UNIVERSITAT DE BARCELONA

SECOND ORDER STOCHASTIC DIFFERENTIAL EQUATIONS WITH DIRICHLET BOUNDARY CONDITIONS

by

David Nualart and *Etienne Pardoux*

AMS Subject Classification: 60H10, 60J25

Mathematics Preprint Series No. 78
March 1990

SECOND ORDER STOCHASTIC DIFFERENTIAL EQUATIONS WITH DIRICHLET BOUNDARY CONDITIONS

by

David Nualart
Facultat de Matemàtiques
Universitat de Barcelona
Gran Via, 585
08007 Barcelona - Spain.

Etienne Pardoux
Mathématiques, URA 225
Université de Provence
13331 Marseille Cedex 3
France

Abstract. We consider the second order stochastic differential equation $\ddot{X}_t + f(X_t, \dot{X}_t) = \dot{W}_t$ where t runs on the interval $[0,1]$, $\{W_t\}$ is an ordinary Brownian motion and we impose the Dirichlet boundary conditions $X(0) = a$ and $X(1) = b$. We show pathwise existence and uniqueness of a solution assuming some smoothness and monotonicity conditions on f, and we study the Markov property of the solution using an extended version of the Girsanov theorem due to Kusuoka.

0. Introduction

In this paper we study a second order stochastic differential equation of the type:

$$\frac{d^2 X_t}{dt^2} + f\Big(X_t, \frac{dX_t}{dt}\Big) = \frac{dW_t}{dt} \quad , \tag{0.1}$$

where the time parameter t runs over the interval [0,1] and we impose the Dirichlet type boundary conditions

$$X_0 = a\,,\ X_1 = b\,,$$

a and b being fixed real numbers. Here $\{W_t\}$ is a one–dimensional Brownian motion starting at zero. First we will give sufficient conditions on the function $f : \mathbb{R}^2 \to \mathbb{R}$ for the existence and uniqueness of a solution for any fixed continuous function $W \in C_0(0,1)$.

Then we will study the Markov property of the solution assuming that W is the trajectory of a Brownian motion. We recall the following types of Markov property:

(1) We say that a stochastic process $\{X_t,\ 0 \le t \le 1\}$ is a ***Markov process*** if for any $t \in [0,1]$ the past and the future of $\{X_s\}$ are conditionally independent, given the present state X_t.

(2) We say that $\{X_t,\ 0 \le t \le 1\}$ is a *Markov field* if for any $0 \le s < t \le 1$, the values of the process inside and outside the interval $[s,t]$ are conditionally independent, given X_s and X_t.

(3) We say that $\{X_t,\ 0 \le t \le 1\}$ is a *germ Markov field* if for any $0 \le s < t \le 1$, the values of the process inside and outside the interval $[s,t]$ are conditionally independent, given the germ σ–field $\bigcap_{\epsilon>0} \sigma(X_u, u \in (s-\epsilon, s+\epsilon) \cup (t-\epsilon, t+\epsilon))$.

Our main result is the following. The solution of (0.1) is a Markov process if f is an affine function, and is not a germ Markov field otherwise. The main tool to study the Markov property is an extended version of the Girsanov theorem due to Kusuoka [2], which allows us to compute conditional expectations under a law under which the Markov property is known to hold.

A similar negative result for first order stochastic differential equations with a more general boundary condition has been obtained in the companion paper [5]. See also Donati–Martin [1] for related results concerning another class of stochastic differential equations with boundary conditions.

The organization of the paper is as follows. Section 1 is devoted to show the existence and uniqueness theorems assuming some smoothness and monotonicity conditions on the function f. In section 2 we compute the Radon–Nikodym derivative using Kusuoka's theorem, and finally we study the Markov property in section 3.

1. Existence and uniqueness of a solution

We denote by $C_0([0,1])$ the set of all continuous functions on $[0,1]$ which vanish at zero. Suppose we are given a locally bounded and measurable function $f : \mathbb{R}^2 \to \mathbb{R}$, an element $W \in C_0([0,1])$, and two real numbers $a, b \in \mathbb{R}$. Our aim is to find a solution for the integral equation

$$\dot{X}_t + \int_0^t f(X_s, \dot{X}_s)\, ds = \dot{X}_0 + W_t, \qquad 0 \le t \le 1 \tag{1.1}$$

with the boundary conditions $X_0 = a$, $X_1 = b$. Observe that the equation (1.1) can be formally written as $\ddot{X}_t + f(X_t, \dot{X}_t) = \dot{W}_t$ and, therefore, it can be regarded as a nonlinear second order differential equation.

In the sequel we set $c = b - a$ and we denote by $Y_t(W)$ (or simply Y_t when there is no confusion) the solution of equation (1.1) for $f \equiv 0$. That means,

$$Y_t = a + ct - t\int_0^1 W_s ds + \int_0^t W_s ds\,, \tag{1.2}$$

and we also have

$$\dot{Y}_t = c - \int_0^1 W_s ds + W_t. \tag{1.2$'$}$$

Notice that the transformation $W \to Y(W)$ from $C_0([0,1])$ into the space $C^1_{a,b}(0,1)$ of continuously differentiable functions Y on (0,1) such that $\lim_{t\downarrow 0} Y(t) = a$ and $\lim_{t\uparrow 1} Y(t) = b$ is bijective and for any $Y \in C^1_{a,b}(0,1)$ we can recover W by the formula $W_t = \dot{Y}_t - \dot{Y}_0$.

In order to solve the equation (1.1) when the function f is non zero, we introduce the mapping $T : C_0([0,1]) \to C_0([0,1])$ defined as follows

$$T(W)_t = W_t + \int_0^t f(Y_s\,,\ \dot{Y}_s)\,ds. \tag{1.3}$$

We remark the following two facts:

(I) If $T(\eta) = W$, then the function $X_t = Y_t(\eta)$ is a solution of the equation (1.1). In fact, we have

$$\begin{aligned}\dot{X}_t = \dot{Y}_t(\eta) &= b - a - \int_0^1 \eta_s\,ds + \eta_t \\ &= \dot{X}_0 + W_t - \int_0^t f(X_s\,,\ \dot{X}_s)\,ds\,.\end{aligned}$$

(II) Conversely, if we are given a solution X_t of equation (1.1), then $T\big(Y^{-1}(X)\big) = W$. Indeed, if we set $Y^{-1}(X) = \eta$, then

$$T(\eta)_t = \eta_t + \int_0^t f\big(Y_s(\eta)\,,\ \dot{Y}_s(\eta)\big)\,ds = \eta_t + W_t + \dot{X}_0 - \dot{X}_t = W_t\,.$$

Consequently, we obtain the following result:

Proposition 1.1. *Suppose that T is a bijection. Then equation (1.1) has the unique solution $X = Y(T^{-1}(W))$.*

We are going to present some sufficient conditions on the function f for the transformation T to be bijective.

Proposition 1.2. *Suppose that f is nonincreasing in each coordinate, locally Lipschitz and with linear growth, then T is bijective.*

Proof: Given $\eta \in C_0([0,1])$ we have to show that there exists a unique function $W \in C_0([0,1])$ such that $T(W) = \eta$. Set $V = \eta - W$. Then V satisfies the differential equation

$$\begin{aligned} \dot{V}_t &= f\Big(t \int_0^1 V_s\, ds - \int_0^t V_s\, ds + \xi_t\,,\ \int_0^1 V_s\, ds - V_t + \rho_t\Big)\,, \\ V_0 &= 0, \end{aligned}$$

where

$$\begin{aligned} \xi_t &= a + ct - t\int_0^1 \eta_s\, ds + \int_0^t n_s\, ds\,, \quad \text{and} \\ \rho_t &= c - \int_0^1 \eta_s\, ds + \eta_t\,. \end{aligned} \tag{1.4}$$

For any $x \in \mathbb{R}$ we consider the differential equation

$$\begin{cases} \dot{V}_t(x) = f\big(tx - \int_0^t V_s(x)\, ds + \xi_t\,,\ x - V_t(x) + \rho_t\big) \\ V_0(x) = 0 \end{cases} \tag{1.5}$$

By a comparison theorem for ordinary differential equations and using the monotonicity properties of f we get that the mapping $x \longmapsto V_t(x)$ is continuous and nonincreasing for each $t \in [0,1]$. Therefore, $\int_0^1 V_t(x)dt$ is a nonincreasing and continuous function of x, and this implies the existence of a unique real number x such that $\int_0^1 V_t(x)dt = x$. This completes the proof of the proposition. Q.E.D.

It is also possible to show that T is bijective assuming that f is Lipschitz and the Lipschitz constant of f is small enough:

Proposition 1.3. *Suppose that f is such that $|f(x,y) - f(\bar{x},\bar{y})| \le K\big(|x-\bar{x}| + |y-\bar{y}|\big)$ with $K < \frac{1}{3}$. Then T is bijective.*

Proof: As in the proof of Proposition 1.2 we denote by $V_t(x)$ the solution of equation (1.5). Then it suffices to check that the mapping $x \longmapsto \int_0^1 V_t(x)\, dt$ has a unique fixed point, which is true because under our assumptions this mapping is a contraction:

$$\begin{aligned} \Big| \int_0^1 V_t(x)\, dt - \int_0^1 V_t(\bar{x})\, dt \Big| &\le \sup_{0\le t\le 1} |V_t(x) - V_t(\bar{x})| \\ &\le \frac{3}{2} K\, |x - \bar{x}| + \frac{3}{2} K \sup_{0\le t\le 1} |V_t(x) - V_t(\bar{x})| \end{aligned}$$

Q.E.D.

To conclude this section we discuss the particular case of an affine function $f(x,y) = \alpha x + \beta y + \gamma$. In this particular case we have the following result.

Proposition 1.4. *Suppose that f is affine. Then there exists a unique solution of equation (1.1) for every $W \in C_0(0,1)$ (that means, T is bijective), if the following condition is satisfied*

$$\int_0^1 \left(\exp\left[(1-s)M\right]\right)_{21} (\alpha\, s + \beta)\, ds \neq 1\,, \tag{1.6}$$

where M denotes the matrix $\left[\begin{smallmatrix} -\beta & -\alpha \\ 1 & 0 \end{smallmatrix}\right]$ and the subindex 21 means that we take the entry of the second row and first column.

