UNIVERSITAT DE BARCELONA

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Rosario Delgado and *Marta Sanz*

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Rosario Delgado * and *Marta Sanz* **

* ETSEIB
Universitat Politècnica de Catalunya
Diagonal, 647
08028 Barcelona
Spain

** Facultat de Matemàtiques
Universitat de Barcelona
Gran Via 585
08007 Barcelona
Spain

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Abstract. In this paper we prove the analogue of the Hu–Meyer formula for random kernels. More precisely, using a suitable notion of trace we give the relation between the multiple Stratonovich integral of a non adapted process and the multiple Skorohod integral.

Key words: Multiple Skorohod integral, Multiple Stratonovich integral, trace.

1. INTRODUCTION

This paper has been motivated by the problem of finding the analogue of the Hu-Meyer formula for random kernels. In [1] the authors present a relation between the multiple Stratonovich integral of deterministic kernels and the multiple Itô-Wiener integral of traces, giving an intuitive explanation for its validity. Recently, different authors have given rigorous proofs of this formula, see for instance [2], [3], [9], [10] and [11].

Consider a non adapted stochastic process $X = \{X_t,\ t \in [0,1]\}$. Under some smoothness requirements, the stochastic integrals of X with respect to a Brownian motion $W = \{W_t,\ t \in [0,1]\}$ in the Skorohod and in the Stratonovich sense can be defined. They usually have been denoted by $\delta(X)$ and $I^s(X)$ respectively. These two notions of integrals can be related by means of a trace type term (see Theorem 7.3 in [6] and Theorem 1.9 in [9]. Formally,

$$I^s(X) = \delta(X) + TX\,. \tag{0.1}$$

In this article we deal with multiparameter stochastic processes $X = \{X_t,\ t \in [0,1]^k\}$. The multiple k–th Skorohod integral $\delta^k(X)$ can be defined as an extension of the one–dimensional parameter case (see for instance [5] and [4]), as well as the k–Stratonovich integral, $I^s_k(X)$. For $k = 2$ the relation between $I^s_2(X)$ and $\delta^2(X)$ has been studied in Section 2.B of [9]. To this end different notions of traces are introduced, say T_1X, TX and $T_{1,2}X$, and it is proved that

$$I^s_2(X) = \delta^2(X) + 2\,\delta(T_1X) + TX + T_{1,2}X\,. \tag{0.2}$$

Both formulae (0.1) and (0.2) are of the Hu–Meyer type for non deterministic kernels.

Our purpose here has been to find an appropiate notion of trace which unifies all the notions explained before, allowing us to relate the multiple Stratonovich and Skorohod integrals of a multiparameter process X. This notion is given in Definition 2.4. The basic result is presented in Theorem 3.1, where the formula

$$I^s_k(X) = \sum_{j=0}^{[\frac{k}{2}]} \sum_{r=0}^{k-2j} \binom{k-2j}{r} \frac{k!}{(k-2j)!\,j!\,2^j}\, \delta^{k-2j-r}\left(T_{j,r}(X)\right) \tag{0.3}$$

is proved. Here $T_{j,r}(X)$ denote the traces, and it should be pointed out that for X deterministic this formula reduces to the Hu–Meyer formula

$$I_k^s(X) = \sum_{j=0}^{[\frac{k}{2}]} \frac{k!}{(k-2j)!\, j!\, 2^j}\, I_{k-2j}\left(Tr^j f\right),$$

(see Remark 3.2).

2. NOTATION AND PRELIMINARY RESULTS

Let T be the interval [0,1]. We will denote by π an arbitrary partition of T, say $\pi = \{0 = t_1 < \ldots < t_{r_\pi} < t_{r_\pi+1} = 1\}$, and by Δ_i the interval $(t_i,\ t_{i+1}]$, $i = 1, \ldots, r_\pi$. The norm of π is defined as $|\pi| = \max\limits_{1 \le i \le r_\pi} (t_{i+1} - t_i)$. For any Borel set A of T, $|A|$ means its Lebesgue measure. We will write $\underline{t} = (t_1, \ldots, t_k)$ for a generic point in T^k, $k \geq 1$.

Consider a standard one–dimensional Brownian motion $\{W_t,\ t \in T\}$ defined on the canonical probability space $(\Omega, \mathcal{F}, P)$, that means $\Omega = \mathcal{C}(T)$, $\mathcal{F} = \mathcal{B}(\Omega)$ and P is the Wiener measure. Consider also a measurable stochastic process $X = \{X_{\underline{t}},\ \underline{t} \in T^k\}$ defined on $(\Omega, \mathcal{F}, P)$, such that $E\left(\int_{T^k} X_{\underline{t}}^2\, d\underline{t}\right) < +\infty$. For any partition π of T we set

$$S_\pi(X) = \sum_{i_1, \ldots, i_k = 1}^{r_\pi} \frac{1}{|\Delta_{i_1}| \ldots |\Delta_{i_k}|} \left(\int_{\Delta_{i_1} \times \ldots \times \Delta_{i_k}} X_{\underline{t}}\, d\underline{t} \right) W(\Delta_{i_1}) \ldots W(\Delta_{i_k}). \qquad (2.1)$$

Definition 2.1. The process X is said to be Stratonovich integrable if the family $\{S_\pi(X),\ \pi \text{ partition of } T\}$ converges in $L^2(\Omega)$ as $|\pi| \longrightarrow 0$. We will call this limit the k–Stratonovich integral of the process X and it will be denoted by $I_k^s(X)$.

