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LOWER SUBDIFFERENTIABILITY OF QUADRATIC FUNCTIONS

by

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Lower subdifferentiability of quadratic functions

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Abstract

In this paper we characterize those quadratic functions whose restrictions to a convex set are boundedly lower subdifferentiable and, for the case of closed hyperbolic convex sets, those which are lower subdifferentiable but not boundedly lower subdifferentiable.

Once characterized, we will study the applicability of the cutting plane algorithm of Plastria to problems where the objective function is quadratic and boundedly lower subdifferentiable.

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1 Introduction

The notion of lower subdifferentiability was introduced by Plastria [12], as a relaxation of the concept of subdifferentiability of convex analysis. The motivation for introducing this new notion was algorithmic, since Plastria

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proved that the classical cutting plane method of Kelley for convex optimization also works, under appropriate assumptions, using lower subgradients to generate the cutting planes [12,13]. He also observed that lower subdifferentiability of a function implies quasiconvexity. Later, it was shown [8,11] that the notion of lower subdifferentiability can be obtained as a particular case of the c-subdifferentiability of Balder [3], within the framework of the generalized conjugation theory of Moreau [10], and conditions were given under which a quasiconvex fuction is lower subdifferentiable [8]. In the latter paper, relations between the lower subdifferential, the tangential of Crouzeix [5] and the quasisubdifferential of Greenberg and Pierskalla [6] were studied. Some further results on lower subdifferentiable functions defined on locally convex spaces can be found in [9]; applications in the field of fractional programming are given in [4].

Based on the above mentioned results on lower subdifferentiability, we regard this concept as a kind of qualified quasiconvexity which, on one hand, is not too restrictive and, on the other, provides a new tool in quasiconvex analysis which, in some aspects, plays a role similar to that played by the subgradient in convex analysis.

Quasiconvexity of quadratic functions has been investigated by several authors (see [16,1] and the references contained therein). Motivated by our belief that lower subdifferentiability is probably the appropriate condition one has to impose to quasiconvex functions in order to obtain a useful theory parallel, to some extent, to convex analysis, we address in this paper to the problem of finding conditions under which the restriction of a quadratic function to a convex domain is lower subdifferentiable. We rely upon the fundamental work of S. Schaible [16] characterizing quasiconvex quadratic functions. When using his results in the text, we refer to the recent book [1] on generalized concavity.

We shall use the following notation. By $\overline{\mathbb{R}}$ we shall denote the extended real line $[-\infty, +\infty]$; by $\mathbb{R}^+$ the set of real nonnegative numbers; the Euclidean scalar product of x and y, vectors of $\mathbb{R}^n$, will be denoted by $x^T y$, where ^{T} indicates transposition. We shall denote by $\| \cdot \|$ the Euclidean norm and by $\mathcal{B}(0; N)$ the closed ball with radius N and center the origin. For nonnecessarily square matrices B we will consider the norm subordinate to the Euclidean vectorial norm, it is,

$$\| B \| = \max_{\| x \| = 1} \| Bx \| = \sqrt{\rho(B^T B)}\ ,$$

where ρ denotes the spectral radius. If the matrix is not square, we will say that it is *orthogonal* if we have $B^T B = I$, where I represents the identity matrix. We will use the notation $\operatorname{diag}(a_1, \ldots, a_n)$ to denote a diagonal matrix having $a_1, \ldots, a_n$ as its diagonal entries.

The same symbol A is used to denote a real matrix $m \times n$ A and the corresponding linear transformation $x \longrightarrow Ax$ from $\mathbb{R}^n$ to $\mathbb{R}^m$. For notions of convex analysis we will use the standard terminology and notation of [14], with the following exceptions: we will denote by $\operatorname{co} K$ and $\overline{\operatorname{co}}\, K$ the convex and closed convex hull of the set $K \subset \mathbb{R}^n$, respectively. A convex set K is solid if $\operatorname{int} K \neq \emptyset$. A function is merely quasiconvex if it is quasiconvex but not convex.

We will consider quadratic functions $Q(x) = \frac{1}{2}x^T A x + b^T x$, where A is a $n \times n$ real symmetric matrix and $b \in \mathbb{R}^n$.

Plastria [12] extended the notion of subdifferentiability as follows:

Definition 1.1 *Let $f : K \subset \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$. We say that f is* lower subdifferentiable *at x if $f(x) \in \mathbb{R}$ and there exists $x^* \in \mathbb{R}^n$ such that*

$$f(y) \geq f(x) + x^{*T}(y - x) \quad \textit{for every } y \in K \textit{ with } f(y) < f(x).$$

Vector x^* is called *lower subgradient of f at x*. The set of lower subgradients of f at x is the *lower subdifferential of f at x* and will be denoted by $\partial^- f(x)$.

Let $f : K \subset \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$. We say that f is *lower subdifferentiable* if it is lower subdifferentiable at each point of K. We will write lower subdifferentiable as l.s.d. If there exists $N > 0$ such that $\partial^- f(x) \cap B(0; N) \neq \emptyset$ for all $x \in K$, we say that f is *boundedly lower subdifferentiable* (b.l.s.d.); in this case, N is called a *b.l.s.d.-bound of f*.

The paper is organized as follows. In Section 2, we give some characterization of b.l.s.d. quadratic functions defined on convex domains. In Section 3, we state necessary conditions for lower subdifferentiability of quadratic functions which, for hyperbolic (in the sense of [2]) closed convex sets, turn out to be also sufficient. Finally, in Section 4 we study the specialization of the cutting plane algorithm of Plastria [12] to the case of a quadratic objective function.

2 Quadratic b.l.s.d. functions

In this section the following lemma will be useful:

Lemma 2.1 *Let $K \subset \mathbb{R}^n$ be a convex set and let $f : \operatorname{cl} K \to \mathbb{R}$ be a continuous function. Then the following statements are equivalent:*

1. f is quasiconvex on ri K,

2. f is quasiconvex on K,

3. f is quasiconvex on cl K.

Proof: Since the implications 3.⇒2.⇒1. are obvious, we only have to prove that 1.⇒3.. Let x, y be two points in $\mathrm{cl}\, K$ and let $\lambda \in [0,1]$. Since, by [14, p.46, th.6.3], $\mathrm{cl}\, K = \mathrm{cl}\,(\mathrm{ri}\, K)$, there exist two sequences $(x_n), (y_n)$ in $\mathrm{ri}\, K$ that converge to x, y respectively. By the continuity of f, the sequences $(f(x_n)), (f(y_n))$ converge to $f(x), f(y)$ respectively. Since f is quasiconvex on $\mathrm{ri}\, K$, we have

$$f((1-\lambda)x_n + \lambda y_n) \leq \max\{f(x_n), f(y_n)\} .$$

Taking limits as $n \to \infty$, we have

$$f((1-\lambda)x + \lambda y) \leq \max\{f(x), f(y)\} ,$$

which concludes the proof. □

Remark. Even in the one-dimensional case, the continuity assumption in the preceding statement can not be replaced by that of lower semicontinuity (consider, e.g., $f : [0,1] \to \mathbb{R}$ defined by $f(0) = f(1) = 0$ and $f(x) = 1$ for $x \in (0,1)$). However, it is easy to see that upper semicontinuity suffices for the above equivalences to hold in the case of functions of one real variable.

However, when $n > 1$, upper semicontinuity is not enough (take, e.g., $f : [-1,1] \times [-1,1] \to \mathbb{R}$ given by $f(x,y) = 0$ if $x \neq 1$, $1 - y^2$ if $x = 1$).

Lemma 2.2 *Let $K \subset \mathbb{R}^n$ be a convex set and let f :* cl $K \to \mathbb{R}$ *be a continuous function. Then the following statements are equivalent:*

1. $f_{|\mathrm{ri}\, K}$ is b.l.s.d. with b.l.s.d. bound N,

2. $f_{|K}$ is b.l.s.d. with b.l.s.d. bound N,

3. $f_{|\mathrm{cl}\, K}$ is b.l.s.d. with b.l.s.d. bound N.

Proof: As in Lemma 2.1, we only need to prove the implication 1.⇒3. Let $x_0 \in \mathrm{cl}\, K$. By [14, p.46, th.6.3], $\mathrm{cl}\, K = \mathrm{cl}\,(\mathrm{ri}\, K)$; hence, there exists a sequence (x_n), with $x_n \in \operatorname{ri} K$, that converges to x_0. Since $f_{|\operatorname{ri} K}$ is b.l.s.d. with b.l.s.d. bound N, there exists a sequence

$$(x_n^*) \text{ with } x_n^* \in \partial^- f_{|\operatorname{ri} K}(x_n) \cap \mathcal{B}(0; N).$$

Since (x_n^*) lies in a compact set, we can assume, without loss of generality, that it converges to some $x^* \in \mathcal{B}(0; N)$.

We shall prove that $x^* \in \partial^- f_{|\operatorname{cl} K}(x_0)$.