Proof: We want to show that there exists a unique function $W \in C_0(0,1)$ such that $T(W) = \eta$, for any given function $\eta \in C_0(0,1)$. Setting, as before, $V = \eta - W$, we want to show the existence and uniqueness of a solution for the equation

$$\begin{cases} \dot{V}_t - \alpha\, t \int_0^1 V_s ds + \alpha \int_0^t V_s ds - \alpha \xi_t - \beta \int_0^1 V_s ds + \beta V_t - \beta \rho_t - \gamma = 0 \\ V_0 = 0\,, \end{cases}$$

where ξ_t and ρ_t are defined by (1.4). Putting $\psi_t = \alpha \xi_t + \beta \rho_t + \gamma$, and $U_t = \int_0^t V_s ds$ we obtain the second order differential equation

$$\begin{cases} \ddot{U}_t - (\alpha t + \beta) U_1 + \alpha U_t + \beta \dot{U}_t - \psi_t = 0 \\ U_0 = \dot{U}_0 = 0\,, \end{cases}$$

which can be written in matrix form as

$$\begin{bmatrix} \ddot{U}_t \\ \dot{U}_t \end{bmatrix} = \begin{bmatrix} -\beta & -\alpha \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{U}_t \\ U_t \end{bmatrix} + \begin{bmatrix} (\alpha t + \beta) U_1 + \psi_t \\ 0 \end{bmatrix}.$$

Consequently,

$$U_t = \int_0^t \Big(\exp\left[(t-s)M\right]\Big)_{21} \Big((\alpha s + \beta) U_1 + \psi_s\Big)\, ds\,,$$

and applying condition (1.6) the value of U_1 is uniquely determined and the desired result is proved. Q.E.D.

Note that if the matrix M has two different real and nonzero eigenvalues λ_1 and λ_2, then the condition (1.6) can be rewritten as

$$-(\alpha + \beta) + (\lambda_1 - \lambda_2)^{-1} \left\{ \beta\left(\frac{e^{\lambda_1}}{\lambda_1} - \frac{e^{\lambda_2}}{\lambda_2}\right) + \alpha\left(e^{\lambda_1} - e^{\lambda_2}\right) \right\} \neq 1.$$

2. Computation of a Radon–Nikodym derivative

Let us now introduce the Wiener measure P on the Borel σ–field $\mathcal{F}$ of $\Omega = C_0([0,1])$ so that the continuous function W in the equation (1.1) becomes a path of the Brownian motion. Our aim is to study the Markov property of the stochastic process $\{X_t\}$ solution of (1.1). This will be the contents of the next section. In this section we will construct a new probability measure Q on $C_0([0,1])$ such that the law of $\{X_t\}$ under P is the same as the law of the process $\{Y_t\}$ given by (1.2) under Q, and we will give an explicit expression for the Radon–Nikodym derivative of Q with respect to P. To do this we will apply the following nonadapted extension of the Girsanov theorem proved by Kusuoka (see Theorem 6.4 of [2]).

Theorem 2.1. *Consider a mapping $T : \Omega \to \Omega$ of the form $T(\omega)_t = \omega_t + \int_0^t K_s(\omega)\, ds$, where K is a measurable function from Ω into the Hilbert space $H = L^2(0,1)$, and suppose that the following conditions are satisfied:*

(i) *T is bijective.*

(ii) *For all $\omega \in \Omega$ there exists a Hilbert–Schmidt operator $DK(\omega)$ from H into itself such that:*

(1) $\|K(\omega + \int_0^\cdot h_s ds) - K(\omega) - DK(\omega)(h)\|_H = o(\|h\|_H)$ for all $\omega \in \Omega$ as $\|h\|_H$ tends to zero.

(2) $h \longmapsto DK(\omega + \int_0^\cdot h_s ds)$ is continuous from H into $L^2([0,1]^2)$ for all ω.

(3) $I + DK(\omega) : H \to H$ is invertible, for all ω.

Then the process $\{W_t + \int_0^t K_s(W)\, ds\}$ is a Wiener process under the probability Q on $C_0(0,1)$ given by

$$\frac{dQ}{dP} = |d_c(-DK)| \exp\Big(- \delta(K) - \frac{1}{2}\int_0^1 K_t^2\, dt\Big), \tag{2.1}$$

where $d_c(-DK)$ denotes the Carleman–Fredholm determinant of the square integrable kernel $DK \in L^2([0,1]^2)$ and $\delta(K)$ is the Skorohod stochastic integral of the process K.

We recall that the Carleman–Ferdholm determinant of a square integrable kernel $B \in L^2([0,1]^2)$ is defined by

$$d_c(B) = 1 + \sum_{n=2}^{\infty} \frac{(-1)^n}{n\,!} \int_{[0,1]^n} \det\, \big(\hat{B}(t_i,t_j)\big)\, dt_1 \cdots dt_n\,,$$

where $\hat{B}(t_i,t_j) = B(t_i,t_j)$ if $i \neq j$ and $\hat{B}(t_i,t_i) = 0$. If B is a square matrix then $d_c(B)$ coincides with $\det(I - B)\exp(\mathrm{tr}B)$. We refer to [10] for a survey of the main properties of this determinant.

On the other hand let us recall briefly the notions of derivation on Wiener space and of Skorohod integral. Let $\mathcal{S}$ denote the subset of $L^2(\Omega)$ consisting of those random variables of the form:

$$F = f\Big(\int_0^1 h_1(t)dW_t, \ldots, \int_0^1 h_n(t)dW_t\Big)$$

where $n \in \mathbb{N}$; $h_1, \ldots, h_n \in L^2(0,1)$; $f \in C_b^\infty(\mathbb{R}^n)$. For $F \in \mathcal{S}$, we set

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big(\int_0^1 h_1(t)dW_t, \ldots, \int_0^1 h_n(t)dW_t\Big) h_i(t)$$

and we denote by $\mathbb{D}^{1,2}$ the completion of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{1,2}$ defined by

$$\|F\|_{1,2}^2 = E(F^2) + E\int_0^1 |D_t F|^2 dt, \quad F \in \mathcal{S}.$$

Then the Skorohod integral is the adjoint of the derivation operator D considered as an unbounded operator from $L^2(\Omega)$ into $L^2(\Omega \times [0,1])$. If a process $u \in L^2(\Omega \times [0,1])$ is Skorohod integrable, the Skorohod integral of u denoted by $\delta(u)$ is determined by the duality relation

$$E(\delta(u)F) = E\Big(\int_0^1 u_t D_t F\, dt\Big), \quad F \in \mathcal{S}.$$

From the proof of Theorem 5.2 in Kusuoka's paper [2] it follows that a process K_t verifying the above condition (ii) is locally Skorohod integrable in the sense that there exists a sequence $\{(\Omega_n, K_n)\}$ such that $\Omega_n \in \mathcal{F}$, $K_n \in L^2(0,1;\mathbb{D}^{1,2})$, $n \in \mathbb{N}$; $\Omega_n \uparrow \Omega$ a.s., as $n \to \infty$ and $K = K_n$ on $\Omega_n \times [0,1]$. Then for every n K_n is Skorohod integrable and $\delta(K)$ is well defined by $\delta(K)(\omega) = \delta(K_n)(\omega)$, $\omega \in \Omega_n$, $n \in \mathbb{N}$ (see [4]). For more information about the operators D and δ, we refer in particular to Nualart-Zakai [6] and Nualart-Pardoux [4].

We are going to apply the above theorem to the particular case $K_t = f(Y_t, \dot{Y}_t)$, where Y_t and $\dot{Y}_t$ are given by the expressions (1.2) and (1.2)'. From the properties of the operator D we deduce that

$$D_s Y_t = -t(1-s) + (t-s)^+ = st - s \wedge t \tag{2.2}$$

$$D_s \dot{Y}_t = -(1-s) + \mathbf{1}_{[0,t]}(s) = s - \mathbf{1}_{[t,1]}(s). \tag{2.3}$$

Therefore, if f is a continuously differentiable function, conditions (ii.1) and (ii.2) of Theorem 2.1 are satisfied, and by the chain rule we get

$$D_s\big(f(Y_t, \dot{Y}_t)\big) = f'_x(Y_t, \dot{Y}_t)(st - s \wedge t) + f'_y(Y_t, \dot{Y}_t)\big(s - \mathbf{1}_{[t,1]}(s)\big). \tag{2.4}$$

In the sequel we will use the notation $\alpha_t = f'_x(Y_t, \dot{Y}_t)$ and $\beta_t = f'_y(Y_t, \dot{Y}_t)$. Moreover, we denote by M_t the matrix $\left[\begin{smallmatrix} -\beta_t & -\alpha_t \\ 1 & 0 \end{smallmatrix}\right]$ and by Φ_t the solution of the linear differential equation

$$\begin{aligned} d\Phi_t &= M_t \Phi_t \, dt \\ \Phi_0 &= I \end{aligned} \tag{2.5}$$

We will also denote by $\Phi(t,s)$ the matrix $\Phi_t \Phi_s^{-1}$. We now have:

Proposition 2.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function such that for $K_t = f(Y_t, \dot{Y}_t)$, the transformation T given by (1.3) is bijective. Assume moreover that*

$$\int_0^1 \Phi_{21}(1,s)(s\alpha_s + \beta_s)\, ds \neq 1 \,. \tag{2.6}$$

Then K_t verifies the conditions of Theorem 2.1.

