UNIVERSITAT DE BARCELONA

FACULTAT DE MATEMÀTIQUES

ON HIGHER DIMENSIONS OF MODULES OVER LOCAL RINGS

by

J-L. GARCIA ROIG

PRE–PRINT N.º 35

ENERO 1986

ON HIGHER DIMENSIONS OF MODULES OVER LOCAL RINGS

J-L. García Roig

Throughout this paper $(R,\mathfrak{m})$ denotes a Noetherian local ring and M a finitely generated R-module.

If $a_1,\dots,a_d$ is a system of parametres for M, we know that the lengths of the Koszul homology modules $H_iK(a_1^n,\dots,a_d^n|M)$, for $i=0,1,\dots,d$, are bounded above in length by polynomials in n of degree at most d-i (cf. [3]). Now, for each i, $0\leq i\leq d$, we can consider the least degree of all polynomials in n which bound the length of $H_iK(a_1^n,\dots,a_d^n|M)$. For instance, if i=0, this least degree is just the dimension d of M. Tosee this, it suffices to bear in mind the chain of inclusions $(a_1,\dots,a_d)^{dn}\subseteq(a_1^n,\dots,a_d^n)\subseteq(a_1,\dots,a_d)^n$ together with the fact that $n\longmapsto \text{length}(M/(a_1,\dots,a_d)^nM)$ is a polynomial function in n of degree d, for $n>>0$. If, on the other hand, i=d, then it is clear that the least degree is zero. In both these cases we see that the least degree considered is the same for all systems of parametres and we can ask whether this is so for the other Koszul homology modules, i.e., whether the least degree considered for $H_iK(a_1^n,\dots,a_d^n|M)$, $0<i<d$, does or does not depend on the particular system of parametres $a_1,\dots,a_d$ chosen for M.

In this paper we show that this least degree is independent of the system of parametres and so, we have been naturally led to the definition of what we have called "higher dimensions" of a module. Similar considerations also hold for the higher Euler-Poincaré characteristics.

I wish to thank Dr. D.Kirby for all his helpful suggestions.

We begin with some introductory lemmas.

Lemma 1. *Let* $a_1,\ldots,a_d$ *and* $b_1,\ldots,b_d$ *be systems of parametres for* M. *Then there exists* $c_1 \in \mathfrak{m}$ *such that both* $c_1,a_2,\ldots,a_d$ *and* $c_1,b_2,\ldots,b_d$ *are systems of parametres for* M.

Proof. It suffices to take c_1 in $\mathfrak{m}$ not belonging to the finite set of primes consisting of the minimal primes $P_1,\ldots,P_r$ of $M/(a_2,\ldots,a_d)M$ and the minimal ones $Q_1,\ldots,Q_s$ of $M/(b_2,\ldots,b_d)M$, all of coheight 1, i.e., submaximal. Such a selection is possible by virtue of Proposition 1.11 of [1]. #

As any permutation of a system of parametres is still a system of parametres, repeated application of lemma 1 allows us to connect any two systems of parametres by a sequence of not more than 2d+1 systems of parametres with the property that two consecutive systems of parametres differ by at most one element.

This fact reduces our problem to the case of two systems of parametres which differ by just one element and which we will denote by $a_1,\dots,a_{d-1},a$ and $a_1,\dots,a_{d-1},b$. With this notation, and writing $\overline{R} = R/_{Ann(M)}$, we have the following

Lemma 2. *The elements* $a_1,\dots,a_{d-1},ab$ *constitute a system of parametres for* M *and, if* $I_n = (a_1^n,\dots,a_{d-1}^n,a^n)\overline{R}$, *and* $J_n = (a_1^n,\dots,a_{d-1}^n,a^nb^n)\overline{R}$, *then there exists a constant* k *such that* $I_{kn} \subseteq J_n \subseteq I_n$, *for all* n.

