose a k-basis of H^oM, $\theta^o,\dots,\theta^N$, such that:

$$\theta^o(\zeta+x_i) \neq 0, \quad \theta^o(\zeta+x_k) \neq 0,$$
$$\theta^1(\zeta+x_i) = 0, \quad \theta^1(\zeta+x_k) \neq 0,$$
$$\theta^r(\zeta+x_i) = \theta^r(\zeta+x_k) = 0 \quad \text{if } r \geq 2.$$

(Note that this is possible because $\psi(\zeta+x_i) \neq \psi(\zeta+x_k)$). The rational functions on X

$$u_r = \theta^r/\theta^o \quad , \qquad r=0,\dots,N$$

are regular at $\zeta+x_i$, $\zeta+x_j$ and $\zeta+x_k$. Moreover, since the symmetry of X acts trivially on H^oM, the functions $u_o,\dots,u_N$ are even.

Consider the subscheme $\tilde{V}_Y \subset X$ defined as in Section 2, with $Y=\{x_o,x_1,x_2\}$. By hypothesis, we have $\tilde{V} \subset \tilde{V}_Y$. The subscheme $\tilde{V}_Y$ is defined at ζ by the functions g_{abc}, $0 \leq a < b < c \leq N$,

$$g_{abc}(x) = \det \begin{pmatrix} u_a(x+x_o) & u_a(x+x_1) & u_a(x+x_2) \\ u_b(x+x_o) & u_b(x+x_1) & u_b(x+x_2) \\ u_c(x+x_o) & u_c(x+x_1) & u_c(x+x_2) \end{pmatrix} .$$

Identifying now $T_X(\zeta)$ with the vector space of invariant vector fields on X, we get, if $D \in T_X(\zeta)$:

$$(Dg_{abc})(\zeta) = \pm 2 \det \begin{pmatrix} (Du_a)(\zeta+x_i) & u_a(\zeta+x_j) & u_a(\zeta+x_k) \\ (Du_b)(\zeta+x_i) & u_b(\zeta+x_j) & u_b(\zeta+x_k) \\ (Du_c)(\zeta+x_i) & u_c(\zeta+x_j) & u_c(\zeta+x_k) \end{pmatrix}$$

(note that the functions Du_r are odd). Since $\zeta+x_i \notin {}_2X$, the map ψ is an immersion at this point, and the foregoing implies

$$\dim T_{\tilde{V}_Y}(\zeta) \leq 1.$$

Therefore $\dim T_{\tilde{V}}(\zeta) = 1$, as was to be shown.

Write $T_{\tilde{V}}(\zeta) = \langle D \rangle$. By our choice of the basis $\theta^0,\dots,\theta^N$, we have: $(Du_1)(\zeta+x_i) \neq 0$. To prove the transversality of $\tilde{\sigma}_i$ and $\tilde{\sigma}_j$ at $\tilde{\sigma}_i(\zeta) = \tilde{\sigma}_j(\zeta)$ we have to check that $(d\tilde{\sigma}_i)_\zeta D \neq (d\tilde{\sigma}_j)_\zeta D$. Now, if $\tilde{u}_1$ denotes the function on $\tilde{P}$ obtained by lifting the rational function X_1/X_0 of $\mathbb{P}^N$, we have

$$((d\tilde{\sigma}_i)_\zeta D)\tilde{u}_1 = (Du_1)(\zeta+x_i) = -(Du_1)(\zeta+x_j) = -((d\tilde{\sigma}_j)_\zeta D)\tilde{u}_1.$$

Since these terms are non zero, we are done, Q.E.D.

To end with Case(a), consider $\{(x_i,x_j)\} \times N \subset Y \times Y \times N$. If $i=j$, then clearly the restriction of $p_2^*(p^*\mathcal{O}_P(\sigma)) \otimes (p_1^*(p^*\mathcal{O}_P(\sigma)))^\vee$ to $\{(x_i,x_j)\} \times N$ is isomorphic with $\mathcal{O}_N$. If $i \neq j$, let x_k be the third point in Y. By the remark preceding (3.17), the restriction of the above sheaf to $\{(x_i,x_j)\} \times N$ is isomorphic with that of the sheaf $p_2^*(p^*\mathcal{O}_P(\sigma_k)) \otimes (p_1^*(p^*\mathcal{O}_P(\sigma_k)))^\vee$, i.e. with

$$R^0_\pi \mathcal{O}_{\sigma_j}(\sigma_k) \otimes R^0_\pi \mathcal{O}_{\sigma_i}(-\sigma_k) = \mathcal{O}_N((-s+x_i)-(-s+x_j)).$$

This finishes the proof of Case (a).