Let $x \in \operatorname{cl} K$ be such that $f(x) < f(x_0)$; then, because of the convergence of (x_n) to x_0 and the continuity of f, there exists n_0 such that $f(x) < f(x_n)$ for every $n \geq n_0$. Hence, if $x \in \operatorname{ri} K$, we have

$$f(x) \geq f(x_n) + x_n^{*T}(x - x_n) \quad \text{for } n \geq n_0 ,$$

whence, taking the limit as $n \to \infty$, we obtain

$$f(x) \geq f(x_0) + x^{*T}(x - x_0) .$$

In the general case, we can write $x = \lim_{n\to\infty} x'_n$, where $x'_n \in \operatorname{ri} K$ and, analogously, we can find n'_0 such that $f(x'_n) < f(x_0)$ for every $n \geq n'_0$. By the preceding result, we have

$$f(x'_n) \geq f(x_0) + x^{*T}(x'_n - x_0) \quad \text{for } n \geq n'_0,$$

whence, taking limits,

$$f(x) \geq f(x_0) + x^{*T}(x - x_0) ,$$

from where we deduce that $x^* \in \partial^- f_{|\operatorname{cl} K}(x_0)$, and then $\partial^- f_{|\operatorname{cl} K}(x_0) \cap \mathcal{B}(0; N) \neq \emptyset$, as we wanted to prove. □

Remark. Lemma 2.2 becomes false when one replaces b.l.s.d. by l.s.d. in its statements, as shown by the function $f : [-1, 1] \to \mathbb{R}$ given by $f(x) = -\sqrt{1 - x^2}$ (see [12, p.39]).

Given a set $K \subset \mathbb{R}^n$, as in [8, p.218], we will denote by $\Phi(K)$ the union of the projections of K onto the hyperplanes whose intersection with K is nonempty, that is,

$$\Phi(K) = \bigcup_{\substack{H \text{ hyperplane} \\ H \cap K \neq \emptyset}} \Pi_H(K),$$

where Π_H denotes projection onto H.

If the set K is bounded, one has

Lemma 2.3 [8, p.218, lemma 4.18] *If $K \subset \mathbb{R}^n$ is bounded, then $\Phi(K)$ is bounded.*

A characterization of merely quasiconvex quadratic functions that are b.l.s.d. appears in the following theorem:

Theorem 2.4 *Let $K \subset \mathbb{R}^n$ be a solid convex set and let $Q(x) = \frac{1}{2}x^T Ax + b^T x$ be merely quasiconvex on K. Then the following statements are equivalent:*

1. *$Q_{|K}$ is b.l.s.d.,*
2. *Q is Lipschitzian on K,*
3. *Q is bounded below on K,*
4. *AK is bounded.*

Proof: Implication 1.$\Rightarrow$2. is true for any (nonnecessarily quadratic) function Q (see [12, p.39, th.2.2]).

2.$\Rightarrow$3. By the continuity of Q, we only need to prove that Q is bounded below on $\operatorname{int} K$. Let $x_0 \in \operatorname{int} K$. By [1, p.182, th.6.6], the matrix

$$A + \frac{1}{2(\delta - Q(x_0))} \nabla Q(x_0) \nabla Q(x_0)^T$$

is positive semidefinite, where $\delta = Q(s)$, s being a stationary point of Q. By [1, p.173, th.6.2], A has exactly one negative eigenvalue λ_1. Let t_1 be a unitary eigenvector of A associated with λ_1 and let N be a Lipschitz constant of Q on K. Then

$$0 \le {t_1}^T A t_1 + \frac{1}{2(\delta - Q(x_0))} {t_1}^T \nabla Q(x_0) \nabla Q(x_0)^T t_1 =$$

$$= \lambda_1 + \frac{1}{2(\delta - Q(x_0))} \left(\nabla Q(x_0)^T t_1\right)^2 \le \lambda_1 + \frac{N^2}{2(\delta - Q(x_0))},$$

hence

$$Q(x_0) \ge \delta + \frac{N^2}{2\lambda_1}.$$

3.$\Rightarrow$4. According to [1, Section 6.1], there exists a bijective affine transformation $x = Py + v$ from $\mathbb{R}^n$ into itself such that the composite function

$$G(y) = Q(Py + v)$$

can be written as $G(y) = \frac{1}{2}y^T \Lambda y + \delta$, where $\Lambda = \operatorname{diag}(-1, 1, \ldots, 1, 0, \ldots, 0)$ and $\delta \in \mathbb{R}$. Let $r = \operatorname{rank} \Lambda$ and $D = \{y \in \mathbb{R}^n \mid x = Py + v \in K\}$. Since Q is bounded below on K , so is G on D . Let m be a strict lower bound of G on D. Without loss of generality, we may assume that, for every $y = (y_1, \ldots, y_n)^T \in D$, we have $y_1 \geq 0$ [1, Section 6.1]. Let

$$E = \{y \in \mathbb{R}^n \mid G(y) \leq m \ , y_1 \geq 0\} \ .$$

By [1, p.173, th.6.2], $m < \delta$ and E is convex. Hence, since D and E are disjoint convex sets, there exist real numbers $\alpha_1, \ldots, \alpha_n, p$ such that $\sum_{i=1}^n {\alpha_i}^2 > 0$,

$$\sum_{i=1}^n \alpha_i y_i \geq p \quad \text{for all} \quad y \in D$$

and

$$\sum_{i=1}^n \alpha_i y_i \leq p \quad \text{for all} \quad y \in E \ .$$

We must have $\alpha_{r+1} = \cdots = \alpha_n = 0$, because G does not depend on $y_{r+1}, \ldots, y_n$. Since $(y_1, 0, \ldots, 0)^T \in E$ for large enough y_1 , we have $\alpha_1 \leq 0$. If we had $\alpha_1 = 0$, then, as

$$\left(\sqrt{M^2 \sum_{i=2}^r {\alpha_i}^2 + 2(\delta - m)}, M\alpha_2, \ldots, M\alpha_r, 0, \ldots, 0\right)^T \in E$$

for any real number M, taking $M > \dfrac{p}{\sum_{i=2}^r {\alpha_i}^2}$ we should obtain a point in E not belonging to the halfspace defined by $\sum_{i=1}^n \alpha_i y_i \leq p$, which is a contradiction. Therefore, $\alpha_1 < 0$. Without loss of generality, we may assume that $\alpha_1 = -1$. Then, for any $y_2, \ldots, y_r \in \mathbb{R}$, the function

$$\beta(y_2, \ldots, y_r) = -\sqrt{\sum_{i=2}^r {y_i}^2 + 2(\delta - m)} + \sum_{i=2}^r \alpha_i y_i$$

is bounded above by p (since $(-\sqrt{\sum_{i=2}^r {y_i}^2 + 2(\delta - m)}, y_2, \ldots, y_r, 0, \ldots, 0)^T \in E$). Let $\alpha = \sqrt{\sum_{i=2}^r {\alpha_i}^2}$. Since

$$\beta(\lambda\alpha_2, \ldots, \lambda\alpha_r) = \frac{\lambda\alpha^2(\alpha^2 - 1) - \frac{2(\delta - m)}{\lambda}}{\alpha^2 + \sqrt{\alpha^2 + \frac{2(\delta - m)}{\lambda^2}}}$$

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must remain bounded above when $\lambda \to +\infty$, we have $\alpha \leq 1$. If we had $\alpha = 1$, by $\lim_{\lambda\to\infty} \beta(\lambda\alpha_2, \ldots, \lambda\alpha_r) = 0$, we should have $p \geq 0$; then, for any $y \in D$, using Cauchy-Schwarz's inequality, we should obtain

$$0 \leq p \leq -y_1 + \sum_{i=2}^{r} \alpha_i y_i \leq -y_1 + \sqrt{\sum_{i=2}^{r} \alpha_i{}^2}\sqrt{\sum_{i=2}^{r} y_i{}^2} =$$

$$= -y_1 + \sqrt{\sum_{i=2}^{r} y_i{}^2} \leq 0$$

(the last inequality being a consequence of the relation $G(y) \leq \delta$ [1, p.174, th.6.3]), implying D to be contained in the hyperplane defined by $-y_1 + \sum_{i=2}^{r} \alpha_i y_i = 0$ and thus contradicting the hypothesis that K is solid. Therefore, $\alpha < 1$.

Given any $y \in D$, using again Cauchy-Schwarz's inequality and the relation $G(y) \leq \delta$, we obtain

$$y_1 \leq \sum_{i=2}^{r} \alpha_i y_i - p \leq \alpha \sqrt{\sum_{i=2}^{r} y_i{}^2} - p \leq \alpha y_1 - p\ ,$$

whence, by $\alpha < 1$, y_1 is bounded on D. Finally, for any $i = 2, \ldots, r$, we have

$$|y_i| \leq \sqrt{\sum_{k=2}^{r} y_k{}^2} \leq y_1\ ;$$

thus, all variables y_i, $i = 2, \ldots, r$, are bounded on D. This proves that ΛD is bounded. Therefore, since

$$\Lambda D = P^T A P D = P^T A(K - v) = P^T A K - P^T A v$$

and P^T is nonsingular, we conclude that AK is bounded.

