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ON TRANSLATION INVARIANCE FOR W_d^r

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§ 0 INTRODUCTION

Let us consider a projective non-singular curve C of genus g.The set of all line bundles of degree d on C is a variety denoted by $Pic^d(C)$.There is a natural action of the Jacobian $JC=Pic^0(C)$ of C (which is an abelian variety) on $Pic^d(C)$ for any d.We may consider the subschemes $W_d^r(C)$ of $Pic^d(C)$ parametrizing those divisors on C of degree d whose space of sections has dimension at least r.Let us write $\mathcal{G}(W_d^r(C))$ for the subgroup of the Jacobian leaving $W_d^r(C)$ invariant under translation.

In his paper [W] (cf Hilfsatz 3) Weil proved that for any curve C and $d\leq g-1$, $\mathcal{G}(W_d^o(C))=0$ (for $d>g$ $W_d^o=Pic^d(C)$ and then obviously $\mathcal{G}(W_d^o(C))=JC$).We could ask whether the same holds for other values of r.The answer is no if we do not impose any restrictions on C.For instance, we could consider a bielliptic curve,that is to say, a curve whith a two to one morphism onto an elliptic curve E.Then $W_4^1(C)$ is the pull-back of $W_2^1(E)$, and so is isomorphic to E and we get $\mathcal{G}(W_4^1(C))\neq E$ (see § 4,5 for more counterexamples).Nevertheless,we have proved that,when C is assumed to be generic (in the sense of moduli) and $W^r_d(C)$ is neither empty nor the whole of $\mathrm{Pic}^d(C)$, then $g(W^r_d(C))=0$.

This paper is organised as follows: In §1 we have gathered a few well-known definitions and results, either to have them at hand or because we have not been able to find a proper reference in the literature. The second paragraph is rather technical; in it we prove that when a curve moves in a good family, the subgroups of the Jacobians leaving the W^r_d's invariant fit toghether This allows us to show $g(W^r_d(C))=0$ in § 3 by reduction to the case of a rational cuspidal curve (see the introduction to § 3 for more details). In § 4 and 5 we study the possible dimensions of $g(W^r_d(C))$ for a fixed r and give examples of curves for which those dimensions are reached. The last paragraph is of a different nature; in it we determine all curves for which $g(W^1_{g-2}(C))\neq 0$. These turn out to be the bielliptic ones Moreover we show that $g(W^1_{g-1}(C))=0$ for all C.

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§ 1 PRELIMINARIES

Let $\mathcal{C} \xrightarrow{p} S$ be a projective morphism of schemes of finite type over the field $\mathbb{C}$ of complex numbers whose fibers are curves of arithmetic genus g which are either non singular or whith cusps as their only singularities. We shall assume that p has disjoint sections $s_1,\ldots s_t$ for t big enough. Under those conditions there exist Picard schemes $\mathrm{Pic}^d(p)$ for every $d\geq 0$ (see [G] V Th.3.1). Moreover, $\mathrm{Pic}^o(p)$ has a natural structure of a group scheme over S and there is a natural action of $\mathrm{Pic}^o(p)$ on $\mathrm{Pic}^d(p)$ for every d.

One can also define a scheme $W^r_d(p)$, a subscheme of $\mathrm{Pic}^d(p)$ which parametrizes the linear equivalence classes of divisors of degree d and dimension at least r (see [A,C] and [F,L]). We recall here its definition.

Let $\mathcal{L}$ be a Poincare bundle over $\mathrm{Pic}^d(p) \times_S \mathcal{C}$. We denote by p_1, p_2 the canonical projections

$$\begin{array}{ccc} \mathrm{Pic}^d(p) \times_S \mathcal{C} & \xrightarrow{p_2} & \mathcal{C} \\ \downarrow p_1 & & \downarrow \\ \mathrm{Pic}^d(p) & \rightarrow & S \end{array}$$

If $t\geq 2g-1-d$, then $E = p_{1*}[\mathcal{L} \otimes p_2^* \mathcal{O}_{\mathcal{C}}(ts_o(S))]$ is a vector bundle on $\mathrm{Pic}^d(p)$ of rank $d+t+1-g$. This follows from the fact that the dimension of the cohomology of the fibers is constant (see [H], Ch III Th(1.2.8)).

We define the divisor on $\mathcal{C}$ $D_t = s_1(S)+\ldots+s_t(S)$

and $F_t = \mathcal{L} \otimes p_2^* \mathcal{O}_{D_t}(ts_o(S))$. We have an exact sequence

$$0 \to \mathcal{L} \otimes p_2^* \mathcal{O}_{\mathcal{C}}(ts_o(S) - D_t) \to \mathcal{L} \otimes p_2^* \mathcal{O}_{\mathcal{C}}(ts_o(S)) \xrightarrow{\bar{\sigma}_t} \mathcal{L} \otimes p_2^* \mathcal{O}_{D_t}(ts_o(S)) \to 0$$

The direct image of $\bar{\sigma}_t$ under p_1 gives rise to a morphism σ_t from E_t to F_t which fits in the direct image of the above sequence:

$$0 \to p_{1*}[\mathcal{L} \otimes p_2^* \mathcal{O}_{\mathcal{C}}(ts_o(S) - D_t)] \to E_t \xrightarrow{\sigma_t} F_t \to \dots$$

(1.1). Definition. One defines $W_d^r(p)$ as the locus where rank $\sigma_t \leq t+d-g-r$ with its natural structure given locally by the vanishing of the minors of the matrices of σ_t.

(1.2). Remark. The construction of $W_d^r(p)$ commutes with base change, that is to say, given a morphism

$$\varphi : T \to S$$

if we consider the pull-back family

$$\begin{tikzcd} \mathcal{C}' = \mathcal{C} \times_S T \arrow[r] \arrow[d, "p'"] & \mathcal{C} \arrow[d, "p"] \\ T \arrow[r, "\varphi"] & S \end{tikzcd}$$

then $W_d^r(p') = W_d^r(p) \times_S T$.

Proof: It is known that $\mathrm{Pic}^d(p')$ is obtained as the pull-back of $\mathrm{Pic}^d(p)$ and that the pull-back of a Poincaré bundle $\mathcal{L}$ on $\mathrm{Pic}^d(p) \times_S \mathcal{C}$ is a Poincaré bundle $\mathcal{L}'$ on $\mathrm{Pic}^d(p') \times_T \mathcal{C}'$ ([G] VI Th.3.3.1).

By definition $W_d^r(p)$ and $W_d^r(p')$ are given by the vanishing of the minors of the morphism of fiber bundles

$$p_{1*}(\mathcal{L}\otimes p_2^*\mathcal{O}_{\mathcal{C}}(ts_o(S))) \to p_{1*}(\mathcal{L}\otimes p_2^*\mathcal{O}_{D_t}(ts_o(S)))$$

and

$$p'_{1*}(\mathcal{L}'\otimes p_2'^*\mathcal{O}_{\mathcal{C}'}(ts'_o(T))) \to p'_{1*}(\mathcal{L}'\otimes p_2'^*\mathcal{O}_{D'_t}(ts'_o(T)))$$

respectively, where $s'_i : T \to \mathcal{C}'$ are obtained from $s_i : S \to \mathcal{C}$ by base extension and D'_t is defined similarly to D_t.

Because of the base change property for Pic^d this last morphism is

$$p'_{1*}(\bar{\varphi}\times \mathrm{Id})^*(\mathcal{L}\otimes p_2^*\mathcal{O}_{\mathcal{C}}(ts_o(S))) \to p'_{1*}(\bar{\varphi}\times \mathrm{Id})^*(\mathcal{L}\otimes p_2^*\mathcal{O}_{D_t}(ts_o(S)))$$

where $\bar{\varphi}$ denotes the morphism $\mathrm{Pic}^d(p') \to \mathrm{Pic}^d(p)$ coming from φ.

Therefore our assertion will be proved if we show that the natural morphisms([H] III Rem.9.1)

$$p'_{1*}((\bar{\varphi}\times \mathrm{Id})^*(\mathcal{L}\otimes p_2^*\mathcal{O}_{\mathcal{C}}(ts_o(S)))) \leftarrow \bar{\varphi}^*(p_{1*}(\mathcal{L}\otimes p_2^*\mathcal{O}_{\mathcal{C}}(ts_o(S)))$$

and

$$p'_{1*}((\bar{\varphi}\times \mathrm{Id})^*(\mathcal{L}\otimes p_2^*\mathcal{O}_{D_t}(ts_o(S)))) \leftarrow \bar{\varphi}^*(p_{1*}\mathcal{L}\otimes p_2^*\mathcal{O}_{D_t}(t(s_o(S)))$$

are isomorphisms. This is true because the above holds over closed points.

