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MALLIAVIN CALCULUS FOR TWO–PARAMETER WIENER FUNCTIONALS

by

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<u>Abstract</u>. In this paper we apply the Malliavin Calculus to deduce the existence and smoothness of density for the solution of stochastic differential equations with respect to a multidimensional two-parameter Wiener process.

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0. Introduction. The aim of this paper is to prove the existence and smoothness of the density of the probability law on $\mathbb{R}^m$ induced by the solution of the stochastic integral equations in the plane

$$X_z^i = x^i + \int_{[0,z]} [A_j^i(X_r)dW_r^j + B^i(X_r)\, dr], \quad i=1,\ldots,m, \qquad (0.1)$$

$z \in \mathbb{R}_+^2$, where $W_z = (W_z^1,\ldots,W_z^d)$ is a d-dimensional two-parameter Wiener process, and assuming some conditions on the coefficients A_j^i and B^i. If these coefficients are globally Lipschitz functions, it is known (cf. Cairoli [2], Hajek [4]) that this system has a unique continuous solution, which has a particular Markov property. There exists a transition semigroup corresponding to these Markov processes, but this semigroup acts on continuous functions over the sets of the form $\{(x,t): x \geq s\} \cup \{(s,y): y \geq t\}$. Then, we cannot expect the probability distribution of X_z to satisfy a second order partial differential equation.

In the case of an ordinary stochastic differential equation with respect to the Brownian motion, Malliavin has developed in [6] probabilistic techniques to show the existence and smoothness of density for the solution of these equations under Hörmander's conditions. Alternative approaches to Malliavin's theory were given by Shigekawa [7], Bismut [1] and Stroock [8]. The extension of Malliavin calculus to the case of two-parameter Wiener functionals is straightforward. In Section 1 we have briefly discussed Shigekawa's presentation of Malliavin Calculus adapted to functionals of a multidimensional Wiener sheet. However when we apply this stochastic calculus to the solution of the system (0.1), some technical difficulties appear, in relation with the following facts:

a) The inner products $\langle DX_z^i, DX_z^k \rangle_H$ (in the notation of Shigekawa) are not solutions of a similar system of equations, because the two-parameter stochastic differentiation rules involve the presence of double integrals over the set $\{(z,z') \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 : z_1 \leq z_1', z_2 \geq z_2'\}$ (cf. [10]).

b) Unlike the one-parameter case, there is no flow of transformations of $\mathbb{R}^m$ naturally associated to the system (0.1). Also, the solution of a linear system cannot be expressed as an exponential (because of the two-parameter stochastic calculus) and is not invertible, in general.

For these reasons, in the development of Malliavin calculus applied to the functional X_z we have avoided the use of Ito's formula, and the fact that X_z^i (and other functionals deduced from X_z) belongs to the space H_∞ (in the notation of Ikeda-Watanabe [5]) is proved by means of a direct approximation method. To do this, a good class of processes is introduced in Section 2, and in Section 3 we show that the process $X = \{X_z, z \in [0,z_o]\}$ is included in this class.

Section 4 is devoted to prove the non-degeneracy condition, following the ideas developed by Stroock in Section 8 of [9]. We remark that the two-dimensional character of the parameter set makes this demonstration a little simpler and, in fact, we do not need estimates for the inverse of the solution of a linear system of equations in the plane. In conclusion, the existence and smoothness of the density of X_z (at any point z outside the axes) is obtained assuming that the vector space spanned by the vector fields $A_1, \dots, A_d$, $A_i^{\nabla} A_j$, $1 \le i,j \le d$, $A_i^{\nabla}(A_j^{\nabla} A_k)$, $1 \le i,j,k \le d$, ..., at the point x is $\mathbb{R}^m$. Here $A_i^{\nabla} A_j$ denotes the covariant derivative of A_j in the direction of A_i. This property is strictly weaker than the restricted Hörmander's conditions, which are expressed in terms of Lie brackets instead of covariant derivatives.

1. Elements of Malliavin calculus. The set of parameters will be $T = [0,s_o] \times [0,t_o]$, with the partial ordering $(s_1,t_1) \le (s_2,t_2)$ if and only if $s_1 \le s_2$ and $t_1 \le t_2$; $(s_1,t_1) < (s_2,t_2)$ means that $s_1 < s_2$ and $t_1 < t_2$. If $z_1 < z_2$, $(z_1,z_2]$ will denote the rectangle $\{z \in T:\ z_1 < z \le z_2\}$. We put $R_z = [0,z]$, and $z_1 \circledast z_2 = (s_1,t_2)$ if $z_1=(s_1,t_1)$ and $z_2=(s_2,t_2)$. The increment of a function $f: R_+^2 \longrightarrow R$ on a rectangle $(z_1,z_2]$ is given by $f((z_1,z_2]) = f(z_1) - f(z_1 \circledast z_2) - f(z_2 \circledast z_1) + f(z_2)$. The Lebesgue measure of a Borel set $B \subset R_+^2$ is denoted by $|B|$.

Our probability space $(\Omega, \mathcal{F}, P)$ is the canonical space associated to the d-dimensional two-parameter Wiener process, that is, Ω is the space of all continuous functions $\omega : T \longrightarrow R^d$ which vanish on the axes, P is the two-parameter Wiener measure and $\mathcal{F}$ is the completion of the Borel σ-field of Ω with respect to P. We also consider the increasing family of σ-fields $\{\mathcal{F}_z,\ z \in T\}$, where $\mathcal{F}_z$ is generated by the functions $\{\omega(r),\ \omega \in \Omega,\ r \le z\}$ and the null sets of $\mathcal{F}$. The family $\{\mathcal{F}_z,\ z \in T\}$ satisfies the usual conditions of [3]. The following subset of Ω plays an important role:

$$H = \{\omega \in \Omega : \text{there exists } \dot{\omega}^i \in L^2(T),\ i=1,\dots,d, \text{ such that } \omega^i(z) = \int_{R_z} \dot{\omega}^i(r)\, dr, \text{ for any } z \in T \text{ and for any } i\}.$$

H is a Hilbert space with the inner product

$$<\omega_1,\omega_2>_H = \int_T \sum_{i=1}^{d} \dot{\omega}_1^i(r)\, \dot{\omega}_2^i(r)\, dr .$$

Any measurable function defined on the Wiener space $(\Omega,\mathcal{F},P)$ is called a Wiener functional. A Wiener functional $F: \Omega \longrightarrow R$ is <u>smooth</u> if there exists some $n \geq 1$ and a C^2-function f on R^n such that

(i) f and its derivatives up to the second order have at most polynomial growth order,

(ii) $F(\omega) = f(\omega(z_1),\dots,\omega(z_n))$ for some $z_1,\dots,z_n \in T$.

Every smooth functional is Fréchet-differentiable, and we have

$$DF(\omega_o)(\omega) = \sum_{j=1}^{d} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i^j} (\omega_o(z_1),\dots,\omega_o(z_n))\, \omega^j(z_i).$$

We also need the operator L defined on smooth functionals as follows:

$$LF(\omega) = \sum_{j=1}^{d} \sum_{i,k=1}^{n} \frac{\partial^2 f}{\partial x_i^j \partial x_k^j} (\omega(z_1),\dots,\omega(z_n))\Gamma(z_i,z_k) - DF(\omega)(\omega),$$

where $\Gamma(z_i,z_k) = (x_i \wedge x_k)(y_i \wedge y_k)$, if $z_i = (x_i,y_i)$, $i=1,..,n$. Note that Γ is the covariance function of the Brownian sheet.

For any $p \geq 1$, L_H^p will denote the space of Wiener functionals $F: \Omega \longrightarrow H$, which are valued on the space H, and such that $E(\|F\|_H^p) < \infty$. If we fix $\omega \in \Omega$ and a smooth functional F, $DF(\omega): H \longrightarrow R$ is a continuous linear map, and, so, it may be considered as an element of H. In this sense we have $DF \in L_H^p$ for any $p \geq 1$.

Let $H(p_1,p_2;p_3)$, $p_1,p_2,p_3 \geq 1$, be the space of real valued Wiener functionals F such that there exists a sequence of smooth functionals $\{F_k, k \geq 1\}$ satisfying:

(a) $F_k \xrightarrow[k\to\infty]{} F$ in L^{p_1},

(b) $\{DF_k, k \geq 1\}$ is a Cauchy sequence in $L_H^{p_2}$, and

(c) $\{LF_k, k \geq 1\}$ is a Cauchy sequence in L^{p_3}.

For a Wiener functional $F \in H(p_1,p_2;p_3)$ we define $DF = \lim_k DF_k$ and $LF = \lim_k LF_k$, and it is proved that these limits are uniquely determined by F. $H(p_1,p_2;p_3)$ is a Banach space with the norm $\|F\|_{p_1} + \|DF\|_{p_2} + \|LF\|_{p_3}$. We set $H_\infty = \bigcap_{p\geq 2} H(p,p;p)$.

