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ON POLYNOMIAL BOUNDS FOR THE KOSZUL HOMOLOGY OF CERTAIN MULTIPLICITY SYSTEMS

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§1. Introduction In this paper R will be a commutative Noetherian local ring.

Let M be a finitely generated R-module and $x_1,\dots,x_r \in R$ a multiplicity system on M, which means that $\mathrm{length}_R(M/IM)<\infty$, where I is the R-ideal generated by $x_1,\dots,x_r$. We will tacitly assume that no x_i is a unit, for, otherwise the functions to be considered in what follows would be all zero.

It is well-known that the function $n \longmapsto \mathrm{length}_R(M/I^nM)$ is polynomial of degree equal to the dimension d of M, for n large (d is necessarily less than or equal to r).

As $I^{rn} \subseteq (x_1^n,\dots,x_r^n)R \subseteq I^n$, we see that the function $n \longmapsto \mathrm{length}_R(M/(x_1^n,\dots,x_r^n)M)$ is bounded above and below by polynomial functions in n of degree d. However $M/(x_1^n,\dots,x_r^n)M$ is just the zeroth homology module of the Koszul complex $K.(x_1^n,\dots,x_r^n|M)$, so part of the above assertion says that the function $n \longmapsto \mathrm{length}_R H_o K(x_1^n,\dots,x_r^n|M)$ is bounded above by a degree d polynomial in n.

In this paper we prove that a similar statement is true for the higher homology modules of the Koszul complex, i.e., that $\text{length}_R H_i K(x_1^n,\dots,x_r^n|M)$ is bounded above by a polynomial in n of degree d, for any $i \geq 0$.

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§2. The higher Koszul homology modules. We start with two technical lemmas.

Lemma 1. *Let M be an R-module and $x_1,\dots,x_r,y$ be elements of R. Then, for any $i \geq 0$,*

$$\text{length}_R H_i K(x_1 y, x_2,\dots,x_r|M) \geq \text{length}_R H_i K(x_1,\dots,x_r|M).$$

Remark. It is not necessary for the lengths involved to be finite.

Proof. The inequality follows from the exact sequence (cf. [4] p.IV-2)

$$0 \longrightarrow \frac{H_i K(x_2,\dots,x_r|M)}{x_1 H_i K(x_2,\dots,x_r|M)} \longrightarrow H_i K(x_1,x_2,\dots,x_r|M) \longrightarrow$$

$$\longrightarrow (0:x_1)_{H_{i-1}K(x_2,\dots,x_r|M)} \longrightarrow 0,$$

and the corresponding one for $H_i K(x_1 y, x_2,\dots,x_r|M)$, by observing that both $(0:x_1)_{H_{i-1}K(x_2,\dots,x_r|M)} \subseteq (0:x_1 y)_{H_{i-1}K(x_2,\dots,x_r|M)}$ and $x_1 H_i K(x_2,\dots,x_r|M) \supseteq x_1 y H_i K(x_2,\dots,x_r|M)$. #

Lemma 2. *If* x *is a non-zero-divisor on* M, *we have*

$$\operatorname{length}_R H_1 K(y_1,\dots,y_r \mid {}^{M}/_{x^n M}) \le n\cdot \operatorname{length}_R H_1 K(y_1,\dots,y_r \mid {}^{M}/_{xM}).$$

Proof. Apply the functor $H_1K(y_1,\dots,y_r|*)$ to the sequence

$$0 \longrightarrow {}^{M}/_{x^{n-1}M} \xrightarrow{\cdot x} {}^{M}/_{x^n M} \longrightarrow {}^{M}/_{xM} \longrightarrow 0,$$

which is exact because x is M-regular, and use induction on n. #

Next, the goal of this paper.

Theorem 3. *Let* M *be a finitely generated* R-*module of dimension* d *and* $x_1,\dots x_r \in R$ *a multiplicity system on* M. *Then, for each* $i \geq 0$, *the function*

$$n \longmapsto \operatorname{length}_R H_i K(x_1^n,\dots,x_r^n|M)$$

is bounded above by a polynomial in n *of degree not greater than* d.

Proof. By induction on d, the case $d=0$ being trivial. Assume $d>0$. As $x_1,\dots,x_r$ is a multiplicity system on M,

$$(x_1,\dots,x_r)R \not\subseteq \bigcup_{\substack{P\in \operatorname{Ass}(M)\\ P \text{ non-maximal}}} P .$$

Thus, by Theorem 124 of [1], there exist $\lambda_2,\dots,\lambda_r$ in R such that $x=x_1+\lambda_2 x_2+\dots+\lambda_r x_r \notin \bigcup_{\substack{P\in \operatorname{Ass}M\\ P \text{ non-maximal}}} P$. In particular

$\dim(0:_M x^n)<d$ for all n, since

$$\mathrm{Ass}(0\underset{M}{:}x^n)=\mathrm{AssHom}({}^R/_{x^nR},M)=V(x^n)\cap \mathrm{Ass}M=V(x)\cap \mathrm{Ass}M,$$

and if $P\in \mathrm{Supp}(M)$ has coheight d, then $x\notin P$.