(1.3). <u>Remark.</u> Every component of $W^r_d(p)$ has dimension at least equal to $\dim S + \rho$, where ρ is the Brill- Noether number $\rho = g-(r+1)(g-d+r)$. In particular, for a given curve C, $W^r_d(C)$ has dimension at least ρ at every closed point. In fact this follows from the definition of $W^r_d(p)$ (see [A,C,G,H] p.83).

(1.4).Remark. If $s \in S$ is such that $\dim W^r_d(\mathcal{C}(s))=\rho$, then

$$W^r_d(\rho) \xrightarrow{\pi} S$$

is flat in a neighborhood of s.

Proof: Because of (1.3), there is a neighborhood S' of s where all fibers of π have dimension ρ. Then, when restricting to the pull-back family $\mathcal{C}'=\mathcal{C}\times_S S' \to S'$, all components of $W^r_d(\rho')$ project onto S' (by (1.3) and (1.2)). Under those conditions, $W_d^r(\rho')$ being the locus where a certain morphism of vector bundles drops rank, it must be Cohen-Macaulay (see for instance [A,C,G,H] Prop.(4.1)). Then we deduce flatness from the criterion in [H] Ch.III Ex.10.9.

(1.5).Lemma. Let C be a cuspidal rational curve, F a torsion free sheaf on C of rank one (i.e. $F\otimes K(C)=K(C)$, where K(C) denotes the field of rational functions on C). If P is a cusp in C, then the fiber F_P of F over P is isomorphic either to the local ring $\mathcal{O}_P$ of C at P or to its maximal ideal M.

Proof: As K(C) is the quotient field of $\mathcal{O}_P$, $F_P\otimes K(C)$ is the symmetrization of F_P in the set $S=\mathcal{O}_P-\{0\}$. Moreover, F_P being torsion free, the morphism $F_P \to S^{-1}F_P$ is injective. Therefore we may assume that F_P is a submodule of K(C).

We shall use the model $\mathcal{O}_P=(\mathbb{C}(t^2,t^3))_{(t^2,t^3)}$, so $M=(t^2,t^3)$ and $K(C)=(\mathbb{C}(t))_{(0)}$.

Then, because F_P has a finite number of generators, there is a k such that

$$t^k F_P \subset R=(\mathbb{C}(t))_{(t)} .$$

So we may assume also $F_P \subset R$.

(1.5.1). Because $K(C)$ is the quotient field of $\mathcal{O}_P$, the condition $F_P \otimes_{\mathcal{O}_P} K(C) \cong K(C)$ is seen to be equivalent to $F_P \cap \mathcal{O}_P \neq 0$.

Now let I be an ideal of $\mathcal{O}_P$, $I \neq 0$. We claim:

(1.5.2). I is isomorphic either to $\mathcal{O}_P$ or to M.

Proof: If $I \not\subset M$, then $I = \mathcal{O}_P$. So we may assume $I \subset M$. Every element of I is of the form

$$\frac{S(t^2, t^3)}{Q(t^2, t^3)}$$

with $Q(0,0) \neq 0$, $S(0,0)=0$. The elements of the form $S(t^2, t^3)$ also generate I.

Let us choose S in I $S=a_k t^k + a_{k+1} t^{k+1} + \ldots + a_m t^m$, $a_k \neq 0$ and k minimal over all $S \in I$. Then

$$(t^2 - t^3 a_{k+1}/a_k) S = t^{k+2}(a_k + ct^2 + \ldots)$$

so, as the element in the parenthesis is a unit in $\mathcal{O}_P$, we have $t^{k+2} \in (S) \subset I$. Then

$$S.t^3 - (a_{k+1} t^2).t^{k+2} = t^{k+3}(a_k + ct^2 + \ldots)$$

so t^{k+3} also belongs to the ideal generated by S, and therefore the same holds for t^{k+r} for $r \geqslant 2$.

If $I=(S)$, it is isomorphic to $\mathcal{O}_P$. So we may assume $I \neq (S)$. Then there must be an element of the form

$b_k t^k + b_{k+1} t^{k+1}$ in I such that $b_k/a_k \neq b_{k+1}/a_{k+1}$. We have therefore that t^{k+1} and t^k belong to I and so $I=(t^k,t^{k+1})$ is isomorphic to M, the isomorphism being given by multiplication with $1/t^{k-2}$.

Write now $I=F_P \cap \mathcal{O}_P$.

Let us assume first that $I=\mathcal{O}_P$. If $I=F_P$ there is nothing to prove. Otherwise, let us consider the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_P \arrow[r] \arrow[d, equal] & R \arrow[r, "g"] & \mathbb{C} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_P \arrow[r] & F_P \arrow[r] \arrow[u, hook] & g(F) \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

where $g(f)=(df)(0)$ is the differential of f at 0. The condition $\mathcal{O}_P \neq F_P$ means that $g(F) \neq 0$, so we must have $g(F)=\mathbb{C}$ and $F=R$. Now R is isomorphic to M as an $\mathcal{O}_P$ module, the isomorphism being given by multiplication by t^2.

If $F_P \subset M$, (1.5.2) gives the desired result. Thus, it only remains to consider the case $F_P \neq F_P \cap \mathcal{O}_P \subset M$. We shall see that this never happens. In fact $F_P \not\subset \mathcal{O}_P$ means there are f in $\mathcal{O}_P$ and g in $\mathcal{O}_P - \{0\}$ such that $f/g \in F_P - \mathcal{O}_P$. Then, $f \in F_P \cap \mathcal{O}_P \subset M$ so $f(0)=0$, $df(0)=0$. We deduce $d(f/g)(0)=0$ which contradicts $f/g \notin \mathcal{O}_P$.

(1.6). Corollary. On a rational cuspidal curve the tensor product of torsion free rank one sheaves is a closed operation.

Proof: From (1.5) and keeping the same notation it suffices to prove that $M \otimes M \cong M$. We have a presentation of M

$$0 \to \mathcal{O}_P \to \mathcal{O}_P \oplus \mathcal{O}_P \xrightarrow{h} M \to 0$$

where $h(f,g)=t^2 f+t^3 g$. By tensoring this exact sequence with M we obtain the result.

§2 CONSTRUCTION OF A SCHEME $\mathcal{G}(W^r_d(p))$.

We keep the notations of §1. Moreover, we shall assume that $W^r_d(p)$ is flat over S. We shall show that there is an S-subscheme of $\mathrm{Pic}^o(p)$ whose closed points correspond to a curve $\mathcal{C}(s)$ in the family p and a point $a \in \mathrm{Pic}^o(\mathcal{C}(s))$ such that $a+W^r_d(\mathcal{C}(s))=W^r_d(\mathcal{C}(s))$. As one could expect, this scheme is in fact a group scheme, has a universal property and behaves well under base change.

In what follows we shall supress the indication to p from the notation where there is no danger of confusion.

Let us write $A=\mathrm{Pic}^o \times_S W^r_d$. Since by hypothesis W^r_d is flat over S and flatness is preserved under base change([H] ChIII (9.2.b)), it follows that A is flat over Pic^o.

We define

$$\tilde{\psi}_d=(p_1,\psi_d):\mathrm{Pic}^o \underset{S}{\times} \mathrm{Pic}^d \to \mathrm{Pic}^o \underset{S}{\times} \mathrm{Pic}^d$$

where ψ_d is the natural action of Pic^o over Pic^d and p_i denotes projection onto the $i^{\underline{th}}$ factor. Then $\tilde{\psi}_d$ is an isomorphism because it has an inverse

$$\tilde{\psi}_d^{-1} = (p_1\ ,\quad \psi_d\ (\psi_- \circ p_1\ , p_2))$$

where ψ_- denotes "inverse" in Pic^o.

We shall denote by B the scheme-theoretic image of A under $\tilde{\psi}_d$. As $\tilde{\psi}_d$ is a Pic^o -isomorphism, B must be flat over Pic^o. So, both A and B induce classifying isomorphisms of Pic^o to the Hilbert scheme of subschemes of Pic^d flat over S

$$Pic^o \underset{\chi_B}{\overset{\chi_A}{\rightrightarrows}} Hilb_{Pic^d/S}$$

(2.1). Definition. Let $\mathcal{G}$ be the maximal subscheme of Pic^o where those two morphisms coincide, i.e. the subscheme defined by the pull-back diagram

$$\begin{array}{ccc} \mathcal{G} & \to & Hilb_{Pic^d/S} \\ \downarrow & & \downarrow \Delta \\ Pic^o & \xrightarrow{(\chi_A, \chi_B)} & Hilb_{Pic^d/S} \times_S Hilb_{Pic^d/S} \end{array}$$

where Δ denotes the diagonal morphism (which is an immersion).