4.$\Rightarrow$1. We shall first assume that K itself is bounded. By Lemma 2.2, it suffices to prove that $Q_{|\mathrm{int}\, K}$ is b.l.s.d.. Let $x_0 \in \mathrm{int}\, K$ be a nonminimal point for Q. Since K is bounded, so is $\Phi(K)$ (see Lemma 2.3) and thus Q is Lipschitzian on $\Phi(K)$. We shall prove that

$$\frac{N}{\| \nabla Q(x_0) \|} \nabla Q(x_0) \in \partial^- Q_{|\mathrm{int}\, K}(x_0)\ ,$$

where N is a Lipschitz constant for Q on $\Phi(K)$. Let $\bar{x} \in \mathrm{int}\, K$ be such that $Q(\bar{x}) < Q(x_0)$ and let x' be the projection of $\bar{x}$ onto the hyperplane defined by the equation $\nabla Q(x_0)^T x = \nabla Q(x_0)^T x_0$. We have $\nabla Q(x_0)^T(x' - x_0) = 0$ and hence, by the pseudoconvexity of Q on $\operatorname{int} K$ [1, p.179, cor.6.4], $Q(x') \geq Q(x_0)$. Since $\bar{x}, x' \in \Phi(K)$ and $\nabla Q(x_0)^T(\bar{x} - x_0) < 0$ (which follows from the pseudoconvexity of Q on $\operatorname{int} K$), we obtain

$$\begin{aligned} Q(\bar{x}) - Q(x_0) &\geq Q(\bar{x}) - Q(x') \geq -N \| \bar{x} - x' \| = \\ &= \frac{N}{\| \nabla Q(x_0) \|} \nabla Q(x_0)^T(\bar{x} - x') = \\ &= \frac{N}{\| \nabla Q(x_0) \|} \nabla Q(x_0)^T(\bar{x} - x_0) , \end{aligned}$$

which proves our assertion.

Let us now consider the general case. Take $P, v, G, \Lambda, \delta, r$ and D as in the proof of the implication 3.⇒4. and let Λ_r be the matrix obtained by taking the first r rows and columns of Λ. Since G is quasiconvex on D, it is easy to prove that the function

$$g(u) = \frac{1}{2} u^T \Lambda_r u$$

is merely quasiconvex on $\Pi_r(D)$, where $\Pi_r : \mathbb{R}^n \to \mathbb{R}^r$ denotes the projection mapping onto the first r components. Since $\Lambda_r \Pi_r(D) = \Pi_r(\Lambda D)$, ΛD is bounded (as AK is bounded and $P^T AK = P^T A(PD + v) = \Lambda D + P^T Av$) and Λ_r is nonsingular, $\Pi_r(D)$ is bounded whence, as it is also a solid convex set, by the preceding proof, $g_{|\Pi_r(D)}$ is b.l.s.d., i.e., there exists $N > 0$ such that

$$\partial^- g_{|\Pi_r(D)}(u_0) \cap \mathcal{B}(0; N) \neq \emptyset \quad \text{for every} \quad u_0 \in \Pi_r(D) \ .$$

Given $x_0 \in K$, for $u_0 = \Pi_r(P^{-1}(x_0 - v))$ we have $u_0 \in \Pi_r(D)$; thus, using the definition of G and the relation $G = g \circ \Pi_r$, we can easily see that

$$(P^{-1})^T(\partial^- g_{|\Pi_r(D)}(u_0) \times \{0\}) \subset \partial^- Q_{|K}(x_0).$$

Hence, there exists $x_0^* \in \partial^- Q_{|K}(x_0)$ that verifies $\| x_0^* \| \leq \| P^{-1} \| \cdot N$, which concludes the proof. □

Remark. The proof of implication 3.⇒1. in the preceding theorem that would be obtained joining the proofs of the implications 3.⇒4. and 4.⇒1. would be rather involved. However, it is rather easy to prove the weaker statement that if Q is bounded below on K then $Q_{|\operatorname{int} K}$ is l.s.d.. Indeed, by [1, p.174, th.6.3], there exists an upper bound δ for Q on K

such that $h(x) = -(\delta - Q(x))^{1/2}$ is a convex function on K ; moreover, δ is a strict upper bound on $\operatorname{int} K$, whence h is differentiable on $\operatorname{int} K$, with $\nabla h(x) = \frac{1}{2}(\delta - Q(x))^{1/2} \nabla Q(x)$. Therefore, for any $x_0, x \in \operatorname{int} K$ one has

$$-(\delta - Q(x))^{1/2} \geq -(\delta - Q(x_0))^{1/2} + \frac{1}{2}(\delta - Q(x_0))^{-1/2} \nabla Q(x_0)^T (x - x_0) .$$

Multiplying by $(\delta - Q(x_0))^{1/2} + (\delta - Q(x))^{1/2}$ one gets, after simplication,

$$Q(x) \geq Q(x_0) + \frac{1}{2}\left(1 + \left(\frac{\delta - Q(x)}{\delta - Q(x_0)}\right)^{1/2}\right) \nabla Q(x_0)^T (x - x_0) .$$

Hence, if $Q(x) < Q(x_0)$ and m is a lower bound of Q on K , one has

$$Q(x) \geq Q(x_0) + \frac{1}{2}\left(1 + \left(\frac{\delta - m}{\delta - Q(x_0)}\right)^{1/2}\right) \nabla Q(x_0)^T (x - x_0) ,$$

which proves that

$$\frac{1}{2}\left(1 + \left(\frac{\delta - m}{\delta - Q(x_0)}\right)^{1/2}\right) \nabla Q(x_0)$$

is a lower subgradient of $Q_{|\operatorname{int} K}$ at x_0 .

Definition 2.1 [2, p.182] *An unbounded convex set K is said to be* hyperbolic *if there exists a bounded set L such that $K \subset L + O^+(\operatorname{cl} K)$ (in other words, K is hyperbolic if it is $O^+(\operatorname{cl} K)$-bounded in the sense of D. T. Luc* [7, p.14]*).*

It is easy to see that it is not restrictive to take L a compact convex set included in aff K.

As proved by J. Bair [2, p.183, prop.5], this is equivalent to saying that the barrier cone of K coincides with the polar of $O^+(\operatorname{cl} K)$ (which is just the barrier cone of $O^+(\operatorname{cl} K)$) or, also, that the barrier cone of K is closed.

The class of hyperbolic convex sets includes all unbounded polyhedral convex sets and convex cones (except $\{0\}$) as particular cases.

The following two results are easily proved:

Proposition 2.5 *Let $K \subset \mathbb{R}^n$ be a nonempty convex set, $L \subset \mathbb{R}^n$ a bounded set and $D \subset \mathbb{R}^n$ a closed convex cone such that $K \subset L + D$. Then*

$$O^+(\operatorname{cl} K) \subset D .$$

Proposition 2.6 *Let $K \subset \mathbb{R}^n$ be a hyperbolic convex set and let A be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then*

$$\mathrm{cl}\, AO^+(\mathrm{cl}\, K) = O^+(\mathrm{cl}\, AK) \ .$$

With these results and Theorem 2.4 one can prove:

Corollary 2.7 *Let $K \subset \mathbb{R}^n$ be a hyperbolic solid convex set and let $Q(x) = \frac{1}{2}x^T Ax + b^T x$ be merely quasiconvex on K. Then $Q_{|K}$ is b.l.s.d. if and only if $AO^+(\mathrm{cl}\, K) = \{0\}$.*

For a general domain, i.e. nonnecessarily solid, we have the following characterization of quadratic functions that are b.l.s.d.:

Corollary 2.8 *Let $K \subset \mathbb{R}^n$ be a convex set and let $Q(x) = \frac{1}{2}x^T Ax + b^T x$ be merely quasiconvex on K . Then the following statements are equivalent:*

1. *$Q_{|K}$ is b.l.s.d.,*
2. *Q is Lipschitzian on K ,*
3. *Q is bounded below on K ,*
4. *The orthogonal projection of AK onto* aff K *is bounded .*

Proof: If $\dim \mathrm{aff}\, K = p$, we can write $\mathrm{aff}\, K = h(\mathbb{R}^p)$, where $h : \mathbb{R}^p \to \mathbb{R}^n$ is defined by $h(y) = By + c$, for some orthogonal matrix B and some $c \in \mathbb{R}^n$. We have that

$$(Q \circ h)(y) = \frac{1}{2}y^T B^T ABy + (Ac + b)^T By + \frac{1}{2}c^T Ac + b^T c$$

is merely quasiconvex on $h^{-1}(K)$, which is a solid convex set; moreover, it is easy to see that each of 1., 2., 3. holds if and only if the corresponding statement holds for $Q \circ h$ on $h^{-1}(K)$. Therefore, in view of Theorem 2.4, it suffices to prove that 4. holds if and only if $B^T ABh^{-1}(K)$ is bounded. We have

$$B^T ABh^{-1}(K) = B^T ABB^T(K - c)$$

and

$$K = h(h^{-1}(K)) = BB^T(K - c) + c,$$

whence $B^T ABh^{-1}(K) = B^T A(K - c)$. Consequently, the boundedness of $B^T ABh^{-1}(K)$ is equivalent to that of $BB^T A(K - c)$ (as B is orthogonal) and, therefore, to 4., since BB^T is just the orthogonal projection mapping onto the subspace parallel to aff K . □

As a consequence of the preceding corollary, one obtains:

Corollary 2.9 *Let $K \subset \mathbb{R}^n$ be a hyperbolic convex set and let $Q(x) = \frac{1}{2}x^TAx + b^Tx$ be merely quasiconvex on K. Then $Q_{|K}$ is b.l.s.d. if and only if the orthogonal projection of $AO^+(\mathrm{cl}\,K)$ onto* aff K *is the vector* 0.

Remark. Using the transformation h employed in the preceding proof and the remark after Theorem 2.4, one can give a simple demonstration of the fact that if Q is bounded below on K then $Q_{|\mathrm{ri}\,K}$ is l.s.d. (which is weaker than implication 3.$\Rightarrow$1. of Corollary 2.8).