Proof: It remains to show that $I + DK$ is invertible. From the Fredholm alternative, it suffices to check that -1 is not an eigenvalue of $DK(\omega)$, for each $\omega \in \Omega$. Let $h \in L^2(0,1)$ such that $(I + DK)h = 0$. Then

$$h_t + \alpha_t \int_0^1 h_s(st - s \wedge t)\, ds + \beta_t \int_0^1 h_s\big(s - \mathbf{1}_{[t,1]}(s)\big)\, ds = 0,$$

which can be written as

$$h_t + t\alpha_t \int_0^1 s\, h_s\, ds - \alpha_t \int_0^t s\, h_s\, ds - t\alpha_t \int_t^1 h_s\, ds + \beta_t \int_0^1 s\, h_s\, ds - \beta_t \int_t^1 h_s\, ds = 0.$$

Setting $g_t = \int_0^t h_s\, ds$ and $U_t = \int_0^t g_s\, ds$ we obtain

$$\dot{g}_t - (t\alpha_t + \beta_t)\int_0^1 g_s\, ds + \alpha_t \int_0^t g_s\, ds + \beta_t g_t = 0\,,$$

and

$$\ddot{U}_t - (t\alpha_t + \beta_t)U_1 + \alpha_t U_t + \beta_t \dot{U}_t = 0\,. \tag{2.7}$$

Then the solution of the second order differential equation (2.7) is given by

$$U_t = U_1 \int_0^t \Phi_{21}(t,s)(s\alpha_s + \beta_s) ds\,, \tag{2.8}$$

and condition (2.6) implies $U_t = 0$ for all t. Q.E.D.

Let us exhibit two examples of functions f verifying the conditions of Proposition 2.2.

(I) We denote by $\mathcal{C}_b$ the class of continuously differentiable functions $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f'_x \leq 0$, $f'_y \leq 0$ and f has linear growth. Then any function $f \in \mathcal{C}_b$ verifies the assumptions of Proposition 2.2. In fact, Proposition 1.2 implies that T is bijective, and (2.6) is satisfied because we have $s\alpha_s + \beta_s \leq 0$ and $\Phi_{21}(t,s) \geq 0$ for all $t \geq s$.

(II) If f is an affine function, then conditions (2.6) and (1.6) are equivalent.

Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a function verifying the hypotheses of Proposition 2.2. Then we can apply Theorem 2.1 and the process $\eta_t = W_t + \int_0^t f(Y_s(W), \dot{Y}_s(W))ds$ is a Wiener process under the probability Q given by (2.1). The equation (1.1) has a unique pathwise solution $X_t(W)$ given by $X(W) = Y(T^{-1}(W))$ (see Proposition 1.1). Consequently the law of the process $X_t(W)$ under the probability P coincides with the law of $Y_t(W)$ under Q. In fact, $P\{X(W) \in B\} = Q\{X(T(W)) \in B\} = Q\{Y(W) \in B\}$ for any Borel subset B of $C^1_{a,b}(0,1)$.

In order to study the Markov property of the process $X_t(W)$ we need an explicit expression for the Radon–Nikodym derivative $J \equiv \frac{dQ}{dP}$. This expression will be given by the next theorem.

Theorem 2.3. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function satisfying the assumptions of Proposition 2.2. Then we have*

$$\frac{dQ}{dP} = |Z_1| \exp\left(\frac{1}{2}\int_0^1 f'_y(Y_t, \dot{Y}_t)\, dt - \int_0^1 f(Y_t, \dot{Y}_t) \circ dW_t - \frac{1}{2}\int_0^1 f(Y_t, \dot{Y}_t)^2\, dt\right), \tag{2.9}$$

where $\int_0^1 f(Y_t, \dot{Y}_t) \circ dW_t$ is an extended Stratonovich integral (see [4]), and Z_1 is the solution at time $t = 1$ of the second order differential equation

$$\begin{cases} \ddot{Z}_t + \beta_t \dot{Z}_t + \alpha_t Z_t = 0 \\ Z_0 = 0, \quad \dot{Z}_0 = 1, \end{cases} \tag{2.10}$$

α_t and β_t being, as before, $f'_x(Y_t, \dot{Y}_t)$ and $f'_y(Y_t, \dot{Y}_t)$.

The main ingredient in the proof of this theorem is the computation of the Carleman–Fredholm determinant of the kernel DK, where $K_t = f(Y_t, \dot{Y}_t)$. The details of this computation are presented in the following lemma.

Lemma 2.4. *Under the assumptions of Theorem 2.3 we have, for $K_t = f(Y_t, \dot{Y}_t)$,*

$$d_c(-DK) = Z_1 \exp\left(\int_0^1 (t\,\alpha_t + \beta_t)(1-t)dt\right). \tag{2.11}$$

Proof: The idea of the proof is to approximate K by a sequence of elementary processes. For each $n \geq 1$ we introduce the orthonormal functions $e_i = \sqrt{n}\,\mathbf{1}_{[t_{i-1},t_i]}$, $t_i = \frac{i}{n}$, $1 \leq i \leq n$.

Then we define

$$K_t^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f\Big(a + c\,t_i - \frac{t_i}{n} \sum_{j=1}^{n} W(t_j) + \frac{1}{n} \sum_{j=1}^{i-1} W(t_i)\,,$$

$$c - \frac{1}{n} \sum_{j=1}^{n} W(t_j) + W(t_{i-1})\Big)\, e_i(t)\,. \tag{2.12}$$

By taking a subsequence if necessary, it holds that

$$\lim_n \int_0^1 \int_0^1 |D_s K_t - D_s K_t^n|^2 ds dt = 0\,,$$

for every W and, consequently, using the continuity in the Hilbert–Schmidt norm of the Carleman–Fredholm determinant, we deduce that $d_c(-DK^n)$ converges to $d_c(-DK)$ as n tends to infinity, for all W. The kernels DK^n are elementary in the sense that they can be expressed as

$$K_t^n = \sum_{i=1}^{n} \Psi_i^n \left(W(e_1), \ldots, W(e_n)\right) e_i(t)\,, \tag{2.13}$$

where $W(e_i) = \int_0^1 e_i(t) W(dt) = \sqrt{n}\left(W(t_i) - W(t_{i-1})\right)$, and the functions $\Psi^n : \mathbb{R}^n \to \mathbb{R}^n$ are given by

$$\begin{aligned}
\Psi_i^n(v_1, \ldots, v_n) &= \frac{1}{\sqrt{n}} f\Big(a + \frac{ci}{n} - \frac{i}{n^2\sqrt{n}} \big(\sum_{j=1}^{n} (v_1 + \cdots + v_j)\big) \\
&+ \frac{1}{n\sqrt{n}} \big(\sum_{j=1}^{i-1} (v_1 + \cdots + v_j)\big)\,, \\
&c - \frac{1}{n\sqrt{n}} \sum_{j=1}^{n} (v_1 + \cdots + v_j) + \frac{1}{\sqrt{n}} (v_1 + \cdots + v_{i-1})\Big) \\
&= \frac{1}{\sqrt{n}} f\, \Big(a + \frac{ci}{n} - \frac{1}{n^2\sqrt{n}} \Big\{ - nv_1 + (i-2n)v_2 + (2i-3n)v_3 \\
&+ \cdots + \big((i-2)i - (i-1)n\big) v_{i-1} \\
&\quad + (-i)(n-i+1)v_i + (-i)(n-i+2)v_{i+1} + \cdots + (-i)v_n \Big\}\,, \\
&\quad c - \frac{1}{n\sqrt{n}} \big(nv_1 + (n-i)v_2 + \cdots + v_n\big) + \frac{1}{n}(v_1 + \cdots + v_{i-1})\Big) \qquad (2.14) \\
&= \frac{1}{\sqrt{n}} f\left(p_i(v), q_i(v)\right).
\end{aligned}$$

From (2.13) we deduce

$$DK^n = \sum_{i,j=1}^{n} \frac{\partial \Psi_i^n}{\partial x_j} \left(W(e_1), \ldots, W(e_n)\right) e_i \otimes e_j\,,$$

and, therefore, the Carleman–Fredholm determinant of $-DK^n$ is equal to that of the Jacobian matrix of Ψ^n composed with the vector $\big(W(e_1),\dots,W(e_n)\big)$. That means

$$d_c(-DK^n)=\det\left(I_n+J\Psi^n\right)\exp\left(-\operatorname{tr}J\Psi^n\right) \tag{2.15}$$

where $J\Psi^n$ denotes the matrix $\left(\frac{\partial\Psi_i^n}{\partial x_j}\left(W(e_1),\dots,W(e_n)\right)\right)$. From (2.14) we get that

$$(J\Psi^n)_{ij}=\begin{cases} i\le j & -\frac{1}{n^2}\left(\frac{i\alpha_i^n}{n}+\beta_i^n\right)(n-j+1)\\ i>j\ge 1\,, & -\frac{1}{n^2}\left(\left(j-\frac{2i}{n}\right)\alpha_i^n-(j-1)\beta_i^n\right)\\ i>j=1\,, & -\frac{1}{n^2}\,\alpha_i^n\,, \end{cases} \tag{2.16}$$

where α_i^n and β_i^n denote, respectively, the functions $f'_x\big(p_i(v),\,q_i(v)\big)$ and $f'_y\big(p_i(v),\,q_i(v)\big)$ evaluated at $v=(W(e_1),\dots,W(e_n))$. Thus the trace of the matrix $J\Psi^n$ is equal to

$$-\frac{1}{n^2}\sum_{i=1}^{n}(n-i+1)\left(\frac{i\alpha_i^n}{n}+\beta_i^n\right),$$

which converges as n tends to infinity to

$$-\int_0^1\left[t(1-t)\,f'_x(Y_t,\,\dot Y_t)+(1-t)\,f'_y(Y_t,\,\dot Y_t)\right]dt\,,$$

and, therefore, the exponential appearing in the equation (2.15) converges to the exponential term in (2.11). So, it only remains to show that

$$\lim_n\,\det\left(I_n+J\Psi^n\right)=Z_1. \tag{2.16}$$

Substracting every column from the next one we obtain that the determinant $\det\left(I_n+J\Psi^n\right)$ is equal to

