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W–ALGEBRAS WHICH ARE BOOLEAN PRODUCTS OF MEMBERS OF SR[1] AND CW–ALGEBRAS

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Abstract: We show that the class of all isomorphic images of Booleans products of members of $\underline{SR}[1]$ is the class of all Archimedean W-algebras. And the class of all isomorphic images of CW-algebras is the class of all W-algebras such that the family of all minimal prime implicative filters is the family of all Stone ultrafilters.

INTRODUCTION AND PURPOSES

W-algebras (or Wajsberg algebras) are the algebraic models of $\aleph_0$-valued Łukasiewicz's Propotional calculus. Indeed, see [7], they are equivalent to MV-algebras introduced by C.Chang in [4] and used in [5] to show the completeness of Łukasiewicz's Propositional Calculus. The advantage of to use W-algebras is that they are defined with the operations "implication" ($\rightarrow$) and "negation" ($\sim$), which have a clear logic signification.

The class of all W-algebras is a variety generated by the W-algebra $\underline{R}$ [1], defined in 1.D., and it has the property that every simple W-algebra is isomorphic to a subalgebra of it. In the other hand, every W-algebra is isomorphic to a subdirect product of CW-algebras (or W-algebras which are chains with the associated partial order defined in (1.12)). The purpose of this paper is to give a characterization of the W-algebras wich are isomorphic to a boolean product of a subalgebras of $\underline{R}$ [1] , or isomorphic to a boolean product of CW-algebras.

In section 1, we give a several well known definitions and results, without proof, on W-algebras which we will need is the paper. In section 2, we define the toplogical Spectrum of any W-algebra, which is a Bounded Stone Space, and we show that it is a Boolean Space if and only if the W-algebra is Archimedean. In section 3, we see that the class of all isomorphic images of boolean products of members of SR[1] is the class of all Archimedean W-algebras, moreover this representation is unique. Finally, in section 4, we prove that the class of all isomorphic images of boolean products of CW-algebras is the class of all W-algebras such that the family of all minimal prime implicative filters is the family of all Stone ultrafilters, and we see that this representation is unique.

To show the last result we could have used the results of R.Cignoli, see[6] , we give a direct proof because the proof using Cignoli's results is very large. A complet study of W-algebras has been made in [8] , unfortunately this work has not published, any way in [7] can be found the properties used in this paper. For the definition and properties of boolean product see [2] , and for its conection with boolean sheaf spaces see [3] .

1. W-ALGEBRAS: DEFINITIONS AND PROPERTIES

Along all the paper $A = (A, \rightarrow, \sim, u)$ represents an algebraic structure of type (2,1,0), we write it $A \in K(2,1,0)$.

1.A. Let $A \in K(2,1,0)$ we say that A is a W-algebra provided that it satisfies the following equations:

(1.1) $u \rightarrow x = x$

(1.2) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$

(1.3) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = u$

(1.5) $(\sim x \rightarrow \sim y) \rightarrow (y \rightarrow x) = u$

By definition the class of all W-algebras is a Variety, we denote it $\underline{W}$.

To short we write $0 = \sim u$, $x^{o} \to y = y$, and for any $n < \omega$, $x^{n} \to y = x \to (x^{n-1} \to y)$. If P is a property, $\underline{W} \models P$ denote that P is valid in all $\underline{W}$. It satisfies:

(1.5) $\underline{W} \models x \to x = u$

(1.6) $\underline{W} \models x \to u = u$

(1.7) $\underline{W} \models x^{n} \to (y^{m} \to z) = y^{m} \to (x^{n} \to z)$, for any $n, m < \omega$.

(1.8) $\underline{W} \models x \to 0 = \sim x$

(1.9) $\underline{W} \models \sim x \to \sim y = y \to x$

(1.10) $\underline{W} \models \sim(\sim x) = x$

1.B. In any W-algebra we can define a lattice structure in the next way. We set:

(1.11) $x \vee y = (x \to y) \to y$ and $x \wedge y = \sim(\sim x \vee \sim y)$.

Then for any $\underset{\sim}{A} \in \underline{W}$, $(A, \wedge, \vee, \sim, 0, u)$ is a De Morgan algebra, where $\wedge$ is the meet, $\vee$ is the join, $\sim$ is the nagation, 0 is the lower bound and u is the upper bound. Moreover the lattice partial order is given by:

(1.12) $x \leqslant y$ if and only if $x \to y = u$

To show the above results it is necessary to see the following properties:

(1.13) $\underline{W} \models x^{n} \to y \leqslant x^{m} \to y$, for any $0 \leqslant n \leqslant m < \omega$

(1.14) $\underline{W} \models (x \wedge y) \to z = (x \to z) \vee (y \to z)$

(1.15) $\underline{W} \models (x \vee y) \to z = (x \to z) \wedge (y \to z)$

(1.16) $\underline{W} \models (x \to y) \vee (y \to x) = u$

(1.17) $\underline{W} \models x \to (y \vee z) = (x \to y) \vee (x \to z)$

(1.18) $\underline{W} \models x \to (y \wedge z) = (x \to y) \wedge (x \to z)$

1.C. $\underline{W}$ is an arithmetical Variety with 2/3 minority term: $m(x,y,z) = ((x \to y) \to z) \wedge \wedge ((z \to y) \to x) \wedge (x \vee y)$, hence it is Congruence distributive and Congruence permutable.

1.D. The Variety $\underline{W}$ is generated by the following algebra:

$\underline{R}[1] = ([0,1], \to, \sim, 1)$, where $[0,1]$ is the unit interval of the totally ordered aditive group of real numbers and $a \to b = \inf\{1, 1-a+b\}$, $\sim a = 1-a$.