3 Quadratic l.s.d. functions

It is untrue that every l.s.d. merely quasiconvex quadratic function on a convex set is necessarily b.l.s.d. For example, take $K = \{x = (x_1, x_2)^T \in \mathbb{R}^2 | x_1 \geq 1\,,\, 1 \leq x_2 \leq 2\}$ and let $Q(x) = -x_1x_2$. It is not difficult to check that $2\,\nabla Q(x) \in \partial^- Q_{|K}(x)$ for every $x \in K$; hence $Q_{|K}$ is l.s.d. However, $Q_{|K}$ is not b.l.s.d., since it is not Lipschitzian.

The characterizations of b.l.s.d. quadratic functions given in the preceding section provide, obviously, sufficient conditions for a quadratic function restricted to a convex domain to be l.s.d.. In this section, we will obtain a necessary condition which, for hyperbolic domains, is also sufficient.

We will need the following result:

Lemma 3.1 *Let $K \subset \mathbb{R}^n$ be a convex set and let $Q(x) = \frac{1}{2}x^TAx + b^Tx$ be quasiconvex on K. Then any local minimum of Q on K is also a global minimum.*

Proof: Let $x_0 \in K$ be a local minimum of Q on K and let $x \in K$. For small $\lambda > 0$, we have $Q((1-\lambda)x_0 + \lambda x) \geq Q(x_0)$. If this inequality holds strictly, by the quasiconvexity of Q, we have $Q(x_0) < Q((1-\lambda)x_0 + \lambda x) \leq Q(x)$. If, instead, $Q((1-\lambda)x_0 + \lambda x) = Q(x_0)$ for any λ in some open interval, then Q is constant on the line joining x_0 and x. In either case,

$$Q(x_0) \leq Q(x),$$

which proves that Q attains a global minimum on K at x_0. $\square$

We will make use of the following notation, taken from the field of multiobjective optimization theory (see, e.g., [15, p.33, Def. 3.1.1]: Given a set

$X \subset \mathbb{R}^n$ and a pointed convex cone $D \subset \mathbb{R}^n$, we define

$$\mathcal{E}(X, D) = X \setminus (X + (D \setminus \{0\}))$$

(i.e., $\mathcal{E}(X, D)$ is the set of minimal points of X with respect to the compatible (with the linear structure) order relation whose nonnegative cone is D).

For the sake of clarity, we will first consider the case when the domain is solid.

Theorem 3.2 *Let $Q(x) = \frac{1}{2}x^T Ax + b^T x$ be merely quasiconvex on a solid convex set $K \subset \mathbb{R}^n$ such that AK is unbounded (in other words, by Theorem 2.4, such that $Q_{|K}$ is not b.l.s.d.). If $Q_{|K}$ is l.s.d., then*

1. *$O^+(\mathrm{cl}\, AK)$ is a half-line $\mathbb{R}^+ s$ (with $s \in \mathbb{R}^n \setminus \{0\}$) orthogonal to $O^+(\mathrm{cl}\, K)$,*
2. *$K \subset \{x \in \mathbb{R}^n \mid s^T x < s^T v\} \cup \{x \in \mathcal{E}(\mathrm{cl}\, K, C) \mid s^T x = s^T v\}$, where*

$$C = (O^+(\mathrm{cl}\, K) \setminus \ker A) \cup \{0\} = O^+(\mathrm{cl}\, K) \cap \mathbb{R}^+ A^{-1} s$$

and $v \in \mathbb{R}^n$ is any vector satisfying $Av + b = 0$.

If K is a closed hyperbolic convex set and conditions 1. and 2. hold then $Q_{|K}$ is l.s.d.

Proof: First let us see that conditions 1. and 2. are necessary.

Let $P, v, G, \Lambda, \delta, r, D, \Pi_r, g$ and Λ_r be as in the proof of Theorem 2.4. Then by [1, Section 6.1], v satisfies $Av + b = 0$ and δ is an upper bound of G on D. As we saw in the cited proof, $\Pi_r(D)$ is a solid convex set and g is merely quasiconvex on $\Pi_r(D)$.

One can easily prove that $G_{|D}$ is l.s.d., but not b.l.s.d.. Moreover, $g_{|\Pi_r(\mathrm{int}\, D)}$ is l.s.d.. Indeed, if $u_0 \in \Pi_r(\mathrm{int}\, D)$, there exists $y_0 \in \mathrm{int}\, D$ such that $\Pi_r y_0 = u_0$. Since G is pseudoconvex, but not convex, on $\mathrm{int}\, D$, we have $\nabla G(y_0) \neq 0$ and thus, by [8, p.217, cor.4.16], $N\nabla G(y_0) \in \partial^- G_{|D}(y_0)$ for some $N \geq 1$. But $G(y)$ does not depend on the last $n - r$ components of y; therefore $N\nabla g(u_0) \in \partial^- g_{|\Pi_r(\mathrm{int}\, D)}(u_0)$.

We shall first see that condition 1. holds.

Since $\Lambda_r \Pi_r$ is a linear transformation, ΛD is unbounded, $\Pi_{r|\Lambda D}$ is an injective map and (as D and $\Lambda_r \Pi_r(D)$ are solid convex sets) $\Lambda_r \Pi_r(\mathrm{int}\, D) = \mathrm{int}\, \Lambda_r \Pi_r(D) = \mathrm{int}\, \Pi_r(\Lambda D)$, we obtain that the set $\Pi_r(\mathrm{int}\, D)$ is unbounded and hence, by [14, p.64, th.8.4], $O^+(\mathrm{cl}\,\Pi_r(\mathrm{int}\,D)) \neq \{0\}$. Thus, we can take $z = (z_1, \ldots, z_r)^T \in \mathbb{R}^r$ such that

$$0 \neq z \in O^+(\mathrm{cl}\,\Pi_r(\mathrm{int}\,D)) = O^+(\mathrm{int}\,\Pi_r(D)) .$$

Let $w = \Lambda_r z$. Without loss of generality, we can suppose that $y_1 \geq 0$ for every $y = (y_1, \ldots, y_n)^T \in D$ ([1, Section 6.1]) and therefore $u_1 \geq 0$ for all $u = (u_1, \ldots, u_r)^T \in \Pi_r(D)$. Then, clearly, $z_1 \geq 0$ and since g is bounded from above by δ on $\Pi_r(D)$, we deduce that $z^T \Lambda_r z \leq 0$. If $z^T \Lambda_r z < 0$, we would have $g(\bar{u} + \lambda z) < g(\bar{u})$ (for $\bar{u} \in \mathrm{int}\,\Pi_r(D) = \Pi_r(\mathrm{int}\,D)$ and large enough λ), but using that $g_{|\Pi_r(\mathrm{int}\,D)}$ is l.s.d., $g(\bar{u} + \lambda z)$ would appear to be minorized by an affine function of λ, for large enough λ, which is impossible if $z^T \Lambda_r z < 0$. Hence

$$z^T \Lambda_r z = 0 .$$

If $z_1 = 0$, fron the above equality we should obtain $z_2 = \cdots = z_r = 0$, a contradiction. Therefore, $z_1 > 0$.

Let us take another $z' = (z'_1, \ldots, z'_r)^T \in O^+(\mathrm{cl}\,\Pi_r(D)) \setminus \{0\}$ and let $w' = \Lambda_r z'$. By the preceding reasoning, we have $z'_1 > 0$ and $z'^T \Lambda z' = 0$ and, since $z + z' \in O^+(\mathrm{cl}\,\Pi_r(D))$, we deduce that $(z + z')^T \Lambda (z + z') = 0$ and thus $z^T \Lambda z' = 0$. Hence,

$$\sum_{i=2}^{r} z_i z'_i = z_1 z'_1 = \sqrt{\sum_{i=2}^{r} z_i^2} \sqrt{\sum_{i=2}^{r} z_i'^2} ,$$

and therefore $z'_i = \alpha z_i$, $i = 2, \ldots, r$ for some $\alpha \in \mathbb{R}^+$. Since $z_1, z'_1 > 0$, we have $\alpha > 0$. Thus, we obtain

$$w' = \Lambda_r z' = \Lambda_r \alpha z = \alpha \Lambda_r z = \alpha w,$$

which shows that $\Lambda_r O^+(\mathrm{cl}\,\Pi_r(D)) = \mathbb{R}^+ w$. From this equality, the fact that $O^+(\mathrm{cl}\,\Pi_r(\Lambda D)) = \Lambda_r O^+(\mathrm{cl}\,\Pi_r(D))$ and the injectivity of $\Pi_{r|\Lambda D}$, one can prove that $O^+(\mathrm{cl}\,\Lambda D) = \mathbb{R}^+ q$ with $q = (w^T, 0)^T \in \mathbb{R}^n \setminus \{0\}$, since ΛD is unbounded. On the other hand, $O^+(\mathrm{cl}\,AK) = (P^T)^{-1}\mathbb{R}^+ q$, i.e., $O^+(\mathrm{cl}\,AK) = \mathbb{R}^+ s$ where $s = (P^T)^{-1} q \in \mathbb{R}^n \setminus \{0\}$.