$$\det\begin{bmatrix} 1-\frac{1}{n}\left(\frac{\alpha_1^n}{n}+\beta_1^n\right) & -\frac{1}{n^2}\left(\frac{\alpha_1^n}{n}+\beta_1^n\right)(n-1) & -\frac{1}{n^2}\left(\frac{\alpha_1^n}{n}+\beta_1^n\right)(n-2) & \cdots\\ -\frac{1}{n^2}\alpha_2^n & 1-\frac{1}{n^2}\left(\frac{2\alpha_2^n}{n}+\beta_2^n\right)(n-1) & -\frac{1}{n^2}\left(\frac{2\alpha_2^n}{n}+\beta_2^n\right)(n-2) & \cdots\\ -\frac{1}{n^2}\alpha_3^n & -\frac{1}{n^2}\left(\left(2-\frac{3}{n}\right)\alpha_3^n-\beta_3^n\right) & 1-\frac{1}{n^2}\left(\frac{3\alpha_3^n}{n}+\beta_3^n\right)(n-2) & \cdots\\ -\frac{1}{n^2}\alpha_4^n & -\frac{1}{n^2}\left(\left(2-\frac{4}{n}\right)\alpha_4^n-\beta_4^n\right) & -\frac{1}{n^2}\left(\left(3-\frac{8}{n}\right)\alpha_4^n-2\beta_4^n\right) & \cdots\\ & \cdots & & \\ -\frac{1}{n^2}\alpha_n^n & -\frac{1}{n^2}(\alpha_n^n-\beta_n^n) & -\frac{1}{n^2}(\alpha_n^n-2\beta_n^n) & \cdots \end{bmatrix}$$

$$=\det\begin{bmatrix} 1-\frac{1}{n^2}\left(\frac{\alpha_1^n}{n}+\beta_1^n\right) & -\frac{1}{n^2}\left(\frac{\alpha_1^n}{n}+\beta_1^n\right) & \cdots\\ -1+\frac{1}{n^2}\left(\left(1-\frac{2}{n}\right)\alpha_2^n+(n-1)\beta_2^n\right) & 1-\frac{1}{n^2}\left(\frac{2\alpha_2^n}{n}+\beta_2^n\right) & \cdots\\ -\frac{1}{n^2}\left(\left(1-\frac{3}{n}\right)\alpha_3^n-\beta_3^n\right) & -1+\frac{1}{n^2}\left(\left(1-\frac{3}{n}\right)\alpha_3^n+(n-1)\beta_3^n\right) & \cdots\\ -\frac{1}{n^2}\left(\left(1-\frac{4}{n}\right)\alpha_4^n-\beta_4^n\right) & \frac{1}{n^2}\left(\left(1-\frac{4}{n}\right)\alpha_4^n-\beta_4^n\right) & \cdots\\ & \cdots & \\ -\frac{1}{n^2}\beta_n^n & -\frac{1}{n^2}\beta_n^n & \cdots \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 0 & \cdots & -\frac{1}{n^2}\left(\frac{\alpha_1^n}{n} + \beta_1^n\right) \\ -2 + \frac{1}{n^2}(\alpha_2^n + n\beta_2^n) & 1 & \cdots & -\frac{1}{n^2}\left(\frac{2\alpha_2^n}{n} + \beta_2^n\right) \\ 1 - \frac{\beta_3^n}{n} & -2 + \frac{1}{n^2}(\alpha_3^n + n\beta_3^n) & \cdots & -\frac{1}{n^2}\left(\frac{3\alpha_3^n}{n} + \beta_3^n\right) \\ 0 & 1 - \frac{\beta_4^n}{n} & \cdots & -\frac{1}{n^2}\left(\frac{4\alpha_4^n}{n} + \beta_4^n\right) \\ & \cdots & & \\ 0 & 0 & \cdots & 1 - \frac{1}{n^2}(\alpha_n^n + \beta_n^n) \end{bmatrix}$$

In conclusion we have to evaluate the determinant of the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & -\sigma_1 \\ \rho_2 & 1 & 0 & 0 & \cdots & -\sigma_2 \\ \pi_3 & \rho_3 & 1 & 0 & \cdots & -\sigma_3 \\ 0 & \pi_4 & \rho_4 & 1 & \cdots & -\sigma_4 \\ & \cdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 - \sigma_n \end{bmatrix}$$

where $\sigma_i = \frac{1}{n^2}\left(\frac{i\alpha_i^n}{n} + \beta_i^n\right)$, $\rho_i = -2 + \frac{1}{n^2}(\alpha_i^n + n\beta_i^n)$ and $\pi_i = 1 - \frac{\beta_i^n}{n}$. Define recursively $\varepsilon_1 = \sigma_1$, $\varepsilon_2 = \sigma_2 - \rho_2\varepsilon_1$ and $\varepsilon_k = \sigma_k - \rho_k\varepsilon_{k-1} - \pi_k\varepsilon_{k-2}$, $3 \le k \le n-1$. Then, multiplying the first $n-1$ columns of the matrix M by $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}$ and addding the result to the last column we obtain that

$$\begin{aligned} \det(I_n + J\Psi_n) &= (1 - \sigma_n) + \rho_n\,\varepsilon_{n-1} + \pi_n\,\varepsilon_{n-2} \\ &= \Big(1 - \frac{1}{n^2}(\alpha_n^n + \beta_n^n)\Big) + \Big(-2 + \frac{1}{n^2}(\alpha_n^n + n\beta_n^n)\Big)\,\varepsilon_{n-1} \\ &+ \big(1 - \frac{\beta_n^n}{n}\big)\,\varepsilon_{n-2}\,. \end{aligned}$$

Notice that $\alpha_{[nt]}^n$ converges uniformly in t to $\alpha_t = f_x'(Y_t, \dot{Y}_t)$ as n tends to infinity, and in the same way $\beta_{[nt]}^n$ converges to β_t. In particular, α_n^n and β_n^n converge, respectively to $f_x'(Y_1, \dot{Y}_1)$ and $f_y'(Y_1, \dot{Y}_1)$. Therefore, in order to show (2.16) it suffices to prove that $\lim_n \varepsilon_{n-1} = \lim_n \varepsilon_{n-2} = 1 - Z_1$. Actually, it holds that $\lim_n \varepsilon_{[nt]} = t - Z_t$ for every $t \in [0,1]$. In fact, making a discretization of the differential equation (2.10) and approximating the coefficients $\alpha_{k/n}$ and $\beta_{k/n}$ by α_k^n and β_k^n, respectively, we obtain the recursive equation

$$n^2(\delta_k - 2\delta_{k-1} + \delta_{k-2}) + \beta_k^n(n\delta_{k-1} - \delta_{k-2}) + \alpha_k^n\delta_{k-1} = 0,$$

where δ_k is an approximation of $Z(k/n)$. If we substitute δ_k for $\frac{k}{n} - \varepsilon_k$ we obtain the recursive equation which defines ε_k. We omit the additional details of this limit argument. The proof of the lemma is now complete. Q.E.D.

Proof of Theorem 2.3: From the expression (2.1) and Lemma 2.4 we deduce that

$$\frac{dQ}{dP} = |Z_1| \exp\Big(\int_0^1 (t\, f_x'(Y_t, \dot{Y}_t) + f_y'(Y_t, \dot{Y}_t))\,(1-t)\,dt$$

$$- \int_0^1 f(Y_t, \dot{Y}_t)\, d W_t - \frac{1}{2} \int_0^1 f(Y_t, \dot{Y}_t)^2\, dt \Big), \tag{2.17}$$

where here $\int_0^1 f(Y_t, \dot{Y}_t)\, d W_t$ denotes the Skorohod integral of $f(Y_t, \dot{Y}_t)$. Now from the results of [4] we know that this Skorohod integral is equal to the extended Stratonovich integral minus a complementary term given by $\frac{1}{2} \int_0^1 \left[D_t^+ (f(Y_t, \dot{Y}_t)) + D_t^- (f(Y_t, \dot{Y}_t))\right] dt$, where $D_t^+ K_t = \lim_{\varepsilon \downarrow 0} D_{t+\varepsilon} K_t$ and $D_t^- K_t = \lim_{\varepsilon \downarrow 0} D_{t-\varepsilon} K_t$. We remark that the process $f(Y_t, \dot{Y}_t)$ belongs to the class $\mathbb{L}^{1,2}_{C,\mathrm{loc}}$ (see [4]) which allows to apply these results. Then, from (2.4) we obtain that

$$\int_0^1 f(Y_t, \dot{Y}_t) d W_t = \int_0^1 f(Y_t, \dot{Y}_t) \circ d W_t - \int_0^1 \left[t(t-1)\, \alpha_t + (t - \frac{1}{2})\beta_t\right] dt, \tag{2.18}$$

and substituting (2.18) into (2.17) we get the desired result. Q.E.D.

3. The Markov property.

In this section we want to study the Markov properties of the process $\{X_t\}$ solution of equation (1.1), where $\{W_t\}$ is a standard Brownian motion. As a solution of a second order stochastic differential equation we might conjecture that this process is 2–Markovian (see, for instance, Russek [9]), that means, the two dimensional process $\{(X_t, \dot{X}_t)\}$ is a Markov process. We first show that this is true for the process $\{Y_t\}$ i.e., when $f \equiv 0$.

Proposition 3.1. *The process $\{(Y_t, \dot{Y}_t),\ 0 \le t \le 1\}$ defined by the equations (1.2) and (1.2)' is a Markov process.*

Proof: Let $\psi(x, y)$ be a real valued bounded and measurable function. Fix $s < t$ and set

$\zeta = \int_0^1 W_t\, dt$. We have to compute the conditional expectation

$$
\begin{aligned}
&E\big(\psi(Y_t, \dot{Y}_t)|(Y_r, \dot{Y}_r),\ 0 \le r \le s\big) \\
&= E\big(\psi(a + ct - t\zeta + \int_0^t W_u du,\ c - \zeta + W_t)\,\big|\,\zeta,\ W_r,\ 0 \le r \le s\big) \\
&= E\big(\psi(a + ct - t\zeta + \int_0^s W_u du + \int_s^t (W_u - W_s) du + (t-s) W_s, \\
&\qquad c - \zeta + W_t - W_s + W_s)|\zeta,\ W_r,\ 0 \le r \le s\big) \\
&= \int_{\mathbb{R}^2} \psi(a + ct - t\zeta + \int_0^s W_u\, du + x + (t-s) W_s,\ c - \zeta + y + W_s) \\
&\cdot N\Big(\Big(\frac{(t-s)^2(3 - 2s - t)}{2(1-s)^3} \int_s^1 (W_u - W_s)\, du, \\
&\frac{3(t-s)(2-s-t)}{2(1-s)^3} \int_s^1 (W_u - W_s)\, du\Big),\ \wedge\Big)\,(dx,\, dy)
\end{aligned}
$$

where $\wedge$ denotes the conditional covariance matrix of the Gaussian vector $(\int_s^t (W_u - W_s)\, du,\ W_t - W_s)$, given $\int_s^1 (W_t - W_s) dt$. Consequently, the above conditional expectation will be a function of the random variables

$$
\begin{aligned}
-t\zeta + \int_0^s W_u\, du + (t-s)\, W_s &= (t-s)\dot{Y}_s + Y_s - ct - a\,, \\
-\zeta + W_s &= \dot{Y}_s - c\,, \\
\int_s^1 (W_u - W_s)\, du &= -\,(1-s)\dot{Y}_s - Y_s + a + c\,,
\end{aligned}
$$

and this implies the Markov property. Q.E.D.

