# UNIVERSITAT DE BARCELONA

## PRYM VARIETIES OF BI-ELLIPTIC CURVES

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## 0.Introduction

Let $\pi : \tilde{C} \longrightarrow C$ be an unramified double cover of a smooth curve of genus $g$. One defines the associated Prym variety as the abelian variety of dimension $g-1$

$$P(\tilde{C},C) = Ker(Nm_{\pi})^0,$$

where $Nm_{\pi} : J(\tilde{C}) \longrightarrow J(C)$ is the induced norm map of Jacobians. The principal polarization on $J(\tilde{C})$ restricts to twice a principal polarization $\Xi$ on $P(\tilde{C},C)$ ([Mu],p.333). In the sequel we shall consider $P(\tilde{C},C)$ always endowed with this canonical principal polarization. We denote by $\mathcal{R}_g$ and $\mathcal{A}_g$ the moduli spaces for the pairs $(\tilde{C},C)$ as above and for principally polarized abelian varieties of dimension $g$ respectively. The morphism:

$$P : \mathcal{R}_g \longrightarrow \mathcal{A}_{g-1}$$

sending $(\tilde{C},C)$ to $P(\tilde{C},C)$ is called the Prym map. Beauville ([Bel]) introduces $\bar{\mathcal{R}}_g \supset \mathcal{R}_g$ parametrizing allowable double covers of stable curves of genus $g$ and he extends P to a proper map

$$\bar{P} : \bar{\mathcal{R}}_g \longrightarrow \mathcal{A}_{g-1}.$$

This map is known to be dominant for $g \leq 6$ and generically injective for $g \geq 7$ ([F-S],[K],[We1],[De1]). On the other hand Donagi associates two allowable double covers to an unramified double cover of a smooth tetragonal curve (i.e.: with a linear series $g^1_4$), the three having the same Prym variety. This construction, called the tetragonal construction, proves that $\bar{P}$ is non-injective for all $g$. Donagi conjectures:

**Tetragonal conjecture**(Donagi,[Do]): If two elements $(\tilde{C},C)$ and $(\tilde{C}',C')$ of $\mathcal{R}_g$ verify $P(\tilde{C},C) \cong P(\tilde{C}',C')$ then $(\tilde{C}',C')$ is obtained from $(\tilde{C},C)$ by succesive applications of the tetragonal construction.

Debarre proves in [De2] that the conjecture is true for the fibre over the Prym variety of a sufficiently general tetragonal curve of genus $g \geq 13$.

We shall say that a smooth curve $C$ is bi-elliptic if it can be represented as a ramified double cover of an elliptic curve. In this paper we study the fibre of the Prym map over a Prym variety associated to a unramified double cover of a generic bi-elliptic curve of genus $g \geq 10$. We shall recall the decomposition of the moduli space for the elements $(\tilde{C}, C) \in \mathcal{R}_g$ with $C$ bi-elliptic ([De3]) and then we shall work separately with generic elements of each component(§2 and §§5,6 and §7 respectively). We obtain a complete description of the fibre of the Prym map, thereby confirming the tetragonal conjecture in almost all the cases (Th.(5.11), (5.19), (6.11) and (6.24)). See (2.12) for further comments on the cases where the results do not agree with the conjecture as stated above. Paragraph 3 is devoted to establish some basic facts about bi-elliptic curves and in §4 we prove a lemma of technical nature.

I am deeply indebted to Gerald E. Welters for his guidance during the preparation of this work. I wish also to stress the influence of the work of O.Debarre in the present paper.

## 1.Notation

Throughout this paper we work over the field of the complex numbers. The constant $g$ will be greater than or equal to 10.

Let $D, D'$ be two divisors on a smooth curve C. We shall use the expression

$$D \equiv D'$$

to say that they are linearly equivalent. We shall denote by $\mathrm{Pic}^d(C)$ the set of isomorphism classes of degree $d$ divisors on $C$. Usually we shall not make difference between a divisor and its equivalence linear class in $\mathrm{Pic}^d(C)$. For two non-negative integers $r, d$ we shall consider the subvarieties of $\mathrm{Pic}^d(C)$:

$$\mathrm{W}^r_d(C) = \{\zeta \in \mathrm{Pic}^d(C)/h^0(\zeta) \geq r+1\}$$

Let $\pi : \tilde{C} \longrightarrow C$ a double cover of a smooth curve either unramified or ramified exactly at the points $\tilde{Q}_1, \ldots, \tilde{Q}_k \in \tilde{C}$. Let $\Delta$ be the discriminant divisor, that is to say

$$\Delta = \sum_{i=1}^{k} \pi(\tilde{Q}_i).$$

Once C is given, the morphism $\pi$ and the curve $\tilde{C}$ are determined by $\Delta$ and a unique element $\xi \in \mathrm{Pic}(C)$ satisfying $2\xi \equiv \Delta$. In fact $\tilde{C} \cong \mathrm{Spec}_C(\mathcal{O}_C \oplus \mathcal{O}_C(-\xi))$ where the $\mathcal{O}_C$-algebra structure is determined by the map $\mathcal{O}_C(-\xi) \otimes \mathcal{O}_C(-\xi) \cong \mathcal{O}_C(-\Delta) \longrightarrow \mathcal{O}_C$ given by multiplication with an equation for $\Delta$. The class $\xi$ verifies $\pi^*(\xi) \equiv \sum_{i=1}^{k} \tilde{Q}_i$. We will refer to $\xi$ and $\Delta$ as the class and the discriminant divisor respectively attached to the cover.

If $A$ is an abelian variety and $n$ is a positive integer, the group of the elements $x \in A$ such that $nx = 0$ will be written by ${}_nA$.

Let $(\tilde{C}, C) \in \mathcal{R}_g$ and P its associated Prym variety (cf Introduction). There is a natural model $(P^*, \Xi^*)$ of $(P, \Xi)$ in $\mathrm{Pic}^{2g-2}(\tilde{C})$ described as follows ([Mu]):

$$P^* = \{\tilde{\zeta} \in \mathrm{Pic}^{2g-2}(\tilde{C}) / Nm_\pi(\tilde{\zeta}) \equiv K_C, \quad h^0(\tilde{\zeta}) \quad even\}$$

$$\Xi^* = \{\tilde{\zeta} \in P^* / h^0(\tilde{\zeta}) \geq 2\}.$$

In these terms, the singular locus of $\Xi$ is described (loc. cit.) as:

$$\mathrm{Sing}\Xi^* \quad = \quad Sing^{\pi}_{st}\Xi^* \quad \cup \quad \mathrm{Sing}^{\pi}_{ex}\Xi^*$$

where

$$\mathrm{Sing}^{\pi}_{st}\Xi^* = \{\tilde{\zeta} \in P^* / h^0(\tilde{\zeta}) \geq 4\}$$

$$\text{and } \mathrm{Sing}^{\pi}_{ex}\Xi^* = \{\tilde{\zeta} \in P^* / \tilde{\zeta} = \pi^*(\zeta) + \tilde{\zeta}_0, \quad h^0(\tilde{\zeta}_0) \geq 1, \quad h^0(\zeta) \geq 2\}.$$

The singularities of the first kind are called stable and the singularities of the second kind are called exceptional. This definition depends on $\pi$.

## 2.Summary of known results

The following facts mostly come from [De 3].

Let $\mathcal{B}_g$ be the moduli space for the bi-elliptic curves of genus g and let $\mathcal{R}_{B,g}$ be the moduli space for unramified double covers of bi-elliptic curves, obtained by the pull-back diagram:

$$\begin{array}{ccc} \mathcal{R}_{B,g} & \longrightarrow & \mathcal{R}_g \\ \downarrow & & \downarrow \\ \mathcal{B}_g & \longrightarrow & \mathcal{M}_g. \end{array}$$

Let us fix an element $(\tilde{C}, C) \in \mathcal{R}_{B,g}$ and let $\varepsilon : C \longrightarrow E$ be a morphism of degree two on a smooth elliptic curve $E$ ($\varepsilon$ is unique up to automorphisms of $E$ if $g \geq 6$). The Galois group of $\tilde{C}$ over $E$ may be identified with either $\frac{Z}{2Z}$ or $\frac{Z}{2Z} \times \frac{Z}{2Z}$. We shall denote by $\mathcal{R}'_{B,g}$ the subset of the elements with Galois group $\frac{Z}{2Z}$.

(2.1).- If $\mathrm{Gal}_E(\tilde{C}) \cong \frac{Z}{2Z} \times \frac{Z}{2Z}$, we write for the elements of the group: $Id, \iota, \iota_1, \iota_2$, where $\iota$ is the involution which interchanges the sheets of the double cover $\pi$. Let $C_1 = \tilde{C}/(\iota_1), C_2 = \tilde{C}/(\iota_2)$ be the quotient curves.

One has a commutative diagram:

(2.2)

$$\begin{array}{ccccc} & & \tilde{C} & & \\ & {}^{\pi}\swarrow & \downarrow{\scriptstyle \pi_1} & \searrow^{\pi_2} & \\ C & & C_1 & & C_2 \\ & {}_{\varepsilon}\searrow & \downarrow{\scriptstyle \varepsilon_1} & \swarrow_{\varepsilon_2} & \\ & & E & & \end{array}$$

where $\pi_1, \pi_2, \varepsilon_1$ and $\varepsilon_2$ are the obvious morphisms. We shall always assume that $g(C_1) \leq g(C_2)$. It is easy to check the equality:

$$g(C_1) + g(C_2) = g + 1;$$

so if $g(C_1) = t+1$, we obtain $g(C_2) = g-t$ and $t+1 \leq g-t$ implies that $t \in \{0, \ldots, [\frac{g-1}{2}]\}$. Let $\mathcal{R}_{B,g,t}$ be the subset of $\mathcal{R}_{B,g}$ consisting of the elements $(\tilde{C}, C)$ with $\mathrm{Gal}_E(\tilde{C}) \cong \frac{Z}{2Z} \times \frac{Z}{2Z}$ and $g(C_1) = t+1, \quad g(C_2) = g-t$.

One finds that $\mathcal{R}'_{B,g}, \mathcal{R}_{B,g,t} \ldots, \mathcal{R}_{B,g,[\frac{g-1}{2}]}$ are the irreducible components of $\mathcal{R}_{B,g}$ and that each one has dimension $2g-2$.

(2.3).- Let $(\tilde{C}, C) \in \mathcal{R}_{B,g,t}$. In relation to the diagram (2.2) we fix the following notation:

i) $\tau, \tau_1$ and $\tau_2$ are the involutions of $C, C_1$ and $C_2$ associated to $\varepsilon, \varepsilon_1$ and $\varepsilon_2$ respectively.

ii) $\bar{P}_1, \ldots, \bar{P}_{2g-2} \in E$ are the discriminant points of $\varepsilon$ and $\bar{\Delta} = \sum_{i=1}^{2g-2} \bar{P}_i$ is the discriminant divisor. We write $P_1, \ldots, P_{2g-2}$ for the corresponding ramification points of $\varepsilon$.

iii) $\bar{\xi} \in \mathrm{Pic}^{g-1}(E)$ is the class associated to $\varepsilon$. Hence $2\bar{\xi} \equiv \bar{\Delta}$.

iv) $\eta \in_2 JC$ is the class associated to $\pi$.

We may assume that $\bar{P}_1, \ldots, \bar{P}_{2t}$ are the discriminant points of $\varepsilon_1$ and that $\bar{P}_{2t+1}, \ldots, \bar{P}_{2g-2}$ are those of $\varepsilon_2$. We shall denote by $\bar{\Delta}_1, \bar{\Delta}_2, \bar{\xi}_1$ and $\bar{\xi}_2$ the discriminant divisors and the classes associated to $\varepsilon_1$ and $\varepsilon_2$ respectively. So:

$$\bar{\Delta} = \bar{\Delta}_1 + \bar{\Delta}_2 \qquad \text{and} \qquad 2\bar{\xi}_1 \equiv \bar{\Delta}_1, \qquad 2\bar{\xi}_2 \equiv \bar{\Delta}_2.$$

(2.4).- It is easy to check the following facts:

i) $\bar{\xi} = \bar{\xi_1} + \bar{\xi_2}$.

ii) $\eta \equiv P_1 + \cdots + P_{2t} - \varepsilon^*(\bar{\xi_1}) \equiv P_{2t+1} + \cdots + P_{2g-2} - \varepsilon^*(\bar{\xi_2})$.

iii) $\tilde{C} \cong C_1 \times_E C_2$.

iv) $\iota_1 \circ \iota_2 = \iota_2 \circ \iota_1 = \iota, \qquad \iota \circ \iota_1 = \iota_1 \circ \iota = \iota_2, \qquad \iota \circ \iota_2 = \iota_2 \circ \iota = \iota_1$.

v) The involutions $\iota_1$ and $\iota_2$ are liftings to $\tilde{C}$ of the involution $\tau$ of $C$. Analogously $\iota$ and $\iota_2$ both lift $\tau_1$ and $\iota$ and $\iota_1$ both lift $\tau_2$.

vi) $\varepsilon_1^*(\bar{\Delta}_2)$ and $\varepsilon_1^*(\bar{\xi_2})$ (resp. $\varepsilon_2^*(\bar{\Delta}_1)$ and $\varepsilon_2^*(\bar{\xi_1})$) are the discriminant divisor and the class associated to $\pi_1$ (resp. $\pi_2$).

(2.5).- We keep the assumption $(\tilde{C}, C) \in \mathcal{R}_{B,g,t}$ and we write $P = P(\tilde{C}, C)$. We have the description:

$$\begin{aligned} \Xi^* = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \quad & \zeta_1 \in W_t^0(C_1), \quad \zeta_2 \in W_{g-t-1}^0(C_2), \\ & Nm_{\varepsilon_1}(\zeta_1) + Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}\} \end{aligned}$$

(2.6).- For $g \geq 7$ define the following subvarieties of $\Xi^*$:

$$\begin{aligned} V = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \quad & \zeta_1 \in W_t^1(C_1), \quad \zeta_2 \in W_{g-t-1}^1(C_2), \\ & Nm_{\varepsilon_1}(\zeta_1) = \bar{\xi_1}, \quad Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi_2}\} \end{aligned}$$

$$\begin{aligned} W_a = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) + \pi^*(\varepsilon^*(\bar{\zeta})) \mid \quad & \zeta_1 \in W_{t-2+a}^0(C_1), \quad \zeta_2 \in W_{g-t-a-3}^0(C_2), \\ & \bar{\zeta} \in \mathrm{Pic}^2(E), \quad Nm_{\varepsilon_1}(\zeta_1) + Nm_{\varepsilon_2}(\zeta_2) + 2\bar{\zeta} = \bar{\xi}\} \end{aligned}$$

where $a \in \{0, 2, -2\}$. Then $\mathrm{Sing}\Xi^* \supseteq V \cup W_{-2} \cup W_0 \cup W_2$ with equality if $\bar{\Delta}$ does not belong to the image of the addition map $\mid \bar{\xi} \mid \times \mid \bar{\xi} \mid \longrightarrow \mid 2\bar{\xi} \mid$ (this happens if $(\tilde{C}, C)$ is general). Otherwise a finite number of new isolated singularities could appear.

(2.7).- The following table contains relevant information to be used in the sequel:

| $t$ | 0 | 1 | 2 | 3 | $\geq 4$ |
|---|---|---|---|---|---|
| $V$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | dim $g-7$ | irred.<br>dim $g-7$ |
| $W_{-2}$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | irred.<br>dim $g-5$ |
| $W_0$ | $\emptyset$ | $\emptyset$ | irred.<br>dim$g-5$ | irred.<br>dim$g-5$ | irred.<br>dim$g-5$ |
| $W_2$ | irred.<br>dim$g-5$ | irred.<br>dim$g-5$ | irred.<br>dim$g-5$ | irred.<br>dim$g-5$ | irred.<br>dim$g-5$ |

As we shall see in (3.4), when $t = 3$ and $(\tilde{C}, C)$ is general V has two components. The singularity corresponding to a element of each one of these varieties is stable for $V$, exceptional for $W_0$ and stable and exceptional for $W_{-2}$ and $W_2$.

By using (2.4.v) the reader can observe that all these varieties are fixed by the reflection with respect to $K_{\tilde{C}}$ (i.e.: fixed by $\iota$).

(2.8).- Consider now the abelian varieties $P_1 = Ker(Nm_{\epsilon_1})$ (if $t \geq 1$) and $P_2 = Ker(Nm_{\epsilon_2})$. We define the morphisms:

$$\varphi : P_1 \times P_2 \longrightarrow P$$

by sending $(\zeta_1, \zeta_2)$ to $\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2)$, if $t \geq 1$, and

$$\psi : P_2 \longrightarrow P$$

by sending $\zeta_2$ to $\pi_2^*(\zeta_2)$ if $t = 0$. Then $\varphi$ and $\psi$ are isogenies and:

$$Ker(\varphi) = \{(\varepsilon_1^*(\bar{\alpha}), \varepsilon_2^*(\bar{\alpha}))/\bar{\alpha} \in_2 JE\}$$

$$Ker(\psi) = \{0, \varepsilon_2^*(\bar{\xi}_1)\}$$

(2.9).- **Remark.** The definitions of $\iota, \tau, \bar{P}_1, \ldots, \bar{P}_{2g-2}, \bar{\Delta}, \xi$ and $\eta$ given in (2.3) make still sense if $(\tilde{C}, C) \in \mathcal{R}'_{B,g}$ and we will use them throughout.

Now we want to apply the tetragonal construction to an element $(\tilde{C}, C) \in \mathcal{R}_{B,g}$. Assuming first that $(\tilde{C}, C) \in \mathcal{R}_{B,g,t}$ and keeping the notation of (2.3), fix a linear series $g^1_2$ on $E$ inducing an involution $\nu$ such that $\nu(\bar{P}_i) \neq \bar{P}_j$ for $1 \leq i \leq 2t < j \leq 2g-2$. Applying the tetragonal construction to $(\tilde{C}, C)$ with respect to $\varepsilon^*(g^1_2)$ one obtains two elements of $\bar{\mathcal{R}}_g$ (cf Introduction) but only one of them lives in $\mathcal{R}_g$. Call this element $(\tilde{C}', C')$. In terms of the data introduced in (2.1), $(\tilde{C}', C')$ can be described by the new set of data:

(2.10)

$$\begin{tikzcd}[column sep=small] & \tilde{C}' \arrow[dl] \arrow[d] \arrow[dr] & \\ C' \arrow[dr] & C_1 \arrow[d, "\nu\cdot\varepsilon_1"] & C_2 \arrow[dl, "\varepsilon_2"] \\ & E & \end{tikzcd}$$

Note that $(\tilde{C}', C') \in \mathcal{R}_{B,g,t}$ and that $(\tilde{C}', C') \cong (\tilde{C}, C)$ if $t = 0$.

(2.11).- In order to describe a certain link between $\mathcal{R}_{B,g,0}$ and $\mathcal{R}'_{B,g}$ we need to say a few words about the "non-smooth cover" attached to elements of $\mathcal{R}_{B,g,0}$ via the tetragonal construction. We write $\mathcal{H}_g$ for the subset of $\bar{\mathcal{R}}_g$ parametrizing the elements $(\tilde{C}, C)$ where $C$ is constructed from a smooth hyperelliptic curve of genus $g-2$ by identifying two pairs of points. Now the non-smooth covers coming from elements of $\mathcal{R}_{B,g,0}$ belong to $\mathcal{H}_g$, and each point of $\mathcal{H}_g$ may be obtained in this way. On the other hand the tetragonal construction attaches two points of $\mathcal{H}_g$ to a cover $(\tilde{C}, C) \in \mathcal{R}'_{B,g}$ with a general $g^1_4$ chosen on $C$, and (again!) the whole $\mathcal{H}_g$ is covered by these elements.