Let $d \in O^+(\mathrm{cl}\,K)$. From the preceding result, one has $\Lambda_r \Pi_r P^{-1} d \in \Lambda_r \Pi_r O^+(\mathrm{cl}\,D) \subset \Lambda_r O^+(\mathrm{cl}\,\Pi_r(D)) = \mathbb{R}^+ w$, whence $\Lambda_r \Pi_r P^{-1} d = \lambda w$ for some $\lambda \geq 0$; so we obtain

$$s^T d = ((P^T)^{-1} q)^T d = q^T P^{-1} d = w^T \Pi_r P^{-1} d = (\Lambda_r z)^T \Pi_r P^{-1} d =$$

$$= z^T \Lambda_r \Pi_r P^{-1} d = \lambda z^T w = \lambda z^T \Lambda_r z = 0\,,$$

which proves that

$$O^+(\mathrm{cl}\, AK) = \mathbb{R}^+ s \text{ is orthogonal to } O^+(\mathrm{cl}\, K).$$

Now let us prove the second condition in the statement.

Let $x \in K$; there exists $y = (u^T, u'^T)^T \in D$ such that $x = Py + v$, with $u \in \Pi_r(D)$. Taking z, w and q as in the preceding paragraphs and since g is bounded from above on $\Pi_r(D)$ and, hence, on $\mathrm{cl}\,\Pi_r(D)$ (by continuity), from $g(u+\lambda z) = \frac{1}{2}u^T\Lambda_r u + \lambda z^T \Lambda_r u + \frac{1}{2}\lambda^2 z^T \Lambda_r z + \delta = g(u) + \lambda z^T \Lambda_r u = g(u) + \lambda w^T u$, one deduces that $w^T u \leq 0$. We have thus obtained

$$\Pi_r(D) \subset \{u \in \mathbb{R}^r \,|\, w^T u \leq 0\}\,;$$

consequently, $s^T(x - v) = q^T P^{-1}(x - v) = q^T y = w^T \Pi_r y \leq 0$, which demonstrates the inclusion

$$K \subset \{x \in \mathbb{R}^n \,|\, s^T x \leq s^T v\}\,.$$

If $AO^+(\mathrm{cl}\, K) = \{0\}$, this latter inclusion is the one in the statement, which concludes the proof in this case.

In the case $AO^+(\mathrm{cl}\, K) \neq \{0\}$, since $AO^+(\mathrm{cl}\, K)$ is a convex cone contained in $O^+(\mathrm{cl}\, AK) = \mathbb{R}^+ s$, we have $AO^+(\mathrm{cl}\, K) = O^+(\mathrm{cl}\, AK)$. Let us now assume that $x_0 \in K$ satisfies $s^T x_0 = s^T v$. We have to prove that $x_0 \in \mathcal{E}(cl\, K, C)$. Suppose, a contrario, that there exists $d \in C \setminus \{0\}$ such that $x_0 - d \in \mathrm{cl}\, K$. Without loss of generality, we can assume that $Ad = s$. If $x_0 - d$ were a local minimum of Q on $\mathrm{cl}\, K$, by Lemmas 2.1 and 3.1, it would also be a global minimum, whence Q would be bounded below on K. But this would contradict Theorem 2.4. Therefore, there exists a sequence $x_k \in \mathrm{cl}\, K$ converging to $x_0 - d$ and satisfying

$$\begin{aligned} Q(x_k) &< Q(x_0 - d) = Q(x_0) - d^T(Ax_0 + b) = Q(x_0) - s^T(x_0 - v) = \\ &= Q(x_0) \qquad (k = 1, 2, \ldots)\,, \end{aligned}$$

(we have used here that $Ad = s$, $d^T s = 0$ and $Av + b = 0$). By the continuity of Q, we can assume that $x_k \in K$. Take any $x^* \in \partial^- Q_{|K}(x_0)$. We have

$$Q(x_k) \geq Q(x_0) + x^{*T}(x_k - x_0) \quad \text{for each } k\,,$$

whence, taking limits, we obtain

$$Q(x_0) = Q(x_0 - d) \geq Q(x_0) - x^{*T} d.$$

Thus, $x^{*T}d \geq 0$. We know that K is solid, so we can choose $x_1 \in \operatorname{int} K$. By $K \subset \{x \in \mathbb{R}^n | \ s^T x \leq s^T v\}$, this x_1 satisfies $s^T x_1 < s^T v$, whence

$$Q(x_1 + \lambda d) = Q(x_1) + \lambda s^T (x_1 - v) \underset{\lambda \to +\infty}{\longrightarrow} -\infty \,.$$

Therefore, for large enough λ , we have $Q(x_1 + \lambda d) < Q(x_0)$ and hence, by $d \in O^+(\operatorname{int} K)$ [14, p.63, cor.8.3.1] and $x^* \in \partial^- Q_{|K}(x_0)$, we obtain

$$\begin{aligned} Q(x_1) + \lambda s^T (x_1 - v) &= Q(x_1 + \lambda d) \geq \\ &\geq Q(x_0) + x^{*T}(x_1 + \lambda d - x_0) \,. \end{aligned}$$

It follows that $\lambda(x^{*T}d - s^T(x_1 - v)) \leq Q(x_1) - Q(x_0) - x^{*T}(x_1 - x_0)$, which cannot hold for large λ unless we have $x^{*T}d - s^T(x_1 - v) \leq 0$. But this yields $x^{*T}d \leq s^T(x_1 - v) < 0$, which is a contradiction. Thus, we must have $x_0 \in \mathcal{E}(cl\, K, C)$, which concludes the proof of condition 2.

Let us now assume that K is a closed hyperbolic convex set. Then, we can reformulate condition 1. as: $AO^+(K) = R^+ s$ for some $s \neq 0$ and this half-line is orthogonal to $O^+(K)$ (by Proposition 2.6). Let $x_0 \in K$, take $\bar{x} \in \operatorname{int} K$ and consider $x_k = \left(1 - \frac{1}{k}\right) x_0 + \frac{1}{k}\bar{x} \in \operatorname{int} K$, for $k \in \mathbb{N}$. Since K is a hyperbolic convex set, we can write $K \subset L + O^+(K)$, where L is a bounded set such that $x_0, \bar{x} \in \operatorname{int} L$. Therefore, every $x \in K$ can be written $x = p_x + r_x$, with $p_x \in L$ and $r_x \in O^+(K)$; in particular, for $x \in K \cap L$ we will take $p_x = x$ and $r_x = 0$.

Given $x \in K$ such that $Q(x) < Q(x_0)$, we will consider the convex compact set

$$\Gamma(x) = \overline{\mathrm{co}}\ ((L \cup \{x\}) \cap K) \,.$$

By Theorem 2.4, we have that $Q_{|\Gamma(x)}$ is b.l.s.d. and hence, by [8, p.217, cor.4.16], there exists a positive number $N(x)$ such that $N(x) \dfrac{\nabla Q(x_k)}{\| \nabla Q(x_k) \|} \in \partial^- Q_{|\Gamma(x)}(x_k)$ (note that $\nabla Q(x_k) \neq 0$, since Q is merely pseudoconvex on $\operatorname{int} K$ ([1, p.179, cor.6.4]), and thus

$$Q(x) \geq Q(x_k) + \frac{N(x)}{\| \nabla Q(x_k) \|} \nabla Q(x_k)^T (x - x_k) \,. \tag{1}$$

We shall first demonstrate that $Q_{|K}$ is l.s.d. at x_0 if $\nabla Q(x_0) \neq 0$. In order to prove it, we only need to check that the expression

$$\alpha(x) = \frac{Q(x_0) - Q(x)}{\nabla Q(x_0)^T (x_0 - x)}$$

is bounded above on $\{x \in K|\ Q(x) < Q(x_0)\}$. First observe that, when x belongs to this set, the denominator that appears in $\alpha(x)$ does not vanish (it is easily verified by taking limits in inequality (1), as $k \to \infty$).

We have that $Ar_x = \lambda_x s$ for some $\lambda_x \geq 0$. A straightforward calculation, using the orthogonality of s to $O^+(K)$ and the equality $Av + b = 0$, gives

$$\alpha(x) = \frac{Q(x_0) - Q(p_x) - \lambda_x(p_x - v)^T s}{\nabla Q(x_0)^T(x_0 - p_x) - \lambda_x(x_0 - v)^T s} \ .$$

If we take $d \in O^+(K)$ with $Ad = s$, that is, such that $d \in C \setminus \{0\}$, we obtain $x_0 + d \in K \setminus \mathcal{E}(K, C)$. Therefore, by condition 2., $s^T x_0 = s^T(x_0 + d) < s^T v$.

Let

$$m = \inf_{p \in L} Q(p),\ h = \inf_{p \in L} p^T s,\ M = \sup_{p \in L} \nabla Q(x_0)^T p \ \text{and}\ \bar{\lambda} = \frac{M - \nabla Q(x_0)^T x_0}{(v - x_0)^T s}$$

(note that, by $x_0 \in L$, $\bar{\lambda} \geq 0$).

If $\lambda_x > \bar{\lambda}$, we have

$$\alpha(x) \leq \frac{Q(x_0) - m - \lambda_x(h - v^T s)}{\nabla Q(x_0)^T x_0 - M - \lambda_x(x_0 - v)^T s} \ .$$

Since this expression tends to $\dfrac{h - v^T s}{(x_0 - v)^T s}$ as $\lambda_x \to +\infty$, $\alpha(x)$ is bounded above on the set $\{x \in K|\ \lambda_x > \bar{\lambda}, Q(x) < Q(x_0)\}$.