Let $F^i \in H_\infty$ for $i=1,\dots,d$, and let $u: R^d \longrightarrow R$ be a twice continuously differentiable function such that u and its first and second derivatives have at most polynomial growth order. If we set $F = (F^1,\dots,F^d)$, then $u \circ F \in H_\infty$, and the following differentiation rules hold

$$D(u\circ F) = (\frac{\partial u}{\partial x_i} \circ F)\ DF^i,$$

and

$$L(u\circ F) = (\frac{\partial^2 u}{\partial x_i \partial x_j} \circ F) \langle DF^i, DF^j\rangle_H + (\frac{\partial u}{\partial x_i} \circ F)\ LF^i.$$

The next result is the two-parameter version of Malliavin's theorem. The proof of this theorem in the one-parameter case can be generalized without any problem.

<u>Theorem 1.1</u> (cf. Ikeda-Watanabe, [5]). Let $F = (F^1,\dots,F^m)$ be an $\mathbb{R}^m$-valued two-parameter Wiener functional. Assume that F satisfies the following two conditions:

(i) $F^i \in H_\infty$, $i=1,\dots,m$ and also the class defined by

$$\mathcal{C}_o = \{F^i, \langle DF^i, DF^j\rangle_H, LF^i;\ i,j=1,\dots,m\}$$

satisfies that $\mathcal{C}_o \subset H_\infty$. Furthermore assuming $\mathcal{C}_{r-1} \subset H_\infty$ we define the class $\mathcal{C}_r$ by

$$\mathcal{C}_r = \mathcal{C}_{r-1} \cup \{\langle DF^i, DG\rangle_H;\ G \in \mathcal{C}_{r-1},\ i=1,\dots,m\}$$

and assume that $\mathcal{C}_r \subset H_\infty$ for every $r=0,1,\dots$

(ii) Setting $Q^{ij} = \langle DF^i, DF^j\rangle_H$ we suppose that $(\det Q)^{-1} \in L^p$ for all $p \geq 1$.

Then, the probability law of F is absolutely continuous with respect to the Lebesgue measure and it has a infinitely differentiable density.

2. A class of two-parameter processes. In order to prove that the solution X_z of the stochastic differential system (0,1) (assuming that the coefficients are smooth and have bounded derivatives) satisfies condition (i) of theorem 1.1, we are going to introduce a rich class of processes $\mathcal{H}_\infty$ which includes the process X_z, and such that $\mathcal{H}_\infty$ has the following properties:

(i) If $F \in \mathcal{H}_\infty$, then $F_z \in H_\infty$ for any $z \in T$.

(ii) If $F, G \in \mathcal{H}_\infty$, then the processes $LF = \{LF_z, z \in T\}$ and $\langle DF, DG\rangle_H = \{\langle DF_z, DG_z\rangle_H, z \in T\}$ are also in $\mathcal{H}_\infty$.

Consider the processes of the form $F_z(\omega) = f(z,\omega(z_1),\dots,\omega(z_n))$ where:

(i) $z_1,\dots,z_n$ are n fixed points of T, not on the axes.

(ii) The functions $f(z,\cdot)$ and all their derivatives have at most polynomial growth order and are continuous functions of z.

(iii) For any $z \in T$, the function $(x_1,\dots,x_n) \longrightarrow f(z,x_1,\dots,x_n)$ depends only on the coordinates $x_i \in \mathbb{R}^d$ such that $z_i \leq z$, $i=1,\dots,n$.

With these assumptions $\{F_z, z \in T\}$ is a continuous and adapted process such that F_z is a smooth functional for each $z \in T$. We will call such processes, <u>smooth processes</u>. Note that for any $h \in H$

$$DF_z(h) = \sum_{j=1}^{d} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i^j} (z,\omega(z_1),\dots,\omega(z_n))\, h^j(z_i) = \int_{R_z} \xi_j(z,r)\dot{h}^j(r)dr,$$

where

$$\xi_j(z,r) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i^j} (z,\omega(z_1),\dots,\omega(z_n)) 1_{R_{z_i}}(r).$$

The process $\xi_j(z,r)$ vanishes unless $r \leq z$. More generally, we define, for $j_1,\dots,j_N \in \{1,\dots,d\}$,

$$\xi_{j_1\dots j_N}(z,r_1,\dots,r_N)$$

$$= \sum_{i_1,\dots,i_N=1}^{n} \frac{\partial^N f}{\partial x_{i_1}^{j_1}\dots\partial x_{i_N}^{j_N}} (z,\omega(z_1),\dots,\omega(z_n))\, 1_{R_{z_{i_1}}}(r_1)\dots 1_{R_{z_{i_N}}}(r_N).$$

These processes vanish unless $z \geq r_1 \vee \dots \vee r_N$, and they will be called the N-th derivatives of F. Observe that for any $h \in H$

$$D\xi_{j_1\ldots j_N}(z,r_1,\ldots,r_N)(h) = \int_{R_z} \xi_{j_1\ldots j_N j}(z,r_1,\ldots,r_N,r)\dot{h}^j(r)\, dr.$$

For any subset $K = \{\varepsilon_1 < \ldots < \varepsilon_\eta\}$ of $\{1,\ldots,N\}$ we put $K^c = \{1,\ldots,N\} - K$, $j(K) = j_{\varepsilon_1} \ldots j_{\varepsilon_\eta}$ and $r(K) = r_{\varepsilon_1},\ldots,r_{\varepsilon_\eta}$. We define

$$I_K \xi_{j(K^c)}(z,r(K^c)) = \sum_{\varepsilon\in K} \sum_{j_\varepsilon=1}^{d} \sum_{i_1,\ldots,i_N=1}^{n} \frac{\partial^N f}{\partial x_{i_1}^{j_1}\ldots\partial x_{i_N}^{j_N}} (z,\omega(z_1),\ldots,\omega(z_n))$$

$$\cdot\left(\prod_{\varepsilon\in K^c} 1_{R_{z_{i_\varepsilon}}}(r_\varepsilon)\right)\left(\prod_{\varepsilon\in K} \omega^{j_\varepsilon}(z_{i_\varepsilon})\right). \qquad (2.1)$$

The expression (2.1) represents a multiple stochastic integral with respect to the Brownian sheet. We remark, however, that these stochastic integrals will not follow the rules of the stochastic calculus because the processes $\xi_{j_1\ldots j_N}(z,\cdot)$ are not adapted. We will write I_i instead of $I_{\{i\}}$, and $I_{\{1,\ldots,N\}}\xi(z)$ for $I_{\{1,\ldots,N\}}\xi_{j(\phi)}(z,r(\phi))$. Note that $I_\phi\xi=\xi$.

For any integer $M \geq 1$ and any real $p \geq 1$, we set

$$\|F\|_{p,M} = [E(\sup_{z\in T} |F_z|^p)]^{1/p}$$

$$+ \sum_{N=1}^{M} \sum_{K\subset\{1,\ldots,N\}} \sup_{r_\varepsilon,\varepsilon\in K^c} [E(\sup_{z\in T} \|I_K \xi(z,r(K^c))\|^p)]^{1/p},$$

where $\|\cdot\|$ denotes the Hilbert-Schmidt norm. Let $\mathcal{H}_{p,M}$ be the closed hull of the family of smooth processes with respect to this norm. The processes of $\mathcal{H}_{p,M}$ are continuous, and if $F \in \mathcal{H}_{p,M}$, then $F_z \in H(p,p;p)$ for any $z\in T$. Set $\mathcal{H}_\infty = \bigcap_{M\geq 1} \bigcap_{p\geq 1} \mathcal{H}_{p,M}$. For any $F\in\mathcal{H}_\infty$ there exists a sequence of smooth processes F^n such that $\lim_n E(\sup_z |F_z^n - F_z|^p) = 0$ for all $p \geq 1$, and such that $\{F^n, n \geq 1\}$ is a Cauchy sequence for all norms $\|\cdot\|_{p,M}$. We will call $\{F^n, n\geq 1\}$ an <u>approximating sequence</u> for the process F.

Proposition 2.1. Suppose that F and G belong to $\mathcal{H}_\infty$. Then, the processes LF and $\langle DF, DG\rangle_H$ are also in $\mathcal{H}_\infty$.

