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CHARACTERIZATIONS OF EFFICIENT SOLUTIONS UNDER POLYHEDRALITY ASSUMPTIONS

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Summary. We give several characterizations of efficient solutions of subsets of R^p (with respect to compatible preorders) in terms of the lexicographical order, under the assumptions that the set of admissible points is a convex polyhedron or the nonnegative cone corresponding to the preorder is polyhedral. We also characterize the lexicographical minimum of a convex polyhedron by means of the componentwise order and unitary lower triangular matrices.

INTRODUCTION

The usual characterizations of efficient points with respect to preference orders via scalarization by vectors in the polar of the nonnegative cone require topological assumptions; usually, the only possible results consist in conditions which are either necessary or sufficient, but not both, for efficiency [see, e.g., Section 3.4 in Sawaragi, Nakayama and Tanino (1985)]. In order to obtain more general characterizations of efficient points, these type of scalarizations are not enough; however, one can get some related characterizations by means of the lexicographical order. Although, conceptually, this is not so simple as scalarization, both methods share the important property that they provide characterizations of efficiency with respect to partial preorders in terms of total preorders. A general result of this kind, for arbitrary real linear spaces, is Theorem 3 of Gorokhovik (1986); all results we shall give in this paper are consequences of a corollary of this theorem, restated as Theorem 1 below. Some other related results concerning efficiency, expressed in terms of the lexicographical order, can be found in Borwein (1980) and Martínez-Legaz (1988); in the latter paper, as well as in Martínez-Legaz and Singer (1987), the lexicographical order has been also used in connection with duality theory in vector optimization. On the other hand, the applications of the lexicographical order relation in optimization theory (in particular, in vector optimization) have been the motivation for a detailed study of its properties in Martínez-Legaz (1984) and Martínez-Legaz and Singer (1990). The purpose of this paper is to exploit these properties in connection with the above mentioned results of Gorokhovik.

We shall give a characterization of efficient solutions of convex polyhedra in terms of matrix multipliers; the case when the polyhedron is explicitly given by a linear inequality system will also be considered. Since we shall no require the nonnegative cone to be closed, we shall be able to apply our general results to characterize lexicographical minima. A second type of characterizations of efficient points will be obtained for the case when the nonnegative cone is polyhedral and the set of admissible points is cone convex; again, we shall distinguish the situation when an (homogeneous) linear inequality system describing the nonnegative cone is known.

DEFINITIONS AND NOTATION

Let $\preceq$ be a preorder on R^p which is compatible with the vector space structure, i.e., a binary relation on R^p which is reflexive and transitive and such that

$$y_1 \preceq y_1', y_2 \preceq y_2' \Rightarrow y_1 + y_2 \preceq y_1' + y_2',$$
$$y \preceq y', \lambda \geq 0 \Rightarrow \lambda y \preceq \lambda y'.$$

The nonnegative cone of $\preceq$ is defined as

$$D = \{y \in R^p \mid 0 \preceq y\};$$

D is indeed a convex cone. In fact, given any convex cone D in R^p there is exactly one compatible preorder $\preceq$ on R^p for which the nonnnegative cone is D; namely, $\preceq$ is defined by

$$y_1 \preceq y_2 \Longleftrightarrow y_2 - y_1 \in D.$$

We shall denote by D_S the set of "strictly positive" (in the sense of $\preceq$) vectors, i.e.,

$$D_S = D \backslash (-D);$$

D_S is again a convex cone (excluding its vertex). It will be assumed that $D_S \neq \emptyset$, i.e., that D is not a linear subspace.