On the other hand, since $A\ \{x \in K|\ \lambda_x \leq \bar{\lambda}\} \subset A\ \mathrm{co}\, L + [0, \bar{\lambda}]\, s$, by Corollary 2.8, we have that $Q_{|E}$ is b.l.s.d., where $E = \mathrm{co}\,\{x \in K|\ \lambda_x \leq \bar{\lambda}\}$. Since $x_k \in \mathrm{int}\, K \cap \mathrm{int}\, L \subset \mathrm{int}\,\{x \in K \mid \lambda_x = 0\} \subset \mathrm{int}\, E$ and $\nabla Q(x_k) \neq 0$, reasoning analogously as we have done to prove that the denominator in $\alpha(x)$ does not vanish, one gets

$$Q(x) \geq Q(x_0) + \frac{N}{\| \nabla Q(x_0) \|} \nabla Q(x_0)^T (x - x_0)$$

for every $x \in E$ such that $Q(x) < Q(x_0)$, where N is a b.l.s.d.-bound of $Q_{|E}$. From this we easily deduce that $\alpha(x)$ is bounded above on $\{x \in K \,|\, \lambda_x \leq \bar{\lambda}\, ,\, Q(x) < Q(x_0)\, \}$. Hence, we have proved that for every $x_0 \in K$ such that $\nabla Q(x_0) \neq 0, \alpha(x)$ is bounded above on the set $\{x \in K|\ Q(x) < Q(x_0)\}$, as we needed in order to show the lower subdifferentiability of $Q_{|K}$ at x_0.

Now, let $x_0 \in K$ be such that $\nabla Q(x_0) = 0$. We have $\nabla Q(x_k) = \frac{1}{k} \nabla Q(\bar{x})$, since $\nabla Q(x)$ is an affine mapping. Let $x \in K$ with $Q(x) < Q(x_0)$. For large enough k, $Q(x) < Q(x_k)$ and, therefore, we can write inequality (1) as

$$Q(x) \geq Q(x_k) + \frac{N(x)}{\| \nabla Q(\bar{x}) \|} \nabla Q(\bar{x})^T (x - x_k),$$

whence, taking limits as $k \to \infty$,

$$Q(x) \geq Q(x_0) + \frac{N(x)}{\| \nabla Q(\bar{x}) \|} \nabla Q(\bar{x})^T (x - x_0).$$

Therefore, $\nabla Q(\bar{x})^T (x - x_0) < 0$.

Analogously to the preceding case, we have to verify that the expression

$$\beta(x) = \frac{Q(x_0) - Q(x)}{\nabla Q(\bar{x})^T (x_0 - x)}$$

is bounded above on $\{x \in K \mid Q(x) < Q(x_0)\}$. Similarly we can write

$$\beta(x) = \frac{Q(x_0) - Q(p_x) - \lambda_x (p_x - v)^T s}{\nabla Q(\bar{x})^T (x_0 - p_x) - \lambda_x (\bar{x} - v)^T s}$$

and, as $\bar{x} \in \operatorname{int} K$, by 2. we have $(\bar{x} - v)^T s < 0$. Defining m, h, M and $\bar{\lambda}$ analogously to the preceding case and using the same reasoning as we used there, we deduce that $\beta(x)$ is bounded above on $\{x \in K \mid \lambda_x > \bar{\lambda},\, Q(x) < Q(x_0)\}$.

It is easy to verify that, similarly to the preceding case, $Q_{|F}$ is b.l.s.d., where $F = \operatorname{co}\{x \in K \mid \lambda_x \leq \bar{\lambda}\}$, and that $\beta(x)$ is bounded above on $\{x \in K \mid \lambda_x \leq \bar{\lambda},\, Q(x) < Q(x_0)\}$, which concludes the proof. □

Remarks. **1.** Since, in the preceding theorem, s is unique up to a multiplication by a positive scalar and C does not depend on s, the set

$$\{x \in \mathbb{R}^n \mid s^T x < s^T v\} \cup \{x \in \mathcal{E}(cl\, K, C) \mid s^T x = s^T v\}$$

is independent of s. It does not depend on the choice of v either, since we can write $s = Ae$ for some $e \in \mathbb{R}^n$ and, therefore, $s^T(x - v) = (Ae)^T(x - v) = e^T(Ax - Av) = e^T(Ax + b)$ and the sign of this expression does not depend on v.

2. If K is not a closed hyperbolic convex set, conditions 1. and 2. of Theorem 3.2 are not sufficient. Let, for example, $K = \{x = (x_1, x_2)^T \in \mathbb{R}^2 \mid x_1 \geq 1,\, x_2 \geq x_1^2\}$ and let $Q(x) = -\frac{1}{2}x_1^2$. One has that K is a solid closed convex set, $O^+(K) = \mathbb{R}^+ \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, Q is merely quasiconvex on K ([1, p.193, th.6.15]) and $O^+(AK) = \mathbb{R}^+ \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. The first condition in the theorem is verified with $s = (-1,0)^T$. Taking $v = 0$, one gets $\left\{ x = (x_1, x_2)^T \mid s^T(x_1, x_2)^T < 0 \right\} = \{x \mid -x_1 < 0\} \supset K$. However, $Q_{|K}$ is not l.s.d. To see this, take, for example, $\bar{x} = (2,5)^T$ and suppose that $Q_{|K}$ is l.s.d. at $\bar{x}$. Since $\bar{x} \in \operatorname{int} K$, there exists $N \geq 1$ such that $N\nabla Q(\bar{x}) \in \partial^- Q_{|K}(\bar{x})$ ([8, p.217, cor.4.16]). For $\lambda > 2$, we have $\begin{pmatrix} \lambda \\ \lambda^2 \end{pmatrix} \in K$ and therefore, as $-\frac{1}{2}\lambda^2 < -2$,

$$-\frac{1}{2}\lambda^2 \geq -2 + (-N\bar{x}_1, 0) \begin{pmatrix} \lambda - 2 \\ \lambda^2 - 5 \end{pmatrix} = -2 - N\bar{x}_1(\lambda - 2) .$$

This expression has to be true for any $\lambda > 2$, but, when $\lambda \longrightarrow +\infty$, we have a contradiction.

Note that, in the preceding example, we have that AK is a closed hyperbolic set, which indicates that, in Theorem 3.2, the hypothesis that K is hyperbolic can not be relaxed to AK hyperbolic.

For the case when K is nonnecessarily solid, one has the following generalization of Theorem 3.2.

Corollary 3.3 *Let $Q(x) = \frac{1}{2}x^T Ax + b^T x$ be merely quasiconvex on a convex set $K \subset \mathbb{R}^n$ such that $\Pi(AK)$ is unbounded, where $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ denotes the orthogonal projection mapping onto the subspace parallel to* $\operatorname{aff} K$ *(in other words, by Corollary 2.8, such that $Q_{|K}$ is not b.l.s.d.).*

If $Q_{|K}$ is l.s.d. then

1. *$O^+(\operatorname{cl}\Pi(AK))$ is a half-line $\mathbb{R}^+ s$ (with $s \in \mathbb{R}^n \setminus \{0\}$) orthogonal to $O^+(\operatorname{cl} K)$,*
2. *$K \subset \{x \in \mathbb{R}^n \mid s^T x < s^T v\} \cup \{x \in \mathcal{E}(\operatorname{cl} K, C) \mid s^T x = s^T v\}$, where*

$$C = (O^+(\operatorname{cl} K) \setminus \ker \Pi A) \cup \{0\} = O^+(\operatorname{cl} K) \cap \mathbb{R}^+(\Pi A)^{-1} s$$

and v is any vector in $\operatorname{aff} K$ *satisfying $\Pi(Av + b) = 0$.*

If K is a closed hyperbolic convex set and conditions 1. and 2. hold then $Q_{|K}$ is l.s.d.

Proof: Let p, h, B and c be as in the proof of Corollary 2.8. Then the quadratic function $Q \circ h$ is merely quasiconvex on $h^{-1}(K)$, which is a solid convex set, and $B^T A B h^{-1}(K)$ is unbounded (otherwise, $\Pi(AK) = BB^T ABh^{-1}(K) + \Pi(Ac)$ would be bounded).

First let us see that conditions 1. and 2. are necessary.

Let $Q_{|K}$ be l.s.d.; it is easy to prove that $Q \circ h_{|h^{-1}(K)}$ is l.s.d. Therefore, by Theorem 3.2,

$$O^+(\operatorname{cl} B^T ABh^{-1}(K)) = \mathbb{R}^+ z$$

for some $z \in \mathbb{R}^p \setminus \{0\}$, orthogonal to $O^+(\operatorname{cl} h^{-1}(K))$, and, also,

$$h^{-1}(K) \subset \{y \in \mathbb{R}^p |\ z^T y < z^T w\} \cup \{y \in \mathcal{E}(\operatorname{cl} h^{-1}(K), \Delta)|\ z^T y = z^T w\} \ ,$$

where

$$\Delta = (O^+(\operatorname{cl} h^{-1}(K)) \setminus \ker B^T AB) \cup \{0\}$$

and $w \in \mathbb{R}^p$ is any vector that satisfies $B^T ABw + B^T(Ac + b) = 0$.