1.E. Let $\underset{\sim}{A} \in \underline{W}$ and $f \subsetneq A$ we say that f is an implicative filter when: $u \in f$; for any $a,b \in A$, $a \in f$ and $a \to b \in f$ implies $b \in f$. The family of all implicative filters of $\underset{\sim}{A}$, which we denote $\mathbb{F}_i(\underset{\sim}{A})$, is an algebraic closure system and it is a subfamily of the family of all lattice filters of the De Morgan algebra defined in 1.B. Hence $\mathbb{F}_i(\underset{\sim}{A})$ is an algebraic lattice where the meeet is the set-theoretic intersection, the join is: $f_1, f_2 \in \mathbb{F}_i(\underset{\sim}{A})$ $f_1 \vee f_2 = F_i(f_1 \cup f_2)$ (F_i is the associated closure oprator to $\mathbb{F}_i(\underset{\sim}{A})$), A is the upper bound and $\{u\}$ is the lower bound. To short we write $F_i(a) = F_i(\{a\})$ and $F_i(X,a) = F_i(X \cup \{a\})$.

From the properties of $\mathbb{F}_i(\underset{\sim}{A})$ we quote the following:

(1.19) (Deduction principle).

$F_i(X,a) = \{b \in A / \ a^n \to b \in F_i(X), \text{ for some } n < \omega\}$.

$F_i(a) = \{b \in A / \ a^n \to b = u, \text{ for some } n < \omega\}$.

(1.20) $\mathbb{F}_i(\underset{\sim}{A})$ has the family of prime implicative filters (prime as lattice filters) as a basis, hence every proper implicative filter is characterized by the set of all prime implicative filters which contain it.

(1.21) If $f \in \mathbb{F}_i(\underset{\sim}{A})$, f is a proper maximal if and only if for any $a \notin f$ there exists $n < \omega$ such that $a^n \to 0 \in f$.

(1.22) For every prime implicative filter there exists a prime implicative filter contained in it which is minimal in the partial ordered set of all prime implicative filters.

1.F. Let $\underset{\sim}{A} \varepsilon \underline{W}$ and let $C(\underset{\sim}{A})$ be the algebraic lattice of all congruence relations of A, then the map:

$\theta : ⓕ_i(\underset{\sim}{A}) \longrightarrow C(\underset{\sim}{A}) : f \longmapsto \theta_f = \{(a,b) \varepsilon AxA / (a \to b) \wedge (b \to a) \varepsilon f\}$

is an order isomophism and its inverse is:

$f : C(\underset{\sim}{A}) \longrightarrow ⓕ_i(\underset{\sim}{A}) : \theta \longmapsto f_\theta = \{a \varepsilon A / (a,1) \varepsilon \theta\}$.

Hence $ⓕ_i(\underset{\sim}{A})$ is a distributive algebraic lattice.

Now we give several properties more:

(1.23) If $\underset{\sim}{A} \varepsilon \underline{W}$ and $\theta \varepsilon C(\underset{\sim}{A})$, the quotien algebra $\underset{\sim}{A}/\theta$ is a chain, with the partial order of 1.B., if and only if f_θ is prime implicative filter.

We call CW-algebra to a W-algebra which is chain, and we denote by $\underline{CW}$ the class of all CW-algebras.

(1.24) If $\underset{\sim}{A} \varepsilon \underline{W}$ and $\theta \varepsilon C(\underset{\sim}{A})$, then $\underset{\sim}{A}/\theta$ is simple if and only if f_θ is a proper maximal implicative filter, or equivalently, $\underset{\sim}{A}/\theta$ is a subalgebra of $\underline{R}[1]$.

2. TOPOLOGICAL SPECTRUM OF A W-ALGEBRA

2.A. Let $\underset{\sim}{A} \varepsilon \underline{W}$, we considere:

$Sp\ \underset{\sim}{A} = \{p \varepsilon ⓕ_i(\underset{\sim}{A}) / p \text{ is prime}\}$,

and for any $a \varepsilon A$:

$S(a) = \{p \varepsilon Sp\ \underset{\sim}{A} / a \varepsilon p\} = \{p \varepsilon Sp\ \underset{\sim}{A} / F_i(a) \subsetneq p\}$

LEMMA 1. For any $\underset{\sim}{A} \varepsilon \underline{W}$ it satisfies:

(2.1) $S(a) = S(b)$ implies $F_i(a) = F_i(b)$, for any $a,b \varepsilon A$

(2.2) $S(a \wedge b) = S(a) \cap S(b)$, for any $a,b \varepsilon A$

(2.3) $S(a \vee b) = S(a) \cup S(b)$, for any $a,b \varepsilon A$

(2.4) $S(u) = Sp\ \underset{\sim}{A}$ and $S(0) = \emptyset$.

PROOF. (2,1) is a consequence of (1.20)

(2.2) and (2.3) are consequences of the fact that every $p \in Sp\, \underset{\sim}{A}$ is a prime lattice filter. (2.4) is trivial.

It is clear that the family $(S(a)/a \in A)$ is a basis for a topology of open sets on $Sp\, \underset{\sim}{A}$. This topological space is called the topological spectrum of $\underset{\sim}{A}$ and we represent it for $Sp_\tau \underset{\sim}{A}$.