(2.2). Proposition. If all fibers of p are non- singular, then $\mathcal{G}$ is projective over S.

Proof: Under these conditions $Pic^o \to S$ is projective ([G] VI, Cor. 4.2.), so, it is enough to show that $\mathcal{G}$ is closed in Pic^o.

The Hilbert scheme of flat subschemes whith a given Hilbert polynomial P of a scheme X projective over S is projective ([G] VI Th. 3.1) and, in particular separated. Let P_1 (P_2) be the Hilbert polynomial corresponding to the fibers of A (B) over Pic^o. As $\mathcal{G}$ is non-empty, we must have $P_1=P_2$ and $\mathcal{G}$ can be obtained by replacing Hilb Pic^d/S by $\mathrm{Hilb}^P\ \mathrm{Pic}^d/S$ in Definition (2.1) Then the diagonal map is a closed immersion and so $\mathcal{G}$ is a closed subset of Pic^o.

(2.3). Remark. The subscheme $\mathcal{G}$ is a final object among those subschemes of Pic^o which leave W_d^r invariant under the natural action. More precisely, given a morphism

$$R \xrightarrow{f} \mathrm{Pic}^o$$

such that

$$R+W_d^r=(W_d^r)_R$$

then f may be factorised in a unique way through

$$\begin{tikzcd} R \arrow[r] \arrow[dr, dotted] & \mathrm{Pic}^o \\ & \mathcal{G} \arrow[u, hook] \end{tikzcd}$$

(Here $(W_d^r)_R$ denotes $R\times_S W_d^r = R\times_{\mathrm{Pic}^o}(\mathrm{Pic}^o \times_S W_d^r)$ and $R+W_d^r$ is the image of $R\times_{\mathrm{Pic}^o}(\mathrm{Pic}^o\times_S W_d^r)$ under the morphism $\mathrm{Id}\times\psi_d$).

Proof: The pull-back of A by f is $(W_d^r)_R$ and that of B is $R\times(R+W_d^r)$. As A and B are flat over Pic^o, these two schemes are flat over R. Therefore we have classifying morphisms

$$R \underset{\chi_{R\times B}}{\overset{\chi_{R\times A}}{\rightrightarrows}} \mathrm{Hilb}\ \mathrm{Pic}^d/S$$

which in fact are the compositions $f\circ\chi_A$ and $f\circ\chi_B$. Since, by hypothesis, $\chi_{R\times A} = \chi_{R\times B}$, the product of those two morphisms factorises through the diagonal morphism

$$\begin{tikzcd} R \arrow[r] \arrow[drr, dotted, bend right] & \mathrm{Pic}^{o} \arrow[r] & \mathrm{Hilb}_{\mathrm{Pic}^{d}/S} \times_{S} \mathrm{Hilb}_{\mathrm{Pic}^{d}/S} \\ & \mathcal{G} \arrow[u] \arrow[r, dashed] & \mathrm{Hilb}_{\mathrm{Pic}^{d}/S} \arrow[u] \end{tikzcd}$$

and we dedude from this the morphism $R \to \mathcal{G}$.

(2.4). <u>Proposition.</u> $\mathcal{G}$ has a natural structure of a group-subscheme of Pic^{o}.

Proof: This follows from Remark (2.3) coupled with the formal properties of Pic^{o} as a group-scheme and those of the action of Pic^{o} on Pic^{d}.

(2.5). <u>Proposition.</u> The construction of $\mathcal{G}$ commutes whith base change. In particular, the restriction of $\mathcal{G}$ to a geometric fiber $\mathcal{C}(t)$ of the family p is the subgroup of the Jacobian of $\mathcal{C}(t)$ of those elements leaving the scheme $W_{d}^{r}(\mathcal{C}(t))$ invariant under translation

Proof: Given a pull-back diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d, "p'"'] & \mathcal{C} \arrow[d, "p"] \\ S' \arrow[r] & S \end{tikzcd}$$

we noted in (1.2) that we have pull-back diagrams

$$\begin{tikzcd} \mathrm{Pic}^{d}(p') \arrow[r] \arrow[d] & \mathrm{Pic}^{d}(p) \arrow[d] \\ S' \arrow[r] & S \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} W_{d}^{r}(p') \arrow[r] \arrow[d] & W_{d}^{r}(p) \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

So, there is a natural morphism

$$\mathrm{Hilb}_{\,Pic^d(p')/S'} \to \mathrm{Hilb}_{\,Pic^d(p)/S} \times_S S'$$

Moreover $A'=A\times_S S'$ and $B'=B\times_S S'$. Therefore the diagram defining $\mathcal{G}(p')$ factors through the pull-back of the diagram defining $\mathcal{G}(p)$. Therefore $\mathcal{G}(p')$ is the pull-back of $\mathcal{G}(p)$.

§3 VANISHING OF $\mathcal{G}(W^r_d(C))$ FOR A GENERIC CURVE

In all of these paragraph, g, r, d will denote three integers $\geqslant 0$ such that the Brill-Noeter number

$$\rho = g-(r+1)(g-d+r)$$

satisfies

$$0 \leqslant \rho < g$$

This is equivalent with the fact that for any genus g curve $W^r_d(C)$ is neither empty nor the whole of Pic^d.

We are going to show that, $\mathcal{G}(W^r_d(C))=0$ for a generic curve C of genus g over $\mathbb{C}$. The proof consists of two parts. In the first one we show that, under suitable conditions, the property $\mathcal{G}(W^r_d(C))=0$ extends to all curves in a neighborhood of C. In the second we prove that $\mathcal{G}(W^r_d(C))=0$, for a cuspidal rational curve C and that, when we deform the curve to a non singular one, we can also deform the group which becomes a member of a projective family. We have encountered here a few difficulties arising from the fact that $Pic^o(C)$ is non compact and that, when we compactify it, it loses its group structure. We have been able to overcome this problem by proving first that the action of $\mathrm{Pic}^{o}(C)$ over $\mathrm{Pic}^{d}(C)$ extends to an action of their compactifications and secondly that the family of groups we defined in § 2 is in fact closed in the compactification of the family of Pic^{o} of the curves.

(3.1). <u>Proposition.</u> Let $\mathcal{G} \xrightarrow{p} T$ be a projective group-scheme, T a non-singular scheme over the field $\mathbb{C}$ such that the fiber $\mathcal{G}(t_o)$ over a closed point $t_o \in T$ is the zero group. Then $\mathcal{G}(t)=0$ in a neighborhood of t .

Proof: No fiber of p can be empty because p has the zero section $e:T \to \mathcal{G}$. As $\mathcal{G}(t_o)$ is zero-dimensional, we may assume , by restricting T if necessary, that all fibers are zero-dimensional. Before we proceed we need to prove the following:

(3.1.1). <u>Lemma.</u> Under the above conditions, if x is a closed point of $\mathcal{G}$ belonging to a component which projects onto T, then $\mathcal{G}$ is non-singular at x and the morphism p is unramified at x.

Proof: Denote by t the point $p(x)$. We have an exact sequence of tangent spaces

$$0 \to T_{\mathcal{G}(t),x} \to T_{\mathcal{G},x} \to T_{T,x}$$

As $\mathcal{G}(t)$ is a group-scheme over $\mathbb{C}$ it is non-singular, so, being zero-dimensional, $T_{\mathcal{G}(t),x} = 0$. Then the above exact sequence proves the second assertion.

The assuption of the lemma is $\dim T \leqslant \dim_x \mathcal{G}$. So we have

$$\dim T \leqslant \dim_x \mathcal{G} \leqslant \dim T_{\mathcal{G},x} \leqslant \dim T$$

and the first assertion of the lemma follows.

Now the lemma gives us that $\mathcal{G}$ is non-singular at the single point x_o of $\mathcal{G}(t_o)$ because x_o belongs to the component G which contains the zero section. So, this must be the only component of $\mathcal{G}$ containing x_o. The morphism being projective, we may restrict T to a neighbor hood of t_o and assume that all components of $\mathcal{G}$ project onto T. Then p is flat ([H] III ex.10.9). As it has finite fibers and is unramified by (3.1.1), it is étale. So $h^o(\mathcal{G}(t),\mathcal{O}_{\mathcal{G}(t)})$ is exactly the number of elements of the fiber $\mathcal{G}(t)$. By assumption it is one for $t=t_o$ and we have already noted that it is always at least one because all fibers contain the zero element. So, (3.1) follows by upper-semicontinuity.