(2.12).- **Remark.** From (2.11) we see that for each element of either $\mathcal{R}_{B,g,0}$ or $\mathcal{R}'_{B,g}$ there exists an element in the other component giving the same Prym variety. This implies that the tetragonal conjecture as stated in the introduction is not true. This suggests the need for extending the tetragonal construction to the case of allowable double covers and for extending the tetragonal conjecture to the proper map $\bar{P}$ in order to keep its chances to hold true.

(2.13).- Although we have no description of $\mathrm{Sing}\Xi^*$ when we are in $\mathcal{R}'_{B,g}$ we deduce from Remark (2.12) and Table (2.7) that $\mathrm{Sing}\Xi^*$ has a unique component of dimension $g-5$ and possibly a finite number of isolated singularities.

(2.14).- Finally we recall two lemmas borrowed from [Mu] and [De2]. First we need a definition. Let $\pi : \tilde{C} \longrightarrow C$ be a double cover of a smooth curve. We shall say that an effective divisor on $\tilde{C}$ is $\pi$-simple if it does not contain inverse images of effective divisors of $C$. Let $\zeta \in \mathrm{Pic}(C)$ be the class attached to $\pi$. With this notation one has:

(2.15).- **Lemma** ([Mu],p.338). If $\mathcal{L}$ is an invertible sheaf on $C$ and $\tilde{D}$ is an effective $\pi$-simple divisor on $\tilde{C}$ there exists an exact sequence:

$$0 \longrightarrow \mathcal{L} \longrightarrow \pi_*(\pi^*(\mathcal{L}) \otimes_{\mathcal{O}_{\tilde{C}}} \mathcal{O}_{\tilde{C}}(\tilde{D})) \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_C} \mathcal{O}_C(Nm_\pi(\tilde{D}) - \zeta) \longrightarrow 0.$$

(2.16).- **Lemma** (Debarre,[De2],p.550). Let $\pi : \tilde{C} \longrightarrow C$ be an allowable double cover of a stable curve $C$, $\tilde{\mathcal{L}}$ an invertible sheaf on $\tilde{C}$ and $D$ a reduced element of $\mid K_C \otimes (Nm_\pi(\tilde{\mathcal{L}}))^{-1} \mid$ with non-singular support. Suppose that $h^0(\tilde{\mathcal{L}} \otimes_{\mathcal{O}_{\tilde{C}}} \mathcal{O}_{\tilde{C}}(\tilde{D})) \geq 1$ for all effective divisor $\tilde{D}$ such that $Nm_\pi(\tilde{D}) = D$. Then $h^0(\tilde{\mathcal{L}}) \geq 1$.

## 3.Some properties of bi-elliptic curves

This section deals with properties of bi-elliptic curves that will be used later on. In a first reading it may be skipped and kept for reference purposes.

Let $\varepsilon : C \longrightarrow E$ be a (2:1) morphism of smooth curves where $E$ is an elliptic curve. We denote by $\bar{\Delta}$ and $\bar{\xi}$ the discriminant divisor and the class determining $\varepsilon$. By Riemann-Hurwitz:

$$\deg\bar{\Delta} = 2g - 2, \qquad \deg\bar{\xi} = g - 1.$$

Let $\tau : C \longrightarrow C$ be the involution which interchanges the points of each fibre.

(3.1).-**Lemma** Let $\bar{A}$, B be effective divisors on $E$ and $C$ respectively. Assume that B is $\varepsilon$-simple (cf (2.14)). Then:

$$\deg(\bar{A}) + \deg(B) < g(C) - 1 \quad \Rightarrow \quad h^0(\varepsilon^*(\bar{A}) + B) = h^0(\bar{A}).$$

PROOF: By applying (2.15) we obtain an exact sequence:

$$0 \longrightarrow \mathcal{O}_E(\bar{A}) \longrightarrow \varepsilon_*(\mathcal{O}_C(\varepsilon^*(\bar{A}) + B)) \longrightarrow \mathcal{O}_E(\bar{A} + Nm_\varepsilon(B) - \bar{\xi}) \longrightarrow 0.$$

Thus:

$$h^0(\varepsilon^*(\bar{A}) + B) \leq h^0(\bar{A}) + h^0(\bar{A} + Nm_\varepsilon(B) - \bar{\xi}) = h^0(\bar{A}).$$

On the other side, one has an injection $\varphi :\mid \bar{A} \mid \longrightarrow \mid \varepsilon^*(\bar{A}) + B \mid$ given by $\varphi(\bar{R}) = \varepsilon^*(\bar{R}) + B$, so $h^0(\varepsilon^*(\bar{A}) + B) = h^0(\bar{A})$. ∎

Note that B is fixed in the linear series $\mid \varepsilon^*(\bar{A}) + B \mid$ and that $\varphi$ is a bijection. In particular, if $B = 0$:

$$\deg(\bar{A}) < g(C) - 1 \quad \Rightarrow \quad h^0(\varepsilon^*(\bar{A})) = h^0(\bar{A})$$

and

$$\mid \varepsilon^*(\bar{A}) \mid = \{\varepsilon^*(\bar{R}) \mid \bar{R} \in \mid \bar{A} \mid\} = \varepsilon^*(\mid \bar{A} \mid).$$

(3.2).- If $g(C) \geq 5$, then $C$ is not trigonal (cf [Te]).

(3.3).- If $g(C) \geq 4$, then $C$ is not hyperelliptic. To see this, take $D \geq 0$ a divisor of degree two on $C$. By (3.1) $h^0(D) = 1$.

(3.4).- Assume that $C$ is general, of genus 4. Then $W_3^1(C)$ has two different points. In fact the canonical model of $C$ is the complete intersection of a cubic and a quadric in $\mathbf{P}^3$. The rulings of the quadric cut on $C$ the linear series $g_3^1$ of this curve. Let $D \geq 0$ be a degree 3 divisor on $C$ such that $h^0(D) = 2$. If $\mid D \mid$ were the unique $g_3^1$ then:

$$D \equiv \tau(D).$$

We write $Q_1, \ldots, Q_6 \in C$ for the ramification points of $\varepsilon$. For each $i \in \{1, \ldots, 6\}$ we find points $x_i, y_i \in C$ such that $D \equiv Q_i + x_i + y_i$, hence:

$$x_i + y_i \equiv \tau(x_i) + \tau(y_i).$$

By (3.3) this is an equality. If $y_i = \tau(x_i)$ then $D \equiv Q_i + \varepsilon^*(\varepsilon(x_i))$ and by (3.1) $h^0(D) = 1$. We deduce that $x_i = \tau(x_i)$, $y_i = \tau(y_i)$. Now, by taking norms, there appear linear equivalences between divisors on $E$ having support on discriminant points. This contradicts the generality of $C$.

(3.5).- Assume that $C$ is general, of genus 3. Then $C$ is not hyperelliptic. Indeed, by using that $C$ can be hyperelliptic in at most one way we can imitate the proof of (3.4).

(3.6).- We consider the following subvarieties of $\mathrm{Pic}^{g(C)-1}(C)$:

$$\begin{aligned}
Z' &= \{\zeta \in \mathrm{Sing}\Theta^* \mid \quad Nm_\varepsilon(\zeta) = \bar{\xi}\} \\
Z'' &= \{\varepsilon^*(\bar{x} + \bar{y}) + \zeta' \mid \quad \bar{x}, \bar{y} \in E, \quad \zeta' \in W^0_{g(C)-5}\} \qquad \text{if} \quad g(C) \geq 5 \\
A &= \{\varepsilon^*(\bar{x}) + \zeta' \mid \quad \bar{x} \in E, \quad \zeta' \in W^0_{g(C)-3}\} \supset Z'' \quad \text{if} \quad g(C) \geq 3.
\end{aligned}$$

**Remarks**: i) If $g(C) \geq 3$, then $A$ is irreducible of dimension $g(C) - 2$.

ii) If $g(C) \geq 6$, then $Z'$ and $Z''$ are irreducible of dimension $g(C)-4$ and $\mathrm{Sing}\Theta^* = Z' \cup Z''$ ([We2],Prop.3.6). If $g(C) = 5$, then the equality holds but $Z'$ is not always irreducible (in fact by [Te] there is a bijection between its components and the bi-elliptic structures on C).

(3.7).- In order to avoid annoying notation for the varieties $V$ and $W_a$ (where $a \in 2, 0, -2$) described in (2.6) we use the definitions of (3.6) applied to $C_1$ and $C_2$. In these terms the descriptions of (2.5) read:

$$\begin{aligned}
V &= \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \quad \zeta_1 \in Z_1', \quad \zeta_2 \in Z_2'\} \\
W_2 &= \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \quad \zeta_1 \in Z_1'', \quad \zeta_2 \in \Theta_2^*, \quad Nm_{\varepsilon_1}(\zeta_1) + Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}\} \\
W_0 &= \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \quad \zeta_1 \in A_1, \quad \zeta_2 \in A_2, \quad Nm_{\varepsilon_1}(\zeta_1) + Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}\} \\
W_{-2} &= \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \quad \zeta_1 \in \Theta_1^*, \quad \zeta_2 \in Z_2'', \quad Nm_{\varepsilon_1}(\zeta_1) + Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}\}.
\end{aligned}$$

(3.8).- **Lemma**([De3],Lemma 5.2.10). Assume $g(C) \geq 6$ and fix $\bar{\lambda} \in \mathrm{Pic}^{g(C)-1}(E)$. Then $\{\zeta \in Z'' \mid Nm_\varepsilon(\zeta) = \bar{\lambda}\}$ is irreducible of dimension $g(C) - 5$.

In particular $Z' \cap Z''$ is irreducible.

The following facts will be used throughout.

(3.9).-**Proposition**. One has the following equalities:

i) If $g(C) \geq 3$ then:

$$\{a \in \mathrm{JC} \quad | \quad a + A \subseteq A\} = \{\varepsilon^*(\bar{\alpha}) \quad | \quad \bar{\alpha} \in \mathrm{Pic}(E)\}.$$

ii) If $g(C) \geq 5$ then:

$$\begin{aligned} &\{a \in \mathrm{JC} \mid \quad a + Z'' \subset A\} = \{a \in \mathrm{JC} \mid \quad a + Z'' \subset \Theta^*\} = \\ =&\{a \in \mathrm{JC} \mid \quad a + Z' \cap Z'' \subset A\} = \{a \in \mathrm{JC} \mid \quad a + Z' \cap Z'' \subset \Theta^*\} = \\ =&\{\varepsilon^*(\bar{x}) - r - s \mid \quad \bar{x} \in E, \quad r, s \in C\}. \end{aligned}$$

iii) If $g(C) \geq 5$ then:

$$\begin{aligned} &\{a \in \mathrm{JC} \mid \quad a + Z'' \subset Z''\} = \{a \in \mathrm{JC} \mid \quad a + Z' \cap Z'' \subset Z''\} = \\ =&\{\varepsilon^*(\bar{\alpha}) \mid \quad \bar{\alpha} \in \mathrm{Pic}^0(E)\}. \end{aligned}$$

PROOF: i). The inclusion of the second member in the first one is clear. Let $a \in JC$ such that $a + A \subset A$. In particular, for all $\bar{x} \in E$ and $D \in C^{(g-3)}$ one has $h^0(a + \varepsilon^*(\bar{x}) + D) > 0$. Then:

$$h^0(a + \varepsilon^*(\bar{x})) > 0.$$

Indeed, otherwise by Riemann-Roch

$$h^0(K_c - \varepsilon^*(\bar{x}) - a) = h^0(\varepsilon^*(\bar{\xi} - \bar{x}) - a) = g(C) - 3.$$

By taking points $x_1, \ldots, x_{g(C)-3}$ such that $x_{i+1}$ is not a base point of the linear series $\mid \varepsilon^*(\bar{\xi} - \bar{x}) - a - x_1 - \cdots - x_i \mid$, one finds an effective divisor $D = \sum_{i=1}^{g(C)-3} x_i$ verifying $h^0(\varepsilon^*(\bar{\xi} - \bar{x}) - a - D) = 0$. So $h^0(a + \varepsilon^*(\bar{x}) + D) = 0$, which is a contradiction.

Now we may write $a \equiv D - \varepsilon^*(\bar{x})$ where $D$ is an effective divisor of degree two verifying

$$h^0(D + \varepsilon^*(\bar{\alpha})) > 0 \text{ for all } \bar{\alpha} \in \mathrm{Pic}^0(E).$$

By applying (2.15) we conclude that $D \in \mathrm{Im}(\varepsilon^*)$, thereby proving i).

ii). All the equalities are an easy consecuence of the following one:

$$\{a \in JC \mid a + Z' \cap Z'' \subset \Theta^*\} = \{\varepsilon^*(\bar{x}) - r - s \mid \bar{x} \in E, \quad r, s \in C\}.$$

This fact was proved by Debarre in [De5]. We give here an sketch of the proof. We only prove the inclusion of the left hand side member in the right hand side member. Write $a \equiv D - \varepsilon^*(\bar{A})$, where $\bar{A} \in \mathrm{Pic}^r(E)$ and $D$ is effective. If we assume that $D$ is $\varepsilon$-simple then $2r \leq g + 1$. In fact it is not necessary to consider the case the case $2r = g + 1$. It suffices to obtain a contradiction if $r \geq 2$.

Suppose that $2r \leq g - 2$. For a generic element $\bar{B} \in \mathrm{Pic}^r(E)$ there exists $D' \geq 0$ such that:

- $D + D'$ is $\varepsilon$-simple.
- $2\bar{B} + Nm_\varepsilon(D') \equiv \bar{\xi}$.

Then $\varepsilon^*(\bar{B}) + D' \in Z' \cap Z''$. By applying (2.15)

$$\begin{aligned} 0 < h^0(a + \varepsilon^*(\bar{B}) + D') &= h^0(D + D' + \varepsilon^*(\bar{B} - \bar{A})) \\ &\leq h^0(\bar{B} - \bar{A}) + h^0(Nm_\varepsilon(D + D') + \bar{B} - \bar{A} - \bar{\xi}) \\ &= h^0(\bar{B} - \bar{A}) + h^0(Nm_\varepsilon(D) - \bar{A} - \bar{B}) \end{aligned}$$

which is a contradiction because $\bar{B}$ is generic. The cases $2r = g - 1, g$ are similar.

Part iii) follows from ii). ∎

## 4.A key lemma

Let $f : \tilde{N} \longrightarrow N$ be a (2:1) morphism of smooth curves with ramification divisor $\sum_{i-1}^{k} \tilde{Q}_i$. We denote by $\sigma$ the involution of $\tilde{N}$ attached to $f$.

We start with a characterization of the line bundles on $\tilde{N}$ invariant by $\sigma$.

Let $\tilde{\mathrm{L}}$ be a line bundle on $\tilde{N}$ with $\sigma^*(\tilde{\mathrm{L}}) \cong \tilde{\mathrm{L}}$. Choose an isomorphism $\varphi$ normalized in such a way that:

$$\sigma^*(\varphi) \circ \varphi = \mathrm{Id}_{\tilde{\mathrm{L}}}.$$

Writing $\tilde{\mathrm{L}}[\tilde{x}]$ for the pointwise fibre of $\tilde{\mathrm{L}}$ over $\tilde{x} \in \tilde{N}$, one obtains isomorphisms:

$$\varphi[\tilde{Q}_i] : \tilde{\mathrm{L}}[\tilde{Q}_i] \longrightarrow \sigma^*(\tilde{\mathrm{L}})[\tilde{Q}_i] = \tilde{\mathrm{L}}[\sigma(\tilde{Q}_i)] = \tilde{\mathrm{L}}[\tilde{Q}_i] \qquad i \in \{1, \ldots, k\}$$

given by multiplication with constants $\lambda_i$ with $\lambda_i^2 = 1$. We attach to $\tilde{\mathrm{L}}$ a vector $v(\tilde{\mathrm{L}}) = (\lambda_1, \ldots, \lambda_k) \in (\mu_2)^k$ which depends on the choice of $\varphi$. The ambiguity disappears when we pass to the quotient modulo $\mu_2$ by the natural action. Then we have an homomorphism of groups:

$$v : \mathrm{Ker}(\sigma^* - 1) \longrightarrow \frac{(\mu_2)^k}{\mu_2}.$$

We use the notation $v(\tilde{D})$ and $v(\tilde{\mathcal{L}})$ for $\tilde{D}$ a divisor and $\tilde{\mathcal{L}}$ an invertible sheaf on $\tilde{N}$.

(4.1).- **Proposition**. There exists a line bundle L on $N$ such that $f^*(\mathrm{L}) \cong \tilde{\mathrm{L}}$ iff $v(\tilde{\mathrm{L}}) = \overline{(1, \ldots, 1)}$.

PROOF: It suffices to use [G],Th.1,p.17. ∎

(4.2).- **Proposition**. Let $\tilde{\mathcal{L}}$ be an invertible sheaf on $\tilde{N}$ such that $\sigma^*(\tilde{\mathcal{L}}) \cong \tilde{\mathcal{L}}$. Then there exists a divisor $\tilde{D}$ on $\tilde{N}$ with $0 \leq \tilde{D} \leq \sum_{i=1}^k \tilde{Q}_i$ and an invertible sheaf $\mathcal{L}$ on $N$ such that

$$f^*(\mathcal{L}) \cong \tilde{\mathcal{L}} \otimes \mathcal{O}_{\tilde{N}}(-\tilde{D})$$

PROOF: By using the exact sequence:

$$0 \longrightarrow \mathcal{O}_{\tilde{C}}(-\tilde{Q}_i) \longrightarrow \mathcal{O}_{\tilde{C}} \longrightarrow \mathcal{O}_{\tilde{Q}_i} \longrightarrow 0$$

and by observing that $\mathcal{O}_{\tilde{C}}(-\tilde{Q}_i) \notin Im(f^*)$ (hence by (4.1) $v(-\tilde{Q}_i) \neq \overline{(1, \ldots, 1)}$) one has $v(-\tilde{Q}_i) = \overline{(1, \ldots, -1, \ldots, 1)}$. Then, by tensoring $\tilde{\mathcal{L}}$ with suitable sheaves $\mathcal{O}_{\tilde{C}}(-\tilde{Q}_i)$ we can make all the coordinates of the corresponding vector be equal. ∎

Let $(\tilde{C}, C) \in \mathcal{R}_{B,g}$. We keep the notations of §2. In particular $\eta \in_2 JC$ is the class determining $\pi : \tilde{C} \longrightarrow C$.

(4.3).- **Corollary**. One has $(\tilde{C}, C) \in \mathcal{R}'_{B,g}$ iff $\tau^*(\eta) \neq \eta$.