We will see that, except in the linear case, the Markov property does not hold for the process $\{X_t\}$. One might think that the Markov field property is better adapted to our equation because we impose fixed values at the boundary points $t = 0$ and $t = 1$. However this is not the case and, as we shall see, the nonlinearity of the function f prevents for any type of Markov property. The main result of this section is the following.

Theorem 3.2. *Let $\{X_t,\ t \in [0,1]\}$ be the solution of equation (1.1) where $\{W_t,\ t \in [0,1]\}$ is an ordinary Brownian motion. Then,*

(i) *If f is an affine function verifying condition (1.6) the process $\{(X_t,\ \dot{X}_t)\}$ is a Markov process.*

(ii) *If f is a five times continuously differentiable function of the class $\mathcal{C}_b$ (i.e. $f'_x \le 0$, $f'_y \le 0$ and f has linear growth) and the process $\{(X_t, \dot{X}_t);\ t \in [0,1]\}$ is a Markov field, then the function f must be affine.*

Before proving this theorem let us show a preliminary technical lemma. We recall (see [4]) that $\mathbb{D}^{1,2}_{\mathrm{loc}}$ is the set of random variables F such that there exists $\{(\Omega_n, F_n)\} \subset \mathcal{F} \times \mathbb{D}^{1,2}$ with $\Omega_n \uparrow \Omega$ a.s. and $F = F_n$ a.s. on Ω_n.

Lemma 3.3. *Let $\mathcal{G}_t$ be the σ–algebra generated by Y_t, $\dot{Y}_t$ and $\int_0^1 W_s\, ds$. That means $\mathcal{G}_t = \sigma\left\{ \int_0^1 W_s ds, W_t, \int_0^t W_s ds \right\}$. Let F be a random variable in the space $\mathbb{D}^{1,2}_{\mathrm{loc}}$ such that $F\,\mathbf{1}_G$ is $\mathcal{G}_t$–measurable for some set $G \in \mathcal{G}_t$. Then there exist random variables $A_t(\omega)$, $B_t(\omega)$, and $C_t(\omega)$ such that*

$$D_\theta F(\omega) = \left[A_t(\omega)\theta + C_t(\omega)\right] \mathbf{1}_{[0,t]}(\theta) + B_t(\omega)(\theta - 1)\, \mathbf{1}_{[t,1]}(\theta)\,,$$

for $dP \times d\theta$ almost all $(\omega, \theta) \in G \times [0,1]$.

Proof: Consider the subspace K of $H = L^2(0,1)$ spanned by the derivatives of the generators of the σ–algebra $\mathcal{G}_t$. This is the so–called tangent space of the σ–algebra $\mathcal{G}_t$ and, in our case, it is the deterministic subspace spanned by (see the expressions (2.2) and (2.3))

$$D_\theta Y_t = t(1-\theta) + (t-\theta)\mathbf{1}_{[0,t]}(\theta) = \theta(t-1)\mathbf{1}_{[0,t]}(\theta) + (\theta-1)t\,\mathbf{1}_{[t,1]}(\theta)\,, \tag{3.1}$$

$$D_\theta \dot{Y}_t = -(1-\theta) + \mathbf{1}_{[0,t]}(\theta) = \theta\,\mathbf{1}_{[0,t]}(\theta) + (\theta-1)\mathbf{1}_{[t,1]}(\theta)\,, \tag{3.2}$$

and

$$D_\theta\Big(\int_0^1 W_t\, dt\Big) = 1 - \theta\,.$$

Thus, K is the three–dimensional subspace generated by $\theta\mathbf{1}_{[0,t]}(\theta)$, $\mathbf{1}_{[0,t]}(\theta)$ and $(\theta - 1)\mathbf{1}_{[t,1]}(\theta)$. Then the fact that the σ–algebra $\mathcal{G}_t$ is generated by a finite number of random variables of the first chaos allows to apply Lemma 4.5 of [5] and to conclude that DF belongs to K a.s. on G, which gives the result. Q.E.D.

Remark 3.4. Let $\Phi(t)$ be the solution of the linear system (2.5). Then, the components of the matrix $\Phi(t)$ satisfy the relations

$$\begin{aligned}
\dot{\Phi}_{11}(t) &= -\beta_t\Phi_{11}(t) - \alpha_t\Phi_{21}(t),\\
\dot{\Phi}_{21}(t) &= \Phi_{11}(t),\\
\dot{\Phi}_{12}(t) &= -\beta_t\Phi_{12}(t) - \alpha_t\Phi_{22}(t),\\
\dot{\Phi}_{22}(t) &= \Phi_{12}(t).
\end{aligned}$$

Observe that the entries of the matrix $\Phi(t)$ are nonnegative and $\Phi_{11}(t) > 0$, $\Phi_{21}(t) > 0$ and $\Phi_{22}(t) > 0$ for $t \in (0,1)$ if we assume $\alpha_t \leq 0$ and $\beta_t \leq 0$. The same results hold for the entries of $\Phi(t,s)$ for $0 < s < t$. The inverse matrix $\Phi^{-1}(t)$ solves the linear system $\dot{\widehat{\Phi^{-1}}}(t) = -\Phi^{-1}(t)M_t$. Consequently, we also have

$$\dot{\Phi}(1,t) = \Phi(1)\dot{\widehat{\Phi^{-1}}}(t) = -\Phi(1)\Phi^{-1}(t)M_t = -\Phi(1,t)M_t,$$

that means

$$\begin{aligned}
\dot{\Phi}_{11}(1,t) &= \beta_t\Phi_{11}(1,t) - \Phi_{12}(1,t),\\
\dot{\Phi}_{12}(1,t) &= \alpha_t\Phi_{11}(1,t),\\
\dot{\Phi}_{21}(1,t) &= \beta_t\Phi_{21}(1,t) - \Phi_{22}(1,t),\\
\dot{\Phi}_{22}(1,t) &= \alpha_t\Phi_{21}(1,t).
\end{aligned}$$

Proof of Theorem 3.3. Let Q be the probability measure on on $C_0([0,1])$ given by Theorem 2.3. From the results of Section 2 we know that the law of the process $\{X_t\}$ under P is the same as the law of $\{Y_t\}$ under Q. Therefore, we can replace the process $\{(X_t, \dot{X}_t)\}$ by $\{(Y_t, \dot{Y}_t)\}$ and the probability P by Q in the statement of the theorem. By Proposition 3.1 we already know that $\{(Y_t, \dot{Y}_t)\}$ is a Markov process under P and now we have to study the Markov property with respect to an equivalent probability measure Q. For any fixed $t \in (0,1)$ and using Theorem 2.3 we can factorize the Radon–Nikodym derivative $J = \frac{dQ}{dP}$ as follows

$$J = |Z_1| L_t L^t\ ,$$

where

$$L_t = \exp\Big(\frac{1}{2}\int_0^t f'_y(Y_s, \dot{Y}_s)\,ds - \int_0^t f(Y_s, \dot{Y}_s)\circ dW_s - \frac{1}{2}\int_0^t f(Y_s, \dot{Y}_s)^2\,ds\Big),$$

$$L^t = \exp\Big(\frac{1}{2}\int_t^1 f'_y(Y_s, \dot{Y}_s)\,ds - \int_t^1 f(Y_s, \dot{Y}_s)\circ dW_s - \frac{1}{2}\int_t^1 f(Y_s, \dot{Y}_s)^2\,ds\Big).$$

We define the σ–algebras

$\mathcal{F}_t = \sigma\left\{(Y_s, \dot{Y}_s),\ 0 \leq s \leq t\right\}$

$\mathcal{F}^t = \sigma\left\{(Y_s, \dot{Y}_s),\ t \leq s \leq 1\right\}$, and $\mathcal{F}_0^t = \mathcal{F}^t \vee \sigma\{Y_0, \dot{Y}_0\} = \mathcal{F}^t \vee \sigma\{\int_0^1 W_t\,dt\}$.

For any random variable ξ integrable with respect to Q we set

$$\wedge_\xi = E_Q(\xi/\mathcal{F}_t) = \frac{E_P(\xi J/\mathcal{F}_t)}{E_P(J/\mathcal{F}_t)}. \tag{3.3}$$

Then we have

$$\wedge_\xi = \frac{E_P(\xi|Z_1|L^t/\mathcal{F}_t)}{E_P(|Z_1|L^t/\mathcal{F}_t)}, \tag{3.4}$$

because L_t is $\mathcal{F}_t$–measurable.

(i) Suppose first that f is an affine function verifying (1.6). In that case Z_1 is deterministic and we get

$$\wedge_\xi = \frac{E_P(\xi L^t/\mathcal{F}_t)}{E_P(L^t/\mathcal{F}_t)}.$$

Then if ξ is $\mathcal{F}^t$–measurable, using the fact that L^t is also $\mathcal{F}^t$–measurable and applying the Markov property of $(Y_t, \dot{Y}_t)$ under P we deduce that $\wedge_\xi$ is $\sigma\{Y_t, \dot{Y}_t\}$–measurable and this implies that $\{(Y_t, \dot{Y}_t);\, t \in [0,1]\}$ is a Markov process under Q.

