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# FOR WHICH JACOBI VARIETIES IS SING $\Theta$ REDUCIBLE?

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PRE–PRINT N.º 17
novembre, 1983

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## Introduction

In this paper by a curve we understand a one-dimensional, projective, non-singular and irreducible scheme over $\mathbb{C}$.

Let $C$ be a curve of genus $g$ and denote by $J$ its jacobian variety.

By Abel's theorem, the points of $J$ may be canonically interpreted as the linear equivalence classes of divisors of degree 0 on $C$, or, once a divisor class of degree $n$ has been fixed, as the linear equivalence classes of divisors of degree $n$ on $C$.

In particular, fixing a base point on the curve, we get a map of the symmetric powers of the curve to the Jacobian

$$C^{(d)} \longrightarrow J$$

By Riemann's Parametrization Theorem, the image of this map for $d=g-1$ (usually denoted by $W^o_{g-1}$ or $W_{g-1}$) is a translate of the theta divisor of the canonical polarization of $J$. Moreover, Riemann's Singularity Theorem says that the singular locus of $\Theta=W^o_{g-1}$ is the set of linear equivalence classes of divisors of dimension at least equal to one ([K] p.184 Corollary). So, its study fits naturally in the theory of special divisors.

It is proved (see [A,C], §2 and [F,L] §2), that the sets $W^r_d$ of points of J which represent linear equivalence classes of divisors of degree d with dim $|D| \geq r$ may be given natural scheme structures when we realize them as the locus where certain homomorphisms of vector bundles on J drop rank. The schemes thus obtained together with the restriction of the universal bundle on $C \times J$ represent the functors that assign to every scheme X the families of divisors of degree d and dimension at least r parametrized by X.

By general results of Fulton and Lazarsfeld [F,L] p.271 corollary), it is known that $W^r_d$ is irreducible for a generic curve if $\rho = g-(r+1)(g-d+r) > 0$. In particular, Sing $\Theta = W^1_{g-1}$ is irreducible for C generic when $g \geq 5$.

When g=4, if C is non-hyperelliptic, it is the intersection of a quadric and a cubic surface in $\mathbb{P}^3$ and $W^1_3$ consists in two points corresponding to the linear series cut by every system of generators of the quadric, which come to coincide when the quadric is a cone ([H] (5.2.2) and (5.5.2)).

When g=3 $W^1_2$ is non-empty just when C is hyperelliptic and for $g \leq 2$ it is always empty.

The aim of this paper is to find all curves for which Sing $\Theta = W^1_{g-1}$ is reducible.

Our result is:

<u>Theorem</u>: Let C be a curve of genus $g \geq 5$. The singular lo-

cus of the theta divisor of its jacobian Sing $\Theta$: $W^1_{g-1}$ with its natural scheme structure is reduced, and it is reducible if and only if C is of one of the following types:

a) trigonal

b) superelliptic, that is to say a double cover of an elliptic curve (also called elliptic-hyperelliptic)

c) an unbranched double covering of a genus 3 curve (and so g=5).

Note that the reducedness statement is false for a genus four curve whose canonical model is contained in a quadric cone. Then the two points of $W^1_3$ for a generic genus four C come to coincide.

The author wishes to thank professor Dr. Gerald E. Welters for his guidance during the preparation of this work.

## I Reducibility of $W^1_{g-1}$

For hyperelliptic curves the locus $W^1_{g-1}$ is well known to be irreducible ([A,M] p. 212). Hence, we shall assume in what follows that C is non hyperelliptic.

We shall use the ideas of Fulton and Lazarsfeld in their proof of the irreducibility of $W^r_d$ (remark 1.8 [FL]). By using their construction of the set of special divisors we get the two following sufficient conditions for the irreducibility of $W^1_{g-1}$.

a) $\dim W^1_{g-1} = \rho = g-4$

b) $\mathrm{cod}_{W^1_{g-1}} (\mathrm{Sing}\ W^1_{g-1}) \geq 2$

Condition a) holds for every nonhyperelliptic curve ([A,M] prop. 8 p. 209 or [M] Th1). Moreover, in this case, as pointed out by Fulton and Lazarefeld in their remark named above and proved by Kempf and Laksov ([K,L] p. 160-162), $W^1_{g-1}$ with its scheme-theoretic structure is Cohen Macaulay, so equidimensional.