Now, from the expression for x, we observe that $(x_1,\ldots,x_r)R = (x,x_2,\ldots,x_r)R$, so that $x,x_2,\ldots,x_r$ is also a multiplicity system on M and that $x^{rn}=\mu_1x_1^n+\ldots+\mu_rx_r^n$ for some $\mu_1,\ldots,\mu_r$ (depending on n). But we have

$$\ell H_1K(x_1^n,\ldots,x_r^n|M) \leq \ell H_1K(\mu_1x_1^n,x_2^n,\ldots,x_r^n|M) =$$

$$= \ell H_1K(x^{rn},x_2^n,\ldots,x_r^n|M),$$

the inequality by virtue of lemma 1 and the equality due to the fact that the invertible matrix

$$\begin{pmatrix} 1 & 0 \ldots 0 \\ \mu_2 & 1 \ldots 0 \\ \ldots & \ldots \\ \mu_r & 0 \ldots 1 \end{pmatrix}$$

together with its exterior powers establish an isomorphism between $K.(x^{nr},x_2^n,\ldots,x_r^n|M)$ and $K.(\mu_1x_1^n,x_2^n,\ldots,x_r^n|M)$ and so it is enough to prove the theorem with x_1 replaced by x^r.

Take t large enough so that in the exact sequence

$$0 \longrightarrow 0\underset{M}{:}x^{rt} \longrightarrow M \longrightarrow {}^M/_{(0\underset{M}{:}x^{rt})}=\overline{M} \longrightarrow 0$$

x^r be $\overline{M}$-regular. From the Koszul homology sequence

$$H_1K(x^{rn},x_2^n,\ldots,x_r^n|0\underset{M}{:}x^{rt}) \longrightarrow$$

$$\longrightarrow H_1K(x^{rn},x_2^n,\ldots,x_r^n|M) \longrightarrow H_1K(x^{rn},x_2,\ldots,x_r^n|\overline{M})$$

we have, for $n \geq t$,

$$\ell H_1K(x^{rn},x_2^n,\ldots,x_r^n|M) \leq$$

$$\leq \ell H_1K(x^{rn},x_2^n,\ldots,x_r^n|0:_M x^{rt})+\ell H_1K(x^{rn},x_2^n,\ldots,x_r^n|\overline{M}).$$

The first summand on the right is bounded by induction by a polynomial in n of degree at most d-1. As to the second, since x is $\overline{M}$-regular, we have

$$\ell H_1K(x^{rn},x_2^n,\ldots,x_r^n|\overline{M})=\ell H_1K(x_2^n,\ldots,x_r^n|\overline{M}/x^{rn}\overline{M}) \quad \text{(by [3]p.368)}$$

$$\leq n\cdot H_1K(x_2^n,\ldots,x_r^n|\overline{M}/x^r\overline{M}) \quad \text{(by lemma 2)},$$

and as $\dim(\overline{M}/x^r\overline{M})<\dim(\overline{M})=\dim(M)$, by induction also,

$\ell H_1K(x_2^n,\ldots,x_r^n|\overline{M}/x^r\overline{M}) \leq$ polynomial in n of degree strictly less than d. This concludes the proof. #

Remarks. 1. In case r=d, for instance, if $x_1,\ldots,x_d$ is a system of parametres for M, then a stronger result is obtained from [2]. There it is proved that $\ell H_iK(x_1^n,\ldots,x_r^n|M)$ is bounded by a polynomial in n of degree at most r-i. So the importance of the theorem above becomes clear when r takes large values.

2. A proof by double induction on d and r can be given to theorem 3 if we bear in mind that [2] ensures the case r=d.

Corollary 4. Let M *and* $x_1,\dots,x_r$ *be as in the theorem. Then the functions*

$$n \longmapsto \chi_i(x_1^n,\dots,x_r^n|M)$$

where χ_i *stands for the i'th Euler-Poincaré characteristic* (cf. [4] App. II), *are bounded above by polynomials in* n *of degree not exceeding the dimension of* M. #

Remark. The results above carry over almost immediately to the case of a semilocal ring by means of the formulae $\text{length}_R(N) = \sum_{\mathfrak{m}\ \text{maximal}} \text{length}_{R_{\mathfrak{m}}}(N_{\mathfrak{m}})$, for any module N of finite length, and $\dim_R M = \sup_{\mathfrak{m}\ \text{maximal}} (\dim_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}))$, together with the fact that taking homology commutes with localisation, i.e., in our case $H_iK(x_1^n,\dots,x_r^n|M)_{\mathfrak{m}} \cong H_iK(x_1^n,\dots,x_r^n|M_{\mathfrak{m}})$. In fact, if the Hilbert-Samuel polynomial function associated with (M,I) has degree g (where $g\le d$), then, for each $i\ge 0$, both $\ell H_iK(x_1^n,\dots,x_r^n|M)$ and $\chi_i(x_1^n,\dots,x_r^n|M)$ are bounded above by polynomials in n of degree at most g.

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