Case (b). Here $s=0$ and, as a set, $Y = -s+Y$ consists of the point $0 \in X$ only. By our assumption, we have $0 \in V$. Then, as in (2.14) (cf (2.25)) we see that $Y \hookrightarrow V$ and that V is smooth at $0 \in V$. We shall identify $Y \hookrightarrow V$ with the divisor $3 \cdot 0$ of N. The map $p\colon Y \times N \to P$ of (3.15) factors through a morphism

$$(3.25) \qquad \bar{p}\colon Y \times N \longrightarrow W,$$

where $W \hookrightarrow P$ is the effective divisor 3σ of P. The map $\bar{p}$ is an isomorphism above all points of N other that $0 \in N$. Its local description at the origin is given by the following

(3.26) LEMMA. For a suitable choice of a local parameter t of N at 0 and a local equation φ for σ at $p(0,0) \in P$ we have, writing $\varepsilon \in m_{Y,0}$ the image of t in $m_{Y,0}$: The morphism of $\hat{\mathcal{O}}_{N,0}$ -algebras

$$\bar{p}^*\colon \hat{\mathcal{O}}_{W,p(0,0)} \longrightarrow \hat{\mathcal{O}}_{Y \times N,(0,0)}$$

can be identified with

$$\hat{\mathcal{O}}_{N,0}[\varphi]/\varphi^3 \longrightarrow \hat{\mathcal{O}}_{N,0}[\varepsilon]/\varepsilon^3,$$

defined by sending φ into $t\varepsilon+\varepsilon^2$.

PROOF. As in the proof of Lemma (3.24), we shall deal with the map $\bar{p}\colon Y \times \bar{N} \to \bar{P}$. Choose $\theta^0,\dots,\theta^N$ a basis for H^0M such that

$$\theta^0(0) \neq 0, \quad \theta^1(0) = \dots = \theta^N(0) = 0.$$

Put $u_r = \theta^r/\theta^o$, $r=1,\dots,N$. These are even rational functions on X, regular at the origin. With the notations of Section 2, suppose that $Y \hookrightarrow X$ is given by the couple (D,D') of constant vector fields on X. We may assume that either $D'=0$, or that D and D' are linearly independent (see (2,8)). The connected component at the origin, Z, of $\frac{1}{2}Y \hookrightarrow X$ is defined by $(\frac{1}{2}D, \frac{1}{2}D')$. (Note also that $Z \hookrightarrow \tilde{N}$.) We have a commutative diagram

$$\begin{array}{ccc} Z & \xleftarrow[\cong]{(\frac{1}{2}D,\frac{1}{2}D')} & \operatorname{Spec} k[\epsilon]/\epsilon^3 \\ {\scriptstyle 2}\downarrow\cong & & \| \\ Y & \xleftarrow[\cong]{(D,D')} & \operatorname{Spec} k[\epsilon]/\epsilon^3 . \end{array}$$

By (2.25) we may assume that

$$(DD'u_1)(0) = 0, \quad (D^2u_1)(0) \neq 0, \quad \text{and} \quad (D^2u_r)(0) = 0 \quad \text{if} \quad r \geq 2$$

(recall that we are assuming $\operatorname{char}(k) \neq 2$). The composite map

$$Y \times \tilde{N} \xrightarrow{\tilde{p}} \tilde{P} \longrightarrow \mathbb{P}^N$$

is defined in a neighbourhood of $(0,0)$ by sending the functions X_i/X_o of $\mathbb{P}^N$ into $u_i+(Du_i)\epsilon+((\frac{1}{2}D^2+D')u_i)\epsilon^2$, $i=1,\dots,N$. It follows in particular that the image $\ell_o \subset \mathbb{P}^N$ of the fibre of $\tilde{P}$ above $0 \in \tilde{N}$ is given by $X_2=\dots=X_N = 0$, and that X_1/X_o is a coordinate function on ℓ_o near the origin. On the other hand, since $(Du_1)(0) = 0$ and $(D^2u_1)(0) \neq 0$, we may take $y=Du_1$ as a parameter of $\tilde{N}$ at 0. At $\tilde{p}(0,0) \in \tilde{P}$ we may choose therefore the following coordinates: the function y, lifted from the base $\tilde{N}$, and the function z gotten by pulling back X_1/X_o from $\mathbb{P}^N$. The map $\tilde{p}$ is described locally at $(0,0)$ by

$$y \mapsto y, \quad z \longmapsto u_1+(Du_1)\varepsilon+((\tfrac{1}{2}D^2+D')u_1)\varepsilon^2.$$