2.B. In order to determine the properties of $Sp_\tau \underset{\sim}{A}$ we considere the set of all principal implicative filters of $\underset{\sim}{A}$, or the compact elements of $\mathbb{F}_i(\underset{\sim}{A})$, which are denoted by $\underline{F}_i^{\underset{\sim}{A}}$. $\underline{F}_i^{\underset{\sim}{A}}$ is the universe of a sublattice of $\mathbb{F}_i(\underset{\sim}{A})$, because we have, for any $a,b \in A$:

$F_i(a \vee b) = F_i(a) \cap F_i(b)$ and $F_i(a \wedge b) = F_i(a) \vee F_i(b)$.

Hence $(\underline{F}_i^{\underset{\sim}{A}}, \cap, \vee)$ is a distributive lattice, moreover it has a lower bound $\{u\} = F_i(u)$ and upper bound $A = F_i(0)$.

Let $Sp^*_\tau(\underset{\sim}{A})$ be the topological space defined on the set of all prime lattice filters of $\underline{F}_i^{\underset{\sim}{A}}$, which are denoted by $Sp^*(\underset{\sim}{A})$, and as a basis of open sets the family $(\widehat{F_i(a)} = \{P \in Sp^*(\underset{\sim}{A}) / F_i(a) \notin P\} / a \in A)$.

It is easy to see that $Sp^*_\tau(\underset{\sim}{A})$ is a Bounded Stone Space (in the sense of [1], pag 79).

THEOREM 2. For any $\underset{\sim}{A} \in \underline{W}$, $Sp_\tau \underset{\sim}{A}$ and $Sp^*_\tau(\underset{\sim}{A})$ are homeomorphic

PROOF. Let h be the correspondence defined:

$h : Sp\, \underset{\sim}{A} \longrightarrow Sp^*(\underset{\sim}{A}) : p \longmapsto h(p) = \{F_i(a) / a \notin p\}$.

That h is a map is a simple comprovation, trivially is one to one.

To see that it is onto, for any $P \in Sp^*(\underset{\sim}{A})$ we define $p = \{a \in A / F_i(a) \notin P\}$ then p is implicative filter, because if $a, a \to b \in p$, then $F_i(a), F_i(a \to b) \notin P$, since P is prime $F_i(a) \vee F_i(a \to b) \notin P$, that is $F_i(a \wedge (a \to b)) \notin P$, by definition of implicative filter $b \in F_i(a \wedge (a \to b))$, that implies $F_i(b) \subsetneq F_i(a \wedge (a \to b))$ and hence $F_i(b) \notin P$, and $b \in p$.

In the other hand if $a \vee b \in p$, then $F(a) \cap F(b) \notin P$, that is $F(a) \notin P$ or $F(b) \notin P$, hence $a \in p$ or $b \in p$. Then p is prime and $p \in \mathrm{Sp}\, \underset{\sim}{A}$. Moreover it is easy to see that $h(p) = P$.

Finally we have to see that h is homeomorphism:

$P \in h(S(a))$ iff $h^{-1}(P) \in S(a)$ iff $a \in h^{-1}(P)$ iff $F_i(a) \notin P$ iff $P \in \widehat{F_i(a)}$, thus $h(S(a)) = \widehat{F_i(a)}$. Similarly we could be shown that $h^{-1}(\widehat{F_i(a)}) = S(a)$

COROLLARY. If $\underset{\sim}{A} \in \underline{W}$, then $\mathrm{Sp}\, \underset{\sim}{A}$ is a Bounded Stone Space.

2.C. To characterize the clopen sets of $\mathrm{Sp}\, \underset{\sim}{A}$ we need to define a special elements of the W-algebras. Let $\underset{\sim}{A} \in \underline{W}$ and $a \in A$, we say that a is *archimedean* when there exists $n < \omega$ such that $(a^n \to 0) \vee a = u$; and we say that a is *boolean* when it has complement, is this case this is $\sim a$. The set of boolean elements of $\underset{\sim}{A}$ is denoted by $B(\underset{\sim}{A})$.

Now we give a previous result:

LEMMA 3. If $\underset{\sim}{A} \in \underline{W}$, and $a \in A$, then for any $n < \omega$, $F_i(a) = F_i(\sim(a^n \to 0))$.

PROOF. We use (1.19). From (1.5), (1.7) and (1.8) we have that for any $n < \omega$:

$$a^n \to (\sim(a^n \to 0)) = a^n \to ((a^n \to 0) \to 0) = (a^n \to 0) \to (a^n \to 0) = u$$

this implies that $\sim(a^n \to 0) \in F_i(a)$. In the other hand using (1.10) (1.9) (1.8) and (1.13): $\sim(a^n \to 0) \to a = \sim a \to (a^n \to 0) = (a \to 0) \to (a^n \to 0) = u$

this shows that $F_i(a) \subseteq F_i(\sim(a^n \to 0))$, and hence the Lemma is true.

THEOREM 4. For any $\underset{\sim}{A} \in \underline{W}$, the following conditions are equivalent:

(i) N is a clopen subset of $\mathrm{Sp}\, \underset{\sim}{A}$.

(ii) $N = S(a)$ for some $a \in A$ archimedean

(iii) $N = S(b)$ for some $b \in B(\underset{\sim}{A})$.