Proof. Let $\{F^n, n\geq 1\}$ be an approximating sequence for the process F. Without loss of generality we may assume that $F^n_z = f_n(z,\omega(z_1), \ldots, \omega(z_n))$. We denote by $\xi^n_{j_1\ldots j_N}$ the N-th derivatives of F^n. Then,

$$LF^n_z = \sum_{j=1}^{d}\sum_{i,k=1}^{n} \frac{\partial^2 f_n}{\partial x_i^j \partial x_k^j}(z,\omega(z_1),\ldots,\omega(z_n))\Gamma(z_i,z_k)$$

$$- \sum_{j=1}^{d}\sum_{i=1}^{n} \frac{\partial f_n}{\partial x_i^j}(z,\omega(z_1),\ldots,\omega(z_n))\,\omega^j(z_i)$$

$$= \sum_{j=1}^{d}\int_{R_z} \xi^n_{jj}(z,r,r)\,dr - I_1\xi^n(z).$$

We have $\lim_{n,m} E(\sup_z |LF^n_z - LF^m_z|^p) = 0$ for all p, because F^n is a Cauchy sequence with respect to the norms $\|\cdot\|_{p,M}$. In consequence, LF_z exists for all $z\in T$, and we may choose a version of the process $\{LF_z, z\in T\}$ such that $\lim_n E(\sup_z |LF_z - LF^n_z|^p) = 0$, for any p. Therefore, it suffices to show that the sequence of smooth processes $\{LF^n, n\geq 1\}$ is Cauchy for all norms $\|\cdot\|_{p,M}$. Denote by $\psi^n_{j_1\ldots j_N}$ the N-th derivatives of LF^n. We have

$$\psi^n_{j_1}(z,r_1) = \sum_{j=1}^{d}\int_{R_z} \xi^n_{jjj_1}(z,r,r,r_1)\,dr - I_1\xi^n_{j_1}(z,r_1) - \xi^n_{j_1}(z,r_1),$$

and, by induction we obtain

$$\psi^n_{j_1\dots j_N}(z,r_1,\dots,r_N) = \sum_{j=1}^d \int_{R_z} \xi^n_{jjj_1\dots j_N}(z,r,r,r_1,\dots,r_N)dr$$

$$- I_1\, \xi^n_{j_1\dots j_N}(z,r_1,\dots,r_N) - N\, \xi^n_{j_1\dots j_N}(z,r_1,\dots,r_N).$$

From this expression it is easy to check that

$$\lim_{n,m} \sup_{r_\epsilon,\epsilon\in K^c} E(\sup_z \|I_K\psi^n - I_K\psi^m\|^p) = 0,$$

for any $K\subset\{1,\dots,N\}$, $N\geq 1$, and $p\geq 1$.

In order to show the second part of the proposition, assume that $G^n_z = g_n(z,\omega(z_1),\dots,\omega(z_n))$ is an approximating sequence for G, with derivatives $\phi^n_{j_1\dots j_N}$, and compute

$$\langle DF^n_z, DG^n_z\rangle_H = \sum_{j=1}^d \sum_{i,k=1}^n \frac{\partial f_n}{\partial x^j_i}(z,\omega(z_1),\dots,\omega(z_n)) \frac{\partial g_n}{\partial x^j_k}(z,\omega(z_1),\dots,\omega(z_n))$$

$$\cdot\Gamma(z_i,z_k)$$

$$= \sum_{j=1}^d \int_{R_z} \xi^n_j(z,r)\, \phi^n_j(z,r)\, dr.$$

As above, we have

$$\lim_{n,m} E(\sup_z |\langle DF^n_z,DG^n_z\rangle_H - \langle DF^m_z,DG^m_z\rangle_H|^p) = 0,$$

for all p. Thus, the random variables F_z and G_z belong to H_∞ for any $z\in T$, and there is a version of the process $\{\langle DF_z,DG_z\rangle_H,\ z\in T\}$ satisfying

$$\lim_n E(\sup_z |\langle DF^n_z,DG^n_z\rangle_H - \langle DF_z,DG_z\rangle_H|^p) = 0$$

for all p. Finally, it remains to show that $\{<DF^n, DG^n>_H,\ n \geq 1\}$ is a Cauchy sequence for all norms $\|\cdot\|_{p,M}$. Let $\beta^n_{j_1 \ldots j_N}$ be the N-th derivatives of the smooth process $<DF^n, DG^n>_H$. We have

$$\beta^n_{j_1 \ldots j_N}(z, r_1, \ldots, r_N) = \sum_{j=1}^{d} \sum_{K \subset \{1,\ldots,N\}} \int_{R_z} \xi^n_{jj(K)}(z, r, r(K)) \phi^n_{jj(K^c)}(z, r, r(K^c)) dr,$$

and, from this expression it is easy to verify that

$$\lim_{n,m} \sup_{r_\varepsilon, \varepsilon \in K^c} E(\sup_z \| I_K \beta^n - I_K \beta^m \|^p) = 0,$$

for any $K \subset \{1,\ldots,N\}$, $N \geq 1$, and $p \geq 1$. □

3. Some results on stochastic differential equations in the plane

Henceforth, the d-dimensional two-parameter Wiener process in the canonical probability space $(\Omega, \mathcal{F}, P)$ will be denoted by $W = \{W_z,\ z \in T\}$. We remember that $T = R_{z_0}$, being $z_0 = (s_0, t_0)$. Let $V = \{V_z,\ z \in T\}$ be a continuous and adapted M-dimensional stochastic processes such that $\beta_p = \sup_{z \in T} E(|V_z|^p) < \infty$ for all $p \geq 1$. Suppose that

$$\sigma : \mathbb{R}^M \times \mathbb{R}^m \longrightarrow \mathbb{R}^m \otimes \mathbb{R}^d \quad \text{and} \quad b : \mathbb{R}^M \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$$

are continuous functions verifying the following properties, for some positive constant K:

(i) $\|\sigma(x,y) - \sigma(x,y')\| + |b(x,y) - b(x,y')| \leq K \quad |y - y'|$, for any $x \in \mathbb{R}^m$; $y, y' \in \mathbb{R}^m$.

(ii) The functions $x \longrightarrow \sigma(x,0)$ and $x \longrightarrow b(x,0)$ have at most polynomial growth order. That means, $\|\sigma(x,0)\| + |b(x,0)| \leq K(1 + |x|^\nu)$ for some integer $\nu \geq 0$.

With these assumptions we have the next result.

Lemma 3.1. Fix $r \in T$ and an $\mathcal{F}_r$-measurable random vector $\alpha = (\alpha^1, \dots, \alpha^m)$ such that $E(|\alpha|^p) < \infty$ for any $p \geq 1$. Then, there is a unique continuous and adapted m-dimensional process $Y = \{Y_z, z \in [r, z_o]\}$ satisfying the stochastic differential system

$$Y_z^i = \alpha^i + \int_{[r,z]} [\sigma_j^i(V_u, Y_u) dW_u^j + b^i(V_u, Y_u) du], \ i = 1, \dots, m. \quad (3.1)$$

Moreover, $E(\sup_{z \in [r, z_o]} |Y_z|^p) \leq C_1$, and $E(|Y(\Delta)|^p) \leq C_2 |\Delta|^{p/2}$, for any $p \geq 2$ and for any rectangle $\Delta = [z_1, z_2] \subset [r, z_o]$, where C_1 and C_2 are positive constants depending on p, z_o, K, $\beta_{p\nu}$ and $E(|\alpha|^p)$.

Proof. Using Picard's iteration scheme we introduce the processes

$$Y_0^i(z) = 0 ,$$

and

$$Y_{n+1}^i(z) = \alpha^i + \int_{[r,z]} [\sigma_j^i(V_u, Y_n(u)) dW_u^j + b^i(V_u, Y_n(u)) du] ,$$

for any $n \geq 0$.

Now, applying Burkholder and Hölder's inequalities and condition (i) we obtain, for any $p \geq 2$,

$$E(\sup_{z \in [r, z_o]} |Y_{n+1}(z) - Y_n(z)|^p) \leq C_p K^p \int_{[r, z_o]} E(|Y_n(u) - Y_{n-1}(u)|^p) \, du .$$

It follows inductively that the above expression is bounded by

$$(C_p K^p |[r, z_o]|)^n (n!)^{-2} \int_{[r, z_o]} E(|Y_1(u)|^p) \, du.$$

In consequence, by condition (ii) we have

$$\sum_n E\Big(\sup_{z\in[r,z_o]} |Y_{n+1}(z)-Y_n(z)|^p\Big) < \infty ,$$

which implies the existence of a continuous process Y satisfying (3.1) , and such that

$$E\Big(\sup_{z\in[r,z_o]} |Y_z|^p\Big) < \infty \quad , \quad \text{for all } p\geq 2.$$

Furthermore, this expectation can be bounded by a constant depending only on p, z_o, K, $\beta_{p\nu}$ and $E(|\alpha|^p)$. The uniqueness of this solution can be proved as usual. Finally, the inequality $E(|Y(\Delta)|^p) \leq C_2|\Delta|^{p/2}$ can be easily derived using first Burkholder and Hölder's inequalities and, secondly, applying conditions (i) and (ii), and the above remark on the quantity $E(\sup_{z\in[r,z_o]} |Y_z|^p)$. □

Observe that in the preceding lemma the process V needs only to be defined on $[r,z_o]$. Also, we remark that the constants C_1 and C_2 do not depend on r.

We are going to state a lemma on the approximation of solutions of equation (3.1) by polygonal paths. For any integer $n\geq 1$ we consider the set $\mathcal{S}^n$ of points $(i2^{-n}s_o,\ j2^{-n}t_o)$, $i,j = 0,1,\dots,2^n$. Define $\phi_n(z) = \sup\{u\in\mathcal{S}^n:\ u\leq z\}$ and $\psi_n(z) = \inf\{u\in\mathcal{S}^n:\ u\geq z\}$ for any $z\in T$.