Let $Y \subset R^p$. A point $\hat{y} \in Y$ is called an efficient (minimal) element of Y with respect to the preorder $\preceq$ if there does not exist an element $y \in Y$ such that $y \preceq \hat{y}$ and $\hat{y} \not\preceq y$ or, equivalently, such that $\hat{y} \in y + D_S$. The set of all efficient elements will be denoted by $\mathcal{E}(Y, D)$. Note that, in the case we have excluded (when D is a subspace), $\mathcal{E}(Y, D) = Y$, i.e., any element of Y is efficient. Y is said to be D-convex if $Y + D$ is a convex set. A function $f : X \to R^p$, X being a convex set in R^n, is called D-convex if for any $x_1, x_2 \in X$ and for any $\lambda \in [0, 1]$,

$$(1 - \lambda) f(x_1) + \lambda f(x_2) - f((1 - \lambda) x_1 + \lambda x_2) \in D.$$

According to Proposition 2.1.21 in Sawaragi, Nakayama and Tanino (1985), if f is D-convex then the set $f(X)$ is D-convex.

The elements of R^r will be considered column vectors, and the superscript ^{T} will denote transpose. Given $K \subset R^r$, by K^0 we shall represent the polar of K, defined as

$$K^0 = \{m \in R^p \mid m^T y \geq 0 \text{ for any } y \in K\}.$$

A point $z = (\xi_1, \dots, \xi_r)^T \in R^r$ is said to be lexicographically less than $z' = (\xi'_1, \dots, \xi'_r)^T \in R^r$ (in symbols, $z <_L z'$) if $z \neq z'$ and if for $k = \min\{i \in \{1, \dots, r\} | \xi_i \neq \xi'_i\}$ we have $\xi_k < \xi'_k$. We write $z \leq_L z'$ if $z <_L z'$ or $z = z'$. The notations $z' >_L z$ and $z' \geq_L z$ will have the corresponding obvious meanings. By $z \geq z'$ we shall denote the componentwise inequality, i.e., $\xi_i \geq \xi'_i$ $(i = 1, \dots, r)$. We shall say that a matrix W is lexicographically nonnegative, $W \geq_L 0$, if its columns are lexicographically nonnegative.

CHARACTERIZATIONS OF EFFICIENT SOLUTIONS

We shall use the following fundamental result, which is essentially Corollary 1 in Gorokhovik (1986).

Theorem 1: If Y is D-convex, for any $\hat{y} \in Y$ the following statements are equivalent:

1°) $\hat{y} \in \mathcal{E}(Y, D)$.

2°) There exist $r \in \{1, \dots, q\}$, with $q = \mathit{codim}\, D \cap (-D)$, and a $r \times p$ matrix A having rank $A = r$ such that

$$\begin{aligned} Ad >_L 0 \quad &\text{for all} \quad d \in D_S\,, \\ Ay \geq_L A\hat{y} \quad &\text{for all} \quad y \in Y\,. \end{aligned}$$

The implication 2°) $\Rightarrow$ 1°) in the preceding theorem does not require the assumption that Y is D-convex and the condition rank $A = r$ is not needed.

The interest of Theorem 1 lies in that it characterizes the efficient solutions of Y as the minimal solutions with respect to certain preorders, associated to $\preceq$, which have the advantage, from the theoretical viewpoint, of being total. For other results relating efficient solutions of multiobjective optimization problems to associated lexicographical problems, see Section 7 in Borwein (1980) and sections 1 and 2 in Martínez-Legaz (1988).

Theorem 2: If Y is a convex polyhedron, for any $\hat{y} \in Y$ the following statements are equivalent:

1°) $\hat{y} \in \mathcal{E}(Y, D)$.

2°) There exist $r \in \{1, \dots, q\}$, with $q = codim\, D \cap (-D)$, and a $r \times p$ matrix M having rank $M = r$ such that

$$M\,d >_L 0 \quad \text{for all} \quad d \in D_S\,,$$
$$My \geq M\hat{y} \quad \text{for all} \quad y \in Y\,.$$

Proof: Since 2°) is stronger than 2°) of Theorem 1, the implication 2°) $\Rightarrow$ 1°) is obvious.