A local equation for $\tilde{\sigma}$ near $\tilde{p}(0,0)$ is given by $\eta = z-u_1$, and this is mapped into $(Du_1)\varepsilon+((\tfrac{1}{2}D^2+D')u_1)\varepsilon^2$ by $\tilde{p}$. We write this as $y\varepsilon+f\varepsilon^2$, where $f = (\tfrac{1}{2}D^2+D')u_1$. Observe that $f(0) = \tfrac{1}{2}$, $(Df)(0) = 0$. The image of the parameter y in Z is given by

$$y(0) + \tfrac{1}{2}(Dy)(0)\varepsilon + ((\tfrac{1}{8}D^2 + \tfrac{1}{2}D')y)(0)\varepsilon^2 = \tfrac{1}{2}\varepsilon.$$

Choose now $\tilde{\varphi} = \eta/f$ as a new local equation for $\tilde{\sigma}$ and $\tilde{t} = y/f$ as a new parameter for $\tilde{N}$ at 0. The image of $\tilde{t}$ in Z is ε, and the image of $\tilde{\varphi}$ by $\tilde{p}$ is $\tilde{t}\varepsilon+\varepsilon^2$. In view of the isomorphism $\hat{\mathcal{O}}_{N,0} \xrightarrow{\cong} \hat{\mathcal{O}}_{\tilde{N},0}$ and $\hat{\mathcal{O}}_{P,p(0,0)} \xrightarrow{\cong} \hat{\mathcal{O}}_{\tilde{P},\tilde{p}(0,0)}$, this finishes the poof of the Lemma.

The proof of (3.17)(ii) in the present case (b) will be settled by showing that

(3.27) $$p^*\mathcal{O}_P(\sigma) \cong \mathcal{O}_{Y\times N}(-\Gamma) \otimes \mathcal{O}_N(2\cdot 0).$$

To begin with, we compute $R^0_\pi \mathcal{O}_W(\sigma)$. We remark that $P = \mathbb{P}R^0_\pi\mathcal{O}_P(\sigma)$, hence the dualizing sheaf for P over N is given by

$$\omega_{P/N} = \mathcal{O}_P(-2\sigma) \otimes \mathcal{O}_N(e),$$

where we have put $e = c_1 R^0_\pi \mathcal{O}_P(\sigma)$. Therefore, the relative dualizing sheaf for W over N is

$$\omega_{W/N} = \omega_{P/N} \otimes N_{W/P} \cong \mathcal{O}_W(\sigma) \otimes \mathcal{O}_N(e),$$

and it follows that $\mathcal{O}_W(\sigma) \cong \omega_{W/N} \otimes \mathcal{O}_N(-e)$. Taking direct images and using relative duality gives:

$$R^0_\pi \mathcal{O}_W(\sigma) = R^0_\pi(\omega_{W/N} \otimes \mathcal{O}_N(-e)) = R^0_\pi(\omega_{W/N}) \otimes \mathcal{O}_N(-e) \cong (R^0_\pi \mathcal{O}_W)^\vee \otimes \mathcal{O}_N(-e).$$

We compute $\mathcal{O}_N(e)$. From Lemma (3.26) we obtain an exact sequence of $\mathcal{O}_N$-modules

$$0 \longrightarrow R^0_\pi \mathcal{O}_{2\sigma} \longrightarrow (k[\varepsilon]/\varepsilon^2) \otimes_k \mathcal{O}_N \longrightarrow \mathcal{O}_0 \longrightarrow 0,$$

$\mathcal{O}_0$ standing for the structure sheaf of the reduced one-point scheme $0 \hookrightarrow N$. Thus $c_1 R^0_\pi \mathcal{O}_{2\sigma} = -0 \in \mathrm{Pic}(N)$. On the other side, by using the exact sequence

$$0 \longrightarrow \mathcal{O}_P(-2\sigma) \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_{2\sigma} \longrightarrow 0,$$

we derive the following one, by taking direct images and using relative duality:

$$0 \longrightarrow \mathcal{O}_N \longrightarrow R^0_\pi \mathcal{O}_{2\sigma} \longrightarrow \mathcal{O}_N(-e) \longrightarrow 0.$$