Remarks

(2.2) In the previous definition the value of $I_k^s(X)$ can be obtained, equivalently, as the limit of $\{S_{\pi(n)},\ n \geq 1\}$ for any increasing sequence of partitions $\{\pi(n),\ n \geq 1\}$ of T such that $|\pi(n)| \longrightarrow 0$ as n tends to infinity.

(2.3) Let $\tilde{X}$ be the symmetrization of X, that means

$$\tilde{X}_{\underline{t}} = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} X_{\sigma(\underline{t})}$$

where $\mathfrak{S}_k$ denotes the group of permutations of $\{1,2,\ldots,k\}$, and if $\underline{t}=(t_1,\ldots,t_k)$, $\sigma(\underline{t})=(t_{\sigma(1)},\ldots,t_{\sigma(k)})$.

Then it is obvious that the process X is k–Stratonovich integrable if and only if $\tilde{X}$ possess the same property, and in this case $I_k^s(X)=I_k^s(\tilde{X})$. Hence, when dealing with the Stratonovich integral, we will only consider as integrands processes $\{X_{\underline{t}},\ \underline{t}=(t_1,\ldots,t_k)\in T^k\}$ which are symmetric in the variables $t_1,\ldots,t_k$.

For any integer $k\geq 1$, D^k denotes the Malliavin k–th derivative operator. Given any real $p>1$ we call $\mathbb{D}^{k,p}$ the set of Wiener functionals F in $Dom\, D^k$ such that

$$\|F\|_{k,p}:=\|F\|_p+\sum_{i=1}^{k}\Big\|\,\|D^iF\|_{L^2(T^i)}\Big\|_p$$

is finite.

The adjoint of the operator D^k is the multiple k–Skorohod integral. It will be denoted by δ^k.

By definition $\mathbb{L}_k^{k,2}$ is the space $L^2(T^k,\mathbb{D}^{k,2})$. That means, $\mathbb{L}_k^{k,2}$ is the class of processes $X\in L^2(T^k\times\Omega)$ such that $X_{\underline{t}}\in\mathbb{D}^{k,2}$ for any $\underline{t}\in T^k$, and there exists a measurable version of

$$\{D^k_{\underline{s}}X_{\underline{t}},\ (\underline{s},\underline{t})\in T^k\times T^k\}\quad\text{such that}$$
$$E\int_{T^k}\int_{T^k}|D^k_{\underline{s}}u_{\underline{t}}|^2\,d\underline{s}\,d\underline{t}<+\infty\,.$$

The space $\mathbb{L}_k^{k,2}$ is included in $Dom\,\delta^k$. If $k=1$ we set $\mathbb{L}^{1,2}$ instead of $\mathbb{L}_1^{1,2}$, and δ instead of δ^1.

We refer the reader to [5] and [4] for an extensive treatment of questions concerning the multiple Skorohod integral.

In the next definition we introduce the notion of *trace* for processes of $\mathbb{L}_k^{k,2}$. As it will be shown in the next section, this is the suitable concept to compare the k–th Stratonovich and Skorohod integrals, and unifies different definitions given in [9], [6].

Definition 2.4. Let X be a symmetric process belonging to $\mathbb{L}_k^{k,2}$. Fix $j\in\{0,1,\ldots,[\frac{k}{2}]\}$ and $r\in\{0,1,\ldots,k-2j\}$. Then X has (j,r)*–trace* if the $\mathbb{L}_{k-2j-r}^{k-2j-r,2}$–limit, as n tends to infinity, of the sequence

$$T_{j,r}^{\pi(n)}(X)=\sum_{i_1,\ldots,i_{j+r}\in\{1,\ldots,r_{\pi(n)}\}}\frac{1}{|\Delta_{i_1}|\ldots|\Delta_{i_{j+r}}|}$$
$$\int_{(\Delta_{i_1})^2\times\ldots\times(\Delta_{i_j})^2\times(\Delta_{i_{j+1}})^2\times\ldots\times(\Delta_{i_{j+r}})^2}D^r_{\underline{s}}X_{\underline{t}}\,dt_1\ldots dt_{2j}\,ds_1\,dt_{2j+1}\ldots ds_r\,dt_{2j+r}\ , \qquad (2.2)$$

$n \geq 1$, exists, where $\{\pi(n), \ n \geq 1\}$ is an increasing sequence of partitions of T such that $|\pi(n)| \longrightarrow 0$, as $n \longrightarrow \infty$.

This limit defines a stochastic process with a $(k-2j-r)$–dimensional parameter which will be denoted by $T_{j,r}(X)$.

Remarks

(2.5) By convention $T_{0,0}(X) = X$.

(2.6) Let f be a symmetric function of $L^2(T^k)$. Then f can also be viewed as an element of $\mathbb{L}_k^{k,2}$, and $D^r f(\underline{t}) = 0$ for any $\underline{t} \in T^k$ and any $r \geq 1$. Therefore $T_{j,r}(f) = 0$ if $r \geq 1$. For $r = 0$ the definition of $T_{j,0}(f)$ coincides with the definition of trace for deterministic kernels given in Definition 3.2 [9].