Our processes will depend on a parameter λ which belongs to an arbitrary set Λ. We consider a map $r:\Lambda\longrightarrow T$. For any λ, let $\{V(z,\lambda),\ z\in[r(\lambda),z_o]\}$ be a continuous and adapted M-dimensional process such that $\sup_{\lambda,z} E(|V(z,\lambda)|^p) < \infty$ for all $p\geq 1$. We also consider a sequence of processes $\{V_n(z,\lambda),\ z\in[r(\lambda),z_o]\}$, $n\geq 1$, with the same properties as $V(z,\lambda)$ and verifying

$$\lim_n \sup_{\lambda,z} E(|V(z,\lambda)-V_n(z,\lambda)|^p) = 0, \quad \text{for all } p\geq 1.$$

We also assume that for all $p\geq 1$, the mapping $z\to V(z,\lambda)$ is continuous in L^p, uniformly with respect to λ. That means, for any $\epsilon>0$, there exists $\delta>0$ such that

$$\sup_{\lambda} \sup_{\substack{z,z'\geq r(\lambda)\\ |z-z'|\leq \delta}} E(|V(z,\lambda) - V(z',\lambda)|^p) < \epsilon . \tag{3.2}$$

Let σ and b be functions satisfying (i) and the next condition (which is stronger than (ii)):

(ii') $\|\sigma(x,y)-\sigma(x',y)\| + |b(x,y)-b(x',y)| \leq K|x-x'|(1+|y|^\nu)$,

for some integer $\nu\geq 0$.

Then we have the following result.

Lemma 3.2. Suppose that for any $\lambda\in\Lambda$ and $n\geq 1$ $\{Y(z,\lambda),\ z\geq r(\lambda)\}$ and $\{Y_n(z,\lambda),\ z\geq r(\lambda)\}$ are the continuous solutions of the stochastic differential systems

$$Y^i(z,\lambda) = \alpha^i(\lambda) + \int_{[r(\lambda),z]} [\sigma^i_j(V(u,\lambda),Y(u,\lambda))dW^j_u + b^i(V(u,\lambda),Y(u,\lambda))du] , \tag{3.3}$$

$$Y^i_n(z,\lambda) = \alpha^i_n(\lambda) + \int_{[\psi_n(r(\lambda))\wedge z,z]} [\sigma^i_j(V_n(\phi_n(u),\lambda),Y_n(\phi_n(u),\lambda))\, dW^j_u$$

$$+ b^i(V_n(\phi_n(u),\lambda),Y_n(\phi_n(u),\lambda))du], \tag{3.4}$$

$i=1,\ldots,m$, where $\alpha(\lambda)$ and $\alpha_n(\lambda)$ are $\mathcal{F}_{r(\lambda)}$-measurable m-dimensional random vectors satisfying $\sup_\lambda E(|\alpha(\lambda)|^p) < \infty$, and $\lim_n \sup_\lambda E(|\alpha(\lambda)-\alpha_n(\lambda)|^p)=$ $=0$, for all $p \geq 1$. Then, with the above hypotheses, we have

$$\lim_n \sup_\lambda E(\sup_{z\geq r(\lambda)} |Y(z,\lambda) - Y_n(z,\lambda)|^p) = 0,$$

for all $p \geq 1$.

Proof. Using Burkholder and Hölder's inequalities we have

$$E(\sup_{z\in[r(\lambda),z_o]} |Y(z,\lambda)-Y_n(z,\lambda)|^p)$$

$$\leq C(p,z_o) \Big\{ E(|\alpha(\lambda)-\alpha_n(\lambda)|^p)$$

$$+ \int_{[\psi_n(r(\lambda)),z_o]} E(\|\sigma(V(u,\lambda),Y(u,\lambda))-\sigma(V_n(\phi_n(u),\lambda),Y_n(\phi_n(u),\lambda))\|^p)du$$

$$+ \int_{[\psi_n(r(\lambda)),z_o]} E(|b(V(u,\lambda),Y(u,\lambda)) - b(V_n(\phi_n(u),\lambda),Y_n(\phi_n(u),\lambda))|^p)du$$

$$+ \int_{[r(\lambda),z_o]-[\psi_n(r(\lambda)),z_o]} E(\|\sigma(V(u,\lambda),Y(u,\lambda))\|^p+|b(V(u,\lambda),Y(u,\lambda))|^p)du\Big\}$$

$$= C(p,z_o) \{\gamma_1+\gamma_2+\gamma_3+\gamma_4\},$$

where $C(p,z_o)$ represents a constant which depends only on p and z_o and may be different from one formula to another one. By hypothesis we know that $\lim_n \sup_\lambda \gamma_1 = 0$. Applying conditions (i) and (ii') we deduce the following majorations for the second term.

$$\gamma_2 \leq C(p,z_o) \int_{[\psi_n(r(\lambda)),z_o]} E(\|\sigma(V(u,\lambda),Y(u,\lambda))-\sigma(V(\phi_n(u),\lambda),Y(u,\lambda))\|^p$$

$$+ \|\sigma(V(\phi_n(u),\lambda),Y(u,\lambda)) - \sigma(V_n(\phi_n(u),\lambda),Y(u,\lambda))\|^p$$

$$+ \|\sigma(V_n(\phi_n(u),\lambda),Y(u,\lambda)) - \sigma(V_n(\phi_n(u),\lambda),Y(\phi_n(u),\lambda))\|^p$$

$$+ \|\sigma(V_n(\phi_n(u),\lambda),Y(\phi_n(u),\lambda)) - \sigma(V_n(\phi_n(u),\lambda),Y_n(\phi_n(u),\lambda))\|^p)\, du$$

$$\leq C(p,z_o)K^p \Big\{ \sup_{\substack{u,u'\geq r(\lambda) \\ |u-u'|\leq \delta_n}} [E(|V(u,\lambda)-V(u',\lambda)|^{2p})]^{\frac{1}{2}}(1+ \sup_{u\geq r(\lambda)} [E(|Y(u,\lambda)|^{2\nu p})]^{\frac{1}{2}})$$

$$+ \sup_{u\geq r(\lambda)} [E(|V(u,\lambda)-V_n(u,\lambda)|^{2p})]^{\frac{1}{2}}(1 + \sup_{u\geq r(\lambda)} [E(|Y(u,\lambda)|^{2\nu p})]^{\frac{1}{2}}$$

$$+ \sup_{\substack{u,u'\geq r(\lambda) \\ |u-u'|\leq \delta_n}} E(|Y(u,\lambda)-Y(u',\lambda)|^p)$$

$$+ \int_{[r(\lambda),z_o]} E(\sup_{u'\in[r(\lambda),u]} |Y(u',\lambda)-Y_n(u',\lambda)|^p)du \Big\}, \tag{3.5}$$

where $\delta_n = (s_o \vee t_o)2^{-n}$. A similar bound can be derived for γ_3. Finally

$$\gamma_4 \leq C(p,z_o)K^p\delta_n \Big\{ \sup_{u\geq r(\lambda)} E(|Y(u,\lambda)|^p) + 1 + \sup_{u\geq r(\lambda)} E(|V(u,\lambda)|^p)\Big\}.$$

From lemma 3.1 we know that $\sup_\lambda \sup_{u\geq r(\lambda)} E(|Y(u,\lambda)|^p) < \infty$, and $\lim_n \sup_\lambda \sup_{\substack{u,u'\geq r(\lambda) \\ |u-u'|\leq \delta_n}} E(|Y(u,\lambda)-Y(u',\lambda)|^p) = 0$, for all $p \geq 1$.

Therefore, $\lim_n \sup_\lambda \gamma_4 = 0$, and the first three sumands of the expression (3.5) converge also to zero when n tends to infinity, uniformly with respect to λ, using (for the first two terms) the conditions that we have imposed to the processes V and V_n. Thus by Gronwall's lemma the proof of lemma follows easily. □

We remark that the process $Y(z,\lambda)$ solution of (3.3) satisfies the continuity property (3.2). The preceding lemmas will be useful in proving that the solution of a system of stochastic differential equations in the plane belongs to the class of two-parameter processes $\mathcal{H}_\infty$ introduced in Section 2.

Proposition 3.3. Consider the m-dimensional continuous process $X = \{X_z,\ z \in T\}$ given by the system of stochastic differential equations

$$X_z^i = x^i + \int_{R_z} [A_j^i(X_r)dW_r^j + B^i(X_r)dr], \qquad i=1,\dots,m,$$

where $x \in \mathbb{R}^m$, and the functions A_j^i, B^i have bounded derivatives of all orders greater than or equal to one. Then, the process X belongs to $\mathcal{H}_\infty$.