(3.2). <u>Corollary.</u> If C_o is a non-singular curve such that $\dim W^r_d(C_o)=o$ and $\mathcal{G}(W^r_d(C_o))=0$, then $\mathcal{G}(W^r_d(C))=0$ for all curves C in an open neighborhood of C_o in the moduli space.

Proof:Use (1.4),(2.1),(2.2),(2.4) and (3.1) together whith the well-known existence theorem for a family containing C_o and projecting onto an open neighborhood of C_o in the moduli space of curves.

(3.3).Let $\mathcal{C} \xrightarrow{p} T, T \neq \{t_o\}$,be a projective flat morphism of finite presentation whith $\mathcal{C}$ and T non-singular irreducible schemes over the complex field such that the fiber over the point t_o is a rational curve whith g cusps and the fiber over any closed point $t \neq t_o$ is a non-singular curve of genus g.We shall assume moreover that p has disjoint sections $s_o \ldots s_t$.For the existence of such a family see [E,H] p.394 and apply the usual technical device to construct the sections(see for instance [C],(1.2)).

Since for a cuspidal curve $W^r_d(C_o)$ has dimension ρ , by Remark (1.4),we may also assume that $W^r_d(p)$ is flat over T and construct $\mathcal{G}$ as in (2.1).

By restricting the natural action of $Pic^o(p)$ over $Pic^d(p)$,we then have a morphism

$$\psi_d : \mathcal{G} \times W^r_d(p) \to W^r_d(p)$$

Under our conditions,we may apply the theory of Altman and Kleiman ([A,K] 1 and 2) showing the existence of projective T-schemes $\overline{Pic^d(p)}$ which compactify the schemes $Pic^d(p)$.They represent the equivalence classes of flat families of sheaves over $\mathcal{C}$ whose geometric fibers are torsion free rank one sheaves on the fibers of p with Euler characteristic d+1-g. Moreover, there is a universal sheaf $\overline{\mathcal{L}}_d$ on $\overline{\mathrm{Pic}^d(p)} \times \mathcal{C}$ normalized along a section of p (see [K] for a summary of results).

(3.4). Lemma. The action of $\mathrm{Pic}^o(p)$ over $\mathrm{Pic}^d(p)$ extends to an action of $\overline{\mathrm{Pic}^o(p)}$ over $\overline{\mathrm{Pic}^d(p)}$

$$\begin{array}{ccc} \mathrm{Pic}^o(p) \times_T \mathrm{Pic}^d(p) & \to & \mathrm{Pic}^d(p) \\ \downarrow & & \downarrow \\ \overline{\mathrm{Pic}^o(p)} \times_T \overline{\mathrm{Pic}^d(p)} & \xrightarrow{\overline{\psi}_d} & \overline{\mathrm{Pic}^d(p)} \end{array}$$

Proof: For a cuspidal curve the tensor product of torsion-free rank one sheaves is again a torsion-free rank one sheaf (cf.(1.6)). Obviously the same holds for non-singular curves.

Let us consider the scheme $\overline{\mathrm{Pic}^o(p)} \times_T \overline{\mathrm{Pic}^d(p)} \times_T \mathcal{C}$ and denote by p_j (resp. p_{kj}) the projection onto the factor j (resp. onto the product of the factors k and j). Then $p_{13}^* \overline{\mathcal{L}}_o \otimes p_{23}^* \overline{\mathcal{L}}_d$ is a flat sheaf over this scheme because $\overline{\mathcal{L}}_o$ is flat over $\overline{\mathrm{Pic}^o(p)} \times_T \mathcal{C}$ and $\overline{\mathcal{L}}_d$ is flat over $\overline{\mathrm{Pic}^d(p)} \times_T \mathcal{C}$.

By restricting this sheaf to a fiber of the projection p we get

$$(p_{13}^* \overline{\mathcal{L}}_o \otimes p_{23}^* \overline{\mathcal{L}}_d)|_{\{L_o\} \times \{L_d\} \times \mathcal{C}(t)} = \overline{\mathcal{L}}_o|_{\{L_o\} \times \mathcal{C}(t)} \otimes \overline{\mathcal{L}}_d|_{\{L_d\} \times \mathcal{C}(t)}$$

which is a torsion-free rank one sheaf on $\mathcal{C}(t)$ as mentioned above.

Because the family is flat, the Hilbert polynomials over the fibers are constant ([H] III p.261-262) and on the generic non-singular fibers they are $nd+1-g$. Then we apply the universal property of $\mathrm{Pic}^d(p)$ to obtain the morphism .

We can copy the construction in (1.1) replacing $\mathrm{Pic}^d(p)$ by $\overline{\mathrm{Pic}^d(p)}$ and the Poincaré bundle $\mathcal{L}$ by $\overline{\mathcal{L}_d}$ We get in this way a closed subscheme $\overline{W^r_d(p)}$ of $\overline{\mathrm{Pic}^d(p)}$ which parametrizes those torsion-free rank one sheaves on the fibers of p whose space of sections has dimension at least $r+1$.

(3.5). <u>Lemma.</u> No point of $\overline{\mathrm{Pic}^o(C_o)} - \mathrm{Pic}^o(C_o)$ leaves $\overline{W^r_d(C_o)}$ invariant under the action $\overline{\psi_d}$ of (3.4).

Proof: By the results of Eisenbud and Harris ([E,H] Cor.(4.4)), we have

$$\overline{W^r_d(C_o)} = W^r_d(C_o) \cup \bigcup_{\substack{i_1 \dots i_j \\ j \geq 1}} W^r_d(C_{i_1 \dots i_j})$$

where $C_{i_1 \dots i_j}$ is the normalization of C_o at the cusps $P_{i_1} \dots P_{i_j}$ and the elements of $W^r_d(C_{i_1 \dots i_j})$ are interpreted as those torsion-free rank one sheaves on C_o which fail to be locally free precisely at the cusps $P_{i_1} \dots P_{i_j}$.

Let j_o be the maximum integer j such that

$$W^r_{d-j}(C_{i_1 \dots i_j}) \neq \emptyset$$

As the curves $C_{i_1 \ldots i_j}$ are rational cuspidal curves of genus $g-j$, this is equivalent (by [E,H] Th.5.1) to

$$0 \leq \rho(d-j,r,g-j)$$

and because

$$\rho(d-j,r,g-j)= \rho(d,r,g)-j$$

we have

$$j \leq \rho(d,r,g) < g$$

the last inequality following from our initial hypothesis on d,r,g.

Let L be an element in $\overline{\mathrm{Pic}^o(C_o)}-\mathrm{Pic}^o(C_o)$. As L corresponds to a non- invertible sheaf on C_o, it fails to be locally free at at least one of the cusps, say P_g. We want to show

$$L+W^r_d(C_o) \not\subset \overline{W^r_d(C_o)}$$

Let L_o be an element in $\overline{W^r_d(C_o)}$ corresponding to a point in $W^r_{d-j_o}(C_{1\ldots j_o})$. Then $L\otimes L_o$ fails to be locally free at the cusps $P_1 \ldots P_{j_o}$ and P_g. Therefore, by definition of j_o, it does not belong to $W^r_d(C_o)$.

(3.6). <u>Lemma.</u> $\mathcal{G}(C_o)=0$.

Proof: Let us consider the scheme-theoretic closure $\widetilde{\mathcal{G}(C_o)}$ of $\mathcal{G}(C_o)$ in $\overline{\mathrm{Pic}^o(C_o)}$ (see [E.G.A] p.324-325). Because $\mathcal{G}(C_o)$ is a group-scheme over $\mathbb{C}$, it is reduced and so is $\widetilde{\mathcal{G}(C_o)}$. Moreover, by definition, $\mathcal{G}(C_o)$ is dense in $\widetilde{\mathcal{G}(C_o)}$.

We also have that $W^r_d(C_o)$ is dense in $\overline{W^r_d(C_o)}$ and reduced ([E,H] Th.4.5 and Th.5.1)

By construction, $\mathcal{G}(C_o)$ leaves $W^r_d(C_o)$ invariant. So by continuity, we get a factorization of $\overline{\psi}_d$ in the following way:

$$\begin{array}{ccc}
\mathcal{G}(C_o) \times W^r_d(C_o) & \longrightarrow & W^r_d(C_o) \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{G}(C_o)} \times \overline{W^r_d(C_o)} & \longrightarrow & \overline{W^r_d(C_o)} \\
\downarrow & & \downarrow \\
\overline{\mathrm{Pic}^o(C_o)} \times \overline{\mathrm{Pic}^d(C_o)} & \xrightarrow{\overline{\psi}_d} & \overline{\mathrm{Pic}^d(C_o)}
\end{array}$$

Now, by Lemma (3.5), we have $\widetilde{\mathcal{G}(C_o)} \subset \mathrm{Pic}^o(C_o) \cong \mathbb{C}^g$. As $\widetilde{\mathcal{G}(C_o)}$ is proper and $\mathbb{C}^g$ is affine, $\widetilde{\mathcal{G}(C_o)}$ is finite. Moreover, $\mathcal{G}(C_o) \subset \mathbb{C}^g$ implies $\widetilde{\mathcal{G}(C_o)} = \mathcal{G}(C_o)$. So, $\mathcal{G}(C_o)$ being a group subscheme of $\mathbb{C}^g$ which is torsion-free, it is trivial.