(2.7) Let $k = 1$ and $X \in \mathbb{L}^{1,2}$. Then $T_{0,1}(X)$ exist if and only if X is Stratonovich integrable, and in this case

$$I_1^s(X) = \delta(X) + T_{0,1}(X) \tag{2.3}$$

(see Theorem 1.9 [9]. Moreover, under additional hypothesis on X, $T_{0,1}(X)$ can be written as an integral involving the derivative operator D (see Theorem 7.3 [6]).

(2.8) As in Proposition 1.8 [9] (see also Proposition 2.1 [11]) the existence of the (j,r)–trace of X, $T_{j,r}(X)$ can be given in terms of the existence of traces for the kernels of the Wiener–chaos decomposition of the random variable $X_{\underline{t}}$. The precise statement is as follows. Let $X \in \mathbb{L}_k^{k,2}$. Consider the Wiener–chaos expansion of the L^2–random variable $X_{\underline{t}}$, say

$$X_{\underline{t}} = \sum_{m=0}^{\infty} I_m\left(f_m(\,\cdot\,,\underline{t})\right),$$

where $f_m \in L^2(T^{m+k})$, $m \geq 0$, are symmetric in the first m coordinates. Then

$$D^r_{\underline{s}} X_{\underline{t}} = \sum_{m=r}^{\infty} \frac{m!}{(m-r)!} I_{m-r}\left(f_m(\,\cdot\,,\underline{s},\underline{t})\right)$$

in the $L^2(T^{k+r} \times \Omega)$ convergence.

Denote by $\tau^{j+r} f_m$ the $L^2\left(T^{m+k-2j-2r}\right)$–limit, if it exists, of the sums

$$\sum_{i_1,\dots,i_{j+r} \in \{1,\dots,r_{\pi(n)}\}} \frac{1}{|\Delta_{i_1}|\dots|\Delta_{i_{j+r}}|} \int_{(\Delta_{i_1})^2 \times \dots \times (\Delta_{i_j})^2 \times (\Delta_{i_{j+1}})^2 \times \dots \times (\Delta_{i_{j+r}})^2}$$
$$f_m\left(\,\cdot\,,\underline{s},\underline{t}\right) dt_1 \dots dt_{2j}\, ds_1\, dt_{2j+1} \dots ds_r\, dt_{2j+r}\,.$$

Then, if X has $(j,r)-$trace, $\tau^{j+r} f_m$ exists and

$$T_{j,r}(X) = \sum_{m=r}^{\infty} \frac{m!}{(m-r)!} I_{m-r}\left(\tau^{j+r} f_m\right).$$

3. HU-MEYER FORMULA FOR RANDOM KERNELS

The aim of this section is to prove the following result.

Theorem 3.1. Let $X = \{X_{\underline{t}}, \underline{t} \in T^k\}$ be a symmetric process belonging to $\mathbb{L}_k^{k,2}$. Let $j \in \{0, \ldots, [\frac{k}{2}]\}$, $r \in \{0, \ldots, k-2j\}$ and assume that all traces $T_{j,r}(X)$ exist. Then every $T_{j,r}(X)$, which is an $L^2([0,1]^{k-2j-r} \times \Omega)$ process, is $(k-2j-r)-$Skorohod integrable. Furthermore, X is $k-$Stratonovich integrable and

$$I_k^s(X) = \sum_{j=0}^{[\frac{k}{2}]} \sum_{r=0}^{k-2j} \binom{k-2j}{r} \frac{k!}{(k-2j)!\, j!\, 2^j} \delta^{k-2j-r}\left(T_{j,r}(X)\right). \tag{3.1}$$

Remarks

(3.2) Let $f \in L^2(T^k)$ be a symmetric function, and assume that the traces $T_{j,0}(f)$, $j = 1, \ldots, [\frac{k}{2}]$, exist. Then the multiple Stratonovich–Itô–Wiener integral, $I_k^s(f)$, exists and

$$I_k^s(f) = \sum_{j=0}^{[\frac{k}{2}]} \frac{k!}{(k-2j)!\, j!\, 2^j} I_{k-2j}\left(T_{j,0}(f)\right). \tag{3.2}$$

where $I_m(\cdot)$ denotes the multiple Itô–Wiener integral.

The formula (3.2) has been established by Hu–Meyer in [1] (see also [2], [3], [9], [10] and [11]).

(3.3) For $k = 1$, formula (3.1) reduces to (2.3). For $k = 2$, the result has been proved in Proposition 2.7 [9].

Before giving the proof of Theorem 3.1 we quote some known results that will be used in the sequel.

Lemma 3.4. Let $X \in \mathbb{L}_k^{k,2}$. Then, for any Borel set $B \subset T$ and any $r \in \{1, \ldots, k\}$, the random variable $\int_{B\times\ldots\times B} X_{\underline{t}}\, d\underline{t}$ belongs to $Dom\ D^r$ and

$$D_{\underline{s}}^r\left(\int_{B\times\ldots\times B} X_{\underline{t}}\, d\underline{t}\right) = \int_{B\times\ldots\times B} D_{\underline{s}}^r X_{\underline{t}}\, d\underline{t}. \tag{3.3}$$

The proof is straightforward and will be omitted.