Proof. For any $n \geq 1$ we introduce the process $X_n = \{X_n(z),\ z \in T\}$ defined by

$$X_n^i(z) = x^i + \int_{R_z} [A_j^i(X_n(\phi_n(r)))dW_r^j + B^i(X_n(\phi_n(r)))dr], \qquad i=1,\dots,m.$$

Notice that this is a recursive system and, therefore, X_n^i, $i=1,\dots,m$, are smooth processes. We are going to prove that $\{X_n^i,\ n \geq 1\}$ is an approximating sequence for X^i, $i=1,\dots,m$. Denote by $\xi^{(n)i}_{j_1\dots j_N}(z,r_1,\dots,r_N)$ the N-th derivatives of the process X_n^i. First, by lemma 3.2 we have $\lim_n E(\sup_z |X_n(z)-X(z)|^p) = 0$ for all $p \geq 1$. If $z \in (0,s_o] \times (0,t_o]$ and $u = \sup\{v \in \mathcal{S}^n:\ v < z\}$, then $X_n^i(z)$ is given by

$$X_n^i(z) = X_n^i(z \otimes u) + X_n^i(u \otimes z) - X_n^i(u) + A_h^i(X_n(u))W^h((u,z]) + B^i(X_n(u))|(u,z]|.$$

As a consequence, we obtain

$$\xi_j^{(n)i}(z,r) = \xi_j^{(n)i}(z \circledast u,r)+\xi_j^{(n)i}(u \circledast z,r)-\xi_j^{(n)i}(u,r)$$
$$+ \frac{\partial A_h^i}{\partial x_k}(X_n(u))\xi_j^{(n)k}(u,r)W^h((u,z]) + A_h^i(X_n(u))\delta_j^h 1_{(u,z]}(r)$$
$$+ \frac{\partial B^i}{\partial x_k}(X_n(u))\xi_j^{(n)k}(u,r)|(u,z]|.$$

Therefore, $\xi_j^{(n)i}(u,r)$ is the solution of the following system of equations

$$\xi_j^{(n)i}(z,r) = A_j^i(X_n(\phi_n(r))) + \int_{[\psi_n(r)\wedge z,z]}\left[\frac{\partial A_h^i}{\partial x_k}(X_n(\phi_n(u)))\xi_j^{(n)k}(\phi_n(u),r)dW_u^h\right.$$
$$\left.+ \frac{\partial B^i}{\partial x_k}(X_n(\phi_n(u)))\xi_j^{(n)k}(\phi_n(u),r)dr\right], \quad i=1,\dots,m;\ j=1,\dots,d. \tag{3.6}$$

Then, if we introduce the processes $\{\xi_j^i(z,r),\ z \geq r\}$ defined by the system

$$\xi_j^i(z,r) = A_j^i(X_r) + \int_{[r,z]}\left[\frac{\partial A_h^i}{\partial x_k}(X_u)\ \xi_j^k(u,r)dW_u^h + \frac{\partial B^i}{\partial x_k}(X_u)\xi_j^k(u,r)du\right], \tag{3.7}$$

applying lemma 3.2 we obtain

$$\lim_n \sup_r E(\sup_{z\geq r} \| \xi(z,r) - \xi^{(n)}(z,r)\|^p) = 0,$$

for all p.

We need a similar result for the derivatives $\xi_{j_1\dots j_N}^{(n)i}$ of arbitrary order and for the stochastic integrals $I_K\,\xi^{(n)}$ introduced in Section 2. First we will see that the successive derivatives of the smooth processes X_n^i can be deduced by a recursive argument. To do this, define

$$\alpha^{(n)i}_{hj_1\dots j_N}(u,r_1,\dots,r_N) = \sum \frac{\partial^\nu A^i_h}{\partial x_{k_1}\dots\partial x_{k_\nu}}(X_n(u))\xi^{(n)k_1}_{j(I_1)}(u,r(I_1))\dots\xi^{(n)k_\nu}_{j(I_\nu)}(u,r(I_\nu)),$$

and

$$\beta^{(n)i}_{j_1\dots j_N}(u,r_1,\dots,r_N) = \sum \frac{\partial^\nu B^i}{\partial x_{k_1}\dots\partial x_{k_\nu}}(X_n(u))\xi^{(n)k_1}_{j(I_1)}(u,r(I_1))\dots\xi^{(n)k_\nu}_{j(I_\nu)}(u,r(I_\nu)),$$

where the sums are extended to the set of all partitions $\{1,\dots,N\} = I_1\cup\dots\cup I_\nu$, and we have employed the notations of Section 2. We also set $\alpha^{(n)i}_j = A^i_j(X_n(u))$. Then, we can write the following formula for the N-th derivatives of X^i_n,

$$\xi^{(n)i}_{j_1\dots j_N}(z,r_1,\dots,r_N) = \sum_{\varepsilon=1}^{N} \alpha^{(n)i}_{j_\varepsilon j_1\dots j_{\varepsilon-1}j_{\varepsilon+1}\dots j_N}(\phi_n(r_\varepsilon),r_1,\dots,r_{\varepsilon-1},r_{\varepsilon+1},\dots,r_N)$$

$$+\int_{[\psi_n(r_1\vee\dots\vee r_N)\wedge z,z]}[\alpha^{(n)i}_{hj_1\dots j_N}(\phi_n(u),r_1,\dots,r_N)dW^h_u$$

$$+\beta^{(n)i}_{j_1\dots j_N}(\phi_n(u),r_1,\dots,r_N)du]. \qquad (3.8)$$

This expression can be proved by induction on N. For N=1 it reduces to formula (3.6). Suppose that (3.8) is true for N. Remark that for any $g\in H$ we have

$$D(\alpha^{(n)i}_{hj_1\dots j_N}(z,r_1,\dots,r_N))(g) = \int_{R_z}\alpha^{(n)i}_{hj_1\dots j_N j}(z,r_1,\dots,r_N,r)\dot{g}^j(r)dr,$$

and

$$D(\beta^{(n)i}_{j_1\dots j_N}(z,r_1,\dots,r_N))(g) = \int_{R_z} \beta^{(n)i}_{j_1\dots j_N j}(z,r_1,\dots,r_N,r)\dot{g}^j(r)dr.$$

In consequence, we obtain

$$D(\xi^{(n)i}_{j_1\dots j_N}(z,r_1,\dots,r_N))(g) = \sum_{\varepsilon=1}^{N}\int_{R_z} \alpha^{(n)i}_{j_\varepsilon j_1\dots j_{\varepsilon-1}j_{\varepsilon+1}\dots j_N j}(\phi_n(r_\varepsilon),r_1,$$

$$\dots,r_{\varepsilon-1},r_{\varepsilon+1},\dots,r_N,r)\dot{g}^j(r)dr$$

$$+ \int_{R_z} \alpha^{(n)i}_{jj_1\dots j_N}(\phi_n(r),r_1,\dots,r_N)\ \dot{g}^j(r)dr$$

$$+ \int_{R_z}\int_{[\psi_n(r_1\vee\dots\vee r_N\vee r)\wedge z,z]} [\alpha^{(n)i}_{hj_1\dots j_N j}(\phi_n(u),r_1,\dots,r_N,r)dW^h_u$$

$$+ \beta^{(n)i}_{j_1\dots j_N j}\ (\phi_n(u),r_1,\dots,r_N,r)du]\ \dot{g}^j(r)\ dr\ ,$$

which implies that (3.8) holds for N+1.

Now, for any $r_1,\dots,r_N$, we introduce the processes $\{\xi^i_{j_1\dots j_N}(z,r_1,\dots,r_N),\ z\geq r_1\vee\dots\vee r_N\}$, $i=1,\dots,m$, $j_1,\dots,j_N\in\{1,\dots,d\}$, given by the stochastic differential systems

$$\xi^i_{j_1\dots j_N}(z,r_1,\dots,r_N) = \sum_{\varepsilon=1}^{N} \alpha^i_{j_\varepsilon j_1\dots j_{\varepsilon-1}j_{\varepsilon+1}\dots j_N}(r_\varepsilon,r_1,\dots,r_{\varepsilon-1},r_{\varepsilon+1},$$

$$\dots,r_N) + \int_{[(r_1\vee\dots\vee r_N)\wedge z,z]}[\alpha^i_{hj_1\dots j_N}(u,r_1,\dots,r_N)dW^h_u$$

$$+ \beta^i_{j_1\dots j_N}(u,r_1,\dots,r_N)du]\ ,$$

being

$$\alpha^{i}_{hj_1\ldots j_N}(u,r_1,\ldots,r_N) = \sum \frac{\partial^{\nu} A^{i}_{h}}{\partial x_{k_1}\ldots\partial x_{k_\nu}}(X_u)\xi^{k_1}_{j(I_1)}(u,r(I_1))$$

$$\ldots \xi^{k_\nu}_{j(I_\nu)}(u,r(I_\nu)),$$

$$\beta^{i}_{j_1\ldots j_N}(u,r_1,\ldots,r_N) = \sum \frac{\partial^{\nu} B^{i}}{\partial x_{k_1}\ldots\partial x_{k_\nu}}(X_u)\xi^{k_1}_{j(I_1)}(u,r(I_1))$$

$$\ldots \xi^{k_\nu}_{j(I_\nu)}(u,r(I_\nu)),$$

and $\alpha^{i}_{j}(u) = A^{i}_{j}(X_u)$.

Here we have used the same notations as above.

We claim that

$$\lim_{n} \sup_{r_1,\ldots,r_N} E(\sup_{z\geq r_1\vee\ldots\vee r_N} \|\xi(z,r_1,\ldots,r_N)-\xi^{(n)}(z,r_1,\ldots,r_N)\|^{p}) = 0, \tag{3.9}$$

for any p.