Assume 1°) and let r and A be as in Theorem 1. There exists a $\ell \times p$ matrix B and a vector $b \in R^{\ell}$, for some ℓ, such that $Y = \{y \in R^p \,|\, By \geq b\}$. Since the inequality $Ay \geq_L A\hat{y}$ is a consequence of $By \geq b$, by Corollary 4.1 in Martínez-Legaz (1984) there exist $k \in \{1, \dots r\}$ and a $k \times \ell$ matrix $W \geq_L 0$ such that $WB = A_k$ and $Wb \geq_L A_k\hat{y}$, A_k denoting the matrix obtained by deleting the $r-k$ last rows of A; furthermore, if $k < r$ one has $Wb >_L A_k\hat{y}$. In our case, we must have $Wb = A_k\hat{y}$ and hence $k = r$ and $A_k = A$. Indeed, otherwise we should obtain $A_k\hat{y} <_L \; Wb \leq_L WB\hat{y} = A_k\hat{y}$, which is absurd; here we have used Corollary 2.3 in Martínez-Legaz (1984) according to which when one multiplies a lexicographically nonnegative matrix by a componentwise nonnegative vector the result is lexicographically nonnegative. By Corollary 2.1 in Martínez-Legaz (1984), there exist a unitary lower triangular $r \times r$ matrix (unitary in the sense that the diagonal elements are 1's) L and a termwise nonnegative $r \times \ell$ matrix P such that $W = LP$. Let $\overline{M} = PB$. Since L^{-1} is also unitary and lower triangular and $\overline{M} = PB = L^{-1}WB = L^{-1}A$, for any $d \in D_S$ we have $\overline{M}d = L^{-1}Ad >_L 0$; $\overline{M}y = PBy \geq Pb = L^{-1}Wb = L^{-1}A\hat{y} = \overline{M}\hat{y}$. Suppose now that rank $\overline{M} < r$ and let $\overline{M} = (m_1, \dots, m_r)^T$. Then, there exists $i \in \{1, \dots, r\}$ such that m_i depends linearly on $m_1, \dots, m_{i-1}$ (if $i = 1$, by this we mean that $m_1 = 0$). It is easy to check that the matrix $\tilde{M} = (m_1, \dots, m_{i-1}, m_i, \dots, m_r)^T$, obtained from $\overline{M}$ by deleting its i-th row, also satisfies the relations $\tilde{M}\,d <_L 0$ for all $d \in D_S$ and $\tilde{M}y \geq \tilde{M}\hat{y}$ for all $y \in Y$. By succesively eleminating all linearly dependent rows in the same way, we finally arrive at a matrix M satisfying all conditions in 2°).

The preceding theorem can be regarded as a characterization of efficient solutions by simultaneous scalarization. Indeed, if $M = (m_1, \ldots, m_r)^T$, the condition $M_y \geq M\hat{y}$ for all $y \in Y$ means that

$$m_i^T \hat{y} = \min_{y \in Y} m_i^T y \qquad (i = 1, \ldots, r).$$

On the other hand, the condition $M\, d >_L 0$ for all $d \in D_S$ is slightly more complicated since it involves the lexicographical order; it can be expressed by saying that $Md \neq 0$ for all $d \in D_S$ and

$$m_i \in (D \cap L_i)^0 \qquad (i = 1, \ldots, r),$$

with $L_i = \{y \in R^p \mid m_j^T y \geq 0 \quad (j = 1, \ldots, i-1)\}$. The interest of Theorem 2 lies in that it requires no assumption on the convex cone D, since, under the additional hypothesis that D is pointed and closed, according to Theorem 3.1.7 in Sawaragi, Nakayama and Tanino (1985) any efficient solution is properly efficient in the sense of Benson (1979) and therefore one has the strongest result that the efficient solutions are characterized via scalarization by vectors in the strict polar cone of D [see theorems 3.4.1 and 3.4.2 in Sawaragi, Nakayama and Tanino (1985)].