Proof. Neither a nor b belong to any minimal prime of $M/_{(a_1,\dots,a_{d-1})M}$, so the same happens to ab and thus, $a_1,\dots,a_{d-1},ab$ is a system of parametres for M, or what amounts to the same, for $\overline{R}$. This implies that the radical of $J_1 = (a_1,\dots,a_{d-1},ab)\overline{R}$ is $\overline{\mathfrak{m}}$ (= image of $\mathfrak{m}$ in $\overline{R}$), and as $a \in \mathfrak{m}$, we have $\overline{a}^t \in J_1$ for some t, where $\overline{a}$ stands for the coset defined by a in $\overline{R}$. Consequently, for all n,

$$\overline{a}^{tdn} \in J_1^{dn} \subseteq J_n ,$$

and so, it suffices to take $k = td$. #

We still need another technical lemma.

Lemma 3. *If* $a \in R$, *then we have:*

i) $\ell_M(0:a^n) \leqslant n \cdot \ell_M(0:a)$, *and*

ii) $\ell(M/_{a^nM}) \leqslant n \cdot \ell(M/_{aM})$.

$\underline{\text{Proof}}$. By induction on n. From the exact sequence

$$R/_{aR} \xrightarrow{\cdot a^{n-1}} R/_{a^nR} \longrightarrow R/_{a^{n-1}R} \longrightarrow 0,$$

if we apply $\text{Hom}_R(*,M)$, we get i), for $\text{Hom}_R(R/_{a^nR},M)=0\underset{M}{:}a^n$, and if we apply $*\underset{R}{\otimes}M$, we get ii), for $R/_{a^nR}\underset{R}{\otimes}M=M/_{a^nM}$. #

Remark. Though lemma 3 is quite general, we will be interested only in the case where the lengths involved are finite. Moreover, lemma 3 can be generalised in the form:

i) $\ell(0\underset{M}{:}ab)\leq\ell(0\underset{M}{:}a)+\ell(0\underset{M}{:}b)$, and

ii) $\ell(M/_{abM})\leq\ell(M/_{aM})+\ell(M/_{bM})$.

Next the aim of this paper.

Lemma 4. For $i>0$ and all n, we have the inequality

$$\ell H_iK(a_1^n,\ldots,a_{d-1}^n,b^n|M)\leq k\cdot\ell H_iK(a_1^n,\ldots,a_{d-1}^n,a^n|M),$$

the constant k being that of lemma 2.

Proof. Consider the exact sequences (cf. [6] p.IV-2)

$$0 \longrightarrow H_oK(x|H_iK(a_1^n,\ldots,a_{d-1}^n|M)) \longrightarrow H_iK(a_1^n,\ldots,a_{d-1}^n,x|M)$$

$$\longrightarrow H_1K(x|H_{i-1}K(a_1^n,\ldots,a_{d-1}^n|M)) \longrightarrow 0,$$

with $x=a^n$ or a^nb^n. Thus our lemma will be proved if similar inequalities as the one stated in the lemma hold for the side terms of the above exact sequences.

Now, from lemma 2 we can write, for some λ in R (depending on n),

$$a^{kn} \equiv \lambda a^n b^n \text{ modulo}(a_1^n, \ldots, a_{d-1}^n, \mathrm{Ann}M).$$

As $(a_1^n, \ldots, a_{d-1}^n, \mathrm{Ann}M)$ kills the homology modules of the Koszul complex $K(a_1^n, \ldots, a_{d-1}^n | M)$ (see [6] p.IV-7), for the right hand term, we have

$$\ell H_1 K(a^n b^n | H_{i-1} K(a_1^n, \ldots, a_{d-1}^n | M)) = \ell(0 : a^n b^n)_{H_{i-1} K(a_1^n, \ldots, a_{d-1}^n | M)} \leqslant$$

$$\leqslant \ell(0 : \lambda a^n b^n)_{H_{i-1} K(a_1^n, \ldots, a_{d-1}^n | M)} = \ell(0 : a^{kn})_{H_{i-1} K(a_1^n, \ldots, a_{d-1}^n | M)},$$

and this last length, by lemma 3 i), is less than or equal to

$$k \cdot \ell(0 : a^n)_{H_{i-1} K(a_1^n, \ldots, a_{d-1}^n | M)} = k \cdot \ell H_1 K(a^n | H_{i-1} K(a_1^n, \ldots, a_{d-1}^n | M)).$$