We shall denote by $\tilde{\mathcal{G}}$ the scheme-theoretic closure of $(\mathcal{G}(p))_{red}$ inside $\overline{\mathrm{Pic}^d(p)}$.

(3.7) Lemma. $\mathcal{G}(p)_{red} \times_T W^r_d(p)$ is dense in $\tilde{\mathcal{G}} \times_T W^r_d$

Proof: Let y be a point in $\tilde{\mathcal{G}} \times_T W^r_d(p) - \mathcal{G}(p) \times_T W^r_d(p)$ and x,w the projections of y in $\tilde{\mathcal{G}}$ and $W^r_d(p)$ respectively. We have that $p_o(x) = p_d(w) = t$ where $p_i : \overline{\mathrm{Pic}^i(p)} \longrightarrow T$ is the structural morphism.

If $x=0$, x belongs to the component which contains the zero section $0 : T \longrightarrow \tilde{\mathcal{G}}$ and this component dominates T. If $x \neq 0$, then by Lemma (3.6), $x \notin \widetilde{\mathcal{G}(C_o)}$. So, in either case, there is a component G of $\tilde{\mathcal{G}}$ containing x such that $p_o(G)=T'\neq\{t_o\}$ is a closed irreducible subset of T.

We know that $W^r_d(p)x_T T'$ coincides whith $W^r_d(p')$; here p' is obtained by pulling back p to T' (cf.(1.2)). As all fibers of $W^r_d(p)$ over the closed points of T have dimension ρ (by (3.3)), the same must be true for the fibers of $W^r_d(p')$ over the closed points of T'. Hence we deduce that all the components of $W^r_d(p')$ have dimension equal to $\dim T'+\rho$ and dominate T' (see (1.4)). Because $W^r_d(p')$ is dense in $\overline{W^r_d(p')}$ (see [E,H] p.394) the same is valid too for the components of $\overline{W^r_d(p')}$. Let W be a component of $\overline{W^r_d(p')}$ containing w. Then, there is a closed irreducible set in Gx_T, W containing y and such that its projections on $\mathcal{G}(p)$ and $\overline{Pic^d}(p)$ are G and W respectively (see [E.G.A] Ch.I (3.4)). This proves the lemma.

(3.8). <u>Lemma.</u> $\tilde{\mathcal{G}}$ coincides whith $\mathcal{G}(p)_{red}$ and so $\mathcal{G}(p)$ is projective.

Proof: By applying (3.7), we get a factorization of the restriction of ψ_d to $(\tilde{\mathcal{G}}x_T W^r_d(p))_{red}$ as shown in the diagram below. Then, because of (3.5), it follows that $\tilde{\mathcal{G}}=(\mathcal{G}(p))_{red}$. So, $\mathcal{G}(p)$ is a closed set of $Pic^o(p)$ and in particular it is projective.

$$\begin{array}{ccc}
(\tilde{\mathcal{G}} \times_T W^r_d(p)_{red})_{red} & \longrightarrow & W^r_d(p)_{red} \\
\downarrow & & \downarrow \\
(\tilde{\mathcal{G}} \times_T \overline{W^r_d(p)_{red}})_{red} & \dashrightarrow & \overline{W^r_d(p)_{red}} \\
\downarrow & & \downarrow \\
\overline{\mathrm{Pic}^0(p)} \times_T \overline{\mathrm{Pic}^d(p)} & \longrightarrow & \overline{\mathrm{Pic}^d(p)}
\end{array}$$

(3.9). Theorem. For a generic (non-singular) curve of genus g, the only translation leaving $W^r_d(C)$ invariant is the identity $(0 \leqslant \rho(d,r,g) \leqslant g-1)$.

Proof: By (3.1), (3.3), (3.8) and (3.6) there are non-singular curves whith $\mathcal{G}(W^r_d(C))=0$ and such that $\dim W^r_d(C)=\rho$. Then use Corollary (3.2).

4. POSSIBLE DIMENSIONS OF THE GROUPS $\mathcal{G}(W^r_d(C))$.

We show here that, for any curve C, $0 \leqslant \dim \mathcal{G}(W^r_d(C)) \leqslant r$. Conversely, for any s such that $1 \leqslant s \leqslant r$ we construct curves such that $\dim \mathcal{G}(W^r_d(C))=s$. For $s=0$, we have already seen (cf. § 3) that for a generic curve $\mathcal{G}(W^r_d(C))=0$, so $\mathcal{G}(W^r_d(C))$ is zero dimensional. We have not been able to find an example of a curve whith $\mathcal{G}(W^r_d(C))$ zero dimensional but not trivial (see (5.8)).

(4.1). Lemma. If $a+(W^r_k)_{red} \subset (W^r_k)_{red}$ and $r+g-k \geqslant 1$ (or equivalently $W^r_k \neq JC$), then $a+(W^r_t)_{red} \subset (W^r_t)_{red}$ $\forall\ t \leqslant k$.

Proof: Let $L \in W^r_t$, $D_t \in |L|$ and D_{k-t} be a generic effective divisor of degree k-t.Then $\mathcal{O}_C(D_t+D_{k-t}) \in W^r_k$.So,by hypothesis, $\mathcal{O}_C(a+D_t+D_{k-t}) \in W^r_k$.By Riemann-Roch this is equivalent to

$$h^o(K-a-D_t-D_{k-t}) \geqslant r+g-k \geqslant 1$$

So,D_{k-t} being generic in $C^{(k-t)}$

$$h^o(K-a-D_t) \geqslant r+g-k+(k-t)=r+g-t$$

which again by Riemann-Roch gives

$$h^o(a+D_t) \geqslant r+1$$

or equivalently $\mathcal{O}_C(a+D_t) \in W^r_t$.

(4.2).<u>Proposition.</u>If $\rho(d,r,g)<g$,then $\dim \mathcal{G}(W^r_d) \leqslant r$.

Proof:Let d_o be the minimum integer d such that $W^r_d \neq \phi$.Under those conditions Fulton,Harris and Lazarsfeld proved (see [A,C,G,H] p.329) :dim $(W^r_{d_o}) \leqslant r$.Moreover, by (4.1), $\mathcal{G}(W^r_d)$ leaves the support of $W^r_{d_o}$ invariant.So, when we choose an element w in $W^r_{d_o}$,we get an injection

$$\begin{array}{ccc} \mathcal{G}(W^r_d) & \to & W^r_{d_o} \\ a & \to & a+w \end{array}$$

Hence,we deduce $\dim \mathcal{G}(W^r_d) \leqslant \dim W^r_{d_o} \leqslant r$

(4.3).<u>Proposition.</u>Let C be a generic genus h curve $1 \leqslant h \leqslant r$,$C \xrightarrow{p} C_o$ a (ramified) double covering of C such that $g(C) \geqslant 6r+13$.Then W^r_{2h+2r} (C) is reduced and coincides whith the pull-back of $\mathrm{Pic}^{h+r}(C_o)$ under the morphism induced by p.So $\mathcal{G}(W^r_{2k+2r})$ is the subgroup of dimension h in $\mathrm{Pic}^{o}(C)$ obtained as the pull-back of $\mathrm{Pic}^{o}(C_o)$.

Proof: By Riemann-Roch $W^{r}_{h+r}(C_o)=\mathrm{Pic}^{h+r}(C_o)$. So, in order to prove the proposition we must show that

a) if $L\in W^{r}_{2h+2r}(C)$, there exists a $L_o \in W^{r}_{h+r}(C_o)$ such that $L=p^{*}L_o$;

b) $W^{r}_{2h+2r}(C)$ is reduced.

Proof of a): Let $L\in W_{2h+2r}(C)$. Then any subspace of dimension r+1 in $H^{o}(L)$ determines a morphism f of C onto a non-degenerate curve $\tilde{C}$ of $\mathbb{P}^{r}$.

$$C \to \tilde{C} \to \mathbb{P}^{r}$$

We shall use the following two classical lemmas (see for instance [A,C,G,H] p.116 and p.366 resp.).