PROOF: By (2.4.ii), $\tau^*(\eta) = \eta$ when $(\tilde{C}, C) \notin \mathcal{R}'_{B,g}$. Conversely suppose $\tau^*(\eta) = \eta$. Applying (4.2) we may write:

$$\eta = D - \varepsilon^*(\bar{A}) \qquad \text{with} \qquad 0 \leq D \leq \sum_{i=1}^{2g-2} P_i.$$

Let $C_1$ (resp. $C_2$) be the double cover on $E$ given by the class of $\bar{A}$ (resp.$\xi - \bar{A}$) and the discriminant divisor $Nm_\varepsilon(D)$ (resp. $\bar{\Delta} - Nm_\varepsilon(D)$). Observe that:

$$\varepsilon^*(Nm_\varepsilon(\eta)) = 2\eta = 0.$$

So due to the injectivity of $\varepsilon^*$:

$$Nm_\varepsilon(\eta) = 0 \qquad \text{and} \qquad 2\bar{A} \equiv Nm_\varepsilon(D).$$

Then $\tilde{C} \cong C_1 \times_E C_2$ and $C \cong \tilde{C}/(\iota_1 \circ \iota_2)$, $\iota_1$ and $\iota_2$ being the involutions of $\tilde{C}$ attached to the projections on $C_1$ and $C_2$ respectively. Hence $(\tilde{C}, C) \in \mathcal{R}_{B,g,t}$ for some t. ∎

(4.4).- **Lemma**. Assume $t > 0$. We consider the commutative diagram:

$$\begin{array}{ccc} JC_1 & \xrightarrow{\pi_1^*} & J\tilde{C} \\ {\scriptstyle \varepsilon_1^*}\uparrow & & \uparrow{\scriptstyle \pi_2^*} \\ JE & \xrightarrow[\varepsilon_2^*]{} & JC_2 \end{array}$$

Then:

$$Im(\pi_1^*) \cap Im(\pi_2^*) = \{\pi^*(\varepsilon^*(\bar{\alpha})) \mid \bar{\alpha} \in \mathrm{Pic}^0(E)\}$$

PROOF: Fix $\tilde{\beta} \in Im(\pi_1^*) \cap Im(\pi_2^*)$ and $\beta_1 \in \mathrm{JC}_1$, $\beta_2 \in \mathrm{JC}_2$ such that $\tilde{\beta} = \pi_1^*(\beta_1) = \pi_2^*(\beta_2)$. By the assumption $t > 0$, the morphisms $\pi_1$ and $\pi_2$ are ramified, hence $\pi_1^*$ and $\pi_2^*$ are injective. Thus we need to find an element $\bar{\beta} \in \mathrm{Pic}^0(E)$ such that $\beta_1 = \pi_1^*(\bar{\beta})$ and $\beta_2 = \pi_2^*(\bar{\beta})$. To see this observe that $\iota_2^*(\tilde{\beta}) = \tilde{\beta}$. On the other side from (2.4.v) one has:

$$\iota_2^*(\tilde{\beta}) = \iota_2^*(\pi_1^*(\beta_1)) = \pi_1^*(\tau_1^*(\beta_1))$$

therefore

$$\pi_1^*(\beta_1) = \tilde{\beta} = \pi_1^*(\tau_1^*(\beta_1))$$

and

$$\beta_1 = \tau_1^*(\beta_1).$$

In a similar way we get $\beta_2 = \tau_2^*(\beta_2)$. By applying (4.2), there exist divisors $D_1$ on $C_1$, $D_2$ on $C_2$ and classes $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathrm{Pic}^0(E)$ such that:

(4.5) $$\beta_1 \equiv \varepsilon_1^*(\bar{\alpha}_1) - D_1, \quad \beta_2 \equiv \varepsilon_2^*(\bar{\alpha}_2) - D_2$$

where $0 \leq D_i \leq$ ramification divisor of $\varepsilon_i$, $\quad i = 1, 2$.

Hence:

$$\pi^*(\varepsilon^*(\bar{\alpha}_1 - \bar{\alpha}_2)) \equiv \pi_1^*(D_1) - \pi_2^*(D_2)$$

Let $R_1$ and $R_2$ be the effective divisors on $C$ such that

$$\pi^*(R_1) = \pi_1^*(D_1), \quad \pi^*(R_2) = \pi_2^*(D_2)$$

thus

$$0 \leq R_1 \leq \sum_{i=1}^{2t} P_i, \quad 0 \leq R_2 \leq \sum_{i=2t+1}^{2g-2} P_i.$$

From

$$\pi^*(\varepsilon^*(\bar{\alpha}_1 - \bar{\alpha}_2)) \equiv \pi^*(R_1 - R_2),$$

two possibilities appear:

$$\begin{aligned} \text{either} \quad \text{i)} \quad & \varepsilon^*(\bar{\alpha}_1 - \bar{\alpha}_2) \equiv R_1 - R_2 \\ \text{or} \quad \text{ii)} \quad & \varepsilon^*(\bar{\alpha}_1 - \bar{\alpha}_2) \equiv R_1 - R_2 + \eta. \end{aligned}$$

We first suppose i). From (4.1) we have $v(R_1 - R_2) = \overline{(1, \ldots, 1)}$, i.e.: $v(R_1) = v(R_2)$. By applying the proof of (4.2) we can compute these vectors:

$$\begin{aligned} v(R_1) &= \overline{(\lambda_1, \ldots, \lambda_{2t}, 1, \ldots, 1)} \quad \text{with} \quad \lambda_i = -1 \quad \text{iff} \quad P_i \in \mathrm{Supp}(R_1) \\ v(R_2) &= \overline{(1, \ldots, 1, \lambda_{2t+1}, \ldots, \lambda_{2g-2})} \quad \text{with} \quad \lambda_i = -1 \quad \text{iff} \quad P_i \in \mathrm{Supp}(R_2) \end{aligned}$$

We conclude that $\lambda_1 = \cdots = \lambda_{2t} = \lambda_{2t+1} = \cdots = \lambda_{2g-2}$, that is to say, either $R_1 = R_2 = 0$ or $R_1 = \sum_{i=1}^{2t} P_i$, $\quad R_2 = \sum_{i=2t+1}^{2g-2} P_i$. If $R_1 = R_2 = 0$, then $D_1 = D_2 = 0$ and we finish by taking $\bar{\beta} = \bar{\alpha}_1 = \bar{\alpha}_2$. Similarly, if $R_1 = \sum_{i=1}^{2t} P_i$, $\quad R_2 = \sum_{i=2t+1}^{2g-2} P_i$ we get $D_1 \equiv \varepsilon_1^*(\bar{\xi}_1)$ and $D_2 \equiv \varepsilon_2^*(\bar{\xi}_2)$ (see (2.3)). By replacing in (4.5):

$$\beta_1 = \varepsilon_1^*(\bar{\alpha}_1 - \bar{\xi}_1), \quad \beta_2 = \varepsilon_2^*(\bar{\alpha}_2 - \bar{\xi}_2).$$

On the other side, by (2.4.ii):

$$\varepsilon^*(\bar{\alpha}_1 - \bar{\alpha}_2) \equiv \sum_{i=1}^{2t} P_i - \sum_{i=2t+1}^{2g-2} P_i \equiv \varepsilon^*(\bar{\xi}_1 - \bar{\xi}_2)$$

and one finally obtains $\bar{\beta} = \bar{\alpha}_1 - \bar{\xi}_1 = \bar{\alpha}_2 - \bar{\xi}_2$.

In the case ii) we can imitate the above proof by replacing $\eta$ by the expression of (2.4.ii). ∎

## 5. The components $\mathcal{R}_{B,g,t}$ when $t \geq 3$

In the first half of this paragraph $(\tilde{C}, C)$ is an element of $\mathcal{R}_{B,g,t}$ with $t \geq 4$ and $P = P(\tilde{C}, C)$. We keep the notations of §1 and 2. In particular $g \geq 10$.

In order to describe the fibre of the Prym map over $P$ we shall use ideas from [Wel] and [De2]. Via geometric constructions we recover essential information that almost gives the initial data. To make the construction intrinsical we will need to recognize some components of $\mathrm{Sing}\Xi^*$. For instance we can recognize $V$ by its dimension. Our first goal will be to recover from $P$ other components.

Recalling the descriptions of (3.7) one has:

(5.1).- **Proposition.** The variety $W_{-2} \cap W_2$ is irreducible of dimension $g-9$ and one has the equality:

$$W_{-2} \cap W_2 = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in Z_1'', \zeta_2 \in Z_2'', Nm_{\varepsilon_1}(\zeta_1) + Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}\}.$$

PROOF: We check first the equality. Clearly the second member is contained in the first one. To see the opposite inclusion we apply (4.4). Indeed, suppose that

$$\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) = \pi_1^*(\zeta_1') + \pi_2^*(\zeta_2') \tag{5.2}$$

where

$$\zeta_1 \in \Theta_1^*, \quad \zeta_2 \in Z_2'', \quad \zeta_1' \in Z_1'', \quad \zeta_2' \in \Theta_2^*$$
$$\text{and} \quad Nm_{\varepsilon_1}(\zeta_1) + Nm_{\varepsilon_2}(\zeta_2) = Nm_{\varepsilon_1}(\zeta_1') + Nm_{\varepsilon_2}(\zeta_2') = \bar{\xi}.$$

Then:

$$\pi_1^*(\zeta_1 - \zeta_1') = \pi_2^*(\zeta_2' - \zeta_2).$$

By (4.4) there exists $\bar{\alpha} \in \mathrm{Pic}^0(E)$ such that:

$$\zeta_1 - \zeta_1' = \varepsilon_1^*(\bar{\alpha})$$
$$\zeta_2' - \zeta_2 = \varepsilon_2^*(\bar{\alpha}).$$

In particular $\zeta_1 = \varepsilon_1^*(\bar{\alpha}) + \zeta_1'$ and replacing this in (5.2) we are done.

Consider now the morphism:

$$\begin{aligned}\Psi : Z_1'' \times Z_2'' &\longrightarrow \mathrm{Pic}^{2g-2}(\tilde{C})\\ (\zeta_1,\zeta_2) &\longrightarrow \pi_1^*(\zeta_1)+\pi_2^*(\zeta_2)\end{aligned}$$

Let us define $T = \{(\zeta_1,\zeta_2) \in Z_1'' \times Z_2'' \mid Nm_{\varepsilon_1}(\zeta_1) + Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}\}$. Clearly $\Psi(T) = W_{-2} \cap W_2$. Since each fibre of the induced map $T \longrightarrow W_{-2} \cap W_2$ is isomorphic to $E$ (use (4.4)) it suffices to prove that $T$ is irreducible of dimension $g-8$. To see this look at the first projection: $T \longrightarrow Z_1''$. Clearly $Z_1''$ is irreducible and by (3.8) the fibres are irreducible of dimension $g-t-5$ (note that $g \geq 10$, $t \geq 4$ and $t+1 \leq g-t$ imply $g-t \geq 6$ ). Thus $T$ is irreducible and $\dim T = \dim Z_1'' + g - t - 5 = t - 3 + g - t - 5 = g - 8$. ∎

(5.3).- **Proposition.** The varieties $W_0 \cap W_{-2}$ and $W_0 \cap W_2$ are both irreducible of dimension $g-7$ and they are described as follows:

$$\begin{aligned}W_0 \cap W_{-2} &= \{\pi_1^*(\zeta_1)+\pi_2^*(\zeta_2) \mid \zeta_1 \in Z_1'', \zeta_2 \in A_2, Nm_{\varepsilon_1}(\zeta_1)+Nm_{\varepsilon_2}(\zeta_2)=\bar{\xi}\}\\ W_0 \cap W_2 &= \{\pi_1^*(\zeta_1)+\pi_2^*(\zeta_2) \mid \zeta_1 \in A_1, \zeta_2 \in Z_2'', Nm_{\varepsilon_1}(\zeta_1)+Nm_{\varepsilon_2}(\zeta_2)=\bar{\xi}\}.\end{aligned}$$

PROOF: By symmetry only one variety has to be considered, for instance $W_0 \cap W_2$. Imitating the proof of (5.1) one finds the equality. The irreducibility and dimension may be obtained as in Loc. cit. replacing $\Psi$ by the morphism:

$$\begin{aligned}\Psi' : A_1 \times Z_2'' &\longrightarrow \mathrm{Pic}^{2g-2}(\tilde{C})\\ (\zeta_1,\zeta_2) &\longrightarrow \pi_1^*(\zeta_1)+\pi_2^*(\zeta_2)\end{aligned}$$

and $T$ by $T' = \{(\zeta_1,\zeta_2) \in A_1 \times Z_2'' \mid Nm_{\varepsilon_1}(\zeta_1)+Nm_{\varepsilon_2}(\zeta_2)=\bar{\xi}\}$. ∎

(5.4).- **Remark.** The second statement of Proposition (5.3) still holds true if $t \geq 2$.

(5.5).- We put

$$\Lambda_a = \{\tilde{x} \in P \mid \tilde{x} + W_0 \cap W_a \subset W_0\}$$

where $a = 2, -2$.

Due to (5.1) and (5.3) we know $W_0$ among the components of $\mathrm{Sing}\Xi^*$ of dimension $g-5$. Therefore we recognize also $\{(W_0 \cap W_2), (W_0 \cap W_{-2})\}$. Combining both facts $\{\Lambda_{-2} \cup \Lambda_2\}$ is intrinsically recovered from $P$. Our next aim is to compute $\Lambda_{-2}$ and $\Lambda_2$.

(5.6).- **Proposition.** One has the equalities:

$$i)\quad \Lambda_{-2} = \{\pi_1^*(\varepsilon_1^*(\bar{x}) - r - s) \mid \bar{x} \in E, \quad r, s \in C_1, \quad 2\bar{x} \equiv \varepsilon_1(r) + \varepsilon_1(s)\}$$
$$ii)\quad \Lambda_{2} = \{\pi_2^*(\varepsilon_2^*(\bar{x}) - r - s) \mid \bar{x} \in E, \quad r, s \in C_2, \quad 2\bar{x} \equiv \varepsilon_2(r) + \varepsilon_2(s)\}.$$

PROOF: We only prove the second one, the first one being equivalent. Looking at (5.3) it is easy to check that the second member of this equality is contained in the first one (by (2.8) its elements belong to P). We show the opposite inclusion. Fix $\tilde{a} \in \Lambda_2$. By using (2.8) we may write

$$\tilde{a} = \pi_1^*(a_1) + \pi_2^*(a_2) \qquad \text{with } Nm_{\varepsilon_1}(a_1) = Nm_{\varepsilon_2}(a_2) = 0.$$

Let $\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \in W_0 \cap W_2$ where $\zeta_1 \in A_1$, $\quad \zeta_2 \in Z_2''$ and $Nm_{\varepsilon_1}(\zeta_1) + Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}$ (cf (5.3)). Applying Lemma (4.4) there exist elements $\zeta_1' \in A_1$, $\quad \zeta_2' \in A_2$ and $\bar{\alpha} \in \mathrm{Pic}^0(E)$ such that:

$$a_1 + \zeta_1 = \zeta_1' + \varepsilon_1^*(\bar{\alpha}) \in A_1$$
$$a_2 + \zeta_2 = \zeta_2' - \varepsilon_2^*(\bar{\alpha}) \in A_2.$$

Therefore $a_1 + A_1 \subset A_1$ and $a_2 + Z_2'' \subset A_2$. Then by using (3.9.i) and (3.9.ii) we finish the proof. ∎

(5.7).- **Proposition.** Assume $t \geq 4$. The sets $\Lambda_{-2} \cap 2\Lambda_{-2}$ and $\Lambda_2 \cap 2\Lambda_2$ are two symmetric irreducible curves. Their normalizations are $C_1$ and $C_2$ respectively, and $\tau_1$ and $\tau_2$ are the involutions induced by the (-1) map of $P$.

PROOF: We first observe that:

$$2\Lambda_{-2} = \{\pi_1^*(x + y - \tau_1(x) - \tau_1(y)) \mid \quad x, y \in C_1\}$$
$$2\Lambda_{2} = \{\pi_2^*(x + y - \tau_2(x) - \tau_2(y)) \mid \quad x, y \in C_2\}.$$

Now, it suffices to consider the set $\Lambda_{-2} \cap 2\Lambda_{-2}$. One has:

$$\Lambda_{-2} \cap 2\Lambda_{-2} = \{\pi_1^*(x - \tau_1(x)) \mid \quad x \in C_1\}.$$

Indeed, since $\tau_1$ has fixed points, $\pi_1^*(x - \tau_1(x)) \in 2\Lambda_{-2}$ for all $x \in C_1$. Moreover:

$$\pi_1^*(x - \tau_1(x)) = \pi_1^*(\varepsilon_1^*(\varepsilon_1(x)) - 2\tau_1(x)) \in \Lambda_{-2}.$$

So the right hand side member of the equality is contained in the left hand side member. To see the opposite inclusion, take $\bar{x} \in E$ and $r, s \in C_1$ such that $2\bar{x} \equiv \varepsilon_1(r) + \varepsilon_2(s)$ and suppose that $\pi_1^*(\varepsilon_1^*(\bar{x}) - r - s) \in 2\Lambda_{-2}$. We obtain a linear equivalence:

$$\pi_1^*(\varepsilon_1^*(\bar{x}) - r - s) \equiv \pi_1^*(y + z - \tau_1(y) - \tau_1(z))$$

where $y, z \in C_1$. Since $\pi_1^*$ is injective:

$$\varepsilon_1^*(\bar{x}) + \tau_1(y) + \tau_1(z) \equiv y + z + r + s. \tag{5.8}$$

By assumption $t \geq 4$ and then (3.1) implies that $h^0(\varepsilon_1^*(\bar{x}) + \tau_1(y) + \tau_1(z)) = 1$ iff $\tau_1(z) \neq y$. If $y = \tau_1(z)$ the initial element belongs to the right hand side member trivially. Thus we can assume that (5.8) is an equality of divisors and then either $y = \tau_1(z)$ or $y = \tau_1(y)$ or $z = \tau_1(z)$. In any case the inclusion follows.

Now, taking the morphism

$$\begin{aligned} \varphi_1 : C_1 &\longrightarrow \Lambda_{-2} \cap 2\Lambda_{-2} \\ x &\longrightarrow \pi_1^*(x - \tau_1(x)) \end{aligned}$$

the statement follows by observing that $\varphi_1$ is birational ($C_1$ is not hyperelliptic by (3.3)) and that $\varphi_1(\tau_1(x)) = -\varphi_1(x)$ ∎

**(5.9).- Let $\pi' : \tilde{D} \longrightarrow D$ be an unramified double cover of smooth curves such that $P(\tilde{D}, D) \cong P$. Since the singular locus of the theta divisor of $P$ has dimension $g - 5 = \dim P - 4$, $D$ is either trigonal or bi-elliptic (cf. [Mu],p.344). If $D$ is trigonal $P$ is the Jacobian of a curve (cf [Re]). Then, by [Sh] $C$ has to be either hyperelliptic or trigonal, which contradicts either (3.2) or (3.3). Thus $D$ is bi-elliptic.**

**Moreover, looking at the table (2.7) plus the observation of (2.13) we deduce that $(\tilde{D}, D) \in \mathcal{R}_{B,g,s}$ with $s \geq 4$. Let $D_1$ and $D_2$ be the bi-elliptic curves of genus $s + 1$ and $g - s$ attached to $(\tilde{D}, D)$ in the usual way (cf (2.1)). Since as we have seen in (5.7) $(C_1, \tau_1)$ and $(C_2, \tau_2)$ can be recovered from $P$, one has isomorphisms**

$$\varphi_i : D_i \longrightarrow C_i \qquad i = 1, 2$$

**commuting with the corresponding involutions. In particular the base elliptic curve is the same and $s = t$. Summarizing, if the diagram attached to $(\tilde{D}, D)$ is:**

$$\begin{tikzcd} & \tilde{D} \arrow[dl, "\pi'"'] \arrow[d, "\pi_1'"] \arrow[dr, "\pi_2'"] & \\ D \arrow[dr, "\delta'"'] & D_1 \arrow[d, "\varepsilon_1'"] & D_2 \arrow[dl, "\varepsilon_2'"] \\ & E & \end{tikzcd}$$

there exist $\Phi_i \in \operatorname{Aut}(E)$ $(i = 1, 2)$ such that

$$\begin{tikzcd} D_i \arrow[r, "\varphi_i"] \arrow[d, "\varepsilon'_i"'] & C_i \arrow[d, "\varepsilon_i"] \\ E \arrow[r, "\Phi_i"'] & E. \end{tikzcd}$$

Thus we obtain a diagram

$$\begin{tikzcd} & \tilde{D} \arrow[dl] \arrow[d] \arrow[dr] & \\ D \arrow[dr] & C_1 \arrow[d, "\Phi_1^{-1}\circ\varepsilon_1"] & C_2 \arrow[dl, "\Phi_2^{-1}\circ\varepsilon_2"] \\ & E. & \end{tikzcd}$$

Composing with a suitable automorphism of $E$ we get

(5.10)

$$\begin{tikzcd} & \tilde{D} \arrow[dl] \arrow[d] \arrow[dr] & \\ D \arrow[dr] & C_1 \arrow[d, "\Phi\circ\varepsilon_1"] & C_2 \arrow[dl, "\varepsilon_2"] \\ & E, & \end{tikzcd}$$

where $\Phi \in \operatorname{Aut}(E)$ and $\Phi(\bar{P}_i) \neq \bar{P}_j$, for all $1 \leq i \leq 2t < j \leq 2g-2$.