Indeed, this can be shown by induction on N. We have already remarked that (3.9) is true for N=1. Suppose that it holds for N-1. Observe that $\alpha^{i}_{hj_1\ldots j_N}(u,r_1,\ldots,r_N)$ is equal to $\frac{\partial A^{i}_{h}}{\partial x_k}(X_u)\xi^{k}_{j_1\ldots j_N}(u,r_1,\ldots,r_N)$ plus a polynomial function of the derivatives

$$\frac{\partial^{\nu} A^{i}_{h}}{\partial x_{k_1}\ldots\partial x_{k_\nu}}(X_u) \quad \text{with} \quad \nu\geq 2,$$

and the processes $\xi^{k}_{j_1\ldots j_N}(u,r(I))$ with $\mathrm{card}(I)\leq N-1$. Therefore, (3.9) will follow from the induction hypothesis and lemma 3.2 applied to $\lambda =$

$= (r_1,\dots,r_N) \in T^N$, $r(\lambda) = r_1 \vee \dots \vee r_N$, $Y^i(r,\lambda) = \xi^i_{j_1\dots j_N}(z,r_1,\dots,r_N)$ (we may fix the indexes $j_1,\dots,j_N$), and $V(u,\lambda)$ equal to the $m(2^N-2)$-dimensional process whose components are X^i_u and $\xi^i_{j(I)}(u,r(I))$ being $i=1,\dots,m$, $I \subset \{1,\dots,N\}$, $0 < \text{card}(I) < N$. The processes $\alpha(\lambda)$, $\alpha_n(\lambda)$ and $V_n(u,\lambda)$ would be defined in an obvious way.

By the same methód, for any subset K of $\{1,\dots,N\}$ we can prove that

$$\lim_n \sup_{r_\varepsilon, \varepsilon \in K^c} E\Big(\sup_{z \geq \bigvee \{r_\varepsilon, \varepsilon \in K^c\}} \| I_K \xi(z, r(K^c)) - I_K \xi^{(n)}(z, r(K^c)) \|^p \Big) = 0,$$

for any p, being $I_K\xi$ the processes defined by the following system of equations:

$$I_K \xi^i_{j(K^c)}(z, r(K^c)) = \sum_{\varepsilon \notin K} I_K \alpha^i_{j_\varepsilon j(K^c - \{\varepsilon\})}(r_\varepsilon, r(K^c - \{\varepsilon\}))$$

$$+ \sum_{\varepsilon \in K} \int_{[(\bigvee\{r_\varepsilon, \varepsilon \in K^c\}) \wedge z, z]} I_{K-\{\varepsilon\}} \alpha^i_{j_\varepsilon j(K^c)}(r_\varepsilon, r(K^c)) dW^{j_\varepsilon}_{r_\varepsilon}$$

$$+ \int_{[(\bigvee\{r_\varepsilon, \varepsilon \in K^c\}) \wedge z, z]} [I_K \alpha^i_{hj(K^c)}(u, r(K^c)) dW^h_u + I_K \beta^i_{j(K^c)}(u, r(K^c)) du],$$

$$I_K \alpha^i_{hj(K^c)}(u, r(K^c)) = \sum \frac{\partial^\nu A^i_h}{\partial x_{k_1} \dots \partial x_{k_\nu}} (X_u) I_{K \cap I_1} \xi^{k_1}_{j(I_1 - K)}(u, r(I_1 - K))$$
$$\dots I_{K \cap I_\nu} \xi^{k_\nu}_{j(I_\nu - K)}(u, r(I_\nu - K)),$$

and

$$I_K \beta^i_{j(K^c)}(u, r(K^c)) = \sum \frac{\partial^\nu B^i}{\partial x_{k_1} \dots \partial x_{k_\nu}} (X_u) I_{K \cap I_1} \xi^{k_1}_{j(I_1 - K)}(u, r(I_1 - K))$$
$$\dots I_{K \cap I_\nu} \xi^{k_\nu}_{j(I_\nu - K)}(u, r(I_\nu - K)).$$

The proof of the proposition is now complete. □

4. Application of Malliavin calculus to the solution of stochastic differential equations in the plane. In order to state the main result of this section, we need two preliminar lemmas.

Lemma 4.1. Let $Q(\omega)$ be a symmetric non-negative definite $m\times m$ random matrix. Define the random variable $\Lambda = \max_{|v|=1} v^t Q v$. Then for each $p \geq 1$ there exist a universal constant $C = C(p,m)$ such that

$$E[(\det Q)^{-p}] \leq C\{\sup_{|v|=1} E[(v^t Q v)^{-2(p+m+1)}] E[(1+\Lambda)^{2m+6}]\}^{\frac{1}{2}}.$$

Proof. See Stroock [9], page 359.

Lemma 4.2. Let $Y_t = Y_o + M_t + V_t$ be a continuous semimartingale adapted to an increasing family of σ-fields $\{F_t,\ t \geq 0\}$ satisfying the usual conditions. We assume that $M = \{M_t,\ t\geq 0\}$ is a continuous local martingale such that $M_o = 0$ and $\langle M\rangle_t = \int_0^t \alpha_s^2 ds$, and we also assume that $V_t = \int_0^t \gamma_s ds$, where α and γ are progressively measurable processes such that the preceding integrals exist. Let $S: \Omega \longrightarrow [0,\tau]$ be a bounded stopping time and suppose that $\sup\{|\alpha_t(\omega)|, |\gamma_t(\omega)|\} \leq M$ for any $\omega \in \Omega$ and $t \leq S(\omega)$. We fix real numbers $\delta > 4\eta > 0$, $a,b > 0$ and $p \geq 1$. Then, we have

$$P\left\{ \int_0^S Y_t^2 dt \leq a\varepsilon^{\delta},\ \int_0^S \alpha_t^2 dt \geq b\varepsilon^{\eta} \right\} \leq \varepsilon^p,$$

for any $\varepsilon \leq \varepsilon_o$, where ε_o depends on p, M, τ, a, b, δ and η.

Proof. The proof of this lemma follows the same lines as that of theorem 8.26 in Stroock [9]. For the sake of completeness we will give the main arguments of this demonstration. First we will show that for all constants $A>0$ and $B>0$ the next inequality holds

$$P\left\{\int_0^S Y_t^2 dt \le A, \int_0^S \alpha_t^2 dt \ge B\right\} \le 2^{1/2}\exp\left[-2^{-7}M^{-2}(\tau B^{-1/2}+A^{1/2}B^{-1})^{-2}\right]. \qquad (4.1)$$

In order to prove (4.1) we may assume that for some $\beta>0$, $|\alpha_t(\omega)|\ge \beta$ for any ω and t. Indeed, suppose that $\{\hat{B}_t,\ t\ge 0\}$ is a standard Brownian motion independent of $\bigvee_{t\ge 0}\mathcal{F}_t$. Then, the semimartingale $Y_t' = Y_t + \beta\hat{B}_t$ verifies this property, and making $\beta\downarrow 0$ we get the desired result for Y_t. Set $B_t = M(A_t)$, $t\ge 0$, being $A_t = \inf\{s\ge 0: \langle M\rangle_s \ge t\}$. Then $\{B_t, \mathcal{F}_{A_t}, t\ge 0\}$ is a Brownian motion. If $\int_0^S \alpha_t^2 dt \ge B$, we have

$$\int_0^S Y_t^2 dt = \int_0^{\langle M\rangle_S} Y^2(A_s)\alpha^{-2}(A_s)ds \ \ge\ M^{-2}\int_0^B \left(Y_0+B_s+\int_0^{A_s} Y_u du\right)^2 ds,$$

and, therefore,

$$\left(\int_0^S Y_t^2\, dt\right)^{1/2} \ge M^{-1}\left(\int_0^B (Y_0+B_s)^2 ds\right)^{1/2} - M^{-1}\left(\int_0^B\left(\int_0^{A_s} Y_u du\right)^2 ds\right)^{1/2}$$

$$\ge M^{-1}B^{1/2}\left(\sigma_{[0,B]}(B(\cdot)) - \int_0^{A_B}|Y_u|du\right)$$

$$\ge M^{-1}B^{1/2}(\sigma_{[0,B]}(B(\cdot)) - \tau M),$$

where for any real and continuous function f on $[0,B]$, $\sigma^2_{[0,B]}(f)$ denotes the quantity

$$\frac{1}{B}\int_0^B\left[f(s)^2 - \left(\frac{1}{B}\int_0^B f(u)du\right)^2\right]ds.$$

In consequence,

$$P\{ \int_0^S Y_t^2\, dt \le A, \int_0^S \alpha_t^2\, dt \ge B\}$$

$$\le P\{ M^{-2}B(\sigma_{[0,B]}(B(\cdot))-\tau M)^2 \le A\} \le P\{\sigma_{[0,B]}(B(\cdot)) \le M(A^{1/2}B^{-1/2}+\tau)\} ,$$

and applying lemma 8.6 (pag. 343) of Ikeda-Watanabe [5] we obtain the inequality (4.1).