Therefore $c_1 R^0_\pi \mathcal{O}_{2\sigma} = -e \in \mathrm{Pic}(N)$, and hence $\mathcal{O}_N(e) = \mathcal{O}_N(0)$. We obtain finally:

(3.28) $$R^0_\pi \mathcal{O}_W(\sigma) \cong (R^0_\pi \mathcal{O}_W)^\vee \otimes \mathcal{O}_N(-0).$$

The direct image in N of the sheaf $p^* \mathcal{O}_P(\sigma)$ is the $R^0 \mathcal{O}_{Y \times N}$-module

$$(R^0 \mathcal{O}_{Y \times N}) \otimes_{(R^0_\pi \mathcal{O}_W)} (R^0_\pi \mathcal{O}_W(\sigma)).$$

(In writing $R^0 \mathcal{O}_{Y \times N}$, we drop the subscript referring to the unnamed projection map $Y \times N \longrightarrow N$. We recall also that $R^0 \mathcal{O}_{Y \times N}$ is considered as a $R^0_\pi \mathcal{O}_W$-algebra, using the morphism $\tilde{p}$: $Y \times N \longrightarrow W$). Introduce the invertible $R^{\circ}\mathcal{O}_{Y \times N}$-module

$$F = (R^{\circ}\mathcal{O}_{Y \times N}) \otimes_{(R^{\circ}_{\Pi}\mathcal{O}_W)} (R^{\circ}_{\Pi}\mathcal{O}_W)^{\vee}.$$

In view of (3.28), the relation (3.27) is equivalent with the following one, between $R^{\circ}\mathcal{O}_{Y \times N}$-modules:

$$(3.29) \qquad F \otimes_{\mathcal{O}_N} \mathcal{O}_N(-3\cdot 0) \cong R^{\circ}(\mathcal{O}_{Y \times N}(-\Gamma)).$$

The structure map $R^{\circ}_{\Pi}\mathcal{O}_W \longrightarrow R^{\circ}\mathcal{O}_{Y \times N}$ gives, by transposition (as $\mathcal{O}_N$-modules), a morphism of $(R^{\circ}_{\Pi}\mathcal{O}_W)$-modules

$$(3.30) \qquad (R^{\circ}\mathcal{O}_{Y \times N})^{\vee} \longrightarrow (R^{\circ}_{\Pi}\mathcal{O}_W)^{\vee}.$$

Since the map $Y \times N \xrightarrow{\bar{p}} W$ is an isomorphism over $U = N\setminus\{0\} \subset N$, we may take the inverse of (3.30) over U,

$$(R^{\circ}_{\Pi}\mathcal{O}_W)^{\vee}|U \longrightarrow (R^{\circ}\mathcal{O}_{Y \times N})^{\vee}|U,$$

and derive an isomorphism of $R^{\circ}\mathcal{O}_{Y \times N}$-modules:

$$(3.31) \qquad F|U \xrightarrow{\cong} (R^{\circ}\mathcal{O}_{Y \times N})^{\vee}|U.$$

Using Lemma (3.26), a straightforward computation shows that, choosing conveniently isomorphisms $F_0 \cong \mathcal{O}_{N,0}[\epsilon]/\epsilon^3$ and $(R^{\circ}\mathcal{O}_{Y \times N})^{\vee}_0 \cong \mathcal{O}_{N,0}[\epsilon]/\epsilon^3$, the fibre of (3.31) at the generic point of N is given by

$$\epsilon \longmapsto \frac{1}{t^2} - \frac{1}{t^3}\,\epsilon\,.$$

This shows that the restriction of (3.31) to $(\mathcal{F} \otimes_{\mathcal{O}_N} \mathcal{O}_N(-3\cdot 0))|U \hookrightarrow \mathcal{F}|U$ extends to an injection of $R^{\circ}\mathcal{O}_{Y\times N}$-modules

$$\mathcal{F} \otimes_{\mathcal{O}_N} \mathcal{O}_N(-3\cdot 0) \hookrightarrow (R^{\circ}\mathcal{O}_{Y\times N})^{\vee},$$

whose cokernel is the $(R^{\circ}\mathcal{O}_{Y\times N})$-module

$$\mathcal{O}_{N,0}[\epsilon]/(\epsilon^3, t-\epsilon) \cong R^{\circ}\mathcal{O}_{\Gamma}.$$

Using the isomorphism $(R^{\circ}\mathcal{O}_{Y\times N})^{\vee} \cong R^{\circ}\mathcal{O}_{Y\times N}$, this implies (3.29) thereby finishing the proof of Case (b).