Lemma 3.5. (cf. Proposition 3.4 [4] and Lemma 1.2 [9]). Let $F \in Dom\, D^k$ for some integer $k \geq 1$, and h a symmetric function in $L^2(T^k)$. Then $Fh \in \mathbb{L}_k^{k,2}$. Moreover, for any $i = 1, \ldots, k$ the process $\int_{T^i} (D^i_{\underline{s}} F)\; h(\cdot\,, \underline{s})\; d\underline{s}$ belongs to $Dom\; \delta^{k-i}$ and

$$F\; \delta^k h = \sum_{i=0}^{k} \binom{k}{i}\; \delta^{k-i} \left(\int_{T^i} (D^i_{\underline{s}} F)\; h(\cdot\,, \underline{s})\; d\underline{s} \right) , \tag{3.4}$$

with the convention that for $i = 0$, $\int_{T^i} D^i_{\underline{s}}\, F\, h(\cdot\,, \underline{s})\; d\underline{s} = F\, h$.

Lemma 3.6. (cf. formula (3.2) [9]). Let $B_1, \ldots, B_h$ be disjoint Borel sets of T, and $r_1, \ldots, r_h$ positive integers such that $r_1 + \ldots + r_h = m$. Then

$$W(B_1)^{r_1} \cdot \ldots \cdot W(B_h)^{r_h} = \sum_{j=0}^{\left[\frac{m}{2}\right]} \sum_{\substack{k_1 = 0, \ldots, \left[\frac{r_1}{2}\right] \\ \vdots \\ k_h = 0, \ldots, \left[\frac{r_h}{2}\right] \\ k_1 + \ldots + k_h = j}} \frac{r_1 ! \ \ldots \ r_h !}{(r_1 - 2k_1)! \ \ldots \ (r_h - 2k_h)!\; k_1 ! \ \ldots \ k_h !\; 2^j}$$
$$\cdot\, |B_1|^{k_1} \ \ldots \ |B_h|^{k_h}\; I_{m-2j} \left(1_{B_1}^{\otimes (r_1 - 2k_1)} \otimes \cdots \otimes 1_{B_h}^{\otimes (r_h - 2k_h)} \right) . \tag{3.5}$$

Proof of Theorem 3.1. It has to be shown that, for any increasing sequence of partitions $\{\pi(n)\,,\ n \geq 1\}$ of T such that $|\pi(n)| \longrightarrow 0$ as n tends to infinity, the sequence of random variables $\{S_{\pi(n)}(X)\,,\ n \geq 1\}$ defined by (2.1) converges in $L^2(\Omega)$ to the right hand side of (3.1). The proof will be done in two steps, following along the ideas of the proof of theorem 3.4 [9]. However the random character of X adds some difficulties.

(1) Let π be a partition of T, and $\mathcal{G}_\pi$ the σ–field on T generated by the intervals Δ_i, $i = 1, \ldots, r_\pi$, where we use the notation introduced in Section 2. For any positive integer $m \geq 1$ we denote by $\mathcal{E}_m$ the conditional expectation operator on the probability space $(T^m \times \Omega\,,\ \mathcal{B}(T^m) \otimes \mathcal{F}\,,\ \lambda \times P)$ with respect to the product σ–field $\mathcal{G}_\pi \otimes \overset{m}{\ldots} \otimes \mathcal{G}_\pi \otimes \mathcal{F}$. Then a.s.,

$$S_\pi(X) = \sum_{j=0}^{\left[\frac{k}{2}\right]} \sum_{r=0}^{k-2j} \binom{k-2j}{r} \frac{k!}{(k-2j)!\; j!\; 2^j}\; \delta^{k-2j-r} \left(\mathcal{E}_{k-2j-r} \left[T^\pi_{j,r}(X) \right] \right) . \tag{3.6}$$

Indeed,

$$S_\pi(X) = \sum_{\lambda=1}^{k} \sum_{\substack{\alpha_1,\dots,\alpha_\lambda>0 \\ \alpha_1+\dots+\alpha_\lambda=k}} \frac{k\,!}{\alpha_1\,!\dots\alpha_\lambda\,!} \sum_{\substack{i_1,\dots,i_\lambda\in\{1,\dots,r_\pi\} \\ i_1\neq\dots\neq i_\lambda}} \frac{1}{|\Delta_{i_1}|^{\alpha_1}\dots|\Delta_{i_\lambda}|^{\alpha_\lambda}}$$
$$\cdot\left(\int_{(\Delta_{i_1})^{\alpha_1}\times\dots\times(\Delta_{i_\lambda})^{\alpha_\lambda}} X_{\underline{t}}\, d\underline{t}\right) \left(W(\Delta_{i_1})\right)^{\alpha_1}\cdot\ldots\cdot\left(W(\Delta_{i_\lambda})\right)^{\alpha_\lambda}.$$