(4.3.1). Lemma. (Castelnuovo). Let C be a curve of degree d in $\mathbb{P}^{r}$ not contained in any hyperplane and let g be its geometric genus. Then

$$g\leqslant \binom{m}{2}(r-1)+me$$

where $m=\left[\frac{d-1}{r-1}\right]$ (that is to say, the largest integer less than or equal to $\frac{d-1}{r-1}$) and $e=d-1-m(r-1)$.

(4.3.2). Lemma. Let C be a covering of degree d_1 of a curve C_1 of geometric genus g_1 and a covering of degree d_2 of a curve C_2 of geometric genus g_2. Then either the geometric genus of C satisfies

$$g(C)\leqslant (d_1-1)(d_2-1)+g_1d_1+g_2d_2$$

or both coverings are composite with the same involution (i.e. there is a third curve C_3 and a morphism of degree at least two $C \to C_3$ such that the given morphisms factorize

$$
\begin{tikzcd}
 & C_1 \\
C \arrow[ur, dashed] \arrow[r] \arrow[dr] & C_3 \arrow[u] \arrow[d] \\
 & C_2
\end{tikzcd}
$$

If $r > 1$ (4.3.1) tells us that $g(C) > g(\tilde{C})$, so the morphism is not birrational. For $r=1$ this is obviously true. Let k be the degree of f.

As the curve C is non-degenerate in $\mathbb{P}^r$, it has degree at least r. So, we have $\frac{2h+2r}{k} \geqslant r$ and therefore $k \leqslant 4$. We can use now (4.3.1) to prove that the genus of C is bounded by $h+1$. Then, by (4.3.2), we may conclude that f and p are composite with the same involution. As p has degree 2, this means that f factorizes through

$$
\begin{tikzcd}
 & C_o \arrow[d, dashed] \\
C \arrow[ur, "p"] \arrow[r, "f"'] & \tilde{C} \arrow[r] & \mathbb{P}^r
\end{tikzcd}
$$

Thus we can write $L=p^{*}L_o+F$, where L_o is an invertible sheaf of degree at most $h+r$ and dimension at least r in C_o and F is the fixed part of L.

On the other hand, C_o being generic, we have

$$\dim W^{r}_{h+r-1}(C_{o}) = \rho(h+r-1,r,h) < 0$$

that is to say, $W^{r}_{h+r-1}(C_{o}) = \phi$, so L_{o} has degree exactly $h+r$ and $F=0$. This completes the proof of a).

Proof of b): We shall prove that the tangent space to $W^{r}_{2h+2r}(C)$ at the point $L=p^{*}L_{o}$ has dimension h and so, because of a), $W^{r}_{2h+2r}(C)$ is non-singular.

Let us denote by A the ramification divisor of p and let D be a (not necessarily effective) divisor such that $2D=A$ and $p_{*}\mathcal{O}_{C} = \mathcal{O}_{C_{o}} \oplus \mathcal{O}_{C_{o}}(-D)$. Then, by Hurwitz Formula $2\deg D + 2(2h-2) = 2g-2$. So, $\deg D = g-2h+1$.

Since $H^{o}(L) = H^{o}(L_{o}) \oplus H^{o}(L_{o}-D)$ and $\deg(L_{o}-D) < 0$, we have $H^{o}(L) = H^{o}(L_{o})$. (Note that those are spaces of sections of line-bundles on different curves, we hope it is clear to which curve they refer in each case). In particular, this implies $h^{o}(L) = h^{o}(L_{o}) = r+1$, where the last equality has been seen to hold at the end of the proof of a). This allows us to compute the tangent space to $W^{r}_{h+r}(C) \subset \mathrm{Pic}^{d}(C)$ as the subspace orthogonal to the image of the Petri morphism:

$$H^{o}(L) \otimes H^{o}(K-L) \longrightarrow H^{o}(K)$$

where $H^{o}(K)$ is to be interpreted as the dual of the tangent space to $\mathrm{Pic}^{d}(C)$ (see for instance [A,C,G,H] p.189)

By taking into account that $K_{C} = p^{*}K_{C_{o}} + p^{*}D$ ([H] IV Prop.2.3), we get $K_{C}-L = p^{*}(K_{C_{o}}+D-L_{o})$ and so

$$H^0(K_C-L)=H^0(K_{C_o}+D-L_o)\oplus H^0(K_{C_o}-L_o)$$

We may now decompose the Petri morphism in the following way:

$$H^0(L_o)\otimes H^0(K_{C_o}+D-L_o)\to H^0(K_{C_o}+D)$$
$$H^0(L_o)\otimes H^0(K_{C_o}-L_o)\to H^0(K_{C_o})$$

We have $h^0(K_{C_o}-L_o)=0$ because deg $L_o=h+r$ $2h-2=$ deg K_{C_o}. So, the second component of the Petri morphism is zero. Then, as $h^0(K_{C_o})=h$, showing that dim $T_L W^r_{2h+2r}(C)=h$ is equivalent to proving that the morphism

$$H^0(L_o)\otimes H^0(K_{C_o}+D-L_o)\to H(K_{C_o}+D)$$

is onto and that follows from the Castelnuovo Generalized Lemma ([M] Th.2.1).

5. A FURTHER EXAMPLE.

In this paragraph we give an example of a curve C of genus 37 such that the subgroup of JC leaving the set $W^1_{12}(C)$ invariant is zero-dimensional but not trivial.

(5.1). Definition. Let Q_1 and Q_2 be two rank 3 quadrics of $\mathbb{P}^5$ whose vertices are two disjoint 2-planes (whith a suitable choice of coordinates we may assume that their equations are $x_0^2-x_1x_2$ and $x_3^2-x_4x_5$). Let Q be a generic quadric and S a generic cubic hypersurface in $\mathbb{P}^5$. We define C as the complete intersection of Q_1, Q_2, Q and S. By the genericity of Q and S, C is non-singular. It has degree 24 and its canonical divisors are cut by the hypersurfaces of degree 3 in $\mathbb{P}^5$ (i.e. $K = \mathcal{O}_C(3)$) (see [H] II ex.8.4).

(5.2). Lemma. Given k points in $\mathbb{P}^r$ not contained in a hyperplane, for any j such that $1 \leq k \leq jr+1$, there are j hyperplanes of $\mathbb{P}^r$ whose union contains exactly k-1 of those points.

Proof: Use induction on j.

(5.3). Lemma. Given a g^1_k on C whithout fixed points and $k \leq 12$, the divisors of the g^1_k generate a linear space of dimension at most 3 in $\mathbb{P}^5$.

Proof: Let $D \in g^1_k$, $D=P+E$ where P is a point in C and E an effective divisor. By hypothesis, $h^o(E)=h^o(D)-1$. So, by Riemann-Roch, $h^o(K-E)=h^o(K-D)$. This means that any canonical divisor containing E must also contain P.

Let us assume now that D is not contained in any hyperplane of $\mathbb{P}^5$. Then, by (5.2) we could find three hyperplanes of $\mathbb{P}^5$ whose union would contain all points of D except one. This is impossible because the union of the hyperplanes cuts a canonical divisor on C. Hence we have $D \subset \mathbb{P}^4 \subset \mathbb{P}^5$. If D were not contained in a hyperplane of $\mathbb{P}^4$, we could apply (5.2) again and reach a contradiction as above.

(5.4). Lemma. Given a g^1_k without fixed points on C whith $k \geqslant 12$, this must be the non-fixed part of the series cut on C by the rulings of a quadric of $\mathbb{P}^5$ whose rank is at most 4 and which contains C.

Proof: Let us denote by L the line bundle corresponding to the hyperplane section of $\mathbb{P}^5$. By (5.3), $h^o(L-g^1_k)\geqslant 2$. Let us write $L-g^1_k=F+g^r_j$, where F is the fixed part of the series $L-g^1_k$ and $r \geqslant 1$. We choose D_1, D_2 in g^1_k and $\tilde{D}_1$ $\tilde{D}_2$ in g^r_j such that there is no common point to any pair of those four divisors and such that no one of those divisors contains a point of F. As $D_i+\tilde{D}_j+F$ belongs to $|L|$ and C is projectively normal (see for instance [H] II ex.8.4), there must be a hyperplane in $\mathbb{P}^5$ containing this divisor. Let l_{ij} be an equation for this hyperplane. Then, the rational function

$$\frac{l_{11}l_{22}}{l_{12}l_{21}}$$

cuts on C the divisor $D_1+D_2+\tilde{D}_1+\tilde{D}_2+2F-(D_1+\tilde{D}_2+D_2+\tilde{D}_1+2F)=0$, so it must be constant on C:

$$\left.\frac{l_{11}l_{22}}{l_{12}l_{21}}\right|_C = k \neq 0$$

Thus C is contained in the quadric of equation

$$l_{11}l_{22}-kl_{12}l_{21}$$

One of the two (possibly coincident) rulings of this quadric is the set of three-planes whith equations

$$al_{11}+bl_{21}=0$$

$$kal_{12}+bl_{22}=0$$

For b=0 the corresponding plane cuts D_1 +F and for a=0 it cuts D_2+F as we wanted to show.