In particular condition b) can be written as:

$$\dim \mathrm{Sing}\ W^1_{g-1} < g-5.$$

We shall study separately the case $g=5$.

We claim:

<u>Lemma 1</u> If $g \geq 6$ and $\dim \mathrm{sing}\ W^1_{g-1} \geq g-5$ then C is either trigonal or superelliptic.

Moreover, for trigonal and superelliptic curves it is known that $W^1_{g-1}$ is reducible ([A,M] p.188 and [Sh] (2.5.2) p. 212 respectively). So, in case $g > 5$ the lemma yields the irreducibility statement of the theorem.

Proof of lemma 1:

It is known ([Ma] lemma 6 p.164) that $W^2_{g-1} \subset \mathrm{sing}\ W^1_{g-1}$

For a point L in $W^1_{g-1} - W^2_{g-1}$, the tangent space to $W^1_{g-1}$ at $|L|$ may be interpreted as the subspace of the tangent space to the Jacobian $H^0(C,K)^\vee$, ortogonal to the image of the Petri morphism:

$$P: H^o(L) \otimes H^o(K \otimes L^{-1}) \longrightarrow H^o(K)$$

(compare [S,D] p.162 lemma 2.5).

So

$$\text{sing } W^1_{g-1} = W^2_{g-1} \cup A$$

where A denotes the set of equivalence classes of divisors of degree g-1 and dimension one for which the Petri mofphism is not injective.

As C is non-hyperelliptic, dim $W^2_{g-1} < g-5$ ([M] th 1), so we must just study the second term.

Let L belong to A. If L has no fixed points we have an exact sequence ([S,D] lemma 2.6)

$$0 \to H^o(K \otimes L^{-2}) \longrightarrow H^o(L) \otimes H^o(K \otimes L^{-1}) \xrightarrow{P} H^o(K)$$

and, as we are assuming $|L| \in A$, we must have $h^o(K \otimes L^{-2}) \neq 0$

Since $K \otimes L^{-2}$ is a divisor of degree 0, the above condition implies $K = L^2$, i.e., L is a $\theta$-characteristic. As the set of $\Theta$-characteristics on a curve is finite and we are assuming $g > 5$, we deduce: we must look for curves for which the variety A has a component W with dim $W \geq g-5$ and whose generic point L corresponds to a divisor with fixed points, that is to say $W \subset W^1_{g-2} + W_1$ (here and in what follows $W_k$ will stand for $W^o_k$).

As C is non-hyperelliptic dim $W^1_{g-2} \leq g-5$ ([M] th 1) and equality holds only in the following cases ([Mu] Appendix, p. 348).

a) Trigonal curves

b) Superelliptic curves

c) Non-singular plane quintics.

cases a) and b) are allowed in the lemma. As for c) it is easily shown by using the fact that the canonical system is cut by plane conics that sing $W_5^1 = W_5^2$ has only one point. In fact, a divisor in $W_5^1$ corresponds to a linear series with a fixed point $p \in C$ plus the residual divisor cut by the pencil of lines through another point $Q \in C$. If $P \neq Q$ one can check that this divisor has dimension exactly 1 and that the Petri morphism is injective, so it corresponds to a non-singular point in $W_5$. If $P=Q$ we get the point of $W_5$.

Suppose that C does not belong to any of the listed cases. Then A has a component of dimension g-5 if and only if $W_{g-2}^1$ has a component X of dimension g-6 such that the generic point of X is a divisor class of dimension one and the generic point of $X+W_1$ is in A. We shall prove that this is imposible.

<u>Lemma 2</u>. Let L be a divisor with $h^o(L)=2$ and degree g-1 such that for a generic point P of C L+P does not satisfy Petri's condition. Let $\sum_{i=1}^t p_i$ be the fixed part of the divisor K-L. Then $K-L-\sum_{i=1}^t p_i = 2D$ where $\dim D \geq 1$.

Granted lemma 2 let t be the number of fixed points of a generic divisor k-L. As L describes X, $X-L-\sum_{i=1}^t p_i = 2D$ moves in a subvariety X' of $W_{g-t}^2$ and D describes a non-empty subvariety Y of $W_{(g-t)/2}^1$.