Another version of Theorem 2, expressed in terms of an inequality system describing Y, is given in the next theorem:

Theorem 3: If $Y = \{y \in R^p \mid By \geq b\}$, with B being a $\ell \times p$ matrix and $b \in R^\ell$, for any $\hat{y} \in Y$ the following statements are equivalent:

1°) $\hat{y} \in \mathcal{E}(Y, D)$.

2°) There exist $r \in \{1, \ldots, q\}$, with $q = \mathit{codim}\, D \cap (-D)$, and a termwise nonnegative $r \times \ell$ matrix P having rank $P = r$ such that

$$P\,B\,d >_L 0 \qquad \text{for all} \quad d \in D_S,$$
$$P\,B\,\hat{y} = P\,b.$$

Proof: If 2°) holds, define $M = P\,B$. Then, clearly $Md >_L 0$ for all $d \in D_S$ and, for any $y \in Y$, $My = P\,B\,y \geq P\,b = PB\hat{y} = M\hat{y}$; moreover, by deleting rows of M as in the proof of Theorem 2, if necessary, we can assume that rank $M = r$. Hence, by Theorem 2 we have 1°).

Conversely, assuming 1°) we can take A, W, L, P and $\overline{M}$ as in the proof of Theorem 2. Thus, for any $d \in D_S$ we have $PBd = \overline{M}d >_L 0$. Moreover,

$$PB\hat{y} = \overline{M}\hat{y} = L^{-1}A\hat{y} = L^{-1}Wb = Pb.$$

Hence, P satisfies the conditions in 2°) with the possible exception of rank $P = r$, but this can also be achieved by the already used "deleting rows" procedure.

Let us now consider the multiobjective optimization problem (P) consisting in minimizing $f(x), x \in X$, with $X \subset R^n$ and $f : X \longrightarrow R^p$, the minimization being in the sense of $\preceq$. Thus, (P) has to be understood as the problem of finding the points $\hat{x} \in X$ for which there is no $x \in X$ such that $f(x) \preceq f(\hat{x})$ and $f(\hat{x}) \not\preceq f(x)$. These points will be called efficient; the set of them is just $f^{-1}\Big(\mathcal{E}\big(f(X), D\big)\Big)$. As a particular case of (P) we shall consider the linear problem (LP) in which X is a convex polyhedron and $f(x) = Cx \quad (x \in X)$ for some $p \times n$ matrix C (however, we shall not assume that D is a polyhedral cone). From Theorem 2, we obtain:

Corollary 4: If X is a convex polyhedron, for any $\hat{x} \in X$ the following statements are equivalent:

1°) $\hat{x}$ is an efficient solution to problem (LP).

2°) There exist $r \in \{1, \ldots, q\}$, with $q = codim\, D \cap (-D)$, and a $r \times p$ matrix M having rank $M = r$ such taht

$$\begin{aligned} Md >_L 0 &\quad \text{for all} \quad d \in D_S\,, \\ MCx \geq MC\hat{x} &\quad \text{for all} \quad x \in X\,. \end{aligned}$$

Proof: It suffices to observe that, following Proposition 2.1.15 in Sawaragi, Nakayama and Tanino (1985), the set $f(X)$ is a convex polyhedron.

In the particular case when $\preceq$ coincides with the lexicographical order, theorems 2 and 3 can be restated in the following way:

Theorem 5: If Y is a convex polyhedron, for any $\hat{y} \in Y$ the following statements are equivalent:

1°) $\hat{y}$ is the lexicographical minimum of Y.

2°) There exists a unitary lower triangular $p \times p$ matrix L such that $Ly \geq L\hat{y}$ for all $y \in Y$.