PROOF. (i)$\Rightarrow$(ii). Let N be a clopen subset of Sp $\underset{\sim}{A}$, then N and N^c are compact open, since $Sp_T\underset{\sim}{A}$ is Bounded Stone space, there exist $a, c \in A$ such that $N = S(a)$ and $N^c = S(c)$. We will show that a is archimedean. It is clear that $\emptyset = S(a) \cap S(c) = S(a \wedge c)$, hence $S(0) = S(a \wedge c)$, by (2.1) we have $F_i(a \wedge c) = F(0)$, that is, $0 \in F_i(a \wedge c) =$ $= F_i(\{a,c\})$, hence by (1.19) there exists $n < \omega$ such that $a^n \to 0 \in F_i(c)$ this implies that $F_i(a^n \to 0) \subseteq F_i(c)$. Thus $F_i(a \vee (a^n \to 0)) = F_i(a) \cap F_i(a^n \to 0) \subsetneq$ $\subsetneq F_i(a) \cap F_i(c) = F_i(a \vee c)$. But from $S(a \vee c) = S(a) \cup S(c) = Sp\,\underset{\sim}{A} = S(u)$, we deduce that $F_i(a \vee c) = \{u\}$, hence $F_i(a \vee (a^n \to 0)) = \{u\}$ then $a \vee (a^n \to 0) = u$. This shows that a is archimedean.

(ii)$\Rightarrow$(iii). We suppose (ii), then there exists $n < \omega$ such that $a \vee (a^n \to 0) = u$. We will show that $a^n \to 0$ is boolean, then we will have, by Lemma 3, $S(a) = S(\sim(a^n \to 0))$, with $\sim(a^n \to 0)$ boolean.

Using Lemma 3, we have:

$$\{u\} = F_i(a \vee (a^n \to 0)) = F_i(a) \cap F_i(a^n \to 0) = F_i(\sim(a^n \to 0)) \cap F_i(a^n \to 0) =$$
$$= F_i(\sim(a^n \to 0) \vee (a^n \to 0)).$$

Hence $\sim(a^n \to 0) \vee (a^n \to 0) = u$, Moreover since $(A, \wedge, \vee, \sim, 0, u)$ is a De Morgan algebra, we have that $(a^n \to 0) \wedge \sim(a^n \to 0) = 0$.

(iii) $\Rightarrow$ (i) is trivial.

2.D. Given $\underset{\sim}{A} \in \underline{W}$ we say that $\underset{\sim}{A}$ is <u>archimedean W-algebra</u>, when for any $a \in A$ a is archimedean. $\underline{Wa}$ denotes the class of all Archimedean W-algebras.

THEOREM 5. Let $\underset{\sim}{A} \in \underline{W}$, then the following conditions are equivalent:

(i) $\underset{\sim}{A} \in \underline{Wa}$

(ii) $Sp_T\underset{\sim}{A}$ is Boolean Space

(iii) $Sp_T\underset{\sim}{A}$ is Hausdorff

(iv) $Sp_T\underset{\sim}{A}$ is T_1

PROOF. (i)$\Leftrightarrow$(ii), (ii)$\Rightarrow$(iii), (iii)$\Rightarrow$(iv) are immediate consequences from Theorem 4. (iv)$\Rightarrow$(ii) is satisfied because any Bounded Stone space T_1 is Boolean space.

CROLLARY. Let $\underset{\sim}{A} \in \underline{W}$, then $\underset{\sim}{A} \in \underline{Wa}$ if and only if every prime implicative filter is a proper maximal implicative filters, that is, the family of all proper maximal implicative filters is justly Sp $\underset{\sim}{A}$.

PROOF. By Theorem 5, $\underset{\sim}{A} \in \underline{Wa}$ iff $Sp_T\underset{\sim}{A}$ is T_1, this is equivalent to every two prime implicative filters are equal or non comparables, that is any prime implicative filter is a proper maximal. The fact that any proper maximal implicative filter is prime concludes the proof.

We remark that the class $\underline{Wa}$ is not the class $\underline{W}$, because $\underline{Wa}$ is not definable by means of generalized implications (see [8]) since the direct product of members of $\underline{Wa}$ is not necessarily in $\underline{Wa}$. In the other hand, every Archimedean W-algebra is semisimple, but there exist semisimple W-algebras which are not in $\underline{Wa}$, since the class of all semisimple W-algebra is definable by means of generalized implications (see [7]).

3. BOOLEANS PRODUCTS OF MEMBERS OF $\underline{SR}[1]$.

$\underline{SR}[1]$ will denotes the class of all isomorphic images of subalgebras of $\underline{R}[1]$, that is, $\underline{SR}[1]$ is the class of all simple W-algebras.

3.A. Given a class of algebraic structures $\underline{K}$, and algebra $\underset{\sim}{A}$ of the same type, we say that $\underset{\sim}{A}$ is Boolean product of members of $\underset{\sim}{K}$, when there exists a family $(\underset{\sim}{A}_x / x \in X)$ of $\underline{K}$ such that :

(3.1) X can be endowed with a Boolean Space topology

(3.2) $\underset{\sim}{A}$ is a subdirect product of $(\underset{\sim}{A}_x / x \in X)$

(3.3) The equalizers are clopen subsets of X. That is if $\pi_x : \underset{\sim}{A} \longrightarrow \underset{\sim}{A}_x$ is the canonical projection, then for any $a, b \in A$ the set $[a = b] = \{x \in X \,/\, \pi_x(a) = \pi_x(b)\}$ is clopen subset of X.

(3.4) For any $a, b \in A$ and N clopen subset of X we define: $a_{/N} \cup b_{/N^c}$ the element of $\otimes(A_x / x \in X)$ given by:

$$\pi_x(a_{/N} \cup b_{/N^c}) = \begin{cases} \pi_x(a) & \text{if } x \in N \\ \pi_x(b) & \text{if } x \notin N \end{cases}, \text{ then } a_{/N} \cup b_{/N^c} \in A$$

To represent that $\underset{\sim}{A}$ is Boolean product of the family $(\underset{\sim}{A}_x / x \in X)$ we will write $\underset{\sim}{A} \subseteq_{bp} \otimes(\underset{\sim}{A}_x / x \in X)$. Given a class $\underline{K}$ we will denote by $\Gamma^a(\underline{K})$ the class of all isomorphic images of Boolean products of members of $\underline{K}$.