(5.11).-**Theorem** The tetragonal conjecture holds for the unramified double covers of smooth curves corresponding to generic elements of $\mathcal{R}_{B,g,t}$ with $t \geq 4$ and $g \geq 10$.

PROOF: By (5.9) it only remains to see that each diagram (5.10) can be obtained by applying successively the tetragonal construction starting from the initial element $(\tilde{C}, C)$. By (2.9) it suffices to see the following fact:

**Lemma.** Assume that E is general. Then the set:

$$\Gamma = \{\Phi \in \operatorname{Aut}(E) \mid \Phi(\bar{P}_i) \neq \bar{P}_j, \quad for \quad 1 \leq i \leq 2t < j \leq 2g-2\}$$

is generated multiplicatively by the elements of $\Gamma$ that correspond to the linear series $g^1_2$ of E.

PROOF: Let $\Phi \in \Gamma$. Take a point $\bar{r} \in E$ and put $\bar{s} = \Phi(\bar{r})$. Let $\tilde{\Phi}$ be the associated isomorphism of $JE$. In other words:

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d, "t_1"'] & E \arrow[d, "t_2"] \\ JE \arrow[r, "\tilde{\Phi}"'] & JE \end{tikzcd}$$

where $t_1, t_2$ are the embeddings of $E$ in its Jacobian via translations by $\bar{r}$ and $\bar{s}$ respectively. Since $E$ is general and $\tilde{\Phi}(0) = 0$ one has $\tilde{\Phi} = \pm Id$. Then either $\Phi(\bar{x}) \equiv \bar{x} - \bar{r} + \bar{s}$ for all $\bar{x} \in E$ or $\Phi(\bar{x}) \equiv \bar{r} + \bar{s} - \bar{x}$ for all $\bar{x} \in E$. In the second case $\Phi$ is the automorphism determined by the linear series $\mid \bar{r} + \bar{s} \mid$. Now assume that $\Phi(\bar{x}) \equiv \bar{x} - \bar{r} + \bar{s}$. Let us consider the automorphisms $\Phi_1$ and $\Phi_2$ of $E$ given by the linear series $\mid 2\bar{r} \mid$ and $\mid \bar{r} + \bar{s} \mid$. Then:

$$\Phi_1(\bar{x}) \equiv 2\bar{r} - \bar{x}$$

and

$$\Phi_2(\Phi_1(\bar{x})) \equiv \bar{r} + \bar{s} - \Phi_1(x) \equiv \bar{r} + \bar{s} - \Phi_1(\bar{x}) \equiv \bar{r} + \bar{s} - 2\bar{r} + \bar{x} \equiv \bar{x} - \bar{r} + \bar{s}.$$

So: $\Phi_2 \circ \Phi_1 = \Phi$. Moreover for a general $\bar{r} \in E$ one has $\Phi_1 \in \Gamma$. Since the linear series $\mid \bar{x} + \Phi(\bar{x}) \mid$ varies with $\bar{x}$ when $\Phi$ is not an automorphism associated to a $g^1_2$ we obtain $\Phi_2 \in \Gamma$ also for a general $\bar{r} \in E$. This ends the proof of the lemma. ∎

The rest of the section will be devoted to proving the analogue of the Theorem (5.11) for the component $\mathcal{R}_{B,g,3}$. Let $(\tilde{C}, C)$ be a general element of this component. Since the map

$$\begin{aligned} \mathcal{R}_{B,g,3} &\longrightarrow \mathcal{B}_4 \\ (\tilde{C}, C) &\longrightarrow C_1 \end{aligned}$$

is surjective, we may suppose that $C_1$ is general. Hence we shall assume that $C_1$ has two different linear series $g^1_3$ (cf (3.4)). We shall denote by $\mathcal{D}'$ and $\mathcal{D}''$ two divisors belonging respectively to these linear series. By applying (2.15) one obtains:

$$Nm_{e_1}(\mathcal{D}') \equiv Nm_{e_2}(\mathcal{D}'') \equiv \bar{\xi}_1.$$

From (2.6), (2.7) and (3.7) we have the decomposition:

$$\mathrm{Sing}\Xi^* = V' \cup V''$$

where $V'$ and $V''$ are irreducible varieties and are described as follows:

$$\begin{aligned} V' &= \{\pi_1^*(\mathcal{D}') + \pi_2^*(\zeta_2) \mid \zeta_2 \in Z_2'\} \\ V'' &= \{\pi_1^*(\mathcal{D}'') + \pi_2^*(\zeta_2) \mid \zeta_2 \in Z_2'\}. \end{aligned}$$

(5.12).- **Lemma.** With the above notations:

*i)* $W_2 \cap V' = \{\pi_1^*(\mathcal{D}') + \pi_2^*(\zeta_2) \mid \zeta_2 \in Z_2' \cap Z_2''\}$

*ii)* $W_2 \cap V'' = \{\pi_1^*(\mathcal{D}'') + \pi_2^*(\zeta_2) \mid \zeta_2 \in Z_2' \cap Z_2''\}$

*iii)* $W_0 \cap V' = W_0 \cap V'' = \emptyset$.

PROOF: It suffices to imitate the proof of (5.3). ∎

(5.13).- **Proposition.** One has the equalities (cf (2.8) for notation):

$i)$ $\{\tilde{a} \in P \mid \tilde{a} + W_2 \subset W_2\} = \{\pi_1^*(\beta) \mid Nm_{\varepsilon_1}(\beta) = 0\} = \pi_1^*(P_1)$

$ii)$ $\{\tilde{a} \in P \mid \tilde{a} + W_2 \cap V' \subset W_0\} = \{\pi_1^*(\varepsilon_1^*(\bar{x}) + y - \mathcal{D}') + \pi_2^*(\varepsilon_2^*(\bar{z}) - r - s) \mid$
$\bar{x}, \bar{z} \in E, y \in C_1, r, s \in C_2, 2\bar{x} + \varepsilon_1(y) \equiv \bar{\xi}_1 \quad \text{and} \quad 2\bar{z} \equiv \varepsilon_2(r) + \varepsilon_2(s)\}$

$iii)$ $\{\tilde{a} \in P \mid \tilde{a} + W_2 \cap V'' \subset W_0\} = \{\pi_1^*(\varepsilon_1^*(\bar{x}) + y - \mathcal{D}'') + \pi_2^*(\varepsilon_2^*(\bar{z}) - r - s) \mid$
$\bar{x}, \bar{z} \in E, y \in C_1, r, s \in C_2, 2\bar{x} + \varepsilon_1(y) \equiv \bar{\xi}_1 \quad \text{and} \quad 2\bar{z} \equiv \varepsilon_2(r) + \varepsilon_2(s)\}$

PROOF: We show first the equality i). To prove the inclusion of the right hand side member in the left hand side member we consider $\pi_1^*(\beta) \in \pi_1^*(P_1)$ and we take an element $\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \in W_2$ where $\zeta_1 \in \Theta_2^*$, $\zeta_2 \in Z_2''$ and $Nm_{\varepsilon_1}(\zeta_1) + Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}$. Since the map

$$\mathrm{Pic}^0(E) \times C_1^{(3)} \longrightarrow \mathrm{Pic}^3(C_1)$$

is surjective, we may write

$$\beta + \zeta_1 \equiv \zeta_1' + \varepsilon_1^*(\bar{\rho}), \qquad \text{where} \quad \zeta_1' \in \Theta_1^*, \bar{\rho} \in \mathrm{Pic}^0(E).$$

Then

$$\pi_1^*(\beta) + \pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) = \pi_1^*(\zeta_1') + \pi_2^*(\zeta_2 + \varepsilon_2^*(\bar{\rho})) \in W_2.$$

To see the opposite inclusion take $\tilde{a} = \pi_1^*(a_1) + \pi_2^*(a_2) \in P$ with $a_1 \in P_1, a_2 \in P_2$ (cf. (2.8) for definitions ) and such that $\tilde{a} + W_2 \subset W_2$. By applying Lemma (4.4) as before (see for instance the proof of (5.6)) we get $a_2 + Z_2'' \subset Z_2''$. By (3.9.iii) there exists $\bar{\alpha} \in_2 JE$ such that $a_2 = \varepsilon_2^*(\bar{\alpha})$. Therefore $\tilde{a} \in \pi_1^*(P_1)$.

In ii) only the inclusion of the first member in the second one has to be proved. Take again $\tilde{a} = \pi_1^*(a_1) + \pi_2^*(a_2) \in P$ and assume that $\tilde{a} + W_2 \cap V' \subset W_0$. By using (5.12) and Lemma (4.4) one has:

$$a_1 + \mathcal{D}' \equiv \varepsilon_1^*(\bar{x}) + y$$

where $\bar{x} \in E, y \in C_1$ and

$$a_2 + Z_2' \cap Z_2'' \subset A_2.$$

We end the proof of the proposition by applying (3.9.ii). ∎

(5.14).- Let us define the following sets:

$$\Sigma' = \{\tilde{a} \in P \mid \tilde{a} + W_2 \subset W_2\} \cap \{\tilde{a} \in P \mid \tilde{a} + W_2 \cap V' \subset W_0\}$$
$$\text{and} \quad \Sigma'' = \{\tilde{a} \in P \mid \tilde{a} + W_2 \subset W_2\} \cap \{\tilde{a} \in P \mid \tilde{a} + W_2 \cap V'' \subset W_0\}.$$

By dimension count we may distinguish the components $W_0, W_2$ from the components $V', V''$. The lemma gives a way to differenciate $W_0$ from $W_2$. So we obtain $\{\Sigma', \Sigma''\}$ intrinsically from $P$.

(5.15).- **Proposition**. With the above notations:

$$\Sigma' = \{\pi_1^*(\varepsilon_1^*(\bar{x}) + y - \mathcal{D}') \mid 2\bar{x} + \varepsilon_1(y) \equiv \bar{\xi}_1\}$$
$$\Sigma'' := \{\pi_1^*(\varepsilon_1^*(\bar{x}) + y - \mathcal{D}'') \mid 2\bar{x} + \varepsilon_1(y) \equiv \bar{\xi}_1\}$$

PROOF: It suffices to use Lemma (4.4). ∎

(5.16).- Since

$$\mathcal{D}' + \mathcal{D}'' \equiv K_{C_1} \equiv \varepsilon_1^*(\bar{\xi}_1)$$

it is easy to see that:

$$2\Sigma' = \{\pi_1^*(x - \tau_1(x) + \mathcal{D}'' - \mathcal{D}') \mid x \in C_1\}$$
$$2\Sigma' = \{\pi_1^*(x - \tau_1(x) + \mathcal{D}' - \mathcal{D}'') \mid x \in C_1\}.$$

These curves have normalization $C_1$. On the other side they are not invariant by the (-1)-map of $P$. Our aim is to define an involution on either of them inducing on $C_1$ the involution $\tau_1$. To do this we need the following

**Lemma**. Let $X = \{x - \tau_1(x) \in JC_1 \mid x \in C_1\}$. Then:

$$a + X \subset X \Rightarrow a = 0$$

PROOF: Since $0 \in X$ we may assume that $a = y - \tau_1(y)$ where $y \in C_1$. Now $a + X \subset X$ implies that for all $x \in C_1$ there exists $x' \in C_1$ such that

$$y - \tau_1(y) + x - \tau_1(x) \equiv x' - \tau_1(x').$$

Hence:

$$y + x + \tau_1(x') \equiv x' + \tau_1(y) + \tau_1(x). \tag{5.17}$$

If $h^0(y + x + \tau_1(x')) \geq 2$, then the linear series $\mid y + x + \tau_1(x') \mid$ is one of the $g_3^1$ series on $C_1$. If, for instance, $y + x + \tau_1(x') \equiv \mathcal{D}'$ then

$$\mathcal{D}'' \equiv \tau_1(\mathcal{D}') \equiv \tau_1(y + x + \tau_1(x')) \equiv y + x + \tau_1(x') \equiv \mathcal{D}',$$

which is a contradiction. Now assume that (5.17) is an equality of divisors. By taking $x$ such that $x \neq \tau_1(x)$ and $x \neq \tau_1(y)$ we get $y = \tau_1(y)$ and $a = 0$. ∎

(5.18).- **Corollary**. Let $\tilde{\alpha} \in 2\Sigma'$. There exists a unique element $\tilde{\beta} \in 2\Sigma'$ such that:

$$-(\tilde{\alpha} + 2\Sigma'') = \tilde{\beta} + 2\Sigma''$$

PROOF: We write $\tilde{\alpha} = \pi_1^*(x - \tau_1(x) + \mathcal{D}'' - \mathcal{D}')$ and $\tilde{\beta} = \pi_1^*(y - \tau_1(y) + \mathcal{D}'' - \mathcal{D}')$. Then

$$-(\tilde{\alpha} + 2\Sigma'') = \pi_1^*(\tau_1(x) - x) + \pi_1^*(X)$$

and

$$\tilde{\beta} + 2\Sigma'' = \pi_1^*(y - \tau_1(y)) + \pi_1^*(X)$$

By applying the lemma we find $\tau_1(x) - x = y - \tau_1(y)$. ∎

Putting $\sigma(\tilde{\alpha}) = \tilde{\beta}$ defines an involution of $2\Sigma'$. Consider now the normalization map

$$\begin{aligned} \varphi : C_1 &\longrightarrow 2\Sigma' \\ x &\longrightarrow \pi_1^*(x - \tau_1(x) + \mathcal{D}'' - \mathcal{D}'). \end{aligned}$$

Then $\sigma(\varphi(x)) = \varphi(\tau_1(x))$.

(5.19).-**Theorem.** The tetragonal conjecture holds for the unramified double covers of smooth curves corresponding to generic elements of $\mathcal{R}_{B,g,3}$ with $g \geq 10$.

PROOF: First we observe that the methods used in the first part of this section (i.e.: for $(\tilde{C}, C) \in \mathcal{R}_{B,g,t}$, $t \geq 4$) in order to recover the set of data $(C_2, \tau_2)$ are still valid (cf. (5.4),(5.6.ii) and (5.7)). On the other side we have seen in (5.12), (5.13), (5.14), (5.16) and (5.18) how to recognize intrinsically in $P$ the data $(C_1, \tau_1)$. Then the proof continues as in (5.11). ∎

## 6.The components $\mathcal{R}_{B,g,2}$ and $\mathcal{R}_{B,g,1}$

In this paragraph we wish to prove the analogue of Theorem (5.11) for the components $\mathcal{R}_{B,g,2}$ and $\mathcal{R}_{B,g,1}$. We use essentially the same ideas. Only the way to recover $(C_1, \tau_1)$ needs a new point of view; we shall consider the study of some intersections $\Xi^* \cap \Xi^*{}_{\tilde{a}}$. We keep the assumptions and notations of §1 and §2.

Let us denote by $(\tilde{C}, C)$ a general element of $\mathcal{R}_{B,g,2}$. From (2.6) and (2.7) we may suppose that:

$$\mathrm{Sing}\Xi^* = W_0 \cup W_2.$$

Our first goal is to make a difference between both components.

(6.1).- **Proposition**. One has the equalities:

$$i)\quad \{\tilde{a} \in P \mid \tilde{a} + W_2 \subset W_2\} = \pi_1^*(P_1) \quad (\text{cf } (2.8))$$

$$ii)\quad \{\tilde{a} \in P \mid \tilde{a} + W_0 \subset W_0\} = \pi^*(\varepsilon^*({}_2JE)).$$

In particular $\dim\{\tilde{a} \in P \mid \tilde{a} + W_2 \subset W_2\} = \dim P_1 = 2$ and $\dim\{\tilde{a} \in P \mid \tilde{a} + W_0 \subset W_0\} = 0$.

PROOF: Proceed as in (5.6). ∎

(6.2).- **Remark**. We imitate §5 (see (5.4), (5.5) and (5.7)) and obtain from $P^*$ the curve

$$\Lambda_2 \cap 2\Lambda_2 = \{\pi_2^*(x - \tau_2(x)) \mid x \in C_2\}.$$

By normalizing we recover $(C_2, \tau_2)$.

Now we aim at describing a subvariety of $\pi_1^*(P_1)$ that determines the curve $C_1$.