Proof: It is a consequence of Theorem 2 and the following observations. If $D = \{d \in R^p \mid d \geq_L 0\}$, then $D \cap (-D) = \{0\}$ and hence q of Theorem 2 coincides with p. On the other hand, the condition $Md >_L 0$ for all $d \in D_S = \{d \in R^p \mid d >_L 0\}$ implies, in particular, that $Md \neq 0$ for all $d \neq 0$ and therefore r, the number of rows of M, must be not less than p. Thus, $r = p$. Finally, according to Theorem 2.2 (equivalence $1^\circ \Leftrightarrow 4^\circ$) of Martínez-Legaz and Singer (1990), $Md >_L 0$ holds for any $d >_L 0$ if and only if M is lower triangular and its diagonal elements are positive; by dividing each row of M by the corresponding diagonal element one obtains L as in 2°).

Theorem 6: If $Y = \{y \in R^p \mid By \geq b\}$, with B being a $\ell \times p$ matrix and $b \in R^\ell$, for any $\hat{y} \in Y$ the following statements are equivalent:

1°) $\hat{y}$ is the lexicographical minimum of Y.

2°) There exists a termwise nonnegative $p \times \ell$ matrix P such that PB is unitary and lower triangular and $PB\hat{y} = Pb$.

Proof: See the statement of Theorem 3 and the proof of Theorem 5.

Similarly, as a particular case of Corollary 4 we obtain the following result for the linear lexicographic optimization problem:

Corollary 7: If X is a convex polyhedron, for any $\hat{x} \in X$ the following statements are equivalent:

1°) $\hat{x}$ lexicographically minimizes Cx over X.

2°) There exists a unitary lower triangular $p \times p$ matrix L such that

$$LCx \geq LC\hat{x} \quad \text{for all} \quad x \in X .$$

For other results concerning the linear lexicographic optimization problem, see Isermann H (1982); the nonlinear case has been studied recently by M. Luptáčik and F. Turnovec (1990).

In Theorem 2, we have assumed polyhedrality of Y but no condition has been imposed on the convex cone D. Our next theorem deals with the inverse situation.

Theorem 8: If D is a polyhedral convex cone and Y is D-convex, for any $\hat{y} \in Y$ the following statements are equivalent:

1°) $\hat{y} \in \mathcal{E}(Y, D)$.

2°) There exist $r \in \{1, \ldots, q\}$, with $q = \mathit{codim}\, D \cap (-D)$, and a $r \times p$ matrix M having rank $M = r$ such that

$$Md \geq 0 \quad \text{for all} \quad d \in D \quad \text{and} \quad Md \neq 0 \quad \text{if} \quad d \in D_S ,$$

$$My \geq_L M\hat{y} \quad \text{for all} \quad y \in Y .$$

Proof: The implication 2°) $\Rightarrow$ 1°) follows from that of Theorem 1.

Let us suppose that 1°) holds and let r and A be as in Theorem 1. The polyhedral convex cone D can be represented as $D = \{d \in R^p \mid Bd \geq 0\}$ for some $\ell \times p$ matrix B. Since the lexicographical inequality $Ad >_L 0$ is a consequence of $Bd \geq 0$, by the generalized Farkas Theorem (Proposition 4) of Martínez-Legaz (1984) there exists a $r \times \ell$ nonnegative matrix $W \geq_L 0$ such that $A = WB$. Following Corollary 2.1 in Martínez-Legaz (1984), $W = LP$ for some unitary lower triangular $r \times r$ matrix L and some termwise nonnegative $r \times \ell$ matrix P. Let $M = PB$. Since $M = L^{-1}A$ and L^{-1} is also a unitary lower triangular matrix, for any $y \in Y$ we have $My = L^{-1}Ay \geq_L L^{-1}A\hat{y}$, the last lexicographical inequality following from Corollary 2.3 in Martínez-Legaz (1984). On the other hand, any $d \in D$ satisfies $Md = PBd \geq 0$ and, if $d \in D_S$, $Md = L^{-1}Ad \neq 0$.

The preceding theorem gives a lexicographic characterization of efficient solutions with the lexicographic multiplier matrix having rows in the polar cone D^0.