We proceed in a complete parallel manner with $\ell H_o K(a^n b^n | H_i K(a_1^n, \ldots, a_{d-1}^n | M))$ and get

$$\ell H_o K(a^n b^n | H_i K(a_1^n, \ldots, a_{d-1}^n | M)) \leqslant k \cdot \ell H_o K(a^n | H_i K(a_1^n, \ldots, a_{d-1}^n | M)).$$

These two inequalities establish that

$$\ell H_i K(a_1^n, \ldots, a_{d-1}^n, a^n b^n | M) \leqslant k \cdot \ell H_i K(a_1^n, \ldots, a_{d-1}^n, a^n | M),$$

and the conclusion follows, for, according to lemma 1 of [2],

$$\ell H_i K(a_1^n, \ldots, a_{d-1}^n, b^n | M) \leqslant \ell H_i K(a_1^n, \ldots, a_{d-1}^n, a^n b^n | M). \ \#$$

From these lemmas the proof of the following theorem is immediate.

Theorem 5. Let $a_1,\ldots,a_d$ *be a system of parametres for* M. *If the length of* $H_iK(a_1^n,\ldots,a_d^n|M)$, $i\geq 0$, *is bounded above by a polynomial in* n *of degree* g, *then* $\ell H_iK(b_1^n\ldots,b_d^n|M)$ *is also bounded above by a polynomial in* n *of degree* g, *for any other system of parametres* $b_1,\ldots,b_d$ *for* M. #

This theorem makes possible the following

Definition. The i'th dimension of M $(0\leq i\leq d)$, *denoted by* $\dim_i(M)$ *is the least integer* g *such that for a system of parametres* $a_1,\ldots,a_d$ *of* M, *there exists a polynomial in* n *of degree* g *which bounds the length of* $H_iK(a_1^n,\ldots,a_d^n|M)$.

Observe that this definition is independent of the system of parametres chosen by virtue of theorem 5. Moreover, $\dim_o(M)=\dim(M)$ and, in general, $\dim_i(M)\leq d-i$ (cf. [3]), so that, in particular $\dim_d(M)=0$.

With this terminology, generalised Cohen-Macaulay modules can be characterised by the condition $\dim_1(M)=0$ (cf. [5] Satz 3.3).

The results above carry over to the case of the higher Euler-Poincaré characteristics χ_i, $i>0$, defined by the formula (cf. [6] Ch.IV App.II)

$$\chi_i(a_1,\ldots,a_d|M)=\sum_{j\geq i}(-1)^{j-i}\ell H_jK(a_1,\ldots,a_d|M)$$

where $a_1,\ldots,a_d$ is a system of parametres for M.

Lemma 6. With our previous notations, for $i>0$ and all n, we have the inequality

$$\chi_i(a_1^n,\dots,a_{d-1}^n,b^n|M)\leq k\cdot\chi_i(a_1^n,\dots,a_{d-1}^n,a^n|M),$$

where k is the constant of lemma 2.

Proof. This follows from the expression (see [6]p.IV-56)

$$\chi_i(a_1^n,\dots,a_{d-1}^n,a^nb^n|M) = \ell H_1K(a^nb^n|H_{i-1}K(a_1^n,\dots,a_{d-1}^n|M))+$$

$$+\chi_o(a^nb^n|\chi_i(a_1^n,\dots,a_{d-1}^n|M)).$$

The proof of lemma 4 gives

$$\ell H_1K(a^nb^n|H_{i-1}K(a_1^n,\dots,a_{d-1}^n|M))\leq k\cdot\ell H_1K(a^n|H_{i-1}K(a_1^n,\dots,a_{d-1}^n|M))$$

which we link with the obvious inequality

$$\ell H_1K(b^n|H_{i-1}K(a_1^n,\dots,a_{d-1}^n|M))\leq\ell H_1K(a^nb^n|H_{i-1}K(a_1^n,\dots,a_{d-1}^n|M)).$$