(6.3).- **Proposition**. One has the following equalities:

i) If $\tilde{a} = \pi_2^*(x - \tau_2(x))$, where $x \in C_2$, then

$$\Xi^* \cap \Xi^*_{\tilde{a}} = F \cup X(\tilde{a})$$

where

$$X(\tilde{a}) = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in \Theta_1^*, \zeta_2 \in \Theta_2^*, h^0(\zeta_2 - x) > 0 \text{ and } Nm_{\varepsilon_1}(\zeta_1) + Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}\}$$

is the moving part of this algebraic system and F is the fix part (see below for a description of F).

ii) Let $N = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in \Theta_1^*, Nm_{\varepsilon_1}(\zeta_1) = \bar{\xi}_1, \zeta_2 \in Z_2'\}$. Then:

$$\bigcap_{\tilde{a} \in \Lambda_2 \cap 2\Lambda_2 - \{0\}} X(\tilde{a}) = W_0 \cup W_2 \cup N,$$

and N is the union of the irreducible components distinct from $W_0 \cup W_2$.

iii) If $\tilde{a} = \pi_1^*(a_1)$, where $a_1 \in P_1$, then:

$$\begin{aligned} N \cap \Xi_{\tilde{a}}^* = & \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in \Theta_1^* \cap (\Theta_1^*)_{a_1}, Nm_{\varepsilon_1}(\zeta_1) = \bar{\xi}_1, \zeta_2 \in Z_2'\} \\ & \cup \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in \Theta_1^*, Nm_{\varepsilon_1}(\zeta_1) = \bar{\xi}_1, \zeta_2 \in Z_2' \cap Z_2''\}. \end{aligned}$$

PROOF: i). Let $\tilde{\zeta} = \pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \in \Xi^* \cap \Xi_{\tilde{a}}^*$ with $\tilde{a} = \pi_2^*(x - \tau_2(x))$. By applying Lemma (4.4) we find elements $\zeta_1' \in \Theta_1^*, \zeta_2' \in \Theta_2^*$ and $\bar{\rho} \in \mathrm{Pic}^0(E)$ such that:

$$\begin{aligned} \zeta_1 &\equiv \zeta_1' + \varepsilon_1^*(\bar{\rho}) \\ \tau_2(x) - x + \zeta_2 &\equiv \zeta_2' - \varepsilon_2^*(\bar{\rho}). \end{aligned} \tag{6.4}$$

Suppose first that $\bar{\rho} = 0$. Then

$$\begin{aligned} \zeta_2 \in \Theta_2^* \cap (\Theta_2^*)_{x - \tau_2(x)} = & \{\zeta_2 \in \Theta_2^* \mid h^0(\zeta_2 - x) > 0\} \\ & \cup \{\zeta_2 \in \Theta_2^* \mid h^0(\zeta_2 + \tau_2(x)) \geq 2\}. \end{aligned}$$

If $\zeta_2$ belongs to the second set, by Riemann-Roch one has

$$h^0(K_{C_2} - \zeta_2 - \tau_2(x)) > 0.$$

Define $\bar{\lambda} = \bar{\xi}_2 - Nm_{\varepsilon_2}(\zeta_2)$, $\beta_1 = \zeta_1 - \varepsilon_1^*(\bar{\lambda})$ and $\beta_2 = \zeta_2 + \varepsilon_2^*(\bar{\lambda})$. Then

$$\begin{aligned} h^0(\beta_1) &= h^0(\zeta_1 - \varepsilon_1^*(\bar{\xi}_2 - Nm_{\varepsilon_2}(\zeta_2))) = h^0(-\tau_1^*(\zeta_1) + \varepsilon_1^*(\bar{\xi}_1)) = h^0(\tau_1(\zeta_1)) > 0 \\ h^0(\beta_2 - x) &= h^0(K_{C_2} - \tau_2^*(\zeta_2) - x) = h^0(K_{C_2} - \zeta_2 - \tau_2(x)) > 0. \end{aligned}$$

Therefore $\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) = \pi_1^*(\beta_1) + \pi_2^*(\beta_2) \in X(\tilde{a})$.

On the other hand if $\bar{\rho} \neq 0$ then

$$\zeta_1 \in \Theta_1^* \cap (\Theta_1^*)_{\varepsilon_1^*(\bar{\rho})} = A_1 \cup \{\zeta_1 \in \Theta_1^* \mid Nm_{\varepsilon_1}(\zeta_1) = \bar{\xi}_1 + \bar{\rho}\}.$$

If $Nm_{\varepsilon_1}(\zeta_1) = \bar{\xi}_1 + \bar{\rho}$ then $\bar{\rho} = Nm_{\varepsilon_1}(\zeta_1) - \bar{\xi}_1 = \bar{\xi}_2 - Nm_{\varepsilon_2}(\zeta_2)$ and by replacing in (6.4) one has

$$\tau_2^*(\zeta_2) + x - \tau_2(x) \equiv K_{C_2} - \zeta_2'.$$

Thus $\zeta_2 \in \Theta_2^* \cap (\Theta_2^*)_{x-\tau_2(x)}$ and proceeding as above we conclude that $\zeta_2 \in X(\tilde{a})$. We have proved the inclusion $\Xi^* \cap \Xi_{\tilde{a}}^* \subset F \cup X(\tilde{a})$, where

$$F = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in A_1, \zeta_2 \in \Theta_2^*, Nm_{\varepsilon_1}(\zeta_1) + Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}\}.$$

The inclusion of $X(\tilde{a})$ in the left hand side member is trivial. Take now $\tilde{\zeta} = \pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \in F$. Since the map

$$\begin{array}{ccc} \mathrm{Pic}^0(E) \times C_2^{(g-3)} & \longrightarrow & \mathrm{Pic}^{g-3}(C_2) \\ (\bar{a}, D) & \longrightarrow & \varepsilon_1^*(\bar{a}) + D \end{array}$$

is surjective we can write

$$x - \tau_2(x) + \zeta_2 \equiv D + \varepsilon_2^*(\bar{a})$$

and then $\pi_2^*(x - \tau_2(x)) + \tilde{\zeta} \equiv \pi_1^*(\zeta_1 + \varepsilon_1^*(\bar{a})) + \pi_2^*(D) \in \Xi^*$.

The reader may observe that F and $X(\tilde{a})$ have pure dimension $g-3$ and that $\dim(F \cap X(\tilde{a})) = g-4$ for all $\tilde{a}$. This concludes the proof of i).

ii) The inclusion

$$W_0 \cup W_2 \cup N \subset \bigcap_{\tilde{a} \in \Lambda_2 \cap 2\Lambda_2 - \{0\}} X(\tilde{a})$$

is left to the reader.

To see the opposite inclusion let $\tilde{\zeta} = \pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \in X(\tilde{a})$ for all $\tilde{a} \in \Lambda_2 \cap 2\Lambda_2 - \{0\}$. Then for all $x \in C_2$ there exist $\zeta_1' \in \Theta_1^*, \zeta_2' \in \Theta_2^*$ and $\bar{\rho} \in \mathrm{Pic}^0(E)$ such that

$$\begin{aligned} &h^0(\zeta_2' - x) > 0 \\ &\zeta_1 \equiv \zeta_1' + \varepsilon_1^*(\bar{\rho}) \qquad (6.5) \\ &\zeta_2 \equiv \zeta_2' - \varepsilon_2^*(\bar{\rho}). \end{aligned}$$

Let $T$ be an irreducible component of the fibre of the map

$$\begin{array}{ccc} \mathrm{Pic}^0(E) \times C_2 \times C_2^{(g-4)} & \longrightarrow & \mathrm{Pic}(C_2) \\ (\bar{\rho}, x, D) & \longrightarrow & x + D - \varepsilon_2^*(\bar{\rho}) \end{array}$$

over $\zeta_2$. In these terms the conditions (6.5) say that we may (and will) assume that the projection from $T$ to $C_2$ is surjective. Suppose that the projection $T \longrightarrow \mathrm{Pic}^0(E)$ is constant and let $\bar{\rho}_0$ be the image. Then for all $x \in C_2$ we find an effective divisor $D$ such that:

$$\zeta_2 \equiv x + D - \varepsilon_2^*(\bar{\rho}_0).$$

Therefore $h^0(\zeta_2 + \varepsilon_2^*(\bar{\rho}_0) - x) > 0$ for all $x \in C_2$ and hence $\zeta_2 \in \mathrm{Sing}\Theta_2^* = Z_2' \cup Z_2''$. So $\tilde{\zeta}$ belongs to $W_2 \cup N$.

If $T \longrightarrow \mathrm{Pic}^0(E)$ is surjective we find that

$$h^0(\zeta_2 + \varepsilon_2^*(\bar{\rho})) > 0$$

for all $\bar{\rho} \in \mathrm{Pic}^0(E)$. Hence $\zeta_2 \in A_2$. Moreover one has $h^0(\zeta_1 - \varepsilon_1^*(\bar{\rho})) > 0$ for all $\bar{\rho} \in \mathrm{Pic}^0(E)$. Hence $\zeta_1 \in A_1$ and we are done.

From the descriptions it is clear that N has not components contained in $W_0 \cup W_2$. This finishes the proof of ii).

iii) The inclusion of the right hand side member in the left hand side member is left to the reader. To see the opposite inclusion let $\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2)$ such that $\zeta_1 \in \Theta_1^*, \quad Nm_{\varepsilon_1}(\zeta_1) = \bar{\xi}_1, \quad \zeta_2 \in Z_2'$ and suppose that

$$\pi_1^*(-a_1 + \zeta_1) + \pi_2^*(\zeta_2) \in \Xi^*.$$

Again there exist $\zeta_1' \in \Theta_1^*, \quad \zeta_2' \in \Theta_2^*$ and $\bar{\rho} \in \mathrm{Pic}^0(E)$ with

$$\begin{aligned} -a_1 + \zeta_1 &\equiv \zeta_1' + \varepsilon_1^*(\bar{\rho}) \\ \zeta_2 &\equiv \zeta_2' - \varepsilon_2^*(\bar{\rho}). \end{aligned}$$

If $\bar{\rho} = 0$ then $\zeta_1 \in \Theta_1^* \cap (\Theta_1^*)_{a_1}$. On the other hand $\bar{\rho} \neq 0$ implies that

$$\zeta_2 \in \Theta_2^* \cap (\Theta_2^*)_{-\varepsilon_2^*(\bar{\rho})} = A_2 \cup \{\zeta_2 \in \Theta_2^* \mid Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}_2 - \bar{\rho}\}.$$

Since $\zeta_2 \in Z_2'$, only $\zeta_2 \in A_2$ is possible and then $\zeta_2 \in Z_2' \cap Z_2''$. ∎

(6.6).- We shall define

$$N(\tilde{a}) = \{\pi_1^*(\zeta_1) + \pi_2^*(\zeta_2) \mid \zeta_1 \in \Theta_1^* \cap (\Theta_1^*)_{a_1}, Nm_{\varepsilon_1}(\zeta_1) = \bar{\xi}_1, \zeta_2 \in Z_2'\}.$$

This set is recovered from $N \cap \Xi_{\tilde{a}}^*$ as the union of the components not contained in $W_2$. Our next goal is to distinguish points in $\pi_1^*(P_1)$ looking at the number of components of $N(\tilde{a})$. We will see below that the set $\Theta_1^* \cap (\Theta_1^*)_{a_1} \cap Nm_{\varepsilon_1}^{-1}(\bar{\xi}_1)$ is finite. The cardinal of this set coincides with the number of irreducible components of $N(\tilde{a})$.

(6.7).- Let $D$ be the ample divisor induced by $\Theta_1$ on the abelian surface $P_1$. By Riemann-Roch

$$h^0(D) = \frac{D^2}{2} \qquad \text{and} \qquad h^0(D)^2 = \deg(\lambda_D).$$

By using [Mu1],p.330 we obtain $\deg(\lambda_D) = 4$ and therefore $D^2 = 4$.

(6.8).- Let $x \in C_1$ and let $a_1 = x - \tau_1(x) \in P_1$. One has

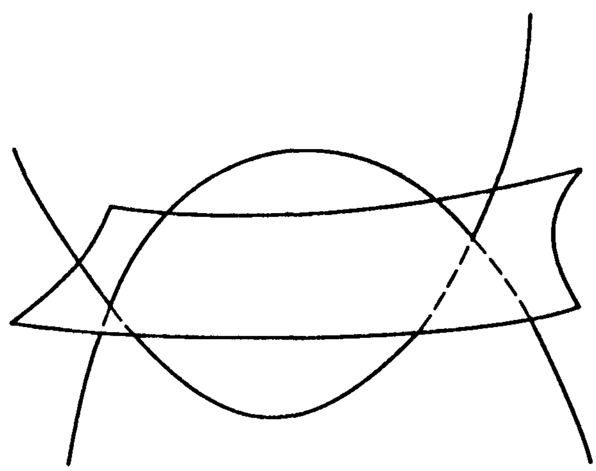
$$\Theta_1^* \cap (\Theta_1^*)_{a_1} = (x + C_1) \cup \{\zeta_1 \in \Theta_1^* \mid h^0(\zeta_1 + \tau_1(x)) = 2\}.$$

The first component meets $Nm_{\varepsilon_1}^{-1}(\bar{\xi}_1)$ at the points $x+y$ and $x+\tau_1(y)$ where $\varepsilon_1(y) = \bar{\xi}_1 - \varepsilon_1(x)$. On the other hand if the divisor $r+s$ verifies $h^0(r+s+\tau_1(x)) = 2$ and $\varepsilon_1(r) + \varepsilon_1(s) \equiv \bar{\xi}_1$ then by Riemann-Roch

$$0 < h^0(K_{C_1} - r - s - \tau_1(x)) = h^0(\tau_1(r) + \tau_1(s) - \tau_1(x)) = h^0(r+s-x).$$

Hence the second component intersects $Nm_{\varepsilon_1}^{-1}(\bar{\xi}_1)$ at the same points.

The picture is:



(6.9).- Let $\Sigma$ be the curve given by the pull-back diagram:

$$\begin{array}{ccc} \Sigma & \longrightarrow & C_1^{(2)} \\ \downarrow & & \downarrow{\scriptstyle \varepsilon_1^{(2)}} \\ \mid \bar{\xi}_1 \mid & \longrightarrow & E^{(2)} \end{array}$$

the horizontal arrows being inclusions. By studying the (4:1) cover $\Sigma \longrightarrow \mid \bar{\xi}_1 \mid$ it is easy to obtain that $p_a(\Sigma) \leq 3$.

We shall denote by $\Sigma_0$ the image of the map

$$\begin{array}{ccc} \Sigma \longrightarrow & & P_1 \\ x+y \longrightarrow & x+y-\tau_1(x)-\tau_1(y). & \end{array}$$

The map $\Sigma \longrightarrow \Sigma_0$ is birational (use Proposition (4.2) to show this) and the curve $\Sigma_0$ has singularities away from the origin.

(6.10).- **Proposition.** One has:

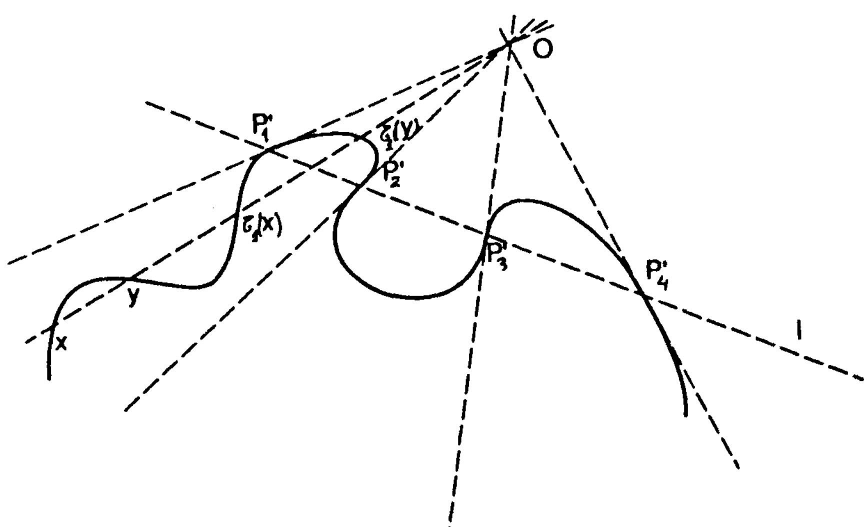
$$\{\tilde{a} \in \pi_1^*(P_1) \mid \text{ number comp. } N(\tilde{a}) < 4\} = \Pi \cup \pi_1^*(\Sigma_0)$$

where $\Pi = \{\pi_1^*(x - \tau_1(x)) \mid x \in C_1\}$.

PROOF: By (6.6) we must study te cardinal of the set $\Theta_1^* \cap (\Theta_1^*)_{a_1} \cap Nm_{\varepsilon_1}^{-1}(\bar{\xi}_1)$ when $a_1 \in P_1$. From (6.8) we have the inclusion of $\Pi$ in the left hand side member. To see the rest of the statement we shall need the following properties of the quartic plane curve $C_1$:

- The lines determined by the divisors $\varepsilon_1^*(\bar{x})$ with $\bar{x} \in E$ all pass through a common point $O \in \mathbf{P}^2$, where $O \notin C_1$. (In fact $O = \mathbf{P}(H^0(E, \mathcal{O}_E(\bar{\xi}_1))^{\perp}) \subset \mathbf{P}H^0(C_1, K_{C_1})^*$).
- The ramification points $P_1', \ldots, P_4'$ of $\varepsilon_1$ belong to a line $l$ and $O \notin l$.
- If $x, y \in C_1$ verify $\varepsilon_1(x) + \varepsilon_1(y) \equiv \bar{\xi}_1$ then $O \in \overline{xy}$.

The picture is:




In fact the involution $\tau_1$ admits the following description: let $x \in C_1$ and take $x' \in l \cap \overline{Ox}$, then $\mid O, x'; x, \tau_1(x) \mid = -1$.

Take now a point $x + y \in \Theta_1^* \cap (\Theta_1^*)_{a_1} \cap Nm_{\varepsilon_1}^{-1}(\bar{\xi}_1)$. The following equalities are well-known:

$$\overline{xy} = \mathbf{P}T_{\Theta_1^*}(x+y) \subset \mathbf{P}T_{JC_1}(x+y) \cong \mathbf{P}H^0(C_1, K_{C_1})^*$$

$$\overline{rs} = \mathbf{P}T_{(\Theta_1^*)_{a_1}}(x+y) \text{ where } r + s \in \mid x + y - a_1 \mid .$$

Since $\varepsilon_1(x) + \varepsilon_1(y) \equiv \varepsilon_1(r) + \varepsilon_1(s) \equiv \bar{\xi}_1$ both lines pass through O. They are equal iff the following equality of divisors holds

$$x + y + \tau_1(x) + \tau_1(y) = r + s + \tau_1(r) + \tau_1(s),$$

that is to say iff $\pi_1^*(a_1) \in \Pi \cup \pi_1^*(\Sigma_0)$.

Assume first that $\pi_1^*(a_1) \notin \Pi \cup \pi_1^*(\Sigma_0)$. In this case the curve $\Theta_1^* \cap (\Theta_1^*)_{a_1}$ is not singular at $x+y$ and it suffices to show that $0 \notin \mathbf{P}T_{P_1}(0)$ in order to obtain transversality in the intersection. Indeed:

$$T_{P_1}(0) = (H^0(C_1, K_{C_1})^-)^* = H^0(E, \mathcal{O}_E(\xi_1))^* = H^0(E, \mathcal{O}_E)^\perp \subset H^0(C_1, K_{C_1})^*.$$

On the other hand, if $s_R$ is an equation for the ramification divisor $R = \sum_{i=1}^4 P_i'$ then the inclusion

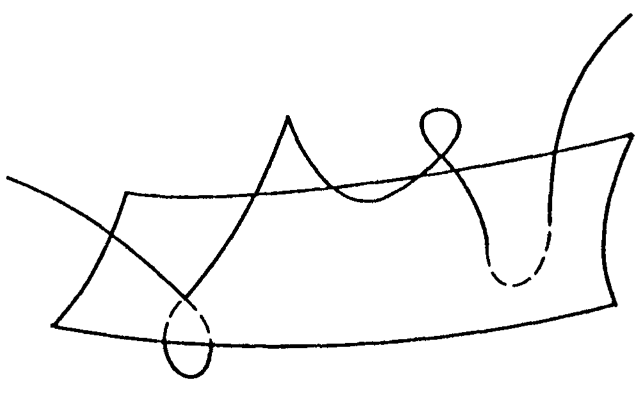
$$\begin{aligned} H^0(E, \mathcal{O}_E) &\hookrightarrow H^0(C_1, K_{C_1}) \\ s \quad &\longrightarrow \varepsilon_1^*(s) s_R \end{aligned}$$

induces an equality $\mathbf{P}H^0(E, \mathcal{O}_E) = \{R\}$. By dualizing we get $0 \notin l = \mathbf{P}T_{P_1}(0)$. Observe in particular that it follows from this that the set $\Theta_1^* \cap (\Theta_1^*)_{a_1} \cap Nm_{\varepsilon_1}^{-1}(\xi_1)$ is finite. Combining (6.7) with the transversality we find

$$\pi_1^*(a_1) \notin \Pi \cup \pi_1^*(\Sigma_0) \Longrightarrow \text{number comp. } N(\tilde{a}) = 4.$$

Finally if $a_1 \in \Sigma_0$ then $\mathbf{P}T_{\Theta_1^*}(x+y) = \mathbf{P}T_{(\Theta_1^*)_{a_1}}(x+y)$. Thus $\Theta_1^* \cap (\Theta_1^*)_{a_1}$ is singular at x+y.