3.B. The main result of this section gives the relation between $\underline{\underline{Wa}}$ and $\Gamma^a(\underline{\underline{SR}}[1])$.

THEOREM 6. $\underline{\underline{Wa}} = \Gamma^a(\underline{\underline{SR}}[1])$.

PROOF. $\underline{\underline{Wa}} \subseteq \Gamma^a(\underline{\underline{SR}}[1])$.

Let $\underset{\sim}{A} \in \underline{\underline{Wa}}$, we considere $X = \mathrm{Sp}\, \underset{\sim}{A}$, with the spectral topology, which is Boolean space. By (1.20) $\bigcap X = \{u\}$, hence $\bigcap(\theta_x / x \in X)$ is the diagonal congruence relation. This implies that $\underset{\sim}{A}$ is isomorphic to a subdirect product of the family $(\underset{\sim}{A}/\theta_x / x \in X)$ (to short we will write $\underset{\sim}{A}_x = \underset{\sim}{A}/\theta_x$, and $[a]_x$ the class of a modul θ_x). We suppose that the isomorphism is the following:

$\partial : A \longrightarrow \otimes(A_x \,/\, x \in X) : a \longmapsto \partial(a) = ([a]_x)_{x \in X}$.

We need on to see that it satisfies (3.3) and (3.4).

(3.3) is immediate because for any $a, b \in A$ $[a = b] = S((a \to b) \wedge (b \to a))$

(3.4) Let $c \in B(\underset{\sim}{A})$, and $a,b \in A$, we considere $d = (c \to a) \wedge (\sim c \to b)$, we will show that $\partial(d) = \partial(a)_{/S(c)} \cup \partial(b)_{/S(c)^c}$.

Let $x \in S(c)$ we need to see that $[d]_x = [a]_x$, that is $(d \to a) \wedge (a \to d) \in x$.

We write to the next of equalities and of inequalities the properties used .

$(a \to d) \wedge (d \to a) = [a \to ((c \to a) \wedge (\sim c \to b))] \wedge [((c \to a) \wedge (\sim c \to b)) \to a] =$

(1.18)(1.14) / $= [(a \to (c \to a)) \wedge (a \to (\sim c \to b))] \wedge [((c \to a) \to a) \vee ((\sim c \to b) \to a)] =$

(1.13)(1.12)(1.11)/ $= u \wedge (a \to (\sim c \to b) \wedge [(c \vee a) \vee ((\sim c \to b) \to a)] \geqslant$

(1.10)(1.9)/ $\geqslant (a \to (\sim b \to c)) \wedge (c \vee a) \geqslant$

(1.13)/ $\geqslant c \wedge (c \vee a) = c$

Since $x \in S(c)$ then $c \in x$. x is a lattice filter then $(a \to d) \wedge (d \to a) \in x$.

Similary,it can to be obtained that for any $x \in S(c)^c = S(\sim c)$, $[d]_x = [b]_x$.

$\Gamma^a(\underline{SR}[1]) \subseteq \underline{Wa}$:

Let $\underset{\sim}{A} \in \Gamma^a(\underline{SR}[1])$ we suppose that the isomorphism which gives a Boolean product representation is:

$\partial: A \hookrightarrow \otimes(A_x / x \in X) : a \mapsto \partial(a) = (a_x)_{x \in X}$,

where $\underset{\sim}{A} \cong \partial(\underset{\sim}{A})$, $\partial(\underset{\sim}{A}) \subseteq_{bp} \otimes(\underset{\sim}{A}_x / x \in X)$ and $\underset{\sim}{A}_x \in \underline{SR}[1]$.

First we observe that $\underset{\sim}{A} \in \underline{W}$, because $\underline{W}$ is a variety and $\underline{SR}[1] \subseteq \underline{W}$. We need to see that any $a \in A$ is archimedean. Fixed $a \in A$, since $\underset{\sim}{A}_x$ is a simple W-algebra for any $x \in X$, if $a_x \neq u_x$, then for any $x \in X$ such that $a_x \neq u_x$ there exists $n(x)$ such that $u_x = a_x^{n(x)} \to 0_x$, that is $\pi_x(u) = \pi_x(\partial(a)^{n(x)} \to 0)$, let $\tilde{X} = \{x \in X / a_x \neq u_x\}$, then we have

$X = (\bigcup_{x \in \tilde{X}} [\partial(a)^{n(x)} \to 0 = u]) \cup [\partial(a) = u]$,

since X is compact , there exist $x_1, \ldots, x_r \in X$ such that

$X = [\partial(a)^{n(x_1)} \to 0 = u] \cup \ldots \cup [\partial(a)^{n(x_r)} \to 0 = u] \cup [\partial(a) = u]$

If $n = \sup\{n(x_1) \ldots n(x_r)\}$,then by (1.13) $\partial(a)^{n(x_i)} \to 0 \leqslant \partial(a)^n \to 0$

hence $X = [\partial(a)^n \to 0 = u] \cup [\partial(a) = u]$ Thus for any $x \in X$ we have $\pi_x((\partial(a)^n \to 0) v(\partial(a))) = \pi_x(\partial(a)^n \to 0) \vee \pi_x(\partial(a)) = u_x$, this implies that $(\partial(a)^n \to 0) \vee \partial(a) = u$, as ∂ is isomorphism, then $(a^n \to 0) \vee a = u$. That is, a is archimedean.