We replace the product $\left(W(\Delta_{i_1})\right)^{\alpha_1}\cdot\ldots\cdot\left(W(\Delta_{i_\lambda})\right)^{\alpha_\lambda}$ by its equivalent expression given in Lemma 3.6. Then if we denote by Σ' the multiple sum

$$\sum_{\lambda=1}^{k} \sum_{\substack{\alpha_1,\dots,\alpha_\lambda>0 \\ \alpha_1+\dots+\alpha_\lambda=k}} \sum_{\substack{i_1,\dots,i_\lambda\in\{1,\dots,r_\pi\} \\ i_1\neq\dots\neq i_\lambda}} \sum_{j=0}^{\left[\frac{k}{2}\right]} \sum_{\substack{k_1=0,\dots,\left[\frac{\alpha_1}{2}\right] \\ \vdots \\ k_\lambda=0,\dots,\left[\frac{\alpha_\lambda}{2}\right] \\ k_1+\dots+k_\lambda=j}},$$

we obtain

$$S_\pi(X) = \sum{}' \frac{k\,!}{k_1!\dots k_\lambda!\,(\alpha_1-2k_1)!\dots(\alpha_\lambda-2k_\lambda)!\,2^j} \frac{1}{|\Delta_{i_1}|^{\alpha_1-k_1}\dots|\Delta_{i_\lambda}|^{\alpha_\lambda-k_\lambda}}$$
$$\cdot\left(\int_{(\Delta_{i_1})^{\alpha_1}\times\dots\times(\Delta_{i_\lambda})^{\alpha_\lambda}} X_{\underline{t}}\, d\underline{t}\right) I_{k-2j}\left(1_{\Delta_{i_1}}^{\otimes(\alpha_1-2k_1)}\otimes\dots\otimes 1_{\Delta_{i_\lambda}}^{\otimes(\alpha_\lambda-2k_\lambda)}\right). \tag{3.7}$$

Set $F=\int_{(\Delta_{i_1})^{\alpha_1}\times\dots\times(\Delta_{i_\lambda})^{\alpha_\lambda}} X_{\underline{t}}\, d\underline{t}$ and $h=\left(1_{\Delta_{i_1}}^{\otimes(\alpha_1-2k_1)}\otimes\dots\otimes 1_{\Delta_{i_\lambda}}^{(\alpha_\lambda-2k_\lambda)}\right)^{\sim}$, where $(\,\cdot\,)^{\sim}$ denotes symmetrization. Then, the hypothesis of Lemma 3.5 are satisfied. Moreover, for any $r\in\{1,\dots,k\}$, $D^r_{\underline{s}}F$ is equal to $\int_{(\Delta_{i_1})^{\alpha_1}\times\dots\times(\Delta_{i_\lambda})^{\alpha_\lambda}} D^r_{\underline{s}}\, X_{\underline{t}}\, d\underline{t}$ (see Lemma 3.4). Therefore, it follows from (3.7) that

$$S_\pi(X) = \sum{}' \sum_{r=0}^{k-2j} \frac{k\,!}{k_1!\dots k_\lambda!\,(\alpha_1-2k_1)!\dots(\alpha_\lambda-2k_\lambda)!\,2^j} \frac{1}{|\Delta_{i_1}|^{\alpha_1-k_1}\dots|\Delta_{i_\lambda}|^{\alpha_\lambda-k_\lambda}}$$
$$\cdot\binom{k-2j}{r}\,\delta^{k-2j-r}\left[\int_{T^r}\left(\int_{(\Delta_{i_1})^{\alpha_1}\times\dots\times(\Delta_{i_\lambda})^{\alpha_\lambda}} D^r_{\underline{s}}\, X_{\underline{t}}\, d\underline{t}\right)\right.$$
$$\left.\cdot\left(1_{\Delta_{i_1}}^{\otimes(\alpha_1-2k_1)}\otimes\dots\otimes 1_{\Delta_{i_\lambda}}^{\otimes(\alpha_\lambda-2k_\lambda)}\right)^{\sim}(\,\cdot\,,\underline{s})\,d\underline{s}\right].$$

We also have, by putting $\gamma_i = \alpha_i - 2\,k_i$, $i = 1, \ldots, \lambda$

$$S_\pi(X) = \sum_{j=0}^{[\frac{k}{2}]} \sum_{r=0}^{k-2j} \binom{k-2j}{r} \delta^{(k-2j-r)} \left\{ \sum_{\lambda=1}^{k-j} \sum_{\substack{k_1,\ldots,k_\lambda \geq 0 \\ k_1+\ldots+k_\lambda = j}} \sum_{\substack{\gamma_1,\ldots,\gamma_\lambda \geq 0 \\ \gamma_1+\ldots+\gamma_\lambda = k-2j \\ k_i+\gamma_i > 0}} \right.$$

$$\frac{k\,!}{k_1! \ldots k_\lambda!\; \gamma_1 \ldots \gamma_\lambda!\; 2^{\,j}} \sum_{\substack{i_1,\ldots,i_\lambda \in \{1,\ldots,r_\pi\} \\ i_1 \neq \ldots \neq i_\lambda}} \frac{1}{|\Delta_{i_1}|^{k_1+\gamma_1} \ldots |\Delta_{i_\lambda}|^{k_\lambda+\gamma_\lambda}}$$

$$\left. \int_{T^r} \left(\int_{(\Delta_{i_1})^{\gamma_1+2k_1} \times \ldots \times (\Delta_{i_\lambda})^{\gamma_\lambda+2k_\lambda}} D^r_{\underline{s}}\, X_{\underline{t}}\; d\underline{t} \right) \left(1^{\otimes \gamma_1}_{\Delta_{i_1}} \otimes \ldots \otimes 1^{\otimes \gamma_\lambda}_{\Delta_{i_\lambda}} \right)^{\sim} (\,\cdot\,, \underline{s})\; d\underline{s} \right\}.$$