(ii) To prove the second assertion of the theorem we suppose that $\{(Y_t,\, \dot{Y}_t);\, t \in [0,1]\}$ is a Markov field under Q. This implies in particular that for any $t \in (0,1)$ and any $\mathcal{F}_0^t$–measurable random variable ξ, integrable with respect to Q, the conditional expectation $\wedge_\xi = E_Q(\xi/\mathcal{F}_t)$ is $\mathcal{G}_t = \sigma\{Y_t,\, \dot{Y}_t,\, \int_0^1 W_t\, dt\}$–measurable. Recall that $f \in \mathcal{C}_b$ implies $f'_x \leq 0$ and $f'_y \leq 0$ and, therefore, $Z_t \geq 0$ for all t because Z_t is given by the equation (2.10). Consequently, we can put $|Z_1| = Z_1$ in the formula (3.4). We can also transform the expression (3.4) by means of a suitable decomposition of the random variable Z_1. Set

$$\begin{bmatrix} \dot{Z}_1 \\ Z_1 \end{bmatrix} = \Phi(1,t) \begin{bmatrix} \dot{Z}_t \\ Z_t \end{bmatrix}, \tag{3.5}$$

that means,

$$Z_1 = \Phi_{21}(1,t)\dot{Z}_t + \Phi_{22}(1,t)Z_t. \tag{3.6}$$

In the sequel we will denote by $\overline{F}$ the conditional expectation of the random variable F under P with respect to the σ–algebra $\mathcal{G}_t$. The random variables Z_t and $\dot{Z}_t$ are $\mathcal{F}_t$–measurable and, on the other hand, $\Phi_{21}(1,t)$, and $\Phi_{22}(1,t)$ are $\mathcal{F}^t$–measurable. Thus, from (3.4) and (3.6) and applying the Markov field property of $(Y_t, \dot{Y}_t)$ under P we deduce that

$$\wedge_\xi = \frac{\overline{\xi\, \Phi_{21}(1,t)L^t}\, \dot{Z}_t + \overline{\xi\, \Phi_{22}(1,t)L^t} Z_t}{\overline{\Phi_{21}(1,t)L^t}\dot{Z}_t + \overline{\Phi_{22}(1,t)L^t} Z_t}$$

and by our hypotheses this expression is $\mathcal{G}_t$–measurable. Therefore we obtain the following equation

$$\{\, \wedge_\xi\, \overline{\Phi_{21}(1,t)L^t} - \overline{\xi\Phi_{21}(1,t)L^t}\}\dot{Z}_t + \{\, \wedge_\xi\, \overline{\Phi_{22}(1,t)L^t} - \overline{\xi\Phi_{22}(1,t)L^t}\}Z_t = 0, \tag{3.7}$$

which is valid for any $\mathcal{F}_0^t$–measurable random variable ξ integrable with respect to Q. We are going to apply this equation to the following random variables

$$\begin{aligned} \xi_1(t) &= [\Phi_{21}(1,t)L^t]^{-1} \\ \xi_2(t) &= [\Phi_{22}(1,t)L^t]^{-1}, \end{aligned} \tag{3.8}$$

assuming that $t \in (0,1)$. First we remark that for $t \in (0,1)$ $\xi_i(t)$ is Q–integrable and nonnegative. In fact, using the equations satisfied by the matrix $\Phi(t,s)$ as a function of t, (see Remark 3.4) we have $0 \le \Phi_{21}(1,t)^{-1} \le (1-t)^{-1}$ and $0 \le \Phi_{22}(1,t)^{-1} \le 1$. Then we define

$$\begin{aligned}
A_t^1 &= \wedge_{\xi_1}(t)\overline{\Phi_{21}(1,t)L^t} - 1\\
B_t^1 &= \wedge_{\xi_1}(t)\overline{\Phi_{22}(1,t)L^t} - \overline{[\Phi_{21}(1,t)]^{-1}\Phi_{22}(1,t)}\\
A_t^2 &= \wedge_{\xi_2}(t)\overline{\Phi_{21}(1,t)L^t} - \overline{\Phi_{21}(1,t)[\Phi_{22}(1,t)]^{-1}}\\
B_t^2 &= \wedge_{\xi_2}(t)\overline{\Phi_{22}(1,t)L^t} - 1
\end{aligned}$$

From (3.7) and (3.8) we deduce

$$A_t^i \dot{Z}_t + B_t^i Z_t = 0,\ i = 1,2. \tag{3.9}$$

Observe that the processes A_t^i and B_t^i are $\mathcal{G}_t$–adapted. For any $t \in (0,1)$ we define the set

$$G_t = \left\{ A_t^1 = A_t^2 = 0 \right\} \in \mathcal{G}_t\,. \tag{3.10}$$

Note that on G_t we also have $B_t^1 = B_t^2 = 0$ because $Z_t > 0$ for $t \in (0,1)$. Then the rest of the proof will be done into several steps.

Step 1: The random variables $\mathbf{1}_{G_t}\frac{\Phi_{21}(1,t)}{\Phi_{22}(1,t)}$ and $\mathbf{1}_{G_t^c}\frac{\Phi_{21}(t)}{\Phi_{11}(t)}$ are $\mathcal{G}_t$–measurable.

Proof of Step 1: The definition of the set G_t leads to the following equalities

$$\begin{aligned}
\frac{1}{\overline{\Phi_{21}(1,t)L^t}} &= \frac{\overline{[\Phi_{21}(1,t)]^{-1}\Phi_{22}(1,t)}}{\overline{\Phi_{22}(1,t)L^t}}\\
\frac{\overline{[\Phi_{22}(1,t)]^{-1}\Phi_{21}(1,t)}}{\overline{\Phi_{21}(1,t)L^t}} &= \frac{1}{\overline{\Phi_{22}(1,t)L^t}},
\end{aligned}$$

a.s. on G_t. Consequently, we obtain $\overline{\Phi_{21}(1,t)[\Phi_{22}(1,t)]^{-1}} = \left\{\overline{[\Phi_{21}(1,t)]^{-1}\Phi_{22}(1,t)}\right\}^{-1}$, and by the strict Jensen inequality applied to the measure space $(G_t, \mathcal{F}|_{G_t}, P)$ we get that the random variable $\mathbf{1}_{G_t}\frac{\Phi_{21}(1,t)}{\Phi_{22}(1,t)}$ is G_t–measurable. Note that $Z_t = \Phi_{21}(t)$, and from Remark 3.4 we deduce $\dot{Z}_t = \Phi_{11}(t)$. Therefore on the set G_t^c we have that the random variable

$$\frac{Z_t}{\dot{Z}_t} = \frac{\Phi_{21}(t)}{\Phi_{11}(t)}$$

is $\mathcal{G}_t$–measurable.

Step 2: Two basic inequalities [(3.18) and (3.19) below].

Proof of Step 2: We define

$$\varphi_t = \frac{\Phi_{21}(1,t)}{\Phi_{22}(1,t)} \qquad \text{and} \qquad \psi_t = \frac{Z_t}{\dot{Z}_t} = \frac{\Phi_{21}(t)}{\Phi_{11}(t)}. \tag{3.11}$$

From the properties of the matrices $\Phi(t,s)$ (see Remark 3.4), we deduce that φ_t and ψ_t are continuously differentiable processes on [0,1], $\varphi_1 = 0$, $\psi_0 = 0$, $\varphi_t > 0$ for $t \in [0,1)$ and $\psi_t > 0$ for $t \in (0,1]$. Also, from the linear differential equations satisfied by $\Phi(t)$ and $\Phi(1,t)$ we can derive Ricatti type differential equations for φ_t and ψ_t. In fact, differentiating with respect to t the equations

$$\Phi_{21}(1,t) = \varphi_t \Phi_{22}(1,t)$$

$$\Phi_{21}(t) = \psi_t \Phi_{11}(t),$$

and using the relations given in Remark 3.4 we obtain

$$\dot{\varphi}_t = \beta_t \varphi_t - \alpha_t \varphi_t^2 - 1; \quad \varphi_1 = 0 \tag{3.12}$$

$$\dot{\psi}_t = \beta_t \psi_t + \alpha_t \psi_t^2 + 1; \quad \psi_0 = 0 \tag{3.13}$$

It is not hard to show that the random variables $\Phi_{21}(1,t)$, $\Phi_{22}(1,t)$, $\Phi_{21}(t)$ and $\Phi_{11}(t)$ belong to the space $\mathbb{D}^{1,2}_{\mathrm{loc}}$, for any $t \in (0,1)$. Consequently, the same is true for the random variables φ_t and ψ_t.