The picture is in this case:



Therefore $a_1 \in \Sigma_0 \Longrightarrow$ number comp. $N(\tilde{a}) < 4$. ∎

(6.11).- **Theorem.** The tetragonal conjecture holds for the unramified double covers of smooth curves corresponding to generic elements of $\mathcal{R}_{B,g,2}$ with $g \geq 10$.

PROOF: In view of the proof of (5.11) it suffices to show how to recognize $(C_1, \tau_1)$ and $(C_2, \tau_2)$ from $P$. By combining (6.1), (6.2), (6.3), (6.6) and (6.10) we recover the set $\Pi \cup \pi_1^*(\Sigma_0)$ intrinsically. By (6.9) one obtains that $\pi_1^*(\Sigma_0)$ is singular away from the origin and it verifies $p_a(\pi_1^*(\Sigma_0)) \leq 3$. Thus if it is irreducible then we distinguish $\Pi$ as the component of the set smooth away from the zero. Otherwise $\Pi$ is the unique component with $p_a$ equal to 3. Now by normalizing the symmetric curve $\Pi$ we obtain $(C_1, \tau_1)$. On the other hand (6.2) says how to recover $(C_2, \tau_2)$. ∎

In the rest of this section $(\tilde{C}, C)$ will be a general element of $\mathcal{R}_{B,g,1}$. By (2.6) and (2.7) we can assume that $\mathrm{Sing}\Xi^* = W_2$ is irreducible of dimension $g-5$.

(6.12).- **Proposition.** One has the following equalities:

$$i) \quad \{\tilde{a} \in P \mid \tilde{a} + W_2 \subset W_2\} = \pi_1^*(P_1).$$

$$ii) \quad \{\tilde{a} \in P \mid \tilde{a} + W_2 \subset \Xi^*\} = \{\pi_1^*(a_1) + \pi_2^*(\varepsilon_2^*(\bar{x}) - r - s) \mid a_1 \in P_1, \bar{x} \in E, \quad r, s \in C_1, \quad 2\bar{x} \equiv \varepsilon_1(r) + \varepsilon_1(s)\}.$$

PROOF: Equality i) may be proved by arguing as in (5.13.i). In equality ii) the inclusion of the second set in the first one is clear. To see the opposite inclusion take $\tilde{a} = \pi_1^*(a_1) + \pi_2^*(a_2) \in P$ where $a_1 \in P_1, \quad a_2 \in P_2$ and such that $\tilde{a} + W_2 \subset \Xi^*$. Let $\tilde{\zeta} = \pi_1^*(x) + \pi_2^*(\zeta_2) \in W_2$, with $x \in C_1, \quad \zeta_2 \in Z_2''$ and $\varepsilon_1(x) + Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}$. By applying Lemma (4.4) one finds elements $x' \in C_1, \zeta_2' \in W_{g-2}^0(C_2)$ and $\bar{\rho} \in \mathrm{Pic}^0(E)$ such that

$$a_1 + x \equiv x' + \varepsilon_1^*(\bar{\rho}) \tag{6.13}$$
$$a_2 + \zeta_2 \equiv \zeta_2' - \varepsilon_2^*(\bar{\rho}).$$

Let us define the following subvariety of $C_1 \times Z_2''$

$$Y = \{(x, \zeta_2) \in C_1 \times Z_2'' \mid \varepsilon_1(x) + Nm_{\varepsilon_2}(\zeta_2) \equiv \bar{\xi}\}.$$

Consider now the morphism:

$$\Psi : \mathrm{Pic}^0(E) \times C_1 \times C_2^{(g-2)} \longrightarrow \mathrm{Pic}^1(C_1) \times \mathrm{Pic}^{g-2}(C_2)$$
$$(\bar{\rho}, x', D) \longrightarrow (x' + \varepsilon_1^*(\bar{\rho}) - a_1, D - \varepsilon_2^*(\bar{\rho}) - a_2).$$

In these terms the equivalences of (6.13) read: $Y \subset \mathrm{Im}(\Psi)$. Since $Y$ is irreducible (it suffices to apply (3.8) to the fibres of the projection map from $Y$ to $C_1$ ) there exists an irreducible component $X$ of $\Psi^{-1}(Y)$ such that the induced map

$$\tilde{\Psi} : X \longrightarrow Y$$

is dominant. If $q : X \longrightarrow \mathrm{Pic}^0(E)$ is the first projection we call $Y_{\bar\rho} := \tilde\Psi(q^{-1}(\bar\rho))$ for all $\bar\rho \in \mathrm{Pic}^0(E)$. Two cases are possible:

$$\begin{aligned} &\textit{either} \quad a) \quad Y_{\bar\rho} = Y \quad \text{for some} \quad \bar\rho \in \mathrm{Pic}^0(E) \\ &\textit{or} \quad b) \quad Y_{\bar\rho} \neq Y \quad \text{for all} \quad \bar\rho \in \mathrm{Pic}^0(E). \end{aligned}$$

In case a) define

$$b_1 = a_1 - \varepsilon_1^*(\bar\rho) \quad \text{and} \quad b_2 = a_2 + \varepsilon_2^*(\bar\rho).$$

Then (6.13) says:

$$h^0(b_1 + x) > 0, \quad h^0(b_2 + \zeta_2) > 0 \quad \text{for all} \quad (x, \zeta_2) \in Y.$$

Hence $b_1 = 0$ and $b_2 + Z_2'' \subset \Theta_2^*$. Therefore by using (3.9.ii) we finish the proof.

In case b) we write $\lambda : Y \longrightarrow C_1 \subset \mathrm{Pic}^1(C_1)$ for the first projection. We claim that $\lambda_{|\, Y_{\bar\rho}}$ is non-surjective for general $\bar\rho \in \mathrm{Pic}^0(E)$. Otherwise for all $x \in C_1$ one finds an element $\zeta_2 \in Z_2''$ such that $(x, \zeta_2) \in Y_{\bar\rho}$. In particular $h^0(a_1 + x - \varepsilon_1^*(\bar\rho)) > 0$ and then $a_1 = \varepsilon_1^*(\bar\rho)$.

Now since for a general $\bar\rho$, $Y_{\bar\rho}$ contains components of codimension 1 in $Y$ we deduce from the claim the following fact: there exists $x_0 \in C_1$ such that $\lambda^{-1}(x_0) \subset Y_{\bar\rho}$. Hence (6.13) reads:

$$h^0(a_1 + x_0 - \varepsilon_1^*(\bar\rho)) > 0 \quad \text{and} \quad h^0(a_2 + \zeta_2 + \varepsilon_2^*(\bar\rho)) > 0$$

for all $\zeta_2 \in Z_2''$ with $Nm_{\varepsilon_2}(\zeta_2) = \bar\xi - \varepsilon_1(x_0)$. In particular

$$a_2 + \varepsilon_2^*(\bar\rho) + \{\zeta_2 \in Z_2'' \mid Nm_{\varepsilon_2}(\zeta_2) \equiv \bar\xi - \varepsilon_1(x_0)\} \subset \Theta_2^*.$$

The proof ends by observing that

$$\{\zeta_2 \in Z_2'' \mid Nm_{\varepsilon_2}(\zeta_2)\} = \varepsilon_2^*(\bar\alpha) + Z_2' \cap Z_2''$$

where $2\bar\alpha = \bar\xi_1 - \varepsilon_1(x_0)$, and applying (3.9.ii). ∎

We shall denote by B the set described in (6.12.ii).

(6.14).- **Proposition.** The abelian variety $\pi_1^*(P_1)$ acts on $B \cap 2B$ by translations on $P$ and the quotient

$$\frac{B \cap 2B}{\pi_1^*(P_1)} \subset \frac{P}{\pi_1^*(P_1)}$$

is a symmetric curve with normalization $C_2$. The reflection on $P$ induces on $C_2$ the involution $\tau_2$.

PROOF: By using the standard arguments of the §5 one has:

$$B \cap 2B = \{\pi_1^*(a_1) + \pi_2^*(x - \tau_2(x)) \mid a_1 \in P_1, x \in C_2\}.$$

Now the morphism

$$\begin{aligned}\lambda : C_2 &\longrightarrow \frac{B \cap 2B}{\pi_1^*(P_1)} \\ x &\longrightarrow \overline{\pi_2^*(x - \tau_2(x))}\end{aligned}$$

is birational and verifies $\lambda(\tau_2(x)) = -\lambda(x)$. ∎

(6.15).-The reader can prove without much work the following properties:

- $P_1 \subset JC_1$ is an elliptic curve .
- The morphism

$$\begin{aligned}\mu : C_1 &\longrightarrow P_1 \\ x &\longrightarrow x - \tau_1(x)\end{aligned}$$

is a double cover with two ramification points inducing on $C_1$ a new bi-elliptic structure. The attached involution $\tau_1'$ is the composition of $\tau_1$ with the hyperelliptic involution.

- We shall write $Q_1$ and $Q_2$ for the fixed points of $\tau_1'$ and $P_1', P_2'$ for the ramification points of $\varepsilon_1$. With the notations of (2.1) :

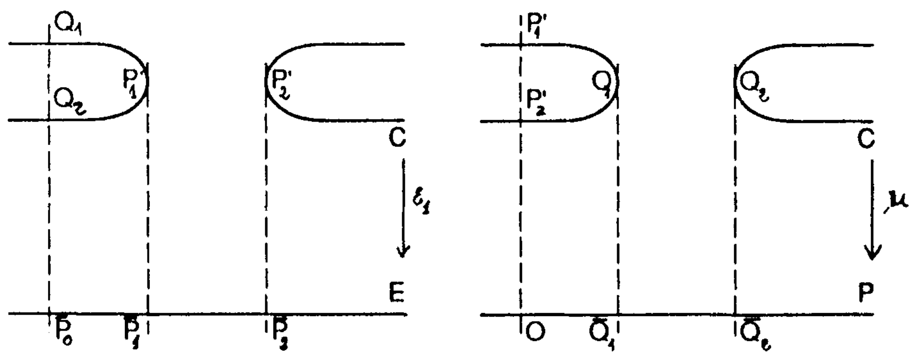
$$Q_1 + Q_2 \equiv P_1' + P_2' \equiv K_{C_1}$$

and

$$\begin{aligned}&\tau_1(Q_1) = Q_2, \quad \varepsilon_1(Q_1) = \varepsilon_1(Q_2) \in \mid \bar{\xi}_1 \mid \\ &\tau_1'(P_1') = P_2', \quad \mu(P_1') = \mu(P_2') = 0.\end{aligned}$$

We write $\bar{Q}_1 = \mu(Q_1)$ and $\bar{Q}_2 = \mu(Q_2)$.

The picture is:




where $|\bar{\xi}_1| = \{\bar{P}_0\}$.

• Note that $\bar{Q}_1 = \mu(Q_1) = Q_1 - \tau_1(Q_1) = -(Q_2 - \tau_1(Q_2)) = -\mu(Q_2) = -\bar{Q}_2$. Moreover $\mu^*(0) = P_1' + P_2' \equiv Q_1 + Q_2$.

Summarizing we obtain (composing with $\pi_1^* : P_1 \longrightarrow \pi_1^*(P_1)$) that $C_1$ can be represented as the double cover of $\pi_1^*(P_1)$ associated to the class of the origin (as a point of the abelian subvariety of $P$) and the discriminant divisor $\pi_1^*(\bar{Q}_1) + \pi_1^*(\bar{Q}_2)$. Since the class is trivially recovered from $\pi_1^*(P_1)$, we only need to find the divisor inside $P$. Moreover the involution $\tau_1$ will appear when composing the canonical involution of $C_1$ with the involution attached to this cover.

(6.16).- **Proposition.** Let $\tilde{a} = \pi_1^*(x - \tau_1(x)) \neq 0$ where $x \in C_1$. Then:

$$\Xi^* \cap \Xi^*_{\tilde{a}} = F' \cup R(\tilde{a})$$

where

$$F' = \{\pi_1^*(y) + \pi_2^*(\zeta_2) \mid y \in C_1, \zeta_2 \in A_2, \varepsilon_1(y) + Nm_{\varepsilon_2}(\zeta_2) \equiv \bar{\xi}\}$$
$$R(\tilde{a}) = \{\pi_1^*(x) + \pi_2^*(\zeta_2) \mid \zeta_2 \in \Theta_2^*, \varepsilon_1(x) + Nm_{\varepsilon_2}(\zeta_2) \equiv \bar{\xi}\}.$$

PROOF: To see the inclusion $F' \cup R(\tilde{a}) \subset \Xi^* \cap \Xi^*_{\tilde{a}}$ compare with (6.3.i). We prove the opposite inclusion. Fix $\tilde{\zeta} = \pi_1^*(y) + \pi_2^*(\zeta_2)$ with $y \in C_1, \zeta_2 \in \Theta_2^*$ and $\varepsilon_1(y) + Nm_{\varepsilon_2}(\zeta_2) \equiv \bar{\xi}$ verifying

$$-\tilde{a} + \tilde{\zeta} \in \Xi^*.$$

By the lemma (4.4) one finds elements $z \in C_1, \zeta_2' \in W^0_{g-2}(C_2)$ and $\bar{\rho} \in \mathrm{Pic}^0(E)$ such that

$$y + \tau_1(x) - x \equiv z + \varepsilon_1^*(\bar{\rho}) \tag{6.17}$$
$$\zeta_2 = \zeta_2' - \varepsilon_2^*(\bar{\rho}).$$

We consider the possibilities:

$$i) \quad \bar{\rho} = 0$$
$$ii) \quad \bar{\rho} \neq 0.$$

If i) occurs one has

$$y + \tau_1(x) \equiv x + z.$$

The equality would lead to $y = x$ (recall that $x \neq \tau_1(x)$ due to $\tilde{a} \neq 0$) and $\tilde{\zeta} \in R(\tilde{a})$. So we only consider the case:

$$y + \tau_1(x) \equiv K_{C_1}. \tag{6.18}$$

Defining $\bar{\alpha} = \bar{\xi}_1 - \varepsilon_1(y)$ one finds

$$y + \varepsilon_1^*(\bar{\alpha}) \equiv y + K_{C_1} - \varepsilon_1^*(\varepsilon_1(y)) \equiv K_{C_1} - \tau_1(y) \equiv \tau_1(K_{C_1} - y) \equiv x. \tag{6.19}$$

Moreover $Nm_{\varepsilon_2}(\zeta_2) \equiv \bar{\xi} - \varepsilon_1(y) \equiv \bar{\xi}_2 + \bar{\alpha}$. Hence

$$h^0(\zeta_2 - \varepsilon^*(\bar{\alpha})) = h^0(K_{C_2} - \tau_2(\zeta_2)) = h^0(\tau_2(\zeta_2)) = h^0(\zeta_2) > 0. \tag{6.20}$$

Combining (6.19) and (6.20) one has:

$$\tilde{\zeta} \equiv \pi_1^*(y + \varepsilon_1^*(\bar{\alpha})) + \pi_2^*(\zeta_2 - \varepsilon_2^*(\bar{\alpha})) \equiv \pi_1^*(x) + \pi_2^*(\zeta_2 - \varepsilon_2^*(\bar{\alpha})) \in R(\tilde{a}).$$

From now on we assume $\bar{\rho} \neq 0$. By (6.17) we get

$$\zeta_2 \in \Theta_2^* \cap (\Theta_2^*)_{-\varepsilon_2^*(\bar{\rho})} = A_2 \cup \{\zeta_2 \in W^0_{g-2}(C_2) \mid Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}_2 - \bar{\rho}\}$$

(cf.[De4],p.9). When $\zeta_2 \in A_2$, clearly $\tilde{\zeta} \in F'$. On the other hand if $Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}_2 - \bar{\rho}$ we deduce

$$\varepsilon_1(y) \equiv \bar{\xi}_1 + \bar{\rho}.$$

By replacing $\bar{\rho}$ in the first linear equivalence of (6.17):

$$\tau_1(x) + K_{C_1} \equiv x + \tau_1(y) + z.$$

Writing $K_{C_1} \equiv z + z'$ we have now $\tau_1(x) + z' \equiv x + \tau_1(y)$. We deduce as before that either $x = y$ or $x + \tau_1(y) \equiv K_{C_1}$. The first possibility leads directly to $\tilde{\zeta} \in R(\tilde{a})$ and the second one implies

$$\tau_1(x) + y \equiv K_{C_1}.$$

Going back to (6.18) we end the proof as in case i). ∎

(6.21).- **Remark.** As a matter of fact the computation of $\Xi^* \cdot \Xi^*_{\tilde a}$ has been made by Debarre in [De5]. He finds:

$$\Xi^* \cdot \Xi^*_{\tilde a} = \{\tilde\zeta \in \Xi^* \mid h^0(\tilde\zeta - \pi_1^*(x)) \geq 1\} \text{ for } \tilde a = \pi_1^*(x - \tau_1(x))$$

and

$$\mathrm{Sing}(\Xi^* \cdot \Xi^*_{\tilde a}) \supset \{\tilde\zeta \in \Xi^* \mid h^0(\tilde\zeta - \pi_1^*(x)) \geq 2\}.$$

We addapt his general proof to our case. First we note that

$$F' \cup R(\tilde a) = \{\tilde\zeta \in \Xi^* \mid h^0(\tilde\zeta - \pi_1^*(x)) \geq 1\}.$$

Indeed, the inclusion of $F' \cup R(\tilde a)$ in the right hand side member comes from the descriptions of (6.16). Next if $\tilde\zeta = \pi_1^*(y) + \pi_2^*(\zeta_2) \in \Xi^*$ with $h^0(\pi_1^*(y - x) + \pi_2^*(\zeta_2)) \geq 1$ one has either that $\pi_2^*(\zeta_2)$ is not $\pi_1$-simple (cf. (2.14)) or

$$\begin{aligned}0 &< h^0(y - x) + h^0(y - x + Nm_{\pi_1}(\pi_2^*(\zeta_2)) - \varepsilon_1^*(\bar\xi_2)) = \\ &= h^0(y - x) + h^0(y - x + \varepsilon_1^*(\bar\xi - \varepsilon_1(y)) - \varepsilon_1^*(\bar\xi_2)) = \\ &= h^0(y - x) + h^0(\varepsilon_1^*(\bar\xi_1) - x - \tau_1(y)).\end{aligned}$$

It is easy to check that the first case implies that $\zeta_2 \in A_2$ and then $\tilde\zeta \in F'$. The second case leads to one of the following two possibilities:

$$\begin{aligned}&i) \quad y = x \\ &ii) \quad x + \tau_1(y) \equiv \varepsilon_1^*(\bar\xi_1) \equiv K_{C_1}.\end{aligned}$$

The recurrent argument starting at (6.18) finishes the proof of the inclusion.