Let $P^{\gamma_1,\ldots,\gamma_\lambda}_{k-2j}$ be the set of permutations of $\{1, \ldots, \lambda\}$ with γ_h repetitions of h, $h = 1, \ldots, \lambda$, $\sum_{h=1}^{\lambda} \gamma_h = k - 2j$. Then

$$S_\pi(X) = \sum_{j=0}^{[\frac{k}{2}]} \sum_{r=0}^{k-2j} \binom{k-2j}{r} \delta^{(k-2j-r)} \left\{ \sum_{\lambda=1}^{k-j} \sum_{\substack{k_1,\ldots,k_\lambda \geq 0 \\ k_1+\ldots+k_\lambda = j}} \sum_{\substack{\gamma_1,\ldots,\gamma_\lambda \geq 0 \\ \gamma_1+\ldots+\gamma_\lambda = k-2j \\ k_i+\gamma_i > 0}} \right.$$

$$\frac{k\,!}{(k-2j)!} \; \frac{1}{k_1! \ldots k_\lambda!\; 2^{\,j}} \sum_{\substack{i_1,\ldots,i_\lambda \in \{1,\ldots,r_\pi\} \\ i_1 \neq \ldots \neq i_\lambda}} \frac{1}{|\Delta_{i_1}|^{k_1+\gamma_1} \ldots |\Delta_{i_\lambda}|^{k_\lambda+\gamma_\lambda}}$$

$$\sum_{\sigma \in P^{\gamma_1 \cdots \gamma_\lambda}_{k-2j}} \left(\int_{\Delta_{i_{\sigma(k-2j-r+1)}} \times \ldots \times \Delta_{i_{\sigma(k-2j)}}} \left(\int_{(\Delta_{i_1})^{\gamma_1+2k_1} \times \ldots \times (\Delta_{i_\lambda})^{\gamma_\lambda+2k_\lambda}} D^r_{\underline{s}}\, X_{\underline{t}}\; d\underline{t} \right) d\underline{s} \right)$$

$$1_{\Delta_{i_{\sigma(1)}}} \otimes \ldots \otimes 1_{\Delta_{i_{\sigma(k-2j-r)}}}\,. \tag{3.8}$$

We prove in the Appendix that

$$\mathcal{E}_{k-2j-r} \left[T^\pi_{j,r} (X) \right] =$$

$$= j\,! \sum_{\lambda=1}^{k-j} \sum_{\substack{k_1,\ldots,k_\lambda \geq 0 \\ k_1+\ldots+k_\lambda = j}} \sum_{\substack{\gamma_1,\ldots,\gamma_\lambda \geq 0 \\ \gamma_1+\ldots+\gamma_\lambda = k-2j \\ k_i+\gamma_i > 0}} \sum_{\substack{i_1,\ldots,i_\lambda \in \{1,\ldots,r_\pi\} \\ i_1 \neq \ldots \neq i_\lambda}} \sum_{\sigma \in P^{\gamma_1 \cdots \gamma_\lambda}_{k-2j}}$$

$$\cdot \frac{1}{k_1! \ldots k_\lambda!} \; \frac{1}{|\Delta_{i_1}|^{k_1+\gamma_1} \ldots |\Delta_{i_\lambda}|^{k_\lambda+\gamma_\lambda}}$$

$$\cdot \left(\int_{\Delta_{i_{\sigma(k-2j-r+1)}} \times \ldots \times \Delta_{i_{\sigma(k-2j)}}} \left(\int_{(\Delta_{i_1})^{\gamma_1+2k_1} \times \ldots \times (\Delta_{i_\lambda})^{\gamma_\lambda+2k_\lambda}} D^r_{\underline{s}}\, X_{\underline{t}}\; d\underline{t} \right) d\underline{s} \right)$$

$$\cdot\, 1_{\Delta_{i_{\sigma(1)}}} \otimes \ldots \otimes 1_{\Delta_{i_{\sigma(k-2j-r)}}}\,. \tag{3.9}$$

Then, in view of (3.8), the equality (3.6) is established.

(2) Let $\{\pi(n)\,,\ n \geq 1\}$ be an increasing sequence of partitions of T such that $|\pi(n)| \longrightarrow 0$, as $n \longrightarrow \infty$. Set $S_n(X) = S_{\pi(n)}(X)$ and $T^n_{j,r}(X) = T^{\pi(n)}_{j,r}(X)$.

By hypothesis, for any $j \in \{0, \dots, [\frac{k}{2}]\}$, $r \in \{0, \dots, k-2j\}$, $i \in \{1, \dots, k-2j-r\}$

$$T^n_{j,r}(X) \xrightarrow[n\to\infty]{} T_{j,r}(X)\,,$$

in $L^2(T^{k-2j-r} \times \Omega)$, and

$$D^i_{\underline{\tau}}\left(T^n_{j,r}(X)\right) \xrightarrow[n\to\infty]{} D^i_{\underline{\tau}}\left(T_{j,r}(X)\right),$$

in $L^2(T^{k-2j-r+i} \times \Omega)$.