Applying the operator D, which commutes with the derivative with respect to the time variable, to the equations (3.12) and (3.13) gives

$$\frac{d}{dt} D_\theta \varphi_t = (\beta_t - 2\varphi_t \alpha_t) D_\theta \varphi_t + \varphi_t D_\theta \beta_t - \varphi_t^2 D_\theta \alpha_t, \qquad D_\theta \varphi_1 = 0,$$
$$\frac{d}{dt} D_\theta \psi_t = (\beta_t + 2\alpha_t \psi_t) D_\theta \psi_t + \psi_t D_\theta \beta_t + \psi_t^2 D_\theta \alpha_t, \qquad D_\theta \psi_0 = 0.$$

These linear differential equations can be solved and we get

$$D_\theta \varphi_t = -\int_t^1 \gamma_{ts} (\varphi_s D_\theta \beta_s - \varphi_s^2 D_\theta \alpha_s) ds \tag{3.14}$$

$$D_\theta \psi_t = \int_0^t \varepsilon_{ts} (\psi_s D_\theta \beta_s + \psi_s^2 D_\theta \alpha_s) ds, \tag{3.15}$$

where

$$\gamma_{ts} = \exp\left(-\int_t^s (\beta_r - 2\varphi_r \alpha_r) dr\right), \qquad \text{and}$$
$$\varepsilon_{ts} = \exp\left(\int_s^t (\beta_r + 2\psi_r \alpha_r) dr\right), \qquad \text{for any} \quad s,t \in [0,1].$$

In order to get a more explicit expression for the derivatives $D_\theta \varphi_t$, $D_\theta \psi_t$ we have to compute $D_\theta \beta_t$ and $D_\theta \alpha_t$. Henceforth we will use the following notations

$$f''_{xx}(s) = f''_{xx}(Y_s, \dot{Y}_s), \; f''_{xy}(s) = f''_{xy}(Y_s, \dot{Y}_s), \; f''_{yy}(s) = f''_{yy}(Y_s, \dot{Y}_s).$$

From (3.1) and (3.2) we obtain

$$D_\theta \alpha_t = \big[(t-1)f''_{xx}(t) + f''_{xy}(t)\big]\theta \mathbf{1}_{[0,t]}(\theta) + \big[tf''_{xx}(t) + f''_{xy}(t)\big](\theta - 1)\mathbf{1}_{[t,1]}(\theta)\,, \qquad (3.16)$$

and

$$D_\theta \beta_t = \big[(t-1)f''_{xy}(t) + f''_{yy}(t)\big]\theta \mathbf{1}_{[0,t]}(\theta) + \big[tf''_{xy}(t) + f''_{yy}(t)\big](\theta - 1)\mathbf{1}_{[t,1]}(\theta)\,. \qquad (3.17)$$

Set

$$\begin{aligned} R_1(s) &= \varphi_s f''_{xy}(s) - \varphi_s^2 f''_{xx}(s) \\ T_1(s) &= \varphi_s f''_{yy}(s) - \varphi_s^2 f''_{xy}(s) \\ R_2(s) &= \psi_s f''_{xy}(s) + \psi_s^2 f''_{xx}(s) \\ T_2(s) &= \psi_s f''_{yy}(s) + \psi_s^2 f''_{xy}(s) \end{aligned}$$

We want to show the following equalities:

$$\int_\theta^1 R_1(s)\gamma_{\theta s} ds + T_1(\theta) = 0, \qquad \text{for} \qquad \omega \in G_t, \quad t \le \theta \le 1, \quad \text{a.e.} \qquad (3.18)$$

$$\int_0^\theta R_2(s)\varepsilon_{\theta s} ds - T_2(\theta) = 0, \qquad \text{for} \qquad \omega \in G_t^c, \quad 0 \le \theta \le t, \quad \text{a.e.} \qquad (3.19).$$

Proof of (3.18) and (3.19):

We will first show the equality (3.18). Using the expressions (3.16) and (3.17) we get the following formula for $D_\theta \varphi_t$, if $\theta \ge t$ and $\omega \in G_t$,

$$D_\theta \varphi_t = -(\theta - 1)\int_t^\theta \gamma_{ts} a_2(s) ds - \theta \int_\theta^1 \gamma_{ts} a_1(s) ds\,, \qquad (3.20)$$

where

$$\begin{aligned} a_1(s) &= (s-1)R_1(s) + T_1(s), \qquad \text{and} \\ a_2(s) &= sR_1(s) + T_1(s). \end{aligned}$$

Now by step 1 of the proof and applying Lemma 3.3 to the random variable φ_t and to the set G_t, there exists a random variable $\Gamma_1(t)$ such that

$$D_\theta \varphi_t = (\theta - 1)\Gamma_1(t)\,, \qquad (3.21)$$

for all $\theta \in [t,1]$, $\omega \in G_t$, a.e. Comparing (3.20) with (3.21) gives by choosing $\theta = t$

$$\Gamma_1(t) = \frac{-t}{1-t}\int_t^1 \gamma_{ts} a_1(s)\, ds\,,$$

and hence

$$(\theta-1)\int_t^\theta \gamma_{ts}a_2(s)ds+\theta\int_\theta^1 \gamma_{ts}a_1(s)ds=\frac{t(\theta-1)}{t-1}\int_t^1 \gamma_{ts}a_1(s)ds\,, \tag{3.22}$$

for $\omega\in G_t$ and $\theta\in[t,1]$, a.e. Multiplying the expression (3.22) by γ_{1t} and differentiating with respect to θ we obtain

$$\begin{aligned}\int_t^\theta \gamma_{1s}a_2(s)ds+(\theta-1)\gamma_{1\theta}a_2(\theta)+\int_\theta^1 \gamma_{1s}a_1(s)ds-\theta\gamma_{1\theta}a_1(\theta)\\ =\frac{t}{t-1}\int_t^1 \gamma_{1s}a_1(s)ds\,.\end{aligned} \tag{3.23}$$

Put $\theta=1$ in (3.23) and observe that $a_1(1)=0$ because $\varphi_1=0$. This implies

$$\int_t^1 \gamma_{1s}\big((t-1)a_2(s)-ta_1(s)\big)ds=0\,. \tag{3.24}$$

From (3.22) (with γ_{ts} replaced by γ_{1s}) and (3.24) we get

$$\int_\theta^1 \gamma_{1s}\big[(\theta-1)a_2(s)-\theta a_1(s)\big]ds=0\,, \tag{3.25}$$

that means

$$\int_\theta^1 \gamma_{1s}\big[(\theta-s)R_1(s)-T_1(s)\big]ds=0\,,$$

which implies (3.18) by differentiating with respect to θ. The proof of the equality (3.19) would follow exactly the same steps. To avoid repetitions we omit the details of this proof.

Step 3: The second derivative f''_{yy} is identically zero.

Proof of Step 3: The equations (3.18) and (3.19) imply that $T_1(s)$ is differentiable on $(t,1)$ for $\omega\in G_t$ a.s., and $T_2(s)$ is differentiable on $(0,t)$ for $\omega\in G_t^c$ a.s. Consequently the quadratic variation of these functions must vanish on these intervals. We can compute these quadratic variations applying the extended Itô–Stratonovich formula (see [4]) to the processes $T_1(s)$ and $T_2(s)$. In this way we obtain

$$f^{(3)}_{yyy}(s)-\varphi_s f^{(3)}_{xyy}(s)=0,\qquad s\in[t,1],\quad \omega\in G_t\quad \text{a.e.}, \tag{3.26}$$

and

$$f^{(3)}_{yyy}(s)+\psi_s f^{(3)}_{xyy}(s)=0,\qquad s\in[0,t],\quad \omega\in G_t^c\quad \text{a.e.} \tag{3.27}$$

Computing again the quadratic variation yields

$$f^{(4)}_{yyyy}(s)-\varphi_s f^{(4)}_{xyyy}(s)=0,\qquad s\in[t,1],\quad \omega\in G_t\quad \text{a.e.}, \tag{3.28}$$

and

$$f^{(4)}_{yyyy}(s) + \psi_s f^{(4)}_{xyyy}(s) = 0, \qquad s \in [0,t], \quad \omega \in G^c_t \quad \text{a.e.} \tag{3.29}$$

Now we can use again the generalized Itô–Stratonovich stochastic calculus to compute the differential of the processes appearing in the preceding equations. In particular, differentiating the left hand sides of the equalities (3.26) and (3.27) yields

$$f^{(4)}_{yyyx}(s)\dot{Y}_s - f^{(4)}_{xyyx}(s)\dot{Y}_s\varphi_s - f^{(3)}_{xyy}(s)\dot{\varphi}_s = 0, \qquad s \in [t,1], \quad \omega \in G_t \quad \text{a.e.}, \tag{3.30}$$

and

$$f^{(4)}_{yyyx}(s)\dot{Y}_s + f^{(4)}_{xyyx}(s)\dot{Y}_s\psi_s + f^{(3)}_{xyy}(s)\dot{\psi}_s = 0, \qquad s \in [0,t], \quad \omega \in G^c_t \quad \text{a.e.} \tag{3.31}$$

Now we compute the quadratic variation of the functions appearing in the equations (3.30) and (3.31) and we get

$$\begin{aligned} f^{(5)}_{yyyyx}(s)\dot{Y}_s - f^{(5)}_{xxyyy}(s)\dot{Y}_s\varphi_s + f^{(4)}_{yyyx}(s) - \varphi_s f^{(4)}_{xyyx}(s) - \dot{\varphi}_s f^{(4)}_{yyyx}(s) \\ - f^{(3)}_{xyy}(s)(\varphi_s f''_{yy}(s) - \varphi_s^2 f''_{xy}(s)) = 0, \end{aligned} \tag{3.32}$$

for $s \in [t,1]$ and $\omega \in G_t$ a.e., and

$$\begin{aligned} f^{(5)}_{yyyyx}(s)\dot{Y}_s + f^{(5)}_{xxyyy}(s)\dot{Y}_s\psi_s + f^{(4)}_{yyyx}(s) + \psi_s f^{(4)}_{xyyx}(s) + \dot{\psi}_s f^{(4)}_{yyyx}(s) \\ + f^{(3)}_{xyy}(s)(\psi_s f''_{yy}(s) + \psi_s^2 f''_{xy}(s)) = 0, \end{aligned} \tag{3.33}$$

for $s \in [0,t]$ and $\omega \in G^c_t$ a.e. Using once more the extended Itô–Stratonovich stochastic calculus, we differentiate the functions appearing in the equations (3.28) and (3.29) and we obtain

$$f^{(5)}_{yyyyx}(s)\dot{Y}_s - f^{(5)}_{xxyyy}(s)\dot{Y}_s\varphi_s - \dot{\varphi}_s f^{(4)}_{yyyx}(s) = 0, \qquad s \in [t,1], \quad \omega \in G_t \quad \text{a.e.}, \tag{3.34}$$

and

$$f^{(5)}_{yyyyx}(s)\dot{Y}_s + f^{(5)}_{xxyyy}(s)\dot{Y}_s\psi_s + \dot{\psi}_s f^{(4)}_{yyyx}(s) = 0, \qquad s \in [0,t], \quad \omega \in G^c_t \quad \text{a.e.} \tag{3.35}$$