It is clear that $\dim X' = \dim Y$. Moreover, as $X \subset X' + W_t$

$$\dim X \leq \dim(X' + W_t) = \dim X' + t$$

So

$$\dim Y = \dim X' \geq \max(\dim X - t, 0) = \max(g-6-t, 0),$$

Therefore

$$\max(g-6-t,0) \leq \dim Y \leq \dim W^1_{(g-t)/2} \leq ((g-t)/2)-3$$

where the last inequality follows from [M] th.1.

We get

$$3 \leq (g-t)/2 \leq 3$$

Therefore $\dim W^1_3 = 0$ and this is impossible as we are assuming that C is neither hyperelliptic nor trigonal. This concludes the proof of lemma 1.

Proof of lemma 2:

As P is generic in C L+P has P as fixed point and $H^0(L+P) = H^0(L)$.

Embedding $H^0(K-L-P)$ in $H^0(K-L)$ we get a commutative diagram

$$\begin{array}{ccccc}
 & & H^0(L+P) \otimes H^0(K-L-P) & \xrightarrow{\ P\ } & H^0(K) \\
 & & \downarrow & & \\
0 \longrightarrow & B \longrightarrow & H^0(L) \otimes H^0(K-L) & \xrightarrow{\ P\ } & H^0(K)
\end{array}$$

where B denotes the Kernel of the Petri morphism.

Condition L+P does not satisfy Petri's condition is written now as

$$B \cap [H^0(L) \otimes H^0(K-L-P)] \neq 0$$

Note that dim B 2. In fact by lemma (1.5.1) in [SD] dim $B \leq 2$ and it cannot be one; otherwise, as B cuts all $H^0(L) \otimes H^0(K-L-P)$ it would be contained in their intersection. But

$$\bigcap_{P\in C} [H^0(L) \otimes H^0(K-L-P)] \quad H^0(L) \otimes [\bigcap_{P\in C} H^0(K-L-P)] \quad 0$$

Now we map C by the complete linear system without fixed points

$$|K-L-\sum_{i=1}^{t} P_i| \text{ in } |K-L-\sum_{i=1}^{t} P_i|^{\vee} \quad (\mathbb{P}^2)^{\vee} .$$

We claim that, under the above conditions, the image curve C is a conic.

For this purpose let us consider the following diagram deduced from above

$$\begin{array}{ccc} & & \mathbb{P}(H^0(L) \otimes H^0(K-L-\Sigma P_i - P) \\ & & \downarrow \\ \mathbb{P}(B) & \hookrightarrow & \mathbb{P}(H^0(L) \otimes H^0(K-L-\Sigma P_i) \end{array}$$

The space $\mathbb{P}(H^0(L) \otimes H^0(K-L-\Sigma P_i)) = \mathbb{P}^5$ is five dimensional and contains one image of Segre's embedding

$$\mathbb{P}^1 \times \mathbb{P}^2 \xrightarrow{\varphi} \mathbb{P}^5$$

where

$$\mathbb{P}^1 = \mathbb{P}(H^0(L)) \qquad \mathbb{P}^2 = \mathbb{P}(H^0(K-L-\Sigma P_i))$$

and $\mathbb{P}(B)$ is a line in this $\mathbb{P}^5$ which does not cut $\mathrm{Im}\,\varphi$ because B does not contain decomponible elements.

Our hypotheses is that for every point $M \in \bar{C} \subset (\mathbb{P}^2)^\nu$, if we denote by $L_M$ the corresponding line in $\mathbb{P}^2$, the three dimensional linear subvariety of $\mathbb{P}^5$ which contains $\varphi(\mathbb{P}^1 \times L_M)$ cuts $\mathbb{P}(B)$ in a point. Call this point $\psi(M)$.

In this way we get a map

$$\psi : \bar{C} \longrightarrow \mathbb{P}(B)$$

Moreover $\psi$ must be injective, otherwise $\mathbb{P}(B)$ would cut the intersection of the linear spaces generated by $\varphi(\mathbb{P}^1 \times L_M)$ and $\varphi(\mathbb{P}^1 \times L_N)$ $N \neq M$ points of $\bar{C}$. This is easily seen to be $\varphi(\mathbb{P}^1 \times (L_N \cap L_M))$ and that would contradict the fact that $\mathbb{P}(B)$ does not cut Im $\varphi$.