(5.5).Lemma.The only quadrics of $\mathbb{P}^5$ of rank$\leqslant$4 containing C are Q_1 and Q_2.

Proof:As C is a complete intersection,the quadrics containing C are those of the two-plane generated by Q_1 Q_2 and Q inside the $\mathbb{P}^{20}$ of all quadrics in $\mathbb{P}^5$ (see [A,C,G,H] p.139).The set of quadrics of rank $\leqslant$4 form a variety of pure codimension 3 in this $\mathbb{P}^{20}$ .Let us denote it by V.The pencil generated by Q_1 and Q_2,say l, cuts V in exactly those two points,as one can check from the equations of Q_1 and Q_2 given in (5.1).We need to prove that one can choose Q in such a way that the two-plane generated by Q_1,Q_2 and Q cuts V in exactly those two points.To this end,let us cut V whith 17 generic hyperplanes containing l.The intersection is a finite number of points in a $\mathbb{P}^3$ containing l,so we can choose a two-plane inside this $\mathbb{P}^3$ containing

1 and avoiding all the other points except Q_1 and Q_2.

(5.6).Lemma.The curve C has exactly two linear series of degree 12 and dimension 1, g^1_{12} and h^1_{12}, which are cut by the rulings of Q_1 and Q_2.

Proof:Use (5.4),(5,5) and the fact that the vertices of Q_1 and Q_2 cannot cut C because of the generic selection of Q and S.

(5.7).Corollary.The subgroup of the Jacobian of C leaving the set W^1_{12} invariant under translation consists of two elements.

Proof:It is clear from (5.6) that $2g^1_{12} = 2h^1_{12}$ .So, $L_o = g^1_{12} - h^1_{12} \neq 0$ leaves $W^1_{12}(C)$ invariant (as a set).

(5.8).Remark.In this case W^1_{12} (C) is non-reduced as one can see by computing its tangent space by means of Petri´s morphism ([A,C,G,H] p.189).We do not know whether the translation whith L_o preserves the scheme structure.

§ 6 .CALCULATION OF $\mathcal{G}(W^1_{g-2})$ AND $\mathcal{G}(W^1_{g-1})$.

We show in this paragraph that $\mathcal{G}(W^1_{g-1}(C))=0$ for all C and that $\mathcal{G}(W^1_{g-2}(C))=0$ for any curve of genus $g\geq 7$ except when C is bielliptic (in which case it is isomorphic to the corresponding associated elliptic curve).

(6.1).<u>Lemma.</u>For a hyperelliptic (resp. trigonal) curve of genus at least 2 (resp. 5) and a k such that $2\leq k\leq g$ (resp. $3\leq k\leq g$), the condition $a+W^1_k=W^1_k$ implies $a=0$.

Proof:By our condition on the genus W^1_2 (resp. W^1_3) consists of a single point (use for instance (4.3.1)). Then by (4.1), we would have $a+g^1_2=g^1_2$ (resp. $a+g^1_3=g^1_3$) and so $a=0$.

(6.2).<u>Lemma.</u>Let C be a curve of genus $g\geq 7$.If we have the set-theoretical equality $a+W^1_{g-2}=W^1_{g-2}$ whith $0\neq a\in JC$, then there is a component of dimension at least g-3 in the inverse image of $W^o_{g-2}-a$ by the morphism

$$C^{(g-2)}\to \mathrm{Pic}^{g-2}(C)$$

Proof:Let us assume the contrary and we shall reach a contradiction.

For a given $L\in W^1_{g-2}$ and $P,Q\in C$, we have $h^o(L-P+Q)\geq 1$ So $W^1_{g-2}+C-C\subset W^o_{g-2}$ .Because of condition $a+W^1_{g-2}=W^1_{g-2}$, we have

$$W^1_{g-2}+C-C-a=W^1_{g-2}+C-C$$

Therefore

$$W^1_{g-2}+C-C\subset W^o_{g-2}\quad (W^o_{g-2}-a)$$

Our assumption means that the dimension of the right hand side is g-4, so the intersection of W^o_{g-2} and $W^o_{g-2}-a$ is proper and the dimension of the left hand side is at most g-4.

On the other hand, for any curve the dimension of any component W of W^1_{g-2} is at least the Brill-Noether number $\rho(g-2,1,g)=g-6$ (cf. (1.3)). As C generates JC, dim(W+C-C)=dim W+2. Therefore, dim W=g-6 and dim(W+C-C)=g-4. The latter equality implies that W+C-C is a component of $W^o_{g-2}(W^o_{g-2}-a)$.

The cohomology classes of any translate of W^1_{g-2} , C and W^o_{g-2} in the cohomology ring of JC are (because of the fact that W^1_{g-2} has the right dimension g-6)

$$\frac{\theta^6}{3!4!}\qquad \frac{\theta^{g-1}}{(g-1)!}\quad \text{and}\quad \frac{\theta^2}{2!}$$

respectively (see [A,C,G,H] p.320). So, the cohomology class of $W^o_{g-2}(W^o_{g-2}-a)$ is given by the intersection

$$\frac{\theta^2}{2}\,\frac{\theta^2}{2}=\frac{\theta^4}{4}$$

and the cohomology class of $W^1_{g-2}+C-C$ is computed by means of the Pontrjagin product

$$\frac{1}{x}\,\frac{\theta^6}{3!4!} * \frac{\theta^{g-1}}{(g-1)!} * \frac{\theta^{g-1}}{(g-1)!} = \frac{1}{x}\,\frac{5(g-5)(g-4)}{4!}\,\theta^4$$

x being the degree of the map

$$W^1_{g-2} \times C-C \to \mathrm{Pic}^{g-2}(C)$$

(we recall that C cannot be hyperelliptic by (6.1) and so the morphism $C-C \to JC$ is birrational).

We are going to compute x.Let L be a generic point in a component W of $W^1_{g-2}(C)$ and P and Q generic points in C.Assume that we have the equality $L+P-Q\equiv L'+P'-Q'$, whith L' in W^1_{g-2} and P',Q' in C.If we had $h^0(L+P+Q')\geq 3$, because L is generic in W and P generic in C we would obtain $\dim W^2_g \geq \dim W+1=g-5$.This is a contradiction because,by Riemann-Roch,$W^2_g\cong W^1_{g-2}$.Therefore,

$$|L+P+Q'|=|L|+P+Q'$$

and because of the hypothesis $L+P+Q'\equiv L'+P'+Q$,also

$$|L'+P'+Q|=|L'|+P'+Q$$

Then,by the genericity of Q, we deduce $Q=Q'$ and therefore $L+P=L'+P'$.

Let F (resp. F') be the fixed part of the series $|L|$ (resp. $|L'|$) and write $L=L_1+F$ (resp. $L'=L_1'+F'$). Because of (6.1) C is not trigonal,so $g-2-\deg F > 3$. Moreover,$|L_1|+F+P=|L'_1|+F'+P'$ implies $F+P=F'+P'$.Therefore, either $P'=P$ or $P'\leq F$.So,the degree x of the morphism

$$W^1_{g-2} \times C-C \to \mathrm{Pic}^{g-2}(C)$$

satisfies $x=\deg F+1\leq g-5$.

Now, from the fact that $W^1_{g-2}+C-C$ is a set-theoretic component of W^o_{g-2} $(W^o_{g-2}-a)$, the above computations and assuming W^1_{g-2} reduced we obtain

$$\frac{5(g-5)(g-4)}{4!(\deg F+1)} \leqslant \frac{1}{4}$$

Therefore

$$g-5 \geqslant (\deg F+1) \geqslant \frac{5}{6}(g-5)(g-4) \geqslant \frac{10}{6}(g-5)$$

which is a contradiction.