Case (c). Write $Y_{red} = \{0,x\}$ and $\mathrm{Spec}\, k[\epsilon]/\epsilon^2 \cong Y_0 \subset Y$. We may assume, without loss of generality, that $(Y_0)_{red} = \{0\}$, i.e., that the non-reduced part of Y is supported at $0 \in X$. With our notations, $s=x$ here, and $(-s+Y)_{red} = \{0,y\}$, with $y=-x$.

The map $p\colon Y\times N \longrightarrow P$ of (3.15) factors through a map

$$\bar{p}\colon Y\times N \longrightarrow W \tag{3.32}$$

onto a divisor $W = 2\sigma+\sigma'$ of P. The morphism $\bar{p}$ is an isomorphism above points of N not mapping to the points 0 or $y=-x$ of X.

We write in $\mathrm{Pic}(P) = \mathrm{Pic}(N) \oplus \mathbb{Z}\sigma$:

$$\sigma' = \sigma + \lambda, \qquad \lambda \in \mathrm{Pic}(N).$$

As in the reduced case, one deduces that $\lambda \in \mathrm{Pic}^{\circ}(N)$ and that the intersection numbers σ^2, $\sigma\cdot\sigma'$ and σ'^2 are all equal to each other.

The map $\bar{p}$ of (3.32) induces a map

(3.33) $\quad \bar{\bar{p}}: Y_o \times N \longrightarrow 2\sigma,$

which is an isomorphism above points of N not mapping to $0 \in X$. This leads to an exact sequence of $\mathcal{O}_N$-modules

$$0 \longrightarrow R^o_\pi \mathcal{O}_{2\sigma} \longrightarrow R^o \mathcal{O}_{Y_o \times N} \longrightarrow \mathcal{D} \longrightarrow 0,$$

the support of $\mathcal{D}$ being contained in the set of points of N mapping to $0 \in X$. We get:

(3.34) $\quad c_1(R^o_\pi \mathcal{O}_{2\sigma}) = -c_1(\mathcal{D}).$

On the other hand, putting, as in the preceding case,

$$e = c_1 R^o_\pi \mathcal{O}_P(\sigma),$$

we deduce as before that

(3.35) $\quad c_1(R^o_\pi \mathcal{O}_{2\sigma}) = -e.$

Taking into account the exact sequence

$$0 \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_P(\sigma) \longrightarrow \mathcal{O}_\sigma(\sigma) \longrightarrow 0,$$

which gives

$$0 \longrightarrow \mathcal{O}_N \longrightarrow R^o_\pi \mathcal{O}_P(\sigma) \longrightarrow R^o_\pi \mathcal{O}_\sigma(\sigma) \longrightarrow 0,$$

we get also

$$(3.36) \qquad e = c_1 R^o_\Pi \mathcal{O}_\sigma(\sigma).$$

Putting (3.34)-(3.36) together, we obtain finally:

$$(3.37) \qquad c_1(\mathcal{D}) = c_1 R^o_\Pi \mathcal{O}_\sigma(\sigma).$$

Recall that, by hypothesis, $(-s+Y)\cap V \neq \emptyset$. This implies that $(-s+Y)_{red} \subset V$. In fact: $y \in V$ if and only if $\sigma \cdot \sigma' > 0$, which is equivalent to $\sigma^2 > 0$, which is equivalent to $0 \in V$, by (3.37).

(3.38) <u>LEMMA</u>. The curve V is smooth at the points 0,y, and $-s+Y \hookrightarrow V$. Moreover, the sections σ and σ' meet transversally at one point (above $y \in N$). The map (3.33) is described above $0 \in N$ as follows: Choosing conveniently a local parameter t of N at 0 and a local equation φ of σ at p(0,0), the morphism of $\hat{\mathcal{O}}_{N,0}$-algebras

$$\bar{\bar{p}}^*: \hat{\mathcal{O}}_{2\sigma,p(0,0)} \longrightarrow \hat{\mathcal{O}}_{Y_o \times N,(0,0)}$$

can be identified with

$$\hat{\mathcal{O}}_{N,0}[\varphi]/\varphi^2 \longrightarrow \hat{\mathcal{O}}_{N,0}[\epsilon]/\epsilon^2,$$

defined by $\varphi \longmapsto t\epsilon$.