So $\bar{C}$ is rational. As C is mapped onto $\bar{C}$ by a complete linear system of dimension two, $\bar{C}$ must be a conic.

Hence we have $K-L-\sum_{i=1}^{t} P_i = 2D$, dim $|D| \geq 1$ as claimed.

## II Reducedness

Before dealing with the irreducibility in case g=5 we want to prove the reducedness statement in general.

Assume first that C is hyperelliptic. For such a curve $W^1_{g-1} \cong W_{g-3}$ as it is easily seen by using the functorial definition of the locus of special divisors. For any curve $W_k$ has the correct dimension $\rho = k$, so it is Cohen Macaulay ([F,L] remark 1.8 and [K,L] p. 160-162)) and it is irreducible. So, to prove the reducedness it is enough to prove that it has a nonsingular point. But that is obvious (for $k \le g$) by taking a point in $W_k - W_k^1$ and computing the tangent space to $W_k$ in that point by means of the Petri morphism.

Were C non hyperelliptic and $W_{g-1}^1$ non reduced then, as we have remarked that $W_{g-1}^1$ is Cohen Macaulay of dimension g-4, one of its components would be contained in the singular locus. So, using the notations above, we would have a component X of dimension g-5 in $W_{g-2}^1$ giving rise to a component for dimension g-4 in A. But then, by lemma 2, for a generic L in X and letting $\sum_{i=1}^{t} P_i$ be the fixed part of K-L we would have $K-L-\sum_{i=1}^{t} P_i = 2D$ $\dim|D| \ge 1$. When L moves in X we would get a non-empty subvariety of dimension g-5-t in $W_{(g-t)/2}^1$

So

$$g-5-t \le \dim W_{(g-t)/2}^1 \le ((g-t)/2)-3$$

where, as before, the last inequality follows from [M] Th. 1.

So we have $(g-t)/2 \le 2$ and this is impossible as C is non hyperelliptic.

## III The case g=5

In this paragraph C will denote a curve of genus 5 neither hyperelliptic nor trigonal. We shall not rule out the superelliptic case, both because we do not need it to apply our method and for the independent interest of its study.

In our situation, by Noether's Theorem and Enriques-Petri's results (see [S D]), C is the complete intersection of three quadrics in $\mathbb{P}^4$.

The linear series of degree four and dimension one on such a curve are cut out by the systems of generators of quadrics of rank 4, i.e singular quadrics, contaning the curve ([A,M] lemma 4 p.192). The quadrics of $\mathbb{P}^4$ containing the curve form a net with base locus C. The condition of degeneration for a quadric is given by the vanishing of the determinant of its associated matrix which is a degree five polynomial. Therefore, the degenerate quadrics containing the curve yield a plane quintic. The variety $W^1_4$ is a double covering of this curve, the couple of points over every point of the quintic corresponding to the two different systems of generators of the quadric. The discriminant points correspond to the quadrics of rank 3 through C.

In Ch VII of [Be], Beauville studied those varieties which are obtained as the complete intersection of three quadrics in $\mathbb{P}^n$ n=2k and their associated double coverings of a plane curve of degree n+1. He proves that both curves have at most ordinary double points. Moreover, the covering map is ramified precisely at the singular points of the covering curve and at those points the covering involution does not interchange the branches. It follows in particular that the covering curve is reducible if and only if the image curve is.

Accordingly, we study the cases where the plane quintic decomposes. Were it to happen, the plane quintic would contain either a line or a non degenerate conic.

If the plane quintic contains a line, this corresponds to a pencil of singular quadrics through C with a common vertex ([Be] lemma 6.8). By proyecting the curve C from this point, we obtain the complete intersection of two quadrics in $\mathbb{P}^3$ (projection of any two quadrics of the pencil). So, the image curve E has degree 4 and genus one. As C has degree eight and the center of projection does not belong to C (otherwise it would be a singular point on C), the morphism is 2 to 1. So C is superelliptic.