3.C. In this part we will show that the representation of archimedean W-algebras by means Boolean products is a good representation in the sense that every $\underset{\sim}{A} \in \underline{Wa}$ is obtained by Booleans products with stalks in $\underline{SR}[1]$ in unique way.

THEOREM 7. Let $\underset{\sim}{A} \subseteq_{bp} \otimes(\underset{\sim}{A}_x \ / \ x \in X)$, where $\underset{\sim}{A}_x \in \underline{SR}[1]$ for any $x \in X$. Then there exist a homeomorphism $h: X \longrightarrow \mathrm{Sp}\, \underset{\sim}{A}$ such that $\underset{\sim}{A}_x \cong \underset{\sim}{A}/\theta_{h(x)}$.

PROOF. We define $h : X \longrightarrow \mathrm{Sp}\, A : x \longmapsto h(x) = \{a \in A/ \ x \in [a = u]\}$. Since: $x \in [a = u]$ iff $\pi_x(a) = \pi_x(u)$ iff $(a,u) \in \theta_{\mathrm{Ker}\, \pi_x}$, thus $\underset{\sim}{A}_x \in SR[1]$ implies that $h(x) = f_{\theta_{\mathrm{Ker}\, \pi_x}} \in \mathrm{Sp}\, \underset{\sim}{A}$, hence h is well defined.

h is one to one, because if $x,y \in X$ and $x \neq y$, then there exist N clopen subset of X such that $x \in N$ and $y \notin N$, let $a = u_{/N} \cup 0_{/N^c}$, then $x \in [a = u] = N$ and $y \notin [a = u]$, hence $a \in h(x)$ and $a \notin h(y)$.

We suppose that h is not onto, then there exists $q \in \mathrm{Sp}\, \underset{\sim}{A}$, such that for any $x \in X$ there exists $b^x \in h(x) \setminus q$. Thus $X = \bigcup([b^x = u] \ / \ x \in X)$. By compacity $X = [b^{x_1} = u] \cup \ldots \cup [b^{x_r} = u]$ for some $x_1, \ldots, x_r \in X$, since $b^{x_i} \leq b^{x_1} \vee \ldots \vee b^{x_r}$, we have $X = [b^{x_1} \vee \ldots \vee b^{x_r} = u]$, hence $b^{x_1} \vee \ldots \vee b^{x_r} = u \in q$. Since $q \in \mathrm{Sp}\, A$ $b^{x_i} \in q$, for some $i \in \{1,\ldots,r\}$, that contradicts the assumption, and h is onto.

h is homeomorphism; if N is clopen subset of X, then $N = [a = u]$ with $a = u_{/N} \cup 0_{/N^c}$, hence we have :

$q \in h([a = u])$ iff $h^{-1}(q) \in [a = u]$ iff $a \in q$, this implies that $h([a = u]) = S(a)$

$x \in h^{-1}(S(a))$ iff $h(x) \in S(a)$ iff $a \in h(x)$ iff $x \in [a = u]$ hence $h^{-1}(S(a)) =$
$= [a = u]$.

In the other hand, since $h(x) = f_{\theta_{\ker \pi_x}}$ we have $\underset{\sim}{A}_x \cong \underset{\sim}{A}/\theta_x$

4. BOOLEAN PRODUCTS OF CW-ALGEBRAS

To give the main results of this section we have to analyze the Stone filters in W-algebras.

4.A. Let $\underset{\sim}{A} \in \underline{W}$, then $B(\underset{\sim}{A})$ is the univers of a Boolean algebra with the operations of A. We represents by $\mathcal{F}B(\underset{\sim}{A})$ the family of all lattice filters of $B(\underset{\sim}{A})$. Given $f \subseteq A$ we say that f is a Stone filter when f is a lattice filter generated by a member of $\mathcal{F}B(A)$. $\mathcal{F}_S(A)$ will denote de family of all Stone filters. An Stone ultrafilter is a proper maximal element of $\mathcal{F}_S(\underset{\sim}{A})$. $U_S(\underset{\sim}{A})$ denotes the family of all Stone ultrafilters of $\underset{\sim}{A}$.

LEMMA 8. If $\underset{\sim}{A} \in \underline{W}$, then it satisfies:

(4.1) Every Stone filter is an implicative filter, i.e. $\mathcal{F}_S(\underset{\sim}{A}) \subseteq \mathcal{F}_i(\underset{\sim}{A})$

(4.2) $f \in U_S(\underset{\sim}{A})$ if and only if $f \in \mathcal{F}_S(\underset{\sim}{A})$ and $f \cap B(A)$ is an ultrafilter of $B(A)$.

PROOF. (4.1): Let $f \in \mathcal{F}_S(\underset{\sim}{A})$, we suppose $a \in f$ and $a \to b \in f$, then there exist $c_1, c_2 \in B(A) \cap f$ such that $c_1 \leq a$ and $c_2 \leq a \to b$. Then $c = c_1 \wedge c_2 \in f \cap B(A)$ and $c \leq a$ and $c \leq a \to b$, by (1.12) $c \to (a \to b) = u$, by (1.7) $a \to (c \to b) = u$, hence $c \leq a \leq c \to b$, that is $c \to (c \to b) = c \to b = u$ (because c is boolean iff $c^2 \to d = c \to d$, for any $d \in A$), then $c \leq b$ and $b \in f$. This show that $f \in \mathcal{F}_i(\underset{\sim}{A})$.