Next, note that $\{\tilde\zeta \in \Xi^* \mid h^0(\tilde\zeta - \pi_1^*(x)) \geq 1\}$ is a special subvariety associated to the linear system $\mid K_C - Nm_\pi(\pi_1^*(x)) \mid$ in the sense of [Be2] (cf. also [We2]). In other words, defining $X$ by the pull-back diagram

$$\begin{array}{ccc} X & \longrightarrow & \tilde C^{(2g-4)} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi^{(2g-4)}} \\ \mid K_C - Nm_\pi(\pi_1^*(x)) \mid & \longrightarrow & C^{(2g-4)} \end{array}$$

the image of $X$ by the morphism

$$\begin{aligned}\varphi : \tilde C^{(2g-4)} &\longrightarrow \mathrm{Pic}^{2g-2}(\tilde C) \\ \tilde D &\longrightarrow \pi_1^*(x) + \tilde D\end{aligned}$$

has two connected components. Only one of these components sits inside $P^*$ and it equals $\Xi^* \cap \Xi^*_{\tilde a}$. In loc. cit., Beauville computes the cohomology class of special subvarieties. In the present case the class is $[\Xi]^2$. So $\Xi^* \cdot \Xi^*_{\tilde a}$ is reduced and the equality of the statement holds.

Suppose now that $\tilde{\zeta} \in \Xi^*$ verifies $h^0(\tilde{\zeta} - \pi_1^*(x)) \geq 2$. We can assume that $h^0(\tilde{\zeta}) = h^0(\tilde{\zeta} - \pi_1^*(x - \tau_1(x))) = 2$ (otherwise either $\tilde{\zeta} \in \mathrm{Sing}\Xi^*$ or $\tilde{\zeta} \in \mathrm{Sing}\Xi_{\tilde{a}}^*$, hence $\tilde{\zeta} \in \mathrm{Sing}(\Xi^* \cdot \Xi_{\tilde{a}}^*)$). It suffices to check that the tangent spaces of the two divisors at the point are equal. Taking bases for both $H^0(\tilde{\zeta})$ and $H^0(\tilde{\zeta} - \pi_1^*(x - \tau_1(x)))$ the reader can apply [Mu], p.343 for this computation.

We remark also that, since on $\Xi^*$ $\varphi$ is one-to-one outside $\{\tilde{\zeta} \in \Xi^* \mid h^0(\tilde{\zeta} - \pi_1^*(x)) \geq 2\}$, a characterization of Welters (cf. [We2] and [Be2]) of the singularities of the special subvarieties gives the inclusion:

(6.22)

$$\mathrm{Sing}(\Xi^* \cdot \Xi_{\tilde{a}}^*) \subset \{\tilde{\zeta} \in \Xi^* \mid h^0(\tilde{\zeta} - \pi_1^*(x)) \geq 2\} \cup \{\pi_1^*(x) + \tilde{D} \text{ such that for } A \geq 0 \text{ maximal with } \pi^*(A) \leq \tilde{D}, \quad h^0(A + \varepsilon^*(\varepsilon_1(x))) > 1\}.$$

The next result allows one to find distinguished points in $\pi_1^*(P_1)$.

(6.23).- **Proposition.** Let $\tilde{a} = \pi_1^*(x - \tau_1(x)) \in \pi_1^*(P_1)$. Then there exist singular points of $\Xi^* \cdot \Xi_{\tilde{a}}^*$ away from $F'$ iff $\varepsilon_1(x) \equiv \bar{\xi}_1$.

PROOF: From (6.21) and (6.22) it suffices to see the following facts:

*i)* If $\varepsilon_1(x) \notin \mid \bar{\xi}_1 \mid$, then $R(\tilde{a}) - F'$ does not intersect the second member of (6.22).

*ii)* If $\varepsilon_1(x) \in \mid \bar{\xi}_1 \mid$, then $(R(\tilde{a}) - F') \cap \{\tilde{\zeta} \in \Xi^* \mid h^0(\tilde{\zeta} - \pi_1^*(x)) \geq 2\} \neq \emptyset$.

To see ii) observe that

$$\{\pi_1^*(x) + \pi_2^*(\zeta_2) \mid \zeta_2 \in Z_2' - Z_2' \cap Z_2'', \varepsilon_1(x) \equiv \bar{\xi}_1\}$$

is contained in the above intersection.

Assume now that $\varepsilon_1(x) \notin \mid \bar{\xi}_1 \mid$ and take $\tilde{\zeta} = \pi_1^*(x) + \pi_2^*(\zeta_2)$ such that $\zeta_2 \notin A_2$. Then

$$h^0(\tilde{\zeta} - \pi_1^*(x)) = h^0(\pi_2^*(\zeta_2)) = h^0(\zeta_2) + h^0(\zeta_2 - \varepsilon_2^*(\bar{\xi}_1)) = h^0(\zeta_2).$$

So $h^0(\tilde{\zeta} - \pi_1^*(x)) \geq 2$ implies $\zeta_2 \in \mathrm{Sing}\Theta_2^* = Z_2' \cup Z_2''$. Since $\zeta_2 \notin A_2 \supset Z_2''$ and $Nm_{\varepsilon_2}(\zeta_2) \neq \bar{\xi}_2$ this is a contradiction.

Suppose finally that there exists a divisor $A \geq 0$ on $C$ such that

$$h^0(\pi_2^*(\zeta_2) - \pi^*(A)) > 0 \text{ and } h^0(A + \varepsilon^*(\varepsilon_1(x))) \geq 2.$$

In particular $A \neq 0$. By using (3.1) the second inequality says that $A$ is not $\varepsilon$-simple. We conclude that we may write

$$\pi_2^*(\zeta_2) \equiv \pi^*(\varepsilon^*(\bar{A})) + \tilde{B}$$

where $\bar{A}$ and $\tilde{B}$ are effective divisors on $E$ and $\tilde{C}$ respectively and $\bar{A}$ is not trivial. Then

$$0 < h^0(\pi_2^*(\zeta_2 - \varepsilon_2^*(\bar{A}))) = h^0(\zeta_2 - \varepsilon_2^*(\bar{A})) + h^0(\zeta_2 - \varepsilon_2^*(\bar{A}) - \varepsilon_2^*(\bar{\xi}_1)),$$

which contradicts that $\zeta_2 \notin A_2$. ∎

(6.24).- **Theorem**. The tetragonal conjecture holds true for the unramified double covers of smooth curves corresponding to generic elements of $\mathcal{R}_{B,g,1}$.

PROOF: Let $(\tilde{C}, C)$ be a generic element of $\mathcal{R}_{B,g,1}$ and suppose that there exists an element $(\tilde{D}, D) \in \mathcal{R}_g$ such that $P(\tilde{C}, C) \cong P(\tilde{D}, D)$. By using the arguments of (5.9) we conclude that $(\tilde{D}, D) \in \mathcal{R}_{B,g}$. Then, from the number of irreducible components of $\mathrm{Sing}\Xi^*$ (cf. (2.7) and (2.13)) we conclude that

$$(\tilde{D}, D) \in \mathcal{R}'_{B,g} \cup \mathcal{R}_{B,g,0} \cup \mathcal{R}_{B,g,1}.$$

As we shall see in (7.1.i) (combined with (2.12)) the property

$$\dim\{\tilde{a} \in P \mid \tilde{a} + \mathrm{Sing}\Xi^* \subset \mathrm{Sing}\Xi^*\} = 1$$

(cf. (6.12.i)) does not hold in the components $\mathcal{R}'_{B,g}$ and $\mathcal{R}_{B,g,0}$. So $(\tilde{D}, D) \in \mathcal{R}_{B,g,1}$. By arguing as in (5.11) it suffices to explain how to recover $(C_1, \tau_1)$ and $(C_2, \tau_2)$ from $P$. The way to do it in the first case comes from Propositions (6.12) and (6.14). In the second case this is done by combining (6.15), (6.16) and (6.23). ∎

## 7.The components $\mathcal{R}_{B,g,0}$ and $\mathcal{R}'_{B,g}$

As we remarked in (2.12) the study of these two components should be done simultaneously. Our approach consist in looking at each component independently and then, comparing the constructions made in each case, to establish a bijection between both of them commuting with the Prym map.

Let $(\tilde{C}, C) \in \mathcal{R}_{B,g,0}$. We keep the notations of §1 and §2. In this section we do not need the assumption of generality. Although the equality $\mathrm{Sing}\Xi^* = W_2$ cannot be used, $W_2$ appears as the unique component of dimension greater than 0. Recall that $t = 0$ implies that $\varepsilon_1$ and $\pi_2$ are unramified. We shall denote by $\lambda$ the non trivial element of $\pi^*(\varepsilon^*({}_2JE))$.

(7.1).- **Proposition.** One has the equalities:

$$i) \quad \{\tilde{a} \in P \mid \tilde{a} + W_2 \subset W_2\} = \{0, \lambda\}$$

$$ii) \quad \{\tilde{a} \in P \mid \tilde{a} + W_2 \subset \Xi^*\} = \{\pi_2^*(\varepsilon_2^*(\bar{x}) - r - s) \mid \bar{x} \in E, \; r, s \in C_2, \quad 2\bar{x} \equiv \varepsilon_2(r) + \varepsilon_2(s)\}.$$

PROOF: In both equalities the inclusion of the right hand side member in the left hand side member is clear. We fix $\tilde{a} \in P$. By (2.8) we may write $\tilde{a} = \pi_2^*(a_2)$ where $a_2 \in P_2$. First we suppose that $\tilde{a} + W_2 \subset W_2$. Since

$$W_2 = \{\pi_2^*(\zeta_2) \mid \zeta_2 \in Z_2'', Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}\}$$

and

$$\mathrm{Ker}(\pi_2^*) = \{0, \varepsilon_2^*(\bar{\xi}_1)\}$$

we deduce that for all $\zeta_2 \in Z_2''$ with $Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}$

$$(7.2) \qquad a_2 + \zeta_2 \in Z_2''.$$

By taking $\alpha = \varepsilon_2^*(\bar{\alpha})$ with $2\bar{\alpha} \equiv \bar{\xi}_1$ we get

$$\alpha + \{\zeta_2 \in Z_2'' \mid Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}\} = Z_2' \cap Z_2''.$$

So (7.2) reads

$$a_2 - \alpha + Z_2' \cap Z_2'' \subset Z_2'.$$

Hence, by (3.9.iii), $a_2 \in P_1 \cap \mathrm{Im}(\varepsilon_1^*)$ and we are done.

Next we assume that $\tilde{a} + W_2 \subset \Xi^*$. We obtain similarly

$$a_2 + \{\zeta_2 \in Z_2'' \mid Nm_{\varepsilon_2}(\zeta_2) = \bar{\xi}\} \subset \Theta_2^* \cup (\Theta_2^*)_{\varepsilon_2^*(\bar{\xi}_1)}.$$

Taking $\alpha$ as before one has:

$$a_2 - \alpha + Z_2' \cap Z_2'' \subset \Theta_2^* \cup (\Theta_2^*)_{\varepsilon_2^*(\bar{\xi}_1)}.$$

By using the irreducibility of $Z_2' \cap Z_2''$ (cf.(3.8)) and (3.9.ii) we end the proof. ∎

Let us denote by S the set described in (7.1.ii). Then

(7.3).- **Proposition** The set S∩2S is a symmetric curve with normalization $C_2$. Moreover $\tau_2$ is the involution induced on $C_2$ by the (-1) map of $P$.

PROOF: We claim that the following equality holds

$$S \cap 2S = \{\pi_2^*(x - \tau_2(x)) \mid x \in C_2\}. \tag{7.4}$$

First we see that all the statements are a consequence of this claim. In fact, only the birationality of the map

$$\begin{aligned} \varphi : C_2 &\longrightarrow S \cap 2S \\ x &\longrightarrow \pi_2^*(x - \tau_2(x)) \end{aligned}$$

needs to be proved. Assume that $\varphi(x) = \varphi(y)$. Then $x + \tau_2(y) - \tau_2(x) - y \in \mathrm{Ker}(\pi_2^*) = \{0, \varepsilon_2^*(\bar{\xi}_1)\}$. Hence:

$$2x + 2\tau_2(y) \equiv 2y + 2\tau_2(x).$$

The equality of divisors would conduce to either $x = y$ or $x = \tau_2(x)$. So we can suppose that $h^0(2x + 2\tau_2(y)) \geq 2$. Since all the linear series $g_4^1$ on $C_2$ come from $g_2^1$'s on $E$ one finds a divisor $\bar{A} \in E^{(2)}$ such that $2x + 2\tau_2(y) = \varepsilon^*(\bar{A})$ and then we have again either $x = y$ or $x = \tau_2(x)$.

In order to prove (7.4) we observe that

$$2S = \{\pi_2^*(\zeta_2 - \tau_2(\zeta_2)) \mid \zeta_2 \in W_2^0(C_2)\}.$$

Suppose that $\pi^*(\varepsilon_2^*(\bar{x}) - r - s) \in 2S$ where $\bar{x} \in E$, $r, s \in C_2$ and $2\bar{x} \equiv \varepsilon_2(r) + \varepsilon_2(s)$. For some points $y, z \in C_2$ one has

$$\begin{aligned} \text{either} \quad & \varepsilon_2^*(\bar{x}) - r - s \equiv y + z - \tau_2(y) - \tau_2(z) \\ \text{or} \quad & \varepsilon_2^*(\bar{x}) - r - s \equiv y + z - \tau_2(y) - \tau_2(z) + \varepsilon_2^*(\bar{\xi}_1). \end{aligned}$$

Since both cases are similar we suppose that

$$\varepsilon_2^*(\bar{x}) + \tau_2(y) + \tau_2(z) \equiv y + z + r + s.$$

If $h^0(\varepsilon_2^*(\bar{x}) + \tau_2(y) + \tau_2(z)) = 2$, then Lemma (3.1) implies that $z = \tau_2(y)$ and one has:

$$\pi_2^*(\varepsilon_2^*(\bar{x}) - r - s) \equiv \pi_2^*(y + z - \tau_2(y) - \tau_2(z)) = 0.$$

Otherwise we get an equality of divisors. The proof ends by looking at the different possibilities. The opposite inclusion is left to the reader. ∎

(7.5).- **Remark.** The data $(C_2, \tau_2)$ do not determine the initial element $(\tilde{C}, C)$. However, by recovering the class $\varepsilon_2^*(\bar{\xi}_1)$, the curve $C_1$ (hence $(\tilde{C}, C)$) may be reconstructed from our information.

(7.6).- **Theorem.** Let $(\tilde{C}, C)$ and $(\tilde{D}, D)$ be two elements of $\mathcal{R}_{B,g,0}$ verifying $P(\tilde{C}, C) \cong P(\tilde{D}, D)$. Then: $(\tilde{C}, C) \cong (\tilde{D}, D)$.

PROOF: By (7.1), (7.3) and (7.5) it suffices to exhibit a way to recover $\varepsilon_2^*(\bar{\xi}_1)$ from $P$. Going back to the proof of (7.3) one finds a morphism:

$$C_2 \longrightarrow P$$

inducing a morphism:

$$j : JC_2 \longrightarrow P.$$

Since $j(\tau_2^*(\alpha)) = -j(\alpha)$ one can factorize j into $j' \circ h$, where

$$h : JC_2 \longrightarrow \frac{JC_2}{\mathrm{Ker}(\tau_2^* - 1)} \cong P_2$$

is the obvious map and $j' = \pi_{2|\,P_2}^*$ (cf [Ma],p.225). Then $\mathrm{Ker}(j') = \{0, \varepsilon_2^*(\bar{\xi}_1)\}$. Hence we obtain $\varepsilon_2^*(\bar{\xi}_1) \in P_2 \subset JC_2$. ∎

Let $(\tilde{C}, C) \in \mathcal{R}'_{B,g}$. We keep the notations and assumptions of §1 and §2 (see specially (2.9)). In particular $g \geq 10$. Recall that by (4.3) one has $\tau^*(\eta) \neq \eta$.

(7.7).- **Proposition.** With the above notations, $\mathrm{Sing}\Xi^*$ has a unique irreducible component of dimension $g - 5$. This component admits the description:

$$W = \{\pi^*(\varepsilon^*(\bar{x} + \bar{y})) + \tilde{\zeta} \in P^* \mid \bar{x}, \bar{y} \in E, \tilde{\zeta} \in W^0_{2g-10}(\tilde{C})\}.$$

PROOF: Acording to (2.13) it only remains to check that $\dim W = g-5$ (note that by definition $W \subset \mathrm{Sing}^{\pi}_{\varepsilon x}\Xi^* \subset \mathrm{Sing}\Xi^*$ (cf.§1)). We consider the morphism:

$$\kappa : E^{(2)} \times \tilde{C}^{(2g-10)} \longrightarrow \mathrm{Pic}^{2g-2}(\tilde{C})$$
$$(\bar{x}+\bar{y}, \tilde{D}) \longrightarrow \tilde{D} + \pi^*(\varepsilon^*(\bar{x}+\bar{y})).$$

Let us define $T = \kappa^{-1}(P^*)$ and $\tilde{\kappa} : T \longrightarrow P^*$ the restriction of $\kappa$. The fibre of the projection of $T$ onto $E^{(2)}$ over a point $\bar{x}+\bar{y}$ is isomorphic to a special subvariety (in the sense of [We2] and [Be2]) associated to the linear system $\mid K_C - 2\varepsilon^*(\bar{x}+\bar{y}) \mid = \mid \varepsilon^*(\bar{\xi} - 2\bar{x} - 2\bar{y}) \mid$ of dimension $g-6$. Hence $\dim T = g-4$. Since $\tilde{\kappa}(T) = W$ and the generic fibre of $\tilde{\kappa}$ has dimension 1 we conclude that $\dim W = g-5$. ∎

(7.8).- **Proposition**. One has the equality:

$$\{\tilde{a} \in P \mid \tilde{a} + W \subset \Xi^*\} = \{\pi^*(\varepsilon^*(\bar{x})) - \tilde{\zeta} \in P \mid \bar{x} \in E, \tilde{\zeta} \in W^0_4(\tilde{C})\}$$

PROOF: Let $\tilde{a} \in P$ such that $\tilde{a} + W \subset \Xi^*$. We wish to see that $\tilde{a}$ is in the right hand side member. By hypothesis

$$h^0(\tilde{a} + \pi^*(\varepsilon^*(\bar{x}+\bar{y})) + \tilde{\zeta}) \geq 2$$

for all $(\bar{x}+\bar{y}, \tilde{\zeta}) \in E^{(2)} \times W^0_{2g-10}(\tilde{C})$ such that $\pi^*(\varepsilon^*(\bar{x}+\bar{y})) + \tilde{\zeta} \in P^*$. By standard facts from the theory of Prym varieties this implies that:

$$h^0(\tilde{a} + \pi^*(\varepsilon^*(\bar{x}+\bar{y})) + \tilde{\zeta}) \geq 1$$

for all $(\bar{x}+\bar{y}, \tilde{\zeta}) \in E^{(2)} \times W^0_{2g-10}(\tilde{C})$ such that $2\varepsilon^*(\bar{x}+\bar{y}) + Nm_\pi(\tilde{\zeta}) \equiv K_C$. By applying (2.16) one has:

(7.9) $$h^0(\tilde{D} + \pi^*(\varepsilon^*(\bar{x}+\bar{y}))) > 0 \quad \text{for all} \quad \bar{x},\bar{y} \in E.$$

In particular we may write $\tilde{a} \equiv \tilde{D} - \pi^*(\varepsilon^*(\bar{x}+\bar{y}))$ where $\tilde{D} \in \tilde{C}^{(8)}$ and $Nm_\pi(\tilde{D}) \equiv 2\varepsilon^*(\bar{x}+\bar{y})$. Then (7.9) reads

$$h^0(\tilde{D} + \pi^*(\varepsilon^*(\bar{\alpha}))) > 0 \quad \text{for all} \quad \bar{\alpha} \in \mathrm{Pic}^0(E).$$

Let us make the decomposition $\tilde{D} = \pi^*(\varepsilon^*(\bar{D}) + D) + \tilde{D}'$ where $\bar{D}, D, \tilde{D}'$ are effective divisors on $E, C$ and $\tilde{C}$ respectively, $D$ is $\varepsilon$-simple and $\tilde{D}'$ is $\pi$-simple (cf. (2.14)). Note that it suffices to see that the divisor $\bar{D}$ is not trivial.