Hence, by Hunt's martingale convergence theorem

$$\mathcal{E}_{k-2j-r}\left[T^n_{j,r}(X)\right] \xrightarrow[n\to\infty]{} E\left\{T_{j,r}(X) \,|\, \mathcal{B}(T^{k-2j-r}) \otimes \mathcal{F}\right\} = T_{j,r}(X)$$

in $L^2(T^{k-2j-r} \times \Omega)$.

Moreover

$$\begin{aligned} D^i_{\underline{\tau}}\left(\mathcal{E}_{k-2j-r}[T^n_{j,r}(X)]\right) &= \mathcal{E}_{k-2j-r}\Big(D^i_{\underline{\tau}}(T^n_{j,r}(X))\Big) \\ &\xrightarrow[n\to\infty]{} D^i_{\underline{\tau}}\left(T_{j,r}(X)\right), \end{aligned}$$

in $L^2\left(T^{k-2j-r+i} \times \Omega\right)$.

Consequently the $\lim\limits_{n\to\infty} \delta^{k-2j-r}\left(\mathcal{E}_{k-2j-r}\left[T^n_{j,r}(X)\right]\right)$ exists in $L^2(\Omega)$ and equals $\delta^{k-2j-r}(T_{j,r}(X))$. Therefore the process X is k–Stratonovich integrable and formula (3.1) is proved. ∎

Appendix

In this final section we prove formula (3.9) for the conditional expectations of the approximations of the traces. We recall that k denotes a natural number, $j \in \{0, \ldots, [\frac{k}{2}]\}$, $r \in \{0, \ldots, k-2j\}$, π is an arbitrary partition of T given by intervals Δ_i, $i = 1, \ldots, r_\pi$ and $\mathcal{G}_\pi$ is the σ–field generated by these intervals. Finally, given a natural number $m \geq 1$, E_m denotes the conditional expectation operator on the probability space $(T^m, \mathcal{B}(T^m), \lambda)$ with respect to the σ–field $\mathcal{G}_\pi \otimes \overset{m}{\ldots} \otimes \mathcal{G}_\pi$.

Lemma 4.1. Let $f(\underline{s}, \underline{t})$, $\underline{s} \in T^r$, $\underline{t} \in T^k$ be a function in $L^1(T^{r+k})$ symmetric in $\underline{t}$. Set

$$tr^\pi_{j,r}(f) = \sum_{i_1, \ldots, i_{j+r} \in \{1, \ldots, r_\pi\}} \frac{1}{|\Delta_{i_1}| \ldots |\Delta_{i_{j+r}}|}$$

$$\int_{(\Delta_{i_1})^2 \times \ldots \times (\Delta_{i_j})^2 \times (\Delta_{i_{j+1}})^2 \times \ldots \times (\Delta_{i_{j+r}})^2} f(\underline{s}, \underline{t})\, dt_1 \ldots dt_{2j}\, ds_1\, dt_{2j+1} \ldots ds_r\, dt_{2j+r}\, .$$

Then

$$E_{k-2j-r}\left[tr^\pi_{j,r}(f)\right] = j! \sum_{\lambda=1}^{k-j} \sum_{\substack{k_1, \ldots, k_\lambda \geq 0 \\ k_1 + \ldots + k_\lambda = j}} \sum_{\substack{\gamma_1, \ldots, \gamma_\lambda \geq 0 \\ \gamma_1 + \ldots + \gamma_\lambda = k-2j \\ k_i + \gamma_i > 0}} \sum_{\substack{i_1, \ldots, i_\lambda \in \{1, \ldots, r_\pi\} \\ i_1 \neq \ldots \neq i_\lambda}} \sum_{\sigma \in P^{\gamma_1 \cdots \gamma_\lambda}_{k-2j}}$$

$$\frac{1}{k_1! \ldots k_\lambda!} \; \frac{1}{|\Delta_{i_1}|^{k_1+\gamma_1} \ldots |\Delta_{i_\lambda}|^{k_\lambda+\gamma_\lambda}}$$

$$\left(\int_{\Delta_{i_{\sigma(k-2j-r+1)}} \times \ldots \times \Delta_{i_{\sigma(k-2j)}}} \left(\int_{(\Delta_{i_1})^{\gamma_1+2k_1} \times \ldots \times (\Delta_{i_\lambda})^{\gamma_\lambda+2k_\lambda}} f(\underline{s}, \underline{t})\, d\underline{t} \right) d\underline{s} \right)$$

$$1_{\Delta_{i_{\sigma(1)}}} \otimes \ldots \otimes 1_{\Delta_{i_{\sigma(k-2j-r)}}}\, , \tag{4.1}$$

where $P^{\gamma_1 \cdots \gamma_\lambda}_{k-2j}$ denotes the set of permutations of $\{1, \ldots, \lambda\}$ with γ_h repetitions of h, $h = 1, \ldots, \lambda$, $\sum_{h=1}^{\lambda} \gamma_h = k - 2j$.

In particular, if X is a symmetric process belonging to $\mathbb{L}^{k,2}_k$ and $f(\underline{s}, \underline{t}) = D^r_{\underline{s}} X_{\underline{t}}$, we obtain (3.9).