Since the projection map onto the subspace parallel to $\operatorname{aff} K$ can be written as BB^T and, by [14, p.73, th.9.1], we have

$$\begin{aligned} O^+(\operatorname{cl}\Pi(AK)) &= O^+(\operatorname{cl}\Pi(ABh^{-1}(K))) = BO^+(\operatorname{cl} B^T ABh^{-1}(K)) = \\ &= \mathbb{R}^+ Bz. \end{aligned}$$

We have seen that $O^+(\operatorname{cl}\Pi(AK)) = \mathbb{R}^+ s$ with $s = Bz \in \mathbb{R}^n \setminus \{0\}$ (since $z \neq 0$ and B is orthogonal). On the other hand, from the equality $O^+(\operatorname{cl} h^{-1}(K)) = B^T O^+(\operatorname{cl} K)$, we deduce that, for $d \in O^+(\operatorname{cl} K)$, $s^T d = z^T B^T d = 0$, since z is orthogonal to $O^+(\operatorname{cl} h^{-1}(K))$. We have thus proved that

$$O^+(\operatorname{cl}\Pi(AK)) = \mathbb{R}^+ s \text{ is orthogonal to } O^+(\operatorname{cl} K).$$

Let, now, $v \in \operatorname{aff} K$ be such that $\Pi(Av+b) = 0$ or, equivalently, $B^T(Av+b) = 0$. There exists $w \in \mathbb{R}^p$ with $Bw + c = v$, namely, $w = B^T(v - c)$. Since $0 = B^T(A(Bw + c) + b) = B^T ABw + B^T(Ac + b)$, we deduce

$$\begin{aligned} K &= h(h^{-1}(K)) = Bh^{-1}(K) + c \ \subset \\ &\subset \{By + c \mid y \in \mathbb{R}^p,\ z^T y < z^T w\} \cup \\ &\quad\ \{By + c \mid y \in \mathcal{E}(\operatorname{cl} h^{-1}(K), \Delta)\ ,\ z^T y = z^T w\} = \\ &= \{x \in \mathbb{R}^n \mid z^T B^T(x - c) < z^T w\} \cup \\ &\quad\ \{x \in \operatorname{aff} K \mid B^T(x - c) \in \mathcal{E}(\operatorname{cl} h^{-1}(K), \Delta),\ z^T B^T(x - c) = z^T w\} = \\ &= \{x \in \mathbb{R}^n \mid s^T x < s^T v\} \cup \\ &\quad\ \{x \in \operatorname{aff} K \mid B^T(x - c) \in \mathcal{E}(\operatorname{cl} h^{-1}(K), \Delta),\ s^T x = s^T v\}. \end{aligned}$$

It only remains to prove that this latter set coincides with the set $\{x \in \mathcal{E}(\mathrm{cl}\, K, C) \mid s^T x = s^T v\}$. To see this, take $x \in \mathrm{aff}\, K$ verifying $B^T(x - c) \in \mathcal{E}(\mathrm{cl}\, h^{-1}(K), \Delta)$. Evidently, $x \in h(\mathrm{cl}\, h^{-1}(K)) = \mathrm{cl}\, K$. Let $d \in C \setminus \{0\} = O^+(\mathrm{cl}\, K) \setminus \ker \Pi A$; we have $B^T d \in O^+(\mathrm{cl}\, h^{-1}(K))$ and $d \notin \ker \Pi A = \ker BB^T A = \ker B^T A$, and thus $B^T ABB^T d = B^T A\Pi(d) = B^T Ad \neq 0$. Hence $B^T d \in \Delta \setminus \{0\}$ and

$$h^{-1}(x - d) = B^T(x - c) - B^T d \notin h^{-1}(\mathrm{cl}\, K) ,$$

which indicates that $x - d \notin \mathrm{cl}\, K$; in this way, we have seen that $x \in \mathcal{E}(\mathrm{cl}\, K, C)$, which proves that conditions 1. and 2. are necessary.

Let us see now that if K is a closed hyperbolic convex set, then conditions 1. and 2. are sufficient for $Q_{|K}$ to be l.s.d. In this case, there exists a bounded set L, which we can suppose to be included in $\mathrm{aff}\, K$, such that $K \subset L + O^+(K)$. We deduce that $h^{-1}(K) \subset h^{-1}(L) + O^+(h^{-1}(K))$. Therefore, $h^{-1}(K)$ is also a closed hyperbolic set.

We know that $O^+(\mathrm{cl}\, \Pi(AK)) = BO^+(\mathrm{cl}\, B^T ABh^{-1}(K))$; hence we have $O^+(\mathrm{cl}\, B^T ABh^{-1}(K)) = \mathbb{R}^+ z$, where $z = B^T s \in \mathbb{R}^p \setminus \{0\}$. On the other hand, by [14, p.73, th.9.1], $O^+(K) = BO^+(h^{-1}(K))$. Taking $d \in O^+(h^{-1}(K))$, we have $z^T d = s^T Bd = 0$, since s is orthogonal to $O^+(K)$. We have seen that

$$O^+(\mathrm{cl}\, B^T ABh^{-1}(K)) = \mathbb{R}^+ z \text{ is orthogonal to } O^+(h^{-1}(K)).$$

Let $w \in \mathbb{R}^p$ with $B^T ABw + B^T(Ac + b) = 0$ and let $v = h(w) \in \mathrm{aff}\, K$. One has $\Pi(Av + b) = BB^T(A(Bw + c) + b) = 0$ and, therefore,

$$\begin{aligned}
h^{-1}(K) \;\subset\; & h^{-1}(\{x \in \mathbb{R}^n \mid s^T x < s^T v\}) \cup \\
& \cup\, h^{-1}(\{x \in \mathcal{E}(K, C) \mid s^T x = s^T v\}) = \\
=\; & \{y \in \mathbb{R}^p \mid s^T(By + c) < s^T v\} \cup \\
& \cup \{y \in \mathbb{R}^p \mid By + c \in \mathcal{E}(K, C) \;,\; s^T(By + c) = s^T v\} \;= \\
=\; & \{y \in \mathbb{R}^p \mid z^T y < z^T w\} \cup \\
& \cup \{y \in \mathbb{R}^p \mid By + c \in \mathcal{E}(K, C) \;,\; z^T y = z^T w\}.
\end{aligned}$$

We want to see now that condition $By + c \in \mathcal{E}(K, C)$ is equivalent to $y \in \mathcal{E}(h^{-1}(K), \Delta)$, where

$$\Delta = (O^+(h^{-1}(K)) \setminus \ker B^T AB) \cup \{0\} .$$

Let $y \in \mathbb{R}^p$ satisfying $By + c \in \mathcal{E}(K, C)$. It is easy to prove that $y \in h^{-1}(K)$ and that, for $e \in \Delta \setminus \{0\} = O^+(h^{-1}(K)) \setminus \ker B^T AB$, $Be \in BO^+(h^{-1}(K)) =$

$O^+(K)$. Moreover, $e \notin \ker B^T AB = \ker BB^T AB = \ker \Pi AB$, it is, $Be \notin \ker \Pi A$. Hence $Be \in O^+(K) \setminus \ker \Pi A = C \setminus \{0\}$ and therefore

$$h(y-e) = B(y-e) + c = By + c - Be \notin K,$$

i.e., $y - e \notin h^{-1}(K)$, which means that $y \in \mathcal{E}(h^{-1}(K), \Delta)$. Thus we are under the conditions of Theorem 3.2, for $Q \circ h$ and the solid closed hyperbolic convex set $h^{-1}(K)$; hence, $Q \circ h_{|h^{-1}(K)}$ is l.s.d. Since

$$B\partial^-(Q \circ h)_{|h^{-1}(K)}(B^T(x_0 - c)) \subset \partial^- Q_{|K}(x_0) \text{ for every } x_0 \in K,$$

we deduce that $Q_{|K}$ is l.s.d., which concludes the proof. □

Remark. Condition 2. in the preceding corollary can be written as

$$K \subset \{x \in \operatorname{aff} K \mid s^T x < s^T v\} \cup \{x \in \mathcal{E}(\operatorname{cl} K, C) \mid s^T x = s^T v\},$$

and, analogously to what happen in Theorem 3.2, this set does not depend on s and v.

4 Minimization of quadratic b.l.s.d. functions

In this section we will consider that the convex sets to which we restrict our functions are solid. This does not mean loss of generality, since for any nonempty convex set $K \subset \mathbb{R}^n$ such that $p = \dim \operatorname{aff} K$, defining h, B and c as in the proof of Corollary 2.8, we have that $h^{-1}(K)$ is a solid convex set and $Q \circ h_{|h^{-1}(K)}$ is b.l.s.d. and if $\bar{y}$ is optimal for g on $h^{-1}(K)$, $h(\bar{y})$ is optimal for Q on K and conversely.

We know that if $K \subset \mathbb{R}^n$ is a solid compact convex set and $Q(x) = \frac{1}{2}x^T Ax + b^T x$ is merely quasiconvex on K, for any $x_0 \in K$ such that $\nabla Q(x_0) \neq 0$, one has that $\frac{N}{\| \nabla Q(x_0) \|} \nabla Q(x_0) \in \partial^- Q_{|K}(x_0)$, where N is a Lipschitz constant of Q on the set $\Phi(K)$ (see Section 2 and the proof of Theorem 2.4). If $\nabla Q(x_0) = 0$, taking $\bar{x} \in \operatorname{int} K$, we have $\frac{N}{\| \nabla Q(\bar{x}) \|} \nabla Q(\bar{x}) \in \partial^- Q_{|K}(x_0)$ (see the proof of Theorem 3.2). Note that, in the latter case, x_0 belongs to the boundary of K, since Q is merely pseudoconvex on $\operatorname{int} K$ (see [1, p.179, cor.6.4]). It is easy to see, using [1, p.174, th.6.3], that the following proposition holds:

Proposition 4.1 *Let $K \subset \mathbb{R}^n$ be a nonempty compact convex set and let $Q(x) = \frac{1}{2}x^T Ax + b^T x$ be merely quasiconvex on K. If $\nabla Q(x_0) = 0$, then x_0 is a maximum of Q on K.*

We want to solve the problem

$$\left.\begin{array}{ll} \min & Q(x) \\ & x \in K \end{array}\right\} \quad (P)$$

where K is a solid convex polytope and Q a quadratic merely quasiconvex function on K. By Theorem 2.4, these hypotheses imply that $Q_{|K}$ is b.l.s.d. We can transform problem (P) into problem

$$\left.\begin{array}{c} \min \ t \\ Q(x) - t \leq 0 \\ x \in K \end{array}\right\}.$$

Next we describe the cutting plane algorithm of Plastria [12, p.48] for minimizing b.l.s.d. functions on polytopes, specialized to the quadratic case. We will denote by N a Lipschitz constant of Q on $\Phi(K)$ and by ε a fixed positive number.