For the case when the polyhedral cone D is given explicitly by an homogeneous linear inequality system, we have:

Theorem 9: If $D = \{d \in R^p \mid Bd \geq 0\}$ for some $\ell \times p$ matrix B and Y is D-convex, for any $\hat{y} \in Y$ the following statements are equivalent:

1°) $\hat{y} \in \mathcal{E}(Y, D)$.

2°) There exist $r \in \{1, \ldots, q\}$, with $q = \mathit{codim}\, D \cap (-D)$, termwise nonnegative matrices P, Q of sizes $r \times \ell$ and $r \times r$, respectively, with rank $P = r$ and a $\ell \times r$ matrix S such that

$$B = (SP - Q)B,$$

$$PBy \geq_L PB\hat{y} \quad \text{for all} \quad y \in Y.$$

Proof: If 2°) holds, define $M = PB$. For all $d \in D$, we have $Md = PBd \geq 0$. Moreover, if $d \in D$ and $Md = 0$ then $Bd = SPBd - QBd = SMd - QBd = -QBd \leq 0$ and hence $d \notin D_S$. Thus, 2°) of Theorem 5 holds and hence we have 1°).

Conversely, assuming 1°) we can take W, L, P and M as in the proof of Theorem 8. If $d \in R^p$ is such that $Bd \geq 0$ and $PBd = 0$ (or $Md = 0$), by 2°) of Theorem 8 we have $d \notin D_S$, i.e., $Bd \leq 0$. Therefore, using Farkas Theorem we obtain the existence of S and Q for which $B = (SP - Q)B$. On the other hand, the relation $PBy \geq_L PB\hat{y}$ is immediate in view of 2°) of Theorem 5 and the equality $M = PB$. Finally, the latter equality and the fact that rank $M = r$ imply that also rank $P = r$.

From Theorem 8 we also obtain:

Corollary 10: If D is a polyhedral convex cone, X is convex and f is D-convex, for any $\hat{x} \in X$ the following statements are equivalent:

1°) $\hat{x}$ is an efficient solution to problem (P).

2°) There exist $r \in \{1, \dots, q\}$, with $q = \mathit{codim}\, D \cap (-D)$, and a $r \times p$ matrix M having rank $M = r$ such that

$$Md \geq 0 \quad \text{for all} \quad d \in D \quad \text{and} \quad Md \neq 0 \quad \text{if} \quad d \in D_S,$$
$$Mf(x) \geq_L Mf(\hat{x}) \quad \text{for all} \quad x \in X.$$

In the preceding corollary, the assumptions on X and f can be replaced by the weaker condition that the set $f(X)$ is convex.

An immediate consequence of Theorem 9 is:

Corollary 11: If $D = \{d \in R^p \mid Bd \geq 0\}$ for some $\ell \times p$ matrix B, X is convex and f is D-convex, for any $\hat{x} \in X$ the following statements are equivalent:

1°) $\hat{x}$ is an efficient solution to problem (P).

2°) There exist $r \in \{1, \dots, q\}$, with $q = \mathit{codim}\, D \cap (-D)$, termwise nonnegative matrices P, Q of sizes $r \times \ell$ and $r \times r$, respectively, with rank $P = r$ and a $\ell \times r$ matrix S such that

$$B = (SP - Q)B,$$
$$PBf(x) \geq_L PBf(\hat{x}) \quad \text{for all} \quad x \in X.$$

We observe that, in the particular case when $D = R^p_+$ (the nonnegative orthant), by taking B as the identity matrix the statements of 2°) of Theorem 9 and Corollary 8 can be simplified by omitting the existence of S and Q such that $B = (SP - Q)B$. Indeed, in this case this equality is satisfied for any nonnegative matrix P if one takes S as the identity matrix and $Q = P$. In this way, one arrives, essentially, at Proposition 1.2 and Corollary 1.3 of Martínez-Legaz (1988), characterizing Pareto optimal solutions.

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