Proof. The conditional expectation $E_{k-2j-r}\left[tr^\pi_{j,r}(f)\right]$ can be developed as follows

$$E_{k-2j-r}\left[tr^{\pi}_{j,r}(f)\right] = \sum_{h_1,\ldots,h_{k-2j-r}\in\{1,\ldots,r_\pi\}} \frac{1}{|\Delta_{h_1}|\ldots|\Delta_{h_{k-2j-r}}|}$$

$$\cdot\left(\int_{\Delta_{h_1}\times\ldots\times\Delta_{h_{k-2j-r}}} \left[tr^{\pi}_{j,r}(f)\right] dt_{2j+r+1}\ldots dt_k\right) 1_{\Delta_{h_1}}\otimes\ldots\otimes 1_{\Delta_{h_{k-2j-r}}}$$

$$= \sum_{h_1,\ldots,h_{k-2j-r}\in\{1,\ldots,r_\pi\}} \sum_{i_1,\ldots,i_{j+r}\in\{1,\ldots,r_\pi\}} \frac{1}{|\Delta_{h_1}|\ldots|\Delta_{h_{k-2j-r}}|\,|\Delta_{i_1}|\ldots|\Delta_{i_{j+r}}|}$$

$$\cdot\left(\int_{\Delta_{i_{j+1}}\times\ldots\times\Delta_{i_{j+r}}}\left(\int_{(\Delta_{i_1})^2\times\ldots\times(\Delta_{i_j})^2\times\Delta_{i_{j+1}}\times\ldots\times\Delta_{i_{j+r}}\times\Delta_{h_1}\times\ldots\times\Delta_{h_{k-2j-r}}} f(\underline{s},\underline{t})\,d\underline{t}\right)d\underline{s}\right)$$

$$1_{\Delta_{h_1}}\otimes\ldots\otimes 1_{\Delta_{h_{k-2j-r}}}$$

$$= \sum_{s=1}^{j} \sum_{\substack{k_1,\ldots k_s>0\\ k_1+\ldots+k_s=j}} \sum_{\substack{i_1,\ldots,i_s\in\{1,\ldots,r_\pi\}\\ i_1\neq\ldots\neq i_s}} \sum_{q=1}^{k-2j} \sum_{\substack{\gamma_1,\ldots,\gamma_q>0\\ \gamma_1+\ldots+\gamma_q=k-2j}} \sum_{\substack{\bar{i}_1,\ldots,\bar{i}_q\in\{1,\ldots,r_\pi\}\\ \bar{i}_1\neq\ldots\neq\bar{i}_q}}$$

$$\frac{j!}{k_1!\ldots k_s!}\quad\frac{(k-2j)!}{\gamma_1!\ldots\gamma_q!}\quad\frac{1}{|\Delta_{i_1}|^{k_1}\ldots|\Delta_{i_s}|^{k_s}\,|\Delta_{\bar{i}_1}|^{\gamma_1}\ldots|\Delta_{\bar{i}_q}|^{\gamma_q}}$$

$$\sum_{\sigma\in P^{\gamma_1\cdots\gamma_q}_{k-2j}} \frac{\gamma_1!\ldots\gamma_q!}{(k-2j)!}\left\{\int_{\Delta_{\bar{i}_{\sigma(k-2j-r+1)}}\times\ldots\times\Delta_{\bar{i}_{\sigma(k-2j)}}}\right.$$

$$\left.\cdot\left(\int_{(\Delta_{i_1})^{2k_1}\times\ldots\times(\Delta_{i_s})^{2k_s}\times(\Delta_{\bar{i}_1})^{\gamma_1}\times\ldots\times(\Delta_{\bar{i}_q})^{\gamma_q}} f(\underline{s},\underline{t})\,d\underline{t}\right)d\underline{s}\right\} 1_{\Delta_{\bar{i}_{\sigma(1)}}}\otimes\ldots\otimes 1_{\Delta_{\bar{i}_{\sigma(k-2j-r)}}}$$

$$= j!\sum_{\lambda=1}^{k-j} \sum_{\substack{k_1,\ldots,k_\lambda\geq 0\\ k_1+\ldots+k_\lambda=j}} \sum_{\substack{\gamma_1,\ldots,\gamma_\lambda\geq 0\\ \gamma_1+\ldots+\gamma_\lambda=k-2j\\ k_i+\gamma_i>0}} \sum_{\substack{i_1,\ldots,i_\lambda\in\{1,\ldots,r_\pi\}\\ i_1\neq\ldots\neq i_\lambda}} \sum_{\sigma\in P^{\gamma_1\cdots\gamma_\lambda}_{k-2j}}$$

$$\frac{1}{k_1!\ldots k_\lambda!}\quad\frac{1}{|\Delta_{i_1}|^{k_1+\gamma_1}\ldots|\Delta_{i_\lambda}|^{k_\lambda+\gamma_\lambda}}$$

$$\cdot\left(\int_{\Delta_{i_{\sigma(k-2j-r+1)}}\times\ldots\times\Delta_{i_{\sigma(k-2j)}}}\left(\int_{(\Delta_{i_1})^{\gamma_1+2k_1}\times\ldots\times(\Delta_{i_\lambda})^{\gamma_\lambda+2k_\lambda}} f(\underline{s},\underline{t})\,d\underline{t}\right)d\underline{s}\right)$$

$$1_{\Delta_{i_{\sigma(1)}}}\otimes\ldots\otimes 1_{\Delta_{i_{\sigma(k-2j-r)}}}\,.$$

Therefore the lemma is proved. ■

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