(4.2) It deduces from the fact that if $f \in U_S(\underset{\sim}{A})$ iff for any $c \in B(A)$, $c \in f$ iff $\sim c \notin f$, and $f \in \mathcal{F}_s(\underset{\sim}{A})$

We denote by $\mathrm{Spm}\underset{\sim}{A}$ the family of all minimal prime implicative filters, in the sense of (1.22). The relation between $U_S(\underset{\sim}{A})$ and $\mathrm{Spm}\underset{\sim}{A}$ is given by the next result.

THEOREM 9. If $\underset{\sim}{A} \in \underline{W}$, then the following conditions are equivalent:

(i) $U_S(\underset{\sim}{A}) \subseteq \mathrm{Sp}\,\underset{\sim}{A}$

(ii) $U_S(\underset{\sim}{A}) = \mathrm{Spm}\underset{\sim}{A}$

PROOF. (ii) $\Rightarrow$ (i) is trivial.

(i) $\Rightarrow$ (ii): Let $f \in U_S(\underset{\sim}{A}) \subsetneq \mathrm{Sp}\,\underset{\sim}{A}$, if $p \in \mathrm{Spm}\underset{\sim}{A}$ is such that $p \subsetneq f$ (it exists by (1.22)), then $(f \setminus p) \cap B(\underset{\sim}{A}) \neq \emptyset$, because if this is not true, for any $c \in f \cap B(\underset{\sim}{A})$, $c \in p$ and $f \subseteq p$. Let $c \in (f \setminus p) \cap B(A)$, since $c \vee \sim c = u \in p$ and $p \in \mathrm{Sp}\ A$, $\sim c \in p \subset f$, that is not posible. Hence $p = f$. This shows that $U_S \subseteq \mathrm{Spm}\underset{\sim}{A}$.

If $q \in \mathrm{SpmA}$, let $f = F_i(q \cap B(\underset{\sim}{A}))$, it is clear that $f \subsetneq q$ and $f \in U_S(\underset{\sim}{A})$ hence $f \in \mathrm{Spm}\underset{\sim}{A}$, that is $f = q$. This shows that $U_S(\underset{\sim}{A}) = \mathrm{Spm}\underset{\sim}{A}$.

4.B. The main result of this section characterizes the algebras which are Boolean products of CW-algebras.

THEOREM 10. $A \in \Gamma^a(\underline{CW})$ if and only if $\underset{\sim}{A} \in \underline{W}$ and $\mathrm{Spm}\underset{\sim}{A} = U_S(\underset{\sim}{A})$.

PROOF. $\Rightarrow$) We suppose that $\underset{\sim}{A} \subsetneq_{bp} \otimes(\underset{\sim}{A}_x /\ x \in X)$, where $\underset{\sim}{A}_x \in \underline{CW}$ for any $x \in X$. Since $\underline{CW} \subsetneq \underline{W}$ and $\underline{W}$ is a Variety,chen $\underset{\sim}{A} \in \underline{W}$. Let $x \in X$, we considere $p_x = f_{\theta_{\ker \pi_x}} = \{a \in A /\ x \in [a = u]\}$. By (1.23) $p_x \in \mathrm{Sp}\,\underset{\sim}{A}$. To show $\mathrm{Spm}\underset{\sim}{A} = U_S(\underset{\sim}{A})$, by Theorem 9, it sufices to see that $P_X = \{p_x /\ x \in X\} = U_S(\underset{\sim}{A})$.

$P_X \subseteq U_S(\underset{\sim}{A})$: Let $x \in X$ and $a \in p_x$, then $x \in [a = u]$. If $c = u_{/[a = u]} \cup 0_{/[a \neq u]}$, then $c \in B(\underset{\sim}{A})$ and $c \leqslant a$ and $[a = u] = [c = u]$, hence $c \in p_x \cap B(\underset{\sim}{A})$. Then $p_x \in \mathbb{F}_S(\underset{\sim}{A})$. Since $p_x \in \mathrm{Sp}\,\underset{\sim}{A}$, $p_x \cap B(\underset{\sim}{A})$ is an ultrafilter of $B(\underset{\sim}{A})$, then $p_x \in U_S(\underset{\sim}{A})$. This shows that $U_S(\underset{\sim}{A}) \supseteq P_X$.

$U_S(A) \subsetneq P_X$: if it is not satisfied, then there exists $f \in U_S(\underset{\sim}{A})$

such that for any $x \in X$, $f \neq p_x$. Then for any $x \in X$, there exists $c^x \in (p_x \setminus f) \cap B(A)$. Thus $X = \cup ([c^x = u] / x \in X)$, since X is Booelan space $X = [c^{x_1} = u] \cup \ldots \cup [c^{x_n} = u]$, for some $x_1, \ldots, x_n \in X$. Since $c^{x_i} \leq c^{x_1} \vee \ldots \vee c^{x_n}$ for any $i \in \{1, \ldots, n\}$, we have $X = [c^{x_1} \vee \ldots \vee c^{x_n} = u]$ hence $c^{x_1} \vee \ldots \vee c^{x_n} = u \in f \cap B(A)$, since $f \cap B(A)$ is ultrafilter of $B(A)$, then there exists $r \in \{1, \ldots, n\}$, such that $c^{x_r} \in f$, that contradicts the assumption. This shows that $P_x \supseteq U_S(A)$.