Substituting (3.34) and (3.35) into (3.32) and (3.33), respectively, and using (3.30) and (3.31) yields

$$f^{(3)}_{xyy}(s)\left(\dot{\varphi}_s - \dot{Y}_s T_1(s)\right) = 0, \qquad s \in [t,1], \quad \omega \in G_t \quad \text{a.e.}, \tag{3.36}$$

and

$$f^{(3)}_{xyy}(s)\left(\dot{\psi}_s - \dot{Y}_s T_2(s)\right) = 0, \qquad s \in [0,t], \quad \omega \in G^c_t \quad \text{a.e.} \tag{3.37}$$

Define $F_1(s) = \dot{\varphi}_s - \dot{Y}_s T_1(s)$ and $F_2(s) = \dot{\psi}_s - \dot{Y}_s T_2(s)$. Note that the quadratic variations of these functions vanish for $s \in [t,1]$, $\omega \in G_t$ and $s \in [0,t]$, $\omega \in G_t^c$, respectively. On the other hand, if we differentiate these functions we obtain:

$$\begin{aligned}\dot{F}_1(s) &= \left(f''_{xy}(s)\varphi_s - f''_{xx}(s)\varphi_s^2\right)\dot{Y}_s \\ &\quad + (\beta_s - 2\alpha_s\varphi_s)\dot{\varphi}_s - \dot{Y}_s\dot{T}_1(s), \qquad s \in [t,1], \quad \omega \in G_t \quad \text{a.e.,}\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}\dot{F}_2(s) &= \left(f''_{xy}(s)\psi_s + f''_{xx}(s)\psi_s^2\right)\dot{Y}_s \\ &\quad + (\beta_s + 2\alpha_s\psi_s)\dot{\psi}_s - \dot{Y}_s\dot{T}_2(s), \qquad s \in [0,t], \quad \omega \in G_t^c \quad \text{a.e.}\end{aligned} \tag{3.39}$$

We come back now to the equations (3.18) and (3.19). If we differentiate the left hand side of these equations we get

$$R_1(s) + T_1(s)(\beta_s - 2\alpha_s\varphi_s) - \dot{T}_1(s) = 0, \qquad s \in [t,1], \quad \omega \in G_t \quad \text{a.e.,} \tag{3.40}$$

$$R_2(s) + T_2(s)(\beta_s + 2\alpha_s\psi_s) - \dot{T}_2(s) = 0, \qquad s \in [0,t], \quad \omega \in G_t^c \quad \text{a.e.} \tag{3.41}$$

So, from (3.38), (3.39), (3.40) and (3.41) we deduce

$$\dot{F}_1(s) = (\beta_s - 2\alpha_s\varphi_s)F_1(s), \qquad s \in [t,1], \quad \omega \in G_t \quad \text{a.e.,} \tag{3.42}$$

$$\dot{F}_2(s) = (\beta_s + 2\alpha_s\psi_s)F_2(s), \qquad s \in [0,t], \quad \omega \in G_t^c \quad \text{a.e.} \tag{3.43}$$

Moreover, $F_1(1) = -1$ and $F_2(0) = 1$. Being the solutions of linear equations, we have $F_1(s) < 0$ a.s. for $s \in [t,1]$ and $\omega \in G_t$, and $F_2(s) > 0$ a.s. for $s \in [0,1]$ and $\omega \in G_t^c$. Consequently, from (3.36) and (3.37) we get $f^{(3)}_{xyy}(s) = 0$ for $s \in [t,1]$ and $\omega \in G_t$ a.s., and also for $s \in [0,t]$ and $\omega \in G_t^c$ a.s. Now from (3.26) and (3.27) we deduce $f^{(3)}_{yyy}(s) = 0$ for the same values of s and ω.

We apply now the generalized Itô–Stratonovich formula to the processes Y_s and $\dot{Y}_s$, and to the function $f''_{yy}(x,y)$, and we get

$$f''_{yy}(Y_1,\dot{Y}_1) = f''_{yy}(Y_t,\dot{Y}_t) + \int_t^1 f^{(3)}_{yyx}(Y_u,\dot{Y}_u)\dot{Y}_u du + \int_t^1 f^{(3)}_{yyy}(Y_u,\dot{Y}_u) \circ dW_u\,,$$

and

$$f''_{yy}(Y_t,\dot{Y}_t) = f''_{yy}(Y_0,\dot{Y}_0) + \int_0^t f^{(3)}_{yyx}(Y_u,\dot{Y}_u)\dot{Y}_u du + \int_0^t f^{(3)}_{yyy}(Y_u,\dot{Y}_u) \circ dW_u\,,$$

Therefore for all t in $(0,1)$ we have $f''_{yy}(Y_t,\dot{Y}_t) = f''_{yy}(Y_1,\dot{Y}_1)$ on G_t a.s. and $f''_{yy}(Y_t,\dot{Y}_t) = f''_{yy}(Y_0,\dot{Y}_0)$ on G_t^c a.s. Putting $s = 1$ in the equation (3.40) and $s = 0$ in (3.41), and using the fact that $\varphi_1 = 0$, $\dot{\varphi}_1 = -1$, $\psi_0 = 0$ and $\dot{\psi}_0 = 1$ we deduce $f''_{yy}(1) = 0$ on G_t and

$f''_{yy}(0) = 0$ on G_t^c. Therefore the process $\{f''_{yy}(t)\}$ vanishes a.s., and we have $f''_{yy}(x,y) = 0$ for all x, y.

Step 4: From Step 3 we know that the function f is of the form $f(x,y) = yf_1(x) + f_2(x)$. We are going to show that $f_1' = 0$ and $f_2'' = 0$. First notice that our hypothesis $f_x' \leq 0$ implies $f_1'(x) = 0$ for all x. On the other hand, the equations (3.18) and (3.19) can be written now in the following form

$$\int_\theta^1 \varphi_s^2 f_2''(Y_s)\gamma_{\theta s} ds = 0, \tag{3.44}$$

for $\theta \in [t,1]$ and $\omega \in G_t$, and

$$\int_0^\theta \psi_s^2 f_2''(Y_s)\varepsilon_{\theta s} ds = 0, \tag{3.45}$$

for $s \in [0,t]$ and $\omega \in G_t^c$. Differentiating these functions, we obtain that the process $f_2''(Y_s)$ is identically zero. Therefore, $f_2''(x) = 0$ for all x, which completes the proof of Step 4. Q.E.D.

Remark 3.4

(1) If the function $f(x,y)$ depends only on the variable x or on the variable y, then the last part of the proof can be simplified. In fact, in these particular cases the equations (3.18) and (3.19) imply directly that the second derivative of f vanishes. Moreover, in this case we only need f to be twice continuously differentiable.

(2) If we assume in part (ii) of Theorem 3.2 that the process $\{(X_t, \dot{X}_t)\}$ is a Markov field for each choice of the boundary conditions $a, b \in \mathbb{R}$ then the last part of the proof can also be simplified and we only need f of class $\mathcal{C}^3$.

Corollary 3.5. *Under the conditions of Part (ii) of Theorem 3.2, if $\{(X_t, \dot{X}_t)\}$ is a germ Markov field, then f is affine.*

Proof : Fix $t > 0$ and for any $\epsilon > 0$ consider the σ -field defined by $\mathcal{G}_\epsilon = \sigma(Y_u, \dot{Y}_u, u \in [0,\epsilon) \cup (t-\epsilon, t+\epsilon))$. It suffices to check that the germ σ-field $\bigcap_{\epsilon>0} \mathcal{G}_\epsilon$ coincides up to zero measure sets with the σ-field generated by Y_0, $\dot{Y}_0$, Y_t and $\dot{Y}_t$, which is $\mathcal{G}_0 = \sigma(\int_0^t W_s ds, \int_t^1 W_s ds, W_t)$. Denote by H_0 the linear span in $L^2([0,1])$ of the functions $s \to \mathbf{1}_{[0,t]}(s)$, $s \to \mathbf{1}_{[0,t]}(s)(t-s)$, and $1-s$. For each $\epsilon > 0$ let H_ϵ be the minimal closed space containing H_0 and the functions $\mathbf{1}_B$, B being a Borel subset of $[0,\epsilon) \cup (t-\epsilon, t+\epsilon)$.

Clearly we have that for every $\epsilon \geq 0$ $\mathcal{G}_\epsilon$ is the σ–field generated by the random variables $\int_0^1 h_t dW_t$, $h \in H_\epsilon$. By Lemma 3.3 of [3] the germ σ–field $\bigcap_{\epsilon>0} \mathcal{G}_\epsilon$ is equal to the σ–field generated by the stochastic integrals of the functions of the space $\bigcap_{\epsilon>0} H_\epsilon$. Consequently, it suffices to check that the intersection of the subspaces H_ϵ is H_0, and this is straightforward. Q.E.D.

REFERENCES

[1] DONATI–MARTIN, C.: Equations différentielles stochastiques dans $\mathbb{R}$ avec conditions au bord. Preprint.

[2] KUSUOKA, S.: The nonlinear transformation of Gaussian measure on Banach space and its absolute continuity (I). J. Fac. Sci. Univ. Tokyo. Sec. IA, 567–597 (1982).

[3] MANDREKAR, V.: Germ–field Markov property for multiparameter processes. Seminaire de Probabilites X. Lecture Notes in Math. 511, 78–85 (1976)

[4] NUALART, D., PARDOUX, E.: Stochastic Calculus with Anticipating Integrands. Probab. Theory Rel. Fields 78, 535–581 (1988).

[5] NUALART, D., PARDOUX, E.: Boundary value problems for stochastic differential equations. Preprint.

[6] NUALART, D., ZAKAI, M.: Generalized stochastic integrals and the Malliavin Calculus. Probab. Theory Rel. Fields 73, 255–280 (1986).

[7] OCONE, D., PARDOUX, E.: Linear Stochastic Differential Equations with Boundary conditions. Probab. Theory Rel. Fields 82, 489-526 (1989).

[8] RAMER, R.: On non–linear transformations of Gaussian measures. J. Funct. Anal. 15, 166–187 (1974).

[9] RUSSEK, A.: Gaussian n–Markovian processes and stochastic boundary value problems. Z. Wahrscheinlichkeittheorie und verw. Gebiete 53, 117–122 (1980).

[10] SMITHIES, F.: Integral Equations. Cambridge Univ. Press, 1958.