Now, to achieve the proof of the lemma we fix an integer $n \ge 1$ and we compute

$$P \left\{ \int_0^S Y_t^2\, dt \le a\varepsilon^{\delta}, \int_0^S \alpha_t^2\, dt \ge b\,\varepsilon^{\eta} \right\}$$

$$\le \sum_{k=1}^{n} P \left\{ \int_{[\frac{k-1}{n}\wedge S,\ \frac{k}{n}\wedge S]} Y_t^2\, dt \le a\varepsilon^{\delta} , \int_{[\frac{k-1}{n}\wedge S,\ \frac{k}{n}\wedge S]} \alpha_t^2\, dt \ge n^{-1}b\varepsilon^{\eta} \right\}$$

$$\le n2^{1/2} \exp [-2^{-7}M^{-2}(\tau b^{-1/2}\varepsilon^{-\eta/2}n^{-1/2} + a^{1/2}\, \varepsilon^{(\delta/2)-\eta}\, n)^{-2}] . \qquad (4.2)$$

We take q such that $q>\eta$ and $\delta > 2q + 2\eta$. Then for any $\varepsilon < 1$ we may choose n such that $n \le \varepsilon^{-q} < n+1$, and (4.2) is bounded by

$$\varepsilon^{-q}2^{1/2}\exp [-2^{-7}M^{-2}(2\tau b^{-1/2}\varepsilon^{(q-\eta)/2} + a^{1/2}\varepsilon^{(\delta/2)-q-\eta})^{-2}],$$

which is less than ε^{p} for any $\varepsilon \le \varepsilon_o$. □

<u>Theorem 4.3.</u> Let $X = \{X_z,\ z\in T\}$ be the continuous solution of the stochastic differential system

$$X_z^i = x^i + \int_{R_z} [A_j^i(X_r)dW_r^j + B^i(X_r)dr], \qquad i=1,\dots,m, \tag{4.3}$$

where $x \in \mathbb{R}^m$, and the functions A_j^i, B^i have bounded derivatives of all orders greater than or equal to one. Assume further that the following property holds:

(P) The vector space spanned by the vector fields $A_1,\dots,A_d$, $A_i^\nabla A_j$, $1 \le i,j \le d$, $A_i^\nabla(A_j^\nabla A_k)$, $1 \le i,j,k \le d$, ..., has full rank at the point x.

Then, for any point $(s,t) \in T$ with $st \neq 0$, the law of the random vector X_{st} admits an infinitely differentiable density function.

Proof. We fix $z = (s,t) \in T$ with $st \neq 0$. We have to check conditions (i) and (ii) of theorem 1.1 for the Wiener functional X_z. The first condition follows from propositions 3.3 and 2.1. In order to prove the second condition we set $Q^{ij} = \langle DX_z^i, DX_z^j \rangle_H$. From the results of Section 3 we know that

$$Q^{ij} = \sum_{h=1}^{d} \int_{R_z} \xi_h^i(z,r)\, \xi_h^j(z,r)dr = \sum_{h=1}^{d} \int_{R_z} \zeta_l^i(z,r)A_h^l(X_r)\zeta_{l'}^j(z,r)A_h^{l'}(X_r)dr, \tag{4.4}$$

where, for any r, the processes $\{\zeta_j^i(z,r),\ z \ge r\}$ are defined as the solution of the stochastic differential system:

$$\zeta_j^i(z,r) = \delta_j^i + \int_{[r,z]} \left[\frac{\partial A_h^i}{\partial x_k}(X_u)\zeta_j^k(u,r)dW_u^h + \frac{\partial B^i}{\partial x_k}(X_u)\zeta_j^k(u,r)du \right]. \tag{4.5}$$

We want to show that $E[(\det Q)^{-p}] < \infty$ for all $p \ge 1$. Set $\Lambda = \max_{|v|=1} v^t Q v \le \|Q\|$. Using the estimates for the moments of the solutions of stochastic differential equations in the plane, obtained in lemma 3.1, we deduce that $E(\|Q\|^p) < \infty$ for any p. Therefore, by lemma 4.1 it suffices to see that $\sup_{|v|=1} E[(v^t Q v)^{-p}] < \infty$.

To this end we are going to show that for all $p \geq 1$ we have

$$P\{v^t Q v \leq \varepsilon\} \leq \varepsilon^p \tag{4.6}$$

for any v such that $|v|=1$ and $\varepsilon \leq \varepsilon_o$, where ε_o depends on p, x, z and the coefficients of system (4.3). Suppose that $0 < \varepsilon < 1$, and using (4.4) compute

$$\begin{aligned}
P\{v^t Q v \leq \varepsilon\} &= P\{\sum_{h=1}^{d} \int_{R_z} (v_i \zeta_j^i(z,r) A_h^j(X_r))^2 dr \leq \varepsilon\} \\
&\leq P\{\sum_{h=1}^{d} \int_0^s \int_{t-\varepsilon^{2/3}}^t (v_i \zeta_j^i(z,r) A_h^j(X_r))^2 dr \leq \varepsilon\} \\
&\leq P\{\sum_{h=1}^{d} \int_0^s (v_i A_h^i(X_{\sigma t}))^2 d\sigma \leq 4\varepsilon^{1/3}\}
\end{aligned}$$

$$+ P\{\sum_{h=1}^{d} \int_0^s (v_i A_h^i(X_{\sigma t}))^2 d\sigma > 4\varepsilon^{1/3}, \sum_{h=1}^{d} \int_0^s \int_{t-\varepsilon^{2/3}}^t (v_i \zeta_j^i(z,r) A_h^j(X_r))^2 dr \leq \varepsilon\}. \tag{4.7}$$

The second probability of this expression is bounded by

$$\begin{aligned}
&P\left\{\sum_{h=1}^{d} \int_0^s \int_{t-\varepsilon^{2/3}}^t (v_i A_h^i(X_{\sigma t}))^2 d\sigma > 4\varepsilon, \sum_{h=1}^{d} \int_0^s \int_{t-\varepsilon^{2/3}}^t (v_i \zeta_j^i(z,r) A_h^j(X_r))^2 dr \leq \varepsilon\right\} \\
&\leq P\left\{\sum_{h=1}^{d} \int_0^s \int_{t-\varepsilon^{2/3}}^t (v_i (\delta_j^i - \zeta_j^i(z,r)) A_h^j(X_r))^2 dr > \varepsilon\right\} \\
&\leq \varepsilon^{-q/3} s^q \sup_{r \in [0,s] \times [t-\varepsilon^{2/3}, t]} E\left(\left|\sum_{h=1}^{d} (v_i (\delta_j^i - \zeta_j^i(z,r)) A_h^j(X_r))^2\right|^q\right) \\
&\leq \varepsilon^{-q/3} s^q \sup_{r \in [0,s] \times [t-\varepsilon^{2/3}, t]} \left[E(\| I_m - \zeta(z,r) \|^{4q})\, E(\| A(X_r) \|^{4q})\right]^{1/2},
\end{aligned}$$

for any $q \geq 1$. Here I_m denotes the identity matrix of order m. In the following C(q) will represent a constant which may depend on q,x,z and the coefficients of system (4.3).

Applying the second inequality deduced in lemma 3.1 to the stochastic differential system (4.5) we obtain

$$\sup_{r\in[0,s]\times[t-\varepsilon^{2/3},t]} E(\| I_m-\zeta(z,r)\|^{4q}) \leq C(q)s^{2q}\varepsilon^{4q/3} .$$

In consequence, the second sumand of expression (4.7) is bounded by $C(q)\varepsilon^{q/3}$, which provides the desired majoration. Then, it suffices to study the term

$$P\{\sum_{h=1}^{d}\int_0^s (v_i A_h^i(X_{\sigma t})^2 d\sigma \leq 4\varepsilon^{1/3}\}. \tag{4.8}$$

Set $\mathcal{G}_0 = \{A_h,\ 1\leq h\leq d\}$ and $\mathcal{G}_j = \{A_h^{\nabla}V,\ 1\leq h\leq d,\ V\in\mathcal{G}_{j-1}\}$ for any $j\geq 1$. By property (P) there exists an integer $j_0\geq 0$ such that the linear span of $\bigcup_{j=0}^{j_0}\mathcal{G}_j$ at the point x has dimension m. This implies that there is an $R>0$ and $c>0$ such that

$$\sum_{j=0}^{j_0}\sum_{V\in\mathcal{G}_j}(v_i V^i(y))^2 \geq c ,$$

for all v and y with $|v|=1$ and $|y-x|<R$. Consider the stopping time S with respect to the family of σ-fields $\{F_{\sigma t},\ \sigma\geq 0\}$ defined as

$$S = \inf\{\sigma\geq 0: \sup_{\substack{\xi\leq\sigma\\ \tau\leq t}} |X_{\xi\tau}-x| \geq R\}\wedge s.$$

For any $j=0,1,\dots,j_0$ we put $m(j) = \frac{1}{3}2^{-4j}$ and we introduce the set

$$E_j = \left\{ \sum_{V \in \mathcal{G}_j} \int_0^S (v_i V^i(X_{\sigma t}))^2 d\sigma \leq 4\, \varepsilon^{m(j)} \right\}.$$

We remark that

$$\left\{ \sum_{h=1}^{d} \int_0^s (v_i A_h^i(X_{\sigma t}))^2\, d\sigma \leq 4\varepsilon^{1/3} \right\} \subset E_o.$$

Consider the decomposition

$$E_o \subset (E_o \cap E_1^c) \cup (E_1 \cap E_2^c) \cup \dots \cup (E_{j_o - 1} \cap E_{j_o}^c) \cup F,$$

where $F = E_o \cap E_1 \cap \dots \cap E_{j_o}$. Then, the probability given in (4.8) is bounded by

$$\sum_{j=1}^{j_o} P(E_{j-1} \cap E_j^c) + P(F)$$

and we are going to estimate each term of this sum. This will be done in two steps:

(i) We can write

$$P(F) \leq P(F \cap \{S \geq \varepsilon^{\beta}\}) + P\{S < \varepsilon^{\beta}\},$$

where $0 < \beta < m(j_o)$. For ε small enough, the intersection $F \cap \{S \geq \varepsilon^{\beta}\}$ is empty. In fact, if $S \geq \varepsilon^{\beta}$ we have

$$\sum_{j=o}^{j_o} \sum_{V \in \mathcal{G}_j} \int_0^S (v_i V^i(X_{\sigma t}))^2 d\sigma \geq c\, \varepsilon^{\beta},$$

whereas on F this integral is bounded by $4(j_o+1)\varepsilon^{m(j_o)}$. Moreover it holds that

$$P\{S<\varepsilon^{\beta}\} \leq P\{\sup_{u\leq(\varepsilon^{\beta},t)} |X_u - x| \geq R\}$$

$$\leq R^{-q} E\left(\sup_{u\leq(\varepsilon^{\beta},t)} \left|\int_{R_u} [A_h(X_r)dW_r^h + B(X_r)dr]\right|^q\right),$$

for any $q \geq 1$. Now, using Burkholder and Hölder inequalities we deduce $P\{S<\varepsilon^{\beta}\} \leq C(q)\ \varepsilon^{q\beta/2}$ for any $q \geq 2$, and, therefore, we have obtained a majoration of the type (4.6) for $P(F)$.

(ii) For any $j = 1,\dots,j_o$ we consider the probability

$$P(E_{j-1} \cap E_j^c)$$

$$= P\left\{\sum_{V\in\mathcal{D}_{j-1}} \int_0^S (v_i\ V^i(X_{\sigma t}))^2 d\sigma \leq 4\varepsilon^{m(j-1)},\ \sum_{A\in\mathcal{D}_j} \int_0^S (v_i V^i(X_{\sigma t}))^2 d\sigma > 4\varepsilon^{m(j)}\right\}$$

$$\leq \sum_{V\in\mathcal{D}_{j-1}} P\left\{\int_0^S (v_i V^i(X_{\sigma t}))^2 d\sigma \leq 4\varepsilon^{m(j-1)},\right.$$
$$\left.\sum_{h=1}^{d} \int_0^S (v_i (A_h^{\nabla} V)^i(X_{\sigma t}))^2\ d\sigma > 4n(j-1)^{-1}\ \varepsilon^{m(j)}\right\}, \tag{4.9}$$

where $n(j) = \text{card}\ \mathcal{D}_j$. We fix $j=1,\dots,j_o$ and a vector field $V \in \mathcal{D}_{j-1}$. Applying Ito's formula in the first coordinate we obtain

$$V^i(X_{uv}) = V^i(x) + \int_{R_{uv}} \frac{\partial V^i}{\partial x_k}(X_{\sigma v})\ A_h^k(X_{\sigma\tau})\ dW_{\sigma\tau}^h$$

$$+\int_{R_{uv}} \left[\frac{\partial V^i}{\partial x_k}(X_{\sigma v})B^k(X_{\sigma\tau}) + \frac{1}{2}\frac{\partial^2 V^i}{\partial x_k \partial x_j}(X_{\sigma v}) \sum_{h=1}^{d} A_h^k(X_{\sigma\tau})A_h^j(X_{\sigma\tau})\right] d\sigma d\tau .$$

Then, by lemma 4.2 we have the following estimation

$$P\Big\{ \int_0^s (v_i V^i(X_{\sigma t}))^2 d\sigma \leq 4\varepsilon^{m(j-1)},\ \sum_{h=1}^d \int_0^s \int_0^t (v_i \frac{\partial V^i}{\partial x_k}(X_{\sigma t})A_h^k(X_{\sigma\tau}))^2 d\sigma d\tau$$

$$\geq \quad n(j-1)^{-1}\ \varepsilon^{3m(j)} \Big\} \leq \ \varepsilon^p, \tag{4.10}$$

for any $p \geq 1$ and $\varepsilon \leq \varepsilon_0$ (ε_0 depending on p). In fact, note that $m(j-1) > 12m(j)$. Finally, we have the following majorations

$$P\Big\{ \sum_{h=1}^d \int_0^s (v_i \frac{\partial V^i}{\partial x_k}(X_{\sigma t})A_h^k(X_{\sigma t}))^2\, d\sigma \ > \ 4n(j-1)^{-1}\ \varepsilon^{m(j)},$$

$$\sum_{h=1}^d \int_0^s \int_0^t (v_i \frac{\partial V^i}{\partial x_k}(X_{\sigma t})A_h^k(X_{\sigma\tau}))^2 d\sigma d\tau \ < \ n(j-1)^{-1}\ \varepsilon^{3m(j)} \Big\}$$

$$\leq P\Big\{ \sum_{h=1}^d \int_0^s \int_{t-\varepsilon^{2m(j)}}^t (v_i \frac{\partial V^i}{\partial x_k}(X_{\sigma t})A_h^k(X_{\sigma t}))^2 d\sigma d\tau \ > \ 4n(j-1)^{-1}\ \varepsilon^{3m(j)},$$

$$\sum_{h=1}^d \int_0^s \int_{t-\varepsilon^{2m(j)}}^t (v_i \frac{\partial V^i}{\partial x_k}(X_{\sigma t})A_h^k(X_{\sigma\tau}))^2 d\sigma d\tau \ < \ n(j-1)^{-1}\ \varepsilon^{3m(j)} \Big\}$$

$$\leq P\Big\{ \sum_{h=1}^d \int_0^s \int_{t-\varepsilon^{2m(j)}}^t (v_i \frac{\partial V^i}{\partial x_k}(X_{\sigma t})(A_h^k(X_{\sigma t}) - A_h^k(X_{\sigma\tau})))^2 d\sigma d\tau \geq n(j-1)^{-1}\varepsilon^{3m(j)} \Big\}$$

$$\leq s^q \varepsilon^{-qm(j)} n(j-1)^q \sup_{\sigma\in[0,s],\, \tau\in[t-\varepsilon^{2m(j)},t]} E(|\sum_{h=1}^d (v_i \frac{\partial V^i}{\partial x_k}(X_{\sigma t})(A_h^k(X_{\sigma t}) - A_h^k(X_{\sigma\tau})))^2|^q)$$

$$\leq s^q \varepsilon^{-qm(j)} n(j-1)^q \sup_{\sigma\in[0,s],\, \tau\in[t-\varepsilon^{2m(j)},t]} \{ E(\|\frac{\partial V}{\partial x}(X_{\sigma t})\|^{4q})$$

$$\cdot E(\|A(X_{\sigma t}) - A(X_{\sigma\tau})\|^{4q})\}^{1/2},$$

for any $q \geq 1$. Using lemma 3.1, this expression is less than or equal to $C(q)\,\varepsilon^{qm(j)}$. This result combined with inequalities (4.9) and (4.10) gives us the desired sort of estimate for the term $P(E_{j-1} \cap E_j^c)$, which achieves the proof of the theorem. □

In the one-parameter case, the existence of a density for the solution of a stochastic differential equation can be proved under Hörmander's condition:

(P') The vector space spanned by $A_1,\dots,A_d$, $[A_i,A_j]$, $1 \leq i,j \leq d$, $[A_i,[A_j,A_k]]$, $1 \leq i,j,k \leq d,\dots$, at the point x is $\mathbb{R}^m$.

Actually, a more general condition using Lie brackets formed with the vector field B as generators would be sufficient. We have been unable to generalize this kind of condition to the two-parameter case.

Remark that hypothesis (P) is weaker than (P') and, in fact, theorem 4.3 can be applied to a family of situations that did not appear in the one-parameter case. Consider, for instance, the following example. Assume that $m \geq 2$, $d=1$, $x=0$, $A_1(x) = (1,x^1,x^2,\dots,x^{m-1})$ and $B=0$. Then property (P') does not hold and, for $m=2$, the one-parameter solution $X_t^1 = W_t^1$, $X_t^2 = \int_0^t W_s^1 dW_s^1 = \frac{1}{2}[(W_t^1)^2 - t]$ satisfies $2X_t^2 = (X_t^1)^2 - t$. However, in the two-parameter case, theorem 4.3 can be used, and, for $z = (s,t)$, $st \neq 0$, the joint distribution of the iterated stochastic integrals $X_z^1 = W_z^1$, $X_z^2 = \int_{R_z} W_r^1\, dW_r^1$, $X_z^3 = \int_{R_z} \left(\int_{R_r} W_u^1 dW_u^1 \right) dW_r^1, \dots, X_1^m = \int_{R_z} X_r^{m-1} dW_r^1$ has an infinitely differentiable density on $\mathbb{R}^m$. Observe that here the stochastic differentiation rules (cf. [10]) claim that $(X_z^1)^2 = 2X_z^2 + 2\int_{R_z \times R_z} 1_{\{(r,r'):\ r_1 \leq r_1',\ r_2 \geq r_2'\}}\, dW_r dW_{r'} + st$, and X_z^2 is not a function of X_z^1.

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