On the other hand, from $a^{kn}\equiv\lambda a^nb^n \bmod(a_1^n,\dots,a_{d-1}^n,\mathrm{Ann}M)$ (cf. Lemma 2), we have the inequality (of nonnegative integers)

$$\chi_o(b^n|\chi_i(a_1^n,\dots,a_{d-1}^n|M))\leq\chi_o(\lambda a^nb^n|\chi_i(a_1^n,\dots,a_{d-1}^n|M))=$$

$$=\chi_o(a^{kn}|\chi_i(a_1^n,\dots,a_{d-1}^n|M))=k\cdot\chi_o(a^n|\chi_i(a_1^n,\dots,a_{d-1}^n|M)).$$

This is enough to conclude the proof. #

Remark. For $i=0$, if $a_1,\dots,a_d$ is a system of parametres for M, $\chi_o(a_1^n,\dots,a_d^n|M)$ is a polynomial in n of degree $d=\dim(M)$. (This is because $\chi_o(a_1^n,\dots,a_d^n|M)=n^d\cdot\chi_o(a_1,\dots,a_d|M)$ and

$\chi_0(a_1,\ldots,a_d|M)>0.$)

Theorem 7. Let $a_1,\ldots,a_d$ be a system of parametres for M. If $\chi_i(a_1^n,\ldots,a_d^n|M)$, for $i\geq 0$, is bounded above by a polynomial in n of degree g, then $\chi_i(b_1^n,\ldots,b_d^n|M)$ is also bounded above by a polynomial in n of degree g, for any other system of parametres $b_1,\ldots,b_d$ of M. #

Theorem 7 makes possible the following

Definition. The i'th Euler-Poincaré dimension of M ($0<i<d$), denoted $\dim_i^*(M)$ is the least integer g such that, for a system of parametres $a_1,\ldots,a_d$ of M, there exists a polynomial in n of degree g which bounds $\chi_i(a_1^n,\ldots,a_d^n|M)$.

Observe that $\dim_0^*(M)=\dim_0(M)=\dim(M)$ and that, in general, $\dim_i^*(M)\leq\dim_i(M)$ as a consequence of the fact that $\ell H_i=\chi_i+\chi_{i+1}$ and that $\chi_i\geq 0$.

In terms of these new higher dimensions, generalised Cohen-Macaulay modules can also be characterised by the condition $\dim_1^*(M)=0$ (cf. [5] Satz 3.3).

Remark. All the statements and proofs of this paper carry over almost immediately to the case the ring R is assumed to be semilocal, in which case, $\mathfrak{m}$ denotes its Jacobson radical.

REFERENCES

1. Atiyah, M.F., MacDonald, I.G., Introduction to Commutative Algebra, Addison-Wesley, 1969.

2. García Roig, J-L., On polynomials bounds for the Koszul homology of certain multiplicity systems, to appear.

3. Kirby, D., An addendum to Lech's limit formula for multiplicities, Bull. London Math. Soc. 16 (1984),281-284.

4. Northcott, D.G., Lessons on Rings, Modules and Multiplicities, Cambridge Univ. Press (1968).

5. Schenzel, P., Ngo Viet Trung, Nguyen Tu Cuong, Verallgemeinerte Cohen-Macaulay Moduln, Math. Nachr. 85 (1978), 57-73.

6. Serre, J-P., Algèbre locale: Multiplicités, Lecture Notes in Math. No. 11, Springer, Berlin (1975).

J-L. García Roig,
Cruz Roja 26,3º4ª,
Hospitalet,
Barcelona,
Spain

Key words, Hilbert polynomial functions, multiplicity system, system of parametres, Koszul complex.

Dipòsit Legal B.: 2.101-1986
BARCELONA – 1986