It only remains to prove that under our conditions W^1_{g-2} is reduced. Let us assume this were not the case. Then by the Cohen-Macaulayness of W^1_{g-2} (cf. [A,C,G,H] Prop.4.1) we would have a component W of W^1_{g-2} contained in the singular locus of W^1_{g-2}. This locus is the union of W^2_{g-2} and those points not in W^2_{g-2} for which the Petri morphism is not injective ([A,C,G,H] p.189). As $\dim W^2_{g-2} \leqslant g-7$ by (6.1) and Martens' Theorem ([A,C,G,H] Th.5.1), our hypothesis would imply that for a generic point L of W the Petri morphism is not injective. Let us write $L=L_1+F$, where F is the fixed part of L. Then the kernel of the Petri morphism is $H^o(K-2L_1-F)$ ([A,C,G,H] p.196). Therefore we have $h^o(K-2L_1-F) \geqslant 1$. Let t be the degree of F. When L moves in W, L moves in a component $V \neq \phi$ of dimension at least $g-6-t$ of W^1_{g-2-t}. As $\dim W^1_{g-2}=g-6$, C is neither hyperelliptic or trigonal nor bielliptic or a plane quintic. Then Mumford's Theorem ([A,C,G,H] p.193) gives dim $V=g-6-t \geq 0$ and so F is generic in C .Therefore $h^0(K-2L) \geq t+1$ and we obtain a component of $W^t_{2g-2-2(g-2-t)}=W^t_{2t+2}$ of dimension at least $g-6-t$

On the other hand, by using Mumford's Theorem again, we obtain dim $W^t_{2t+2} \geq 0$. So, $t=g-6$, deg $L_1=4$ and $h^0(2L_1) \geq 4$.

Moreover, by (4.1), W^1_4 cannot consist of a single point. Let $g^1_4, h^1_4 \in W^1_4$ and consider the product of the morphisms associated to those linear series

$$C \to \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$

Then, by (4.3.1), either the morphism is birrational and $g \leq 9$ or it has degree 2 and C is either bielliptic or hyperelliptic (which contradicts dim $W^1_{g-2}=g-6$). Now by the study of Coppens of quatrigonal curves of genus at least 7 ([C] p.32), the only curves which have more than one g^1_4 one of which satisfies $h^0(2g^1_4) \geq 4$ are of genus 7 and have exactly 2 linear series of degree 4 and dimension 1 g^1_4 and h^1_4 with $h^0(2g^1_4)=4$, $h^0(2h^1_4)=3$. Then the condition $a+W^1_4=W^1_4$ means $a=g^1_4-h^1_4=h^1_4-g^1_4$. So, $2g^1_4=2h^1_4$ which is a contradiction because the space of sections of those bundles have different dimensions.

(6.3). <u>Lemma.</u> Let C be a bielliptic curve, $C \xrightarrow{p} E$ the associated double covering. Then for all k, $4 \leq k \leq g-2$, we have $W^1_k(C)=p^*W^1_2(E)+W^0_{k-4}(C)$ and $\mathcal{G}(W^1_{g-2}) \cong E$.

Proof: Let $L \in W^1_k$ and consider the morphism $C \xrightarrow{f} \mathbb{P}^1$ associated to a two-dimensional subspace of $H^0(L)$. By

(4.3.2),the morphism

$$C \to E \times \mathbb{P}^1$$

cannot be birrational.So,f is composite whith p.Then, g^1_k is a pull-back of a series of dimension at least one in E.So,this series must have degree at least two in E.This proves the first assertion.It follows from this that $p^*\mathrm{Pic}^0(E)\cong E$ leaves W^1_k invariant set-theoretically.The second assertion is that equality also holds scheme-theoretically when k=g-2.To this end it is enough to check that there are no immersed components in W^1_{g-2} (C).Were that to happen,they would have dimension at least equal to the Brill-Noether number g-6 (see [EN] p.202).We shall check that the singular locus of W^1_{g-2} (C) coincides whith W^2_{g-2} (C) and so has dimension g-7 by Mumford´s Theorem.

Let $L\in W^1_{g-2} - W^2_{g-2}$ .Then $L=p^*D+F$ with deg D=2 and no divisor of the form p^*P whith P in E satisfies $p^*P \in F$,for,otherwise L would belong to W^2_{g-2} .Let us consider the canonical immersion of C in $\mathbb{P}^{g-1}$ .There is a point X in $\mathbb{P}^{g-1}-C$ such that the projection from X is the morphism p (A,C,G,H p.269) and we have the diagram

$$(6.4.1)\qquad \begin{tikzcd} C \arrow[r, "p"] \arrow[d] & E \arrow[d] \\ \mathbb{P}^{g-1} \arrow[r] & \mathbb{P}^{g-2} \end{tikzcd}$$

The tangent space to W^1_{g-2} at L is the subspace orthogonal to the image of the Petri morphism (see [A,C,G,H] p.189)

$$H^{o}(L) \otimes H^{o}(K-L) \to H^{o}(K)$$

and the kernel of this morphism is easily proved to be $H^{o}(K-2p^{*}D-F)$ ([A,C,G,H] p.196).Therefore L is a singular point of W^{1}_{g-2} if and only if

$$h^{o}(K-2p^{*}D-F) \geq 2$$

On the other hand,by diagram (6.4.1) and denoting by M the sheaf which gives the immersion of E in $\mathbb{P}^{g-2}$,we have

$$h^{o}(C,K-2p^{*}D-F)=h^{o}(E,M-2D-p(F))=1$$

where the last equality follows from the Riemann-Roch Theorem.

(6.4).Theorem.Let C be a curve of genus $g \geq 7$.Then $\mathcal{g}(W^{1}_{g-2}(C))=0$ if and only if C is bielliptic.

Proof:Necessity is proved in (6.3).We are going to prove sufficiency.Let us assume $a+W^{1}_{g-2}=W^{1}_{g-2}$ .Then by (6.2),we have that the inverse image A of $W^{o}_{g-2}-a$ in $C^{(g-2)}$ has dimension at least g-3.Therefore for a generic divisor D in $C^{(g-3)}$, $D+C \hookrightarrow C^{(g-2)}$ must cut A.This means that there is a point in C,P(D),such that

$$h^{o}(D+P(D)+a) \geq 1$$

By Riemann-Roch this is equivalent to

$$h^{o}(K-a-D-P(D)) \geq 2$$

If for a generic D we had $h^{o}(K-a-D) \geq 3$,then

$$h^{o}(K-a)=h^{o}(K-a-D)+g-3=g$$

and a would be zero contrary to our hypothesis. Thus, for a generic D we have that $h^{o}(K-a-D)=2$ and P(D) is a fixed point of the series $|K-a-D|$.

Let us assume first that the complete linear series $|K-a|$ has no fixed points and consider the morphism associated to it

$$C \rightarrow \tilde{C} \rightarrow |K-a|^{*} \cong \mathbb{P}^{g-2}$$

If f were birrational, $\tilde{C}$ would have degree 2g-2. The condition that $|K-a-D|$ has a fixed point for generic D means that given g-3 generic points of $\tilde{C}$, the linear variety of codimension two in $\mathbb{P}^{g-2}$ which contains them, contains also another point of $\tilde{C}$. This contradicts the principle of general position ([A,C,G,H] p.109). So, f has degree $d \geqslant 2$.

As $\tilde{C}$ is a non-degenerate curve in $\mathbb{P}^{g-2}$, $\tilde{C}$ must have degree at least g-2. We must have

$$d(g-2) \leq d.\deg C = \deg (K-a)=2g-2$$

So

$$2 \leq d \leq \frac{2g-2}{g-2} < 3$$

i.e. d=2 and $\tilde{C}$ has degree g-1. It follows that $\tilde{C}$ is an elliptic curve (use for instance (4.3.1)) and so C is bielliptic.

Let us consider now the case when $|K-a|$ has a fixed point P. Then $h^{o}(K-a-P)=h^{o}(K-a)=g-1$ and, by Riemann-

Roch this means that $h^0(a+P)=1$.So,$a+P\equiv Q$ and $a\equiv Q-P$.

Let us assume that C is not bielliptic.Then,the hypothesis $a+W^1_{g-2}=W^1_{g-2}$ $a\neq 0$,gives that $a=Q-P$,$Q\neq P$,as we have just seen.But,as

$$2a+W^1_{g-2}=a+(a+W^1_{g-2})=W^1_{g-2}$$

we should have either $2a=0$ or $2a\equiv R-S$.The former condition is $2P\equiv 2Q$ and the latter $2P+S\equiv 2Q+R$.So,in the former case C would be hyperelliptic and in the latter trigonal which contradicts (6.1).

(6.5)*.Theorem.*For any curve C $a+W^1_{g-1}(C)=W^1_{g-1}(C)$ implies $a=0$.

Proof:By a result of Welters ([We] Th.5.1),if a is such that $a+W^1_{g-1}(C)\subset W^0_{g-1}$,then $a\in C-C$.Then we can imitate the last part of the proof of (6.4) to obtain that $a=0$.

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