Algorithm

Step 0). Take $x_0 \in \operatorname{int} K$. (Therefore, $\nabla Q(x_0) \neq 0$).

Step 1). Let

$$x_0^* = \frac{N}{\| \nabla Q(x_0) \|} \nabla Q(x_0),$$

which is a lower subgradient of $Q_{|K}$ at x_0.

Set $k = 1$, $q_0 = Q(x_0)$ and $\bar{x}_0 = x_0$.

Step 2). Solve the linear problem

$$(P_k) \quad \left\{ \begin{array}{ll} \min \ t & \\ t \geq x_j^{*T}(x - x_j) + Q(x_j) & j = 0, \ldots, k-1 \\ x \in K & \end{array} \right. .$$

Since K is a compact set, problem (P_k) has an optimal solution (t_k, x_k).

Step 3). If $Q(x_k) < q_{k-1}$ then set $q_k = Q(x_k)$ and $\bar{x}_k = x_k$;

otherwise, set $q_k = q_{k-1}$ and $\bar{x}_k = \bar{x}_{k-1}$.

If $q_k - t_k \leq \varepsilon|t_k|$ then $\bar{x}_k$ is optimal. STOP.

If not, choose x_k^* a lower subgradient of Q at x_k, with norm less than or equal to N.

Increase k by 1 and return to step 2).

The sequence $(t_k)_k$ generated by the algorithm is non-decreasing, by definition of problems (P_k). However, since we can not assure that sequence $(Q(x_k))_k$ were non-decreasing, we have to consider the possibility that, in some iteration, we obtain $\nabla Q(x_k) = 0$. In this case, we can take $x_k^* = \frac{N}{\| \nabla Q(x_0) \|} \nabla Q(x_0)$; when $\nabla Q(x_k) \neq 0$, $x_k^* = \frac{N}{\| \nabla Q(x_k) \|} \nabla Q(x_k)$ can be taken as a lower subgradient. In practice, if we are in the first case, instead of adding the constraint $t \geq x_k^{*T}(x - x_k) + Q(x_k)$ it is preferable to modify the constraint corresponding to $j = 0$ altering, if necessary, its independent term (since both linear inequalities determine parallel halfspaces).

For applying the preceding algorithm we need to know N, a Lipschitz constant of Q on $\Phi(K)$. We can take $N = \sqrt{2}M\| A \| + \| b \|$, since if $K \subset \mathcal{B}(0; M)$ we have $\Phi(K) \subset \mathcal{B}(0; \sqrt{2}M)$ (by the proof of [8, p.218, lemma 4.18]) and therefore, given $x \in \Phi(K)$ we have

$$\| \nabla Q(x) \| = \| Ax + b \| \leq \| A \| \cdot \| x \| + \| b \| \leq \sqrt{2}M\| A \| + \| b \|;$$

On the other hand, K being a compact set, there exists a minimum point of Q on K and sequence $(x_k)_k$ has accumulation points.

An important property of sequence $(t_k)_k$, valid for nonnecessarily quadratic functions, appears in the following proposition. Its proof appears to be not completely obvious, although it appears in [12] as a simple comment.

Proposition 4.2 *For every k, t_k is a lower bound of Q on K.*

Proof: We will do it by induction on k.

Call m the minimum value of Q on K.

For $k = 1$, if x_0 is optimal we have $t_1 \leq Q(x_0)$ (as $(Q(x_0), x_0)$ is a feasible solution to (P_1)). If x_0 is not optimal, taking $\tilde{x}$, a minimum point of Q on K, $Q(\tilde{x}) < Q(x_0)$ and, therefore, $(Q(\tilde{x}), \tilde{x})$ is admissible for (P_1); hence $t_1 \leq Q(\tilde{x}) = m$.

Let $k > 1$; by the induction hypothesis, $t_j \leq m$ for $j = 1, \ldots, k-1$. Since $(Q(x_{k-1}), x_{k-1})$ is admissible for (P_{k-1}) (as (t_{k-1}, x_{k-1}) is and $t_{k-1} \leq m \leq Q(x_{k-1})$), we obtain $Q(x_{k-1}) \geq t_k$.

If x_{k-1} is optimal for (P), then $t_k \leq Q(x_{k-1}) = m$.

If x_{k-1} is not optimal for (P), and $x_0, \ldots, x_{k-2}$ are not optimal either, for any minimal point $\tilde{x}$, we have $Q(\tilde{x}) < Q(x_j)$ for $j = 0, \ldots, k-1$; hence, by $x_j^* \in \partial^- Q_{|K}(x_j)$ for $j = 0, \ldots, k-1$, $(Q(\tilde{x}), \tilde{x})$ is admissible for (P_k) and therefore $m = Q(\tilde{x}) \geq t_k$.

If x_{k-1} is not optimal for (P), but there exists x_j, $j \in \{0, \ldots, k-2\}$, which is optimal, let x_h, with $h \in \{0, \ldots, k-2\}$, be the last optimum appearing in the sequence $x_0, \ldots, x_{k-2}$. We have that (t_h, x_h) is an admissible point for (P_h), whence $(Q(x_h), x_h)$ satisfies the first h constraints of (P_k) and, for $l = h+1, \ldots, k-1$, we have $Q(x_h) < Q(x_l)$. Since $x_l^* \in \partial^- Q_{|K}(x_l)$, $l = h+1, \ldots, k-1$, we obtain that $(Q(x_h), x_h)$ also satisfies the last $k-h-1$ constraints; from this we deduce that $t_k \leq Q(x_h) = m$. □

Using the preceding result, Plastria [12, p.48, th.4.1] proved that every accumulation point of $(x_k)_k$ minimizes Q on K and that the sequence $(t_k)_k$ converges to the minimum value of Q on K (see [12, p.49, th.4.2]).

References

[1] Avriel, M., Diewert, W. E., Schaible, S. and Zang, I.: Generalized Concavity, Plenum Press, New York (1988).

[2] Bair, J.: Liens entre le cone d'ouverture interne et l'internat du cone asymptotique d'un convexe, Bulletin de la Societé Mathématique de Belgique, vol.XXXV, Fasc.II-Ser.B (1983), pp. 177-187.

[3] Balder, E. J.: An extension of duality-stability relations to non convex problems, SIAM J. Control and Optimization, 15 (1977), pp. 329-343.

[4] Boncompte, M. and Martínez-Legaz, J. E.: Fractional programming by lower subdifferentiability techniques, J. Optim. Theory Appl. 68 (1991), pp. 95-116.

[5] Crouzeix, J. P.: Contributions a l'étude des fonctions quasiconvexes, thesis, Université de Clermont-Ferrand II (1977).

[6] Greenberg, H. P. and Pierskalla, W. P.: Quasiconjugate Function and Surrogate Duality, Cahiers du Centre d'études de Rech. Oper. 15 (1973), pp. 437-448.

[7] Luc, D. T.: Theory of Vector Optimization, Springer Verlag (1989).

[8] Martínez-Legaz, J. E.: On lower subdifferentiable functions, in "Trends in Mathematical Optimization" (Proceedings, Internat. Conf. Irsee, 1986), Birkhäuser Verlag, Boston (1988), pp. 197-232.

[9] Martínez Legaz, J. E.: Quasiconvex Duality Theory by Generalized Conjugation Methods, Optimization 19 (1988), pp. 603-652.

[10] Moreau, J. J.: Inf-convolution, sous-additivité, convexité des fonctions numériques, J. Math. pures et appliqués, 49 (1970), pp. 109-154.

[11] Penot, J. P., Volle, M.: Another duality scheme for quasiconvex problems, in "Trends in Mathematical Optimization" (Proceedings, Internat. Conf. Irsee, 1986), Birkhäuser Verlag, Boston (1988), pp. 259-275.

[12] Plastria, F.: Lower Subdifferentiable Functions and Their Minimization by Cutting Planes, J. Optim. Theory Appl. 46 (1985), pp. 37-53.

[13] Plastria, F.: The Minimization of Lower Subdifferentiable Functions under Nonlinear Constraints: An All Feasible Cutting Plane Algorithm, J. Optim. Theory Appl. 57 (1988), pp. 463-484.

[14] Rockafellar, R. T.: Convex Analysis, Princeton University Press, Princeton (1970).

[15] Sawaragi, Y., Nakayama, H. and Tanino, T.: Theory of Multiobjective Optimization, Acad. Press (1985).

[16] Schaible, S.: Quasiconvex, Pseudoconvex and Strictly Pseudoconvex Quadratic Functions, J. Optim. Theory Appl. 35 (1981), pp. 303-338.

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