$\Leftarrow$ Let $A \in W$, such that $U_S(A) = \mathrm{Spm}A$. First we will see that SpmA with the induced topology by the Spectral topology is a Boolean Space. We write $\mathrm{Sm}(a) = S(a) \cap \mathrm{Spm}A$, for any $a \in A$. It is clear that for any $c \in B(A)$, $\mathrm{Sm}(c)$ is a clopen subset of SpmA. Now we will show that for any $a \in A$ there exist $c_a \in B(A)$ such that $\mathrm{Sm}(a) = \mathrm{Sm}(c_a)$. If $\mathrm{Sm}(a) = \emptyset$ then $c_a = 0$. We suppose that $\mathrm{Sm}(a) \neq \emptyset$, then $\bigcap \mathrm{Sm}(a) \in F_S(A)$, and $a \in \bigcap \mathrm{Sm}(a)$, hence there exist $c_a \in \mathrm{Sm}(a)$ such that $c_a \leq a$, then we have :
$x \in \mathrm{Sm}(a)$ implies $c_a \in x$, hence $x \in \mathrm{Sm}(c_a)$. This shows that $\mathrm{Sm}(a) \subseteq \mathrm{Sm}(c_a)$ $x \in \mathrm{Sm}(c_a)$ implies $c_a \in x$, since $c_a \leq x$, then $a \in x$ and $x \in \mathrm{Sm}(a)$. This shows that $\mathrm{Sm}(a) = \mathrm{Sm}(c_a)$. Thus for any $a \in A$, $\mathrm{Sm}(a)$ is clopen subset of Spm(A).

Let $\mathrm{Sp}_\tau B(A)$ be the topological Spectrum of the boolean algebra $B(A)$. WE considere the map $h : \mathrm{Spm}A \longrightarrow \mathrm{Sp}\, B(A) : f \longmapsto f \cap B(A)$, by Lemma 8 this map is one to one and onto, moreover is easy to see that $h(\mathrm{Sm}(c)) = S_{B(A)}(c)$ and $h^{-1}(S_{B(A)}(c)) = \mathrm{Sm}(c)$ for any $c \in B(A)$, hence h is homeomorphism ..Then SpmA is Boolean space with the topology induced by the spectral topology of Sp A..

From (1.23) it is easy to see that $\bigcap \mathrm{Spm}A = \{u\}$, hence A is isomorphic to a subdirect product of $(A/\theta_x / x \in \mathrm{Spm}A)$.
This isomorphism is given by:

$$\partial : A \longrightarrow \otimes(A/\theta_x / x \in \mathrm{Spm}A) : a \longmapsto \partial(a) = ([a]_x)_{x \in \mathrm{Spm}A}.$$

Since $[\partial(a) = \partial(b)]_{\mathrm{Spm}A} = \mathrm{Sm}((a \to b) \wedge (b \to a))$ is clopen subset of SpmA and $\partial(a)_{/Sm(c)} \cup \partial(b)_{/Sm(\sim c)} = \partial((c \to c_a) \wedge (\sim c \to c_b))$, then

$\partial(\underset{\sim}{A}) \subseteq_{bp} \otimes(\underset{\sim}{A}/\theta_x \;/\; x \in X)$, hence $\underset{\sim}{A} \in \Gamma^a(\underline{CW})$

4.C. THEOREM 11. Let $\underset{\sim}{A} \in \underline{W}$ and $\underset{\sim}{A} \subseteq_{bp} (\underset{\sim}{A}_x \;/\; x \in X)$, where $\underset{\sim}{A}_x \in \underline{CW}$, then there exists $h : X \longrightarrow \mathrm{SpmA}$ homeomorphism such that $\underset{\sim}{A}_x \cong \underset{\sim}{A}/\theta_{h(x)}$.

PROOF. Let $h : X \hookrightarrow \mathrm{SpmA} : x \longmapsto h(x) = p_x = \{a \in A \;/\; x \in [a = u]\}$, by the proof of first part of the Theorem 10 h is one to one and onto. By definition $h([a = u]) = Sm(a)$ and $h^{-1}(Sm(a)) = [a = u]$ hence h is homeomorphism. Since $h(x) = f_{\theta_{\ker \pi_x}}$ we have $\underset{\sim}{A}_x \cong \underset{\sim}{A}/\theta_{h(x)}$

Remark: The archimedean W-algebras are specials cases of W-algebras represntables by means of Boolean products of CW-algebras. They are the limit case because the Stone ultafilters are all prime implicative filters.

REFERENCES

[1] Balbes, R and Dwinger,Ph. "Distributive Lattices" University of Missouri Press. 1974

[2] Burris,S and Sankappanavar,H.P."A course in Universal Algebra" Springer-Verlag. New York. 1981

[3] Burris,S and Werner,H "Sheaf construction and their elementary properties" Trans. Amer. Math. Soc. 248(1979) 269-309

[4] Chang,C.C. "Algebraic anlysis of many-values logics" Trans. Amer. Math. Soc. 88(1958) 467-490

[5] Chang,C.C. "A new proof of the completeness of Lukasiewicz axioms" Trans. Amer. Math. Soc. 93(1959) 74-80

[6] Cignoli,R. "The lattice of global sections of sheaves of chains over Boolean spaces" Algebra Universalis 8(1978) 357-373

[7] Font,J.M.; Rodriguez A.J. and Torrens A. "Wajsberg algebras" Stochastica VIII (1984) n 1, 5-31

[8] Rodriguez, A.J. "Un estudio algebraico de los calculos proposicionales de Lukasiewicz" Ph. Doc. Thesis. University of Barcelona.1980

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