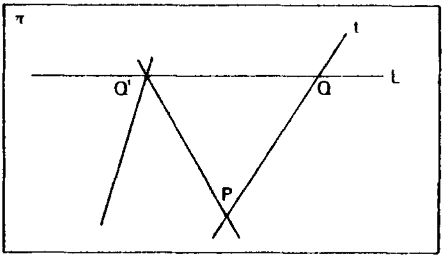
Conversely, when C is syperelliptic the morphism $C \xrightarrow{\varepsilon} E$ induces a morphism $\varepsilon^*$ from the variety of linear series of degree 2 and dimension 1, which is isomorphic to E, into $W^1_4$. So, $W^1_4$ must contain an elliptic curve and by Hurwitz's formula applied to the double covering of the possible components of a plane quintic, this quintic must contain a line.

Assume now that the plane quintic decomposes into a non-singular conic and a cubic. The points of the conic correspond to a family F of rank four quadrics in $\mathbb{P}^4$ containing C whose coefficients are given as quadratic polinomials in a single parameter, that is to say, for a generic point of $\mathbb{P}^4$ there are two such quadrics containing it. Moreover, by applying lemma (6.12) 2 of [Be] to our situation we obtain that the set of vertices of the quadrics in F form a line L.

We project C from L and we want to show that the morphism is 2 to 1. For this purpose we need to show that every 2-plane containing L and a point P of C contains another point. In fact $\pi$ either cuts the quadrics of F in a family of degenerate conics with vertex moving in L and containing P or is contained in them. The latter situation cannot occur for a generic quadric in F. In fact C is the intersection of any three independant quadrics containing it, so, it is the intersection of three quadrics in F. If $\pi$ were contained in a generic quadric in F it would be contained in C, which is imposible.

In the former case, in every degenerate conic one of the lines goes through P and so the set of such lines is a pencil with Kernel in P. As F is a quadratic family, so is the family of conics obtained by section with $\pi$, so the remaining set of lines must constitute another pencil and its Kernel is the point we were looking for.

Moreover, no tangent line t to a point of C may cut L. Assume that this is not the case and call Q the intersection point of t and L. The line t must be tangent to every quadric containing C, in particular to every quadric in F. But this is impossible if the vertex Q' is any point of L different from Q, as the section with $\pi$ is a conic with vertex in Q'.




So, we have proved that the morphism $C \longrightarrow \bar{C}$ is 2:1 and unramified. Hence $\bar{C}$ is nonsingular. Moreover L cannot cut C because the points of L are vertices of quadrics containing C and the curve is the complete intersection of those quadrics. So $\bar{C}$ has degree 4 and it must be a genus 3 curve.

Conversely, if C is an unramified double covering of a genus 3 curve $C \longmapsto \bar{C}$, then the Prym variety associated to this covering is the jacobian of a genus two curve ([Mu] p.344 Th), hence its theta divisor $\Xi$ is a genus two curve. But the linear series in $W^1_4$ whose image in C are canonical divisors are precisely those parametrized by $\Xi$ ([Mu] p.342 Prop.). Hence $W^1_4$ contains an irreducible genus 2 curve. By Hurwitz's formula we deduce that in this case the quintic must contain a conic.

This ends the proof of our theorem.

Note: Possibilities a) and b) in our theorem are, of course, incompatible. In fact, on a canonical model of a curve trigonal and superelliptic at a time, the plane generated by a line containing a trigonal divisor and the center of the superelliptic projection ([Sh] p.211) must contain at least 6 points, so the curve would have a $g_6^3$ and should be either hyperelliptic or of genus 4.

Possibilities b) and c) may occur simultaneously for a genus five curve. Moreover, unlike the higher genus cases where the superelliptic structure is unique ([Sh] p.211), a curve of genus five may have several (five at most) superelliptic structures, each corresponding to a line in the associated quintic. It may also be a double covering of a genus three curve in two different ways (and in this case the curve is necesarily superelliptic because, if the plane quintic contains two non-degenerate conics it must also contain a line).

All the cases listed above actually occur because Beauville's construction recovers the curve from the plane quintic and a $\Theta$-characteristic on its normalization.

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Dipòsit Legal B.: 37.494-1983
BARCELONA – 1983