Furthermore, for a suitable local parameter t of N at y, if $\epsilon \in m_{Y_o,0}$ is its image by the embedding $Y_o \xrightarrow{-s} N$, σ' cuts cut on $Y_o \times N$ the divisor given by the ideal $(t+\epsilon)$ of $\mathcal{O}_{N,y}[\epsilon]/\epsilon^2$.

<u>PROOF</u>. This is a local computation, similar as in the proofs of (3.24) and (3.26), and will be omitted.

Finally, we show that

$$(3.39)\qquad p^*\mathcal{O}_P(\sigma) \cong \mathcal{O}_{Y\times N}(-\Gamma) \otimes \mathcal{O}_N(0+y),$$

and this will finish the proof in Case (c).

Restricting the first member of (3.39) to $\{x\}\times N \hookrightarrow Y\times N$, we obtain $\mathcal{O}_N(y)$. The second member restricts to $\mathcal{O}_N(-\Gamma(x)+0+y)$. Being $\Gamma(x) = 0$, both restrictions are isomorphic.

It remains to investigate the restrictions of these sheaves to $Y_o\times N$. By Lemma (3.38), it follows that $c_1(\mathcal{D}) = \mathcal{O}_N(0) \in \mathrm{Pic}(N)$. Thus, by (3.37)

$$R^o_\Pi \mathcal{O}_\sigma(\sigma) \cong \mathcal{O}_N(0).$$

On the other hand, $R^o_\Pi \mathcal{O}_\sigma(\sigma') \cong \mathcal{O}_N(y)$, thus, having written $\sigma'=\sigma+\lambda$, we derive

$$\mathcal{O}_N(\lambda) \cong R^o_\Pi \mathcal{O}_\sigma(\sigma') \otimes R^o_\Pi \mathcal{O}_\sigma(-\sigma) \cong \mathcal{O}_N(y-0),$$

and hence

$$(3.40)\qquad \mathcal{O}_P(\sigma) \cong \mathcal{O}_P(\sigma') \otimes \mathcal{O}_N(0-y).$$

Replacing $\mathcal{O}_P(\sigma)$ in (3.39) by its value in (3.40), we finally must prove, on $Y_o\times N$, the following isomorphism:

$$(3.41)\qquad (p^*\mathcal{O}_P(\sigma'))|\ Y_o\times N \cong \mathcal{O}_{Y_o\times N}(-\Gamma) \otimes \mathcal{O}_N(2y).$$

Now, with the notations of Lemma (3.38), the divisor σ' cuts out on $Y_o\times N$ a divisor which is defined by the ideal $(t+\epsilon)$ of $\mathcal{O}_{N,y}[\epsilon]/\epsilon^2$, and Γ is given by the ideal $(t-\varepsilon)$ there. Thus the sum of these divisors is given by the ideal $(t^2) = (t+\varepsilon)(t-\varepsilon)$, which also defines the divisor $2(Y_o \times \{y\})$ of $Y_o \times N$. This proves (3.41), hence Proposition (3.17) in this case, and therefore finishes the proof of Theorem (3.1), Q.E.D.

References

1. B.A. DURBROVIN, On S.P. Novikov's conjecture in the theory of Theta functions and nonlinear equations of Kortenveg-De Vries and Kadomcev-Petviašvili type, Sov. Math. Doklady 21 (1980), 469-472.

2. B.A. DUBROVIN, The functions and non-linear equations, Russ. Math. Surveys 36 (1981), 11-92.

3. R.C. GUNNING, Some curves in abelian varieties, Inv. Math. 66 (1982), 377-389.

4. W. HOYT, On products and algebraic families of Jacobian varieties, Annals of Math. 77 (1963), 415-423.

5. D. MUMFORD, Abelian varieties, Bombay, Oxford University Press, 1974.

6. D. MUMFORD, Curves and their jacobians, Ann. Arbor, Univ. of Michigan Press, 1975.

7. D. MUMFORD, On the Equations defining Abelian Varieties, I, Invent. Math. 1 (1966) 287-354.

8. D. MUMFORD, Prym varieties I, in: Contributions to Analysis, New York, Academic Press, 1974, pp. 325-350.

Abstract

We extend the results of R.C. Gunning's paper "Some curves in abelian varieties", Inv. Math. 66 (1982), 377-389, including also degenerate cases of the original hypotheses. Gunning's characterization of Jacobi varieties in terms of trisecants of the Kummer variety leads to similar characterizations in terms of flexes of the Kummer variety.

publicacions
edicions
universitat
de barcelona

Dipòsit Legal B.: 16.496-1983
BARCELONA – 1983