Then by applying (2.15) twice one has:

$$0 < h^0(\tilde{D} + \pi^*(\varepsilon^*(\bar{\alpha}))) = h^0(\pi^*(\varepsilon^*(\bar{D} + \bar{\alpha}) + D) + \tilde{D}') \leq$$
$$\leq h^0(\varepsilon^*(\bar{D} + \bar{\alpha}) + D) + h^0(\varepsilon^*(\bar{D} + \bar{\alpha}) + D + Nm_\pi(\tilde{D}') - \eta)$$

and

$$h^0(\varepsilon^*(\bar{D} + \bar{\alpha}) + D) \leq h^0(\bar{D} + \bar{\alpha}) + h^0(\bar{D} + \bar{\alpha} + Nm_\varepsilon(D) - \bar{\xi}).$$

Since $\deg(\bar{D} + \bar{\alpha} + Nm_\varepsilon(D)) \leq \deg(\tilde{D}) = 8 < g - 1 = \deg(\bar{\xi})$ we obtain

$$0 < h^0(\bar{D} + \bar{\alpha}) + h^0(\varepsilon^*(\bar{D} + \bar{\alpha}) + D + Nm_\pi(\tilde{D}') - \eta).$$

Suppose now that $\deg(\bar{D}) = 0$. Then we have

$$0 < h^0(\varepsilon^*(\bar{\alpha}) + D + Nm_\pi(\tilde{D}') - \eta) \quad \text{for all } \bar{\alpha} \in \mathrm{Pic}^0(E) - \{0\}. \tag{7.10}$$

On the other side $Nm_\pi(\tilde{D}) = 2\varepsilon^*(\bar{x} + \bar{y})$ implies that

$$2D + Nm_\pi(\tilde{D}') \equiv 2\varepsilon^*(\bar{x} + \bar{y}).$$

So (7.10) reads

$$0 < h^0(\varepsilon^*(2\bar{x} + 2\bar{y} + \bar{\alpha}) - D - \eta) \quad \text{for all } \bar{\alpha} \in \mathrm{Pic}^0(E) - \{0\}.$$

Let $D' \geq 0$ such that

$$\varepsilon^*(2\bar{x} + 2\bar{y} + \bar{\alpha}) - D - D' \equiv \eta. \tag{7.11}$$

Then $2(D + D') \in \mid \varepsilon^*(2(2\bar{x} + 2\bar{y} + \bar{\alpha})) \mid$. Since $\deg(2(2\bar{x} + 2\bar{y} + \bar{\alpha})) = 8 < g - 1$, we can apply (3.1). Therefore there exists an effective divisor $\bar{F}$ on $E$ such that

$$2(D + D') = \varepsilon^*(\bar{F}).$$

In particular $D + D'$ is $\tau$-invariant. Looking at (7.11) we conclude that $\tau^*(\eta) = \eta$, which is a contradiction. ∎

Let us denote by $S'$ the set $\{\tilde{a} \in P \mid \tilde{a} + W \subset \Xi^*\}$.

(7.12).- **Proposition**. The following inclusions hold:

$$S' \cap 2S' \subset T' = \{\tilde{D} - \iota^*(\tilde{D}) \in J\check{C} \mid \tilde{D} \in W_2^0(\tilde{C}), Nm_\pi(\tilde{D}) \in \mathrm{Im}(\varepsilon^*)\} \subset S'$$

PROOF: Let us define

$$U = \{\tilde{D} - \iota^*(\tilde{D}) \mid \tilde{D} \in W_4^0(\tilde{C}), Nm_\pi(\tilde{D}) \in \mathrm{Im}(\varepsilon^*)\}$$

By (7.8) one has $2S' \subset U$. So, our statements follow from the claim:

$$U \cap S' = T'.$$

The inclusion $T' \subset U \cap S'$ is clear. We prove the opposite inclusion. Let $\tilde{D} - \iota^*(\tilde{D}) \in U$ and $\bar{r}, \bar{s} \in E$ such that $Nm_\pi(\tilde{D}) = \varepsilon^*(\bar{r} + \bar{s})$. If we suppose that $\tilde{D} - \iota^*(\tilde{D}) \in S'$ then one finds elements $\tilde{D}' \in \tilde{C}^{(4)}$ and $\bar{x} \in E$ such that

$$\iota^*(\tilde{D}) + \tilde{D}' \equiv \tilde{D} + \pi^*(\varepsilon^*(\bar{x})). \tag{7.13}$$

We may write $\tilde{D} = \pi^*(A) + \tilde{B}$ where $\tilde{B} \geq 0$ is $\pi$-simple and $A$ is effective. Looking at the degree of $A$ we have three possibilities:

a) deg($A$)= 2. In this case $\tilde{D} - \iota^*(\tilde{D}) = 0 \in T'$.

b) deg($A$) = 1. Therefore deg($\tilde{B}$) = 2. By replacing in (7.13)

$$\tilde{D}' + \iota^*(\tilde{B}) \equiv \tilde{B} + \pi^*(\varepsilon^*(\bar{x})).$$

The equality of divisors would imply $\tilde{B} \leq \pi^*(\varepsilon^*(\bar{x}))$. Since $\tilde{B}$ is $\pi$-simple, $Nm_\pi(\tilde{B}) = \varepsilon^*(\bar{x})$ and then

$$\tilde{D} - \iota^*(\tilde{D}) \equiv \tilde{B} - \iota^*(\tilde{B}) \in T'.$$

We suppose now that

$$2 \leq h^0(\tilde{B} + \pi^*(\varepsilon^*(\bar{x}))).$$

By applying (2.15)

$$2 \leq h^0(\varepsilon^*(\bar{x})) + h^0(Nm_\pi(\tilde{B}) + \varepsilon^*(\bar{x}) - \eta) = 1 + h^0(Nm_\pi(\tilde{B}) + \varepsilon^*(\bar{x}) - \eta).$$

On the other side $Nm_\pi(\tilde{B}) = Nm_\pi(\tilde{D}) - 2A = \varepsilon^*(\bar{r} + \bar{s}) - 2A$. So

$$0 < h^0(Nm_\pi(\tilde{B}) + \varepsilon^*(\bar{x}) - \eta) = h^0(\varepsilon^*(\bar{r} + \bar{s} + \bar{x}) - 2A - \eta).$$

Arguing as in (7.11) we get $\tau^*(\eta) = \eta$, which is a contradiction.

c) $\deg(A) = 0$. Then $\tilde{D}$ is $\pi$-simple. We go back to (7.13). If there is an equality, then $\tilde{D} = \pi^*(\varepsilon^*(\bar{x}))$. Hence $\tilde{D} - \iota^*(\tilde{D}) = 0$. Otherwise, by applying (2.15)

$$2 \leq h^0(\tilde{D}) + \pi^*(\varepsilon^*(\bar{x}))) \leq 1 + h^0(\varepsilon^*(\bar{x}) + Nm_\pi(\tilde{D})) - \eta).$$

Since $Nm_\pi(\iota^*(\tilde{D})) = \varepsilon^*(\bar{r} + \bar{s})$ one has $h^0(\varepsilon^*(\bar{x} + \bar{r} + \bar{s}) - \eta) > 0$. Again this implies $\tau^*(\eta) = \eta$, which is a contradiction. ∎

(7.14).- Note that the remark (2.12) makes it possible to compare the variety $S'$ with the variety $S$ which appears in the first part of this section. So $S' \cap 2S'$ is a symmetric irreducible curve of genus $g$. Since $T'$ is also a curve we conclude that $S' \cap 2S'$ is an irreducible component of $T'$.

In order to study the curve $T'$ we define $T$ as the variety given by following pull-back diagram:

$$\begin{array}{ccc} T & \longrightarrow & \tilde{C}^{(2)} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi^{(2)}} \\ E & \xrightarrow{\varepsilon^*} & C^{(2)}. \end{array}$$

Note that the morphism

$$\begin{aligned} \tilde{C}^{(2)} &\longrightarrow P \\ \tilde{D} &\longrightarrow \tilde{D} - \iota^*(\tilde{D}) \end{aligned}$$

sends $T$ birationally to $T'$.

We shall denote by $j$ the involution of $T$ induced by $\iota^{(2)}$. The above diagram can be factorized as follows:

$$\begin{array}{ccc} T & \rightarrow & \tilde{C}^{(2)} \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle h_1} \\ T/j & \rightarrow & \tilde{C}^{(2)}/\iota^{(2)} \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle h_2} \\ E & \xrightarrow{\varepsilon^*} & C^{(2)} \end{array}$$

where:

i) the maps $f_1$ and $f_2$ are double covers, $f_1$ is ramified at the points $\pi^*(P_i) \in T$, $i = 1, \ldots, 2g-2$, where $P_i$ are the ramification points of $\varepsilon$ and $f_2$ is unramified. In particular $T/j$ is a smooth curve.

ii) Similarly $h_1, h_2$ are (2:1) morphisms, the ramification locus of $h_1$ is the diagonal $\Delta \subset \tilde{C}^{(2)}$ and $h_1$ is unramified. In particular $\tilde{C}^{(2)}/\iota^{(2)}$ is an smooth surface.

(7.16).- **Lemma**.-The morphism $f_2$ is the double cover associated to the class $\bar{\eta} = Nm_\varepsilon(\eta) \in {}_2JE - \{0\}$.

PROOF: We start by noting that $\bar{\eta} = 0$ implies $\eta + \tau^*(\eta) = 0$ and then $\tau^*(\eta) = \eta$. Next, fixing suitable elements $\zeta$ and $\tilde{\zeta}$ of $C^{(2)}$ and $\tilde{C}^{(2)}$ one has a commutative diagram:

$$\begin{array}{ccc} \tilde{C}^{(2)} & \xrightarrow{-\tilde{\zeta}} & J\tilde{C} \\ {\scriptstyle \pi^{(2)}}\downarrow & & \downarrow{\scriptstyle Nm_\pi} \\ C^{(2)} & \xrightarrow{-\zeta} & JC. \end{array}$$

By applying the functor $\mathrm{Pic}^0$ we obtain:

$$\begin{array}{ccc} \mathrm{Pic}^0(\tilde{C}^{(2)}) & \xleftarrow{\tilde{t}} & \mathrm{Pic}^0(J\tilde{C}) \\ {\scriptstyle \mathrm{Pic}^0(\pi^{(2)})}\uparrow & & \uparrow{\scriptstyle (Nm_\pi)^*} \\ \mathrm{Pic}^0(C^{(2)}) & \xleftarrow{t} & \mathrm{Pic}^0(JC) \end{array}$$

the horizontal arrows being isomorphisms. Since $\pi^*$ and $Nm_\pi$ are dual to each other (cf.[Mu]) and $\mathrm{Ker}(\pi^*) = \{0, \eta\}$ we obtain that $t(\lambda_\Theta(\eta)) \in \mathrm{Ker}(\mathrm{Pic}^0(\pi^{(2)}))$. In this way we get the class defining $h_2$. Now restricting this class to $E$ we end the proof by noting that to restrict this class to $E$ is equivalent to taking the norm of $\eta$. ∎

(7.17).- **Proposition**. $T$ is an irreducible smooth curve of genus $g$.

PROOF: By (7.16) $T/j$ is an irreducible smooth elliptic curve. As we indicate in (7.14) $T'$ contains an irreducible curve of genus $g$, namely $S' \cap 2S'$. Therefore $T$ contains an irreducible component of geometric genus $g$. We call this component X. Since

$$f_{1|X} : X \longrightarrow T/j$$

ramifies in at most $2g-2$ points, by applying Riemann-Hurwitz to the normalization of X we conclude that $f_{1|X}$ ramifies in exactly $2g-2$ points. Thus X is smooth and $X = T$. ∎

(7.18).- **Corollary**. The equality $T' = S' \cap 2S'$ holds. Moreover $T'$ is symmetric and the multiplication by -1 induces on $T$ the involution $j$.

(7.19).- Comparing with the construction made in the first part of the section (from (7.1) to (7.6)) we note that $(T, j)$ play the rôle of $(C_2, \tau_2)$. There we obtained a point of ${}_2(JC_2)$ which allowed us to reconstruct $C_1$.

By translating this to the present context we can conclude that there exists an intrinsical way to recognize a certain element of ${}_2JT$. Moreover this class appears in $\mathrm{Im}(f_1^*)$.

Our next aim is to compute this point in terms of the initial data. To do this we imitate the proof of (7.6).

Let $\gamma : T \longrightarrow P$ be the composition of the normalization map with the inclusion $T' \hookrightarrow P$. Since $\gamma(j(x)) = -j(x)$ the induced map between $JT$ and P factorizes through a morphism

$$\tilde{\gamma} : \mathrm{Ker}(Nm_{f_1}) \longrightarrow P.$$

We want to find the kernel of $\tilde{\gamma}$. Previously:

(7.20).- **Lemma**. Let $\tilde{\zeta} \in \mathrm{Pic}^2(\tilde{C})$. Consider the morphism

$$T \hookrightarrow \tilde{C}^{(2)} \xrightarrow{-\tilde{\zeta}} J\tilde{C}$$

an the induced morphism

$$\nu : JT \longrightarrow J\tilde{C}$$

Then: $\mathrm{Im}(\nu_{|\,\mathrm{Ker}(Nm_{f_1})}) \subset P$ and the restriction

$$\tilde{\nu} : \mathrm{Ker}(Nm_{f_1}) \longrightarrow P$$

is $\tilde{\gamma}$.

PROOF: Straighforward. ∎

Now, since the unique non zero element of $\mathrm{Ker}(\tilde{\gamma})$ appears in $\mathrm{Im}(f_1^*)$ it suffices to study the Kernel of $\nu_{|\,\mathrm{Im}(f_1^*)}$.

An easy computation gives the following result:

(7.21).- **Lemma**. The following diagram commutes:

$$\begin{array}{ccc} JT & \xrightarrow{\nu} & J\tilde{C} \\ {\scriptstyle (f_2\circ f_1)^*}\uparrow & & \uparrow{\scriptstyle (\varepsilon\circ\pi)^*} \\ JE & \xrightarrow{\cdot 2} & JE. \end{array}$$

**Corollary**. $\mathrm{Ker}(\tilde{\gamma}) = (f_2 \circ f_1)^*({}_2JE)$.

PROOF: From Lemma (7.20) we have $\mathrm{Ker}(\tilde{\gamma}) = \mathrm{Ker}\ (\tilde{\nu})$. By applying Lemma (7.21) one finds

$$\mathrm{Ker}(\nu\ |_{\mathrm{Im}((f_2\circ f_1)^*)}) = (f_2 \circ f_1)^*({}_2JE) \subset \mathrm{Ker}(Nm_{f_1}).$$

Since $f_2^*$ is surjective $\mathrm{Im}((f_2 \circ f_1)^*) = \mathrm{Im}(f_1^*)$ and hence $\mathrm{Ker}\ (\tilde{\gamma}) = (f_2 \circ f_1)^*({}_2JE)$. ∎

(7.22).-**Theorem**. Let $(\tilde{C}, C), (\tilde{D}, D) \in \mathcal{R}'_{B,g}$ such that $P(\tilde{C}, C) \cong P(\tilde{D}, D)$. Then $(\tilde{C}, C) \cong (\tilde{D}, D)$.

PROOF: It suffices to show that the initial data are determined by $T, j$ and $f_1^*({}_2JE)$. Indeed the non-zero element of $f_1^*({}_2JE)$ gives a point of ${}_2J(T/j)$ that allows us to recover the morphism $T/j \longrightarrow E$.

Now consider the pull-back diagram

$$\begin{array}{ccc} X & \longrightarrow & T^{(2)} \\ \downarrow & & \downarrow \\ E & \xrightarrow{f_2^*} & (T/j)^{(2)}. \end{array}$$

Then, the morphism

$$\begin{aligned} \tilde{C} &\longrightarrow X \\ \tilde{x} &\longrightarrow (\tilde{x} + \tilde{x}') + (\tilde{x} + \iota(\tilde{x}')) \end{aligned}$$

where $\pi(\tilde{x}) + \pi(\tilde{x}') \in \mathrm{Im}(\varepsilon^*)$, is an isomorphism and the involution $j^{(2)}$ of $T^{(2)}$ induces on $\tilde{C}$ the involution $\iota$. ∎

Finally we point out that the constructions used to prove Theorems (7.6) and (7.22) can be compared in order to obtain a bijection between $\mathcal{R}_{B,g,0}$ and $\mathcal{R}'_{B,g}$ commuting with the Prym map. We explain how this map goes.

Start with an element $(\tilde{C}, C) \in \mathcal{R}_{B,g,0}$. With the notations of §2, observe that $t = 0$ implies that $C_1$ is also elliptic. We call $f_1 : E \longrightarrow C_1$ to the transposed morphism. Then the pull-back diagram

$$\begin{array}{ccc} \tilde{C}' & \longrightarrow & C_2^{(2)} \\ \downarrow & & \downarrow{\scriptstyle \varepsilon_2^{(2)}} \\ C_1 & \xrightarrow{f_1^*} & E^{(2)} \end{array}$$

gives an element $(\tilde{C}', C') \in \mathcal{R}'_{B,g}$, where $C' = \tilde{C}'/\iota$, $\iota$ being the restriction to $\tilde{C}'$ of the involution $\iota^{(2)}$.

Conversely, fix $(\tilde{C}', C') \in \mathcal{R}'_{B,g}$. Suppose that $\varepsilon' : C' \longrightarrow E'$ is a bi-elliptic structure of $C'$.

Construct the pull-back diagram

$$\begin{array}{ccc} T & \longrightarrow & \tilde{C}'^{(2)} \\ \downarrow & & \downarrow \\ E' & \xrightarrow{(\varepsilon')^*} & C'^{(2)} \end{array}$$

The involution $\iota^{(2)}$ restricts to an involution $j$ of $T$. Then $T/j$ is an elliptic curve. We call $\varepsilon_1 : E' \longrightarrow T/j$ to the transposed map. By taking again a pull-back diagram we get

$$\begin{array}{ccc} \tilde{C} & \longrightarrow & T \\ \downarrow & & \downarrow{\scriptstyle \varepsilon_2} \\ E' & \xrightarrow{\varepsilon_1} & T/j \end{array}$$

The curve $\tilde{C}$ has two involutions attached to the projections; call $\iota$ the composition of this involutions. Then $(\tilde{C}, \tilde{C}/\iota) \in \mathcal{R}_{B,g,0}$ is the image of $(\tilde{C}', C')$.

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