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MEASURING THE LACK OF INTEGRABILITY OF THE J_2 PROBLEM FOR EARTH'S SATELLITES

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Abstract. We consider the motion around an oblate primary, keeping only the J_2 term in the expansion of the potential in spherical harmonics. The problem has cylindrical symmetry. It has been suspected for a long time, due to numerical evidences, that the problem is non integrable. This has been proved recently [4]. However, even if the system is non integrable, the size of the stochastic zones can be so small that they can be neglected for all practical purposes. This is what we study here, and we show that for the case of the Earth and considering possible real orbits, i.e., non colliding with the Earth, the effect of the non integrability can be completely neglected.

§ 1. Introduction

Let $q = (x, y, z)^T$ the coordinates of a particle around an oblate planet and p the corresponding momenta. The Hamiltonian of the system is

$$H = \frac{1}{2}(p,p) - \frac{\mu}{r} + J_2 \frac{\mu a_e^2}{r^3} P_2\left(\frac{z}{r}\right), \tag{1}$$

where μ is the gravitational constant, $r = \|q\|$, a_e the equatorial radius of the attracting body, P_2 the Legendre polynomial of second degree and J_2 the coefficient of the zonal harmonic of order two. For the Earth these quantities are, approximately, $a_e = 6378\,\mathrm{km}$, $\mu = 398600\,\mathrm{km}^3\,\mathrm{sec}^{-2}$, $J_2 = 1082 \cdot 10^{-6}$. Due to the cyclicity of the longitude in the Hamiltonian, we can use cylindrical coordinates (ρ, θ, z) and then (1) is expressed as

$$H = \frac{1}{2}\left(P_z^2 + P_\rho^2\right) - \frac{\mu}{r} + \frac{c^2/2}{\rho 2} + \frac{\tilde{J}_2}{r^3}\left(\frac{3}{2}\left(\frac{z}{r}\right)^2 - \frac{1}{2}\right), \tag{2}$$

where c is the component in the z direction of the angular momentum and $\tilde{J}_2 = J_2 a_e^2 \mu$. This is a two degrees of freedom Hamiltonian. The dynamics of (2) can be studied by using an analytical approach or a numerical one. Probably the best thing is to combine both of them. The system obtained from (2) depends on the parameters $\mu, c, \tilde{J}_2$ and on the value of the energy, h.

On the (ρ, p_ρ) plane the available domain is defined by $p_\rho^2 \leq 2h + \frac{2\mu}{\rho} - \frac{c^2}{\rho^2} + \frac{\tilde{J}_2}{\rho^3}$. The boundary $(p_z = 0)$ is an orbit of the Hamiltonian on the equatorial plane. Let $V(\rho) = -\frac{\mu}{\rho} + \frac{c^2/2}{\rho^2} - \frac{\tilde{J}_2/2}{\rho^3}$. Then this orbits is written as $p_\rho^2 = 2(h - V(\rho))$. The function $V(\rho)$ has two extrema provided $c^4 > 6\mu\tilde{J}_2$. We denote them by ρ_1, ρ_2 with $0 < \rho_1 < \rho_2$. Assume $0 > h > V(\rho_2)$. Then ρ can range in two intervals, one of them containing $\rho = 0$ and being skipped in the case of the Earth because it is fully contained inside the Earth. We also assume c not too small to have $V(\rho_1) > 0$, i.e. $c^4 > 8\mu\tilde{J}_2$ (see figure 1).

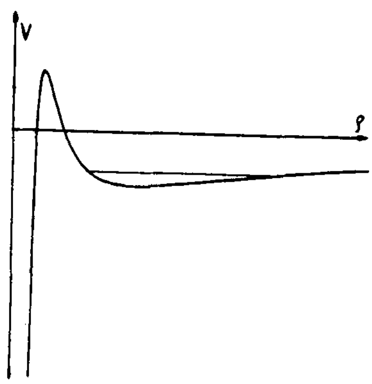

Figure 1

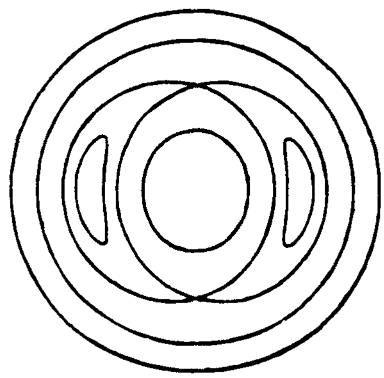

Figure 2

We can study a representation of the motion by using the Poincaré section through $z = 0$. The section is the interior of the available domain described before. If $J_2 = 0$ and $c \neq 0$ the Poincaré map is the identity inside the boundary, and the boundary is an orbit of the flow. Compactifying the Poincaré section (topologically an open disk) to a sphere (by adding one point which represents the boundary orbit) we have the identity on S^2. As J_2 is small and under the previous constrains on c we shall have a near the identity map for the case of the Earth. This map has, at least, two fixed points of elliptic type. There are also invariant curves and chaotic (or stochastic) zones associated to the resonances. The largest chaotic zone is related to the 1 to 1 resonance known to occur near the critical inclination [3]. Figure 2 shows a qualitative picture of the Poincaré map when the two symmetric hyperbolic points are present. In fact the heteroclinic orbits between them do not agree. They create a very narrow stochastic layer whose size we shall bound.

§ 2. The normal form. Hyperbolic fixed points

If J_2 is small and r is bounded away from zero we can compute a useful normal form of (1). As the system has, essentially, two degrees of freedom, as displayed by (2), the normal form is integrable. We learn from [1] that the Hamiltonian can be approximated, to the second order in J_2, by

$$
\begin{aligned}
(3)\quad N = &-\frac{\mu^2}{2L^2} + J_2 n(a_e\mu)^2\left(-\frac{1}{4G^3}+\frac{3H^2}{4G^5}\right)\\
&+\frac{1}{2}J_2^2 n(a_2\mu)^4\left\{\frac{15}{64G^7}-\frac{15H^2}{32G^9}-\frac{105H^4}{64G^{11}}+\frac{1}{L}\left(-\frac{3}{16G^6}+\frac{9H^2}{8G^8}-\frac{27H^4}{16G^{10}}\right)\right.\\
&+\frac{1}{L^2}\left(-\frac{15}{64G^5}+\frac{27H^2}{32G^7}-\frac{15H^4}{64G^9}\right)\\
&+\cos(2g)\left[-\frac{39}{32G^7}+\frac{33H^2}{4G^9}-\frac{225H^4}{32G^{11}}+\frac{G^2+LG+L^2}{L(L+G)}\left(\frac{3}{4G^7}-\frac{9H^2}{2G^9}+\frac{15H^4}{4G^{11}}\right)\right.\\
&\left.\left.+\frac{1}{L^2}\left(\frac{3}{32G^5}-\frac{3H^2}{2G^7}+\frac{45H^4}{32G^9}\right)\right]\right\},
\end{aligned}
$$

where L, G and H are the Delaunay momenta and g is the angle canonically conjugated to G. The variables L, G and H are related to the Keplerian semimajor axis, eccentricity and inclination by

$$
(4)\qquad L(\mu a)^{1/2},\quad G = L(1-e^2)^{1/2},\quad H = G\cos I,
$$

and the problem is considered as an $O(J_2)$ perturbation of the Keplerian motion with mean motion n. We remark that G and H are nothing else then the total angular momentum and the z–component of the angular momentum. Hence $H = c$. The variables L and H being invariant we have that (3) is, essentially, a one degree of freedom Hamiltonian. The fixed points are obtained by setting $\dfrac{\partial N}{\partial G} = 0,\ \dfrac{\partial N}{\partial g} = 0$. In particular one obtains hyperbolic points for

$$
g = \pm\frac{\pi}{L},\ G = \sqrt{5}H\left[1+\frac{J_2}{500}\frac{\mu^2 a_e^2}{H^4}\left(-13+35\frac{H^2}{L^2}+40\frac{H^2/L^2}{1+\sqrt{5}H/L}\right)+O(J_2^2)\right].
$$

It is also possible to obtain the eigenvalues at those points. The hyperbolic points of (3) (concerning the (G,g) couple) are related to the hyperbolic points of the Poincaré section mentioned in section 1. By using (4) and the relation between a and h for the Kepler problem one obtains for the eigenvalues of the hyperbolic points:

$$
(5)\qquad \log\lambda_{\text{Poincaré}} = \frac{2\pi(a_e\mu)^3}{10c^5 5^{5/2}}(3J_2)^{1/2}\left(\frac{3}{c^2}+\frac{30h}{\mu^2}\right)^{1/2} J_2(1+O(J_2)),
$$

provided the factor $\left(3c^{-2}+30h\mu^{-2}\right)^{1/2}$ is not too small (then the higher order terms in J_2 become important).

As said in section 1, the separatrices of figure 2 are, in fact, splitted. In the next section we shall give bounds on the size of the splitting. The expression (5) will play a very important role. It agrees quite well with direct numerical computations of the Poincaré map.

§ 3. The maximal width of the stochastic zone

As shown in [5] for the standard map the width of the stochastic zone is exponentially small with respect to the parameter of that map. In [2] it is proved that the size of the stochastic zone is bounded by expressions of the type $M \exp\left(-\frac{2\pi(\delta-\eta)}{\log\lambda}\right)$, where λ is the eigenvalue at the homoclinic or heteroclinic points of the analytic near the identity area preserving map, η is any positive quantity (to be choosen small), M is a constant which only depends on η (and not on the small parameter, J_2 in our case) and δ is the minimum distance to the real axis of the singularities of the separatrix of some Hamiltonian planar flow (see [2] for the details). In our case the planar flow is essentially a pendulum as it follows from (3) using the fact that G changes by an small amount along the separatrix. So, with the required scalings (see again [2]) one has $\delta = \pi/2$.

The basic idea now is that to have the largest possible values of the stochastic zone we had to use the largest possible value of λ as given by (5). Let D the minimum value allowed to the perigeon distance of an artificial Earth satellite. Typically D can be taken as 6600 km. Then (5) should be maximized under the constrain $q = a(1-e) \geq D$. By using

$$G = \sqrt{5}c\Big(1+O(J_2)\Big), \quad a = -\frac{\mu}{2h}(1+O(J_2)) \quad \text{and} \quad 1-e^2 = \left(\frac{G}{L}\right)^2,$$

one obtains the equivalent constrain (skipping, from now on, the terms $O(J_2)$ when they appear as $1+O(J_2)$)

$$\text{(6)} \qquad \left[1-\left(1+\frac{10c^2h}{\mu^2}\right)^{1/2}\right]\left(\frac{\mu}{-2h}\right) \geq D.$$

To maximize (5) it is enough to maximize the factor depending on c and h, i.e. $c^{-5}\left(\frac{3}{c^2}+\frac{30h}{\mu^2}\right)^{1/2}$. By using (6) this factor is bounded by

$$\text{(7)} \qquad \frac{\left(1+\frac{2Dh}{\mu}\right)h^3}{\left[\left(1+\frac{2Dh}{\mu}\right)^2-1\right]^3},$$

where we have skipped numerical factors or factors depending only on μ. Let $w = 2Dh\mu^{-1}$. The maximum of (7) is obtained for $w = -1/2$ and this implies $h = -\frac{\mu}{4D}$, $c = \left(\frac{3\mu D}{10}\right)^{1/2}$ and $e = \frac{1}{2}$. Hence

$$\text{(8)} \qquad \max\ \log\lambda_{\text{Poincaré}} = \frac{4\pi}{9\sqrt{5}}\left(\frac{a_e}{D}\right)^3 J_2^{3/2}.$$

Using $D = 6600$ km this amounts to $2\cdot 10^{-5}$. As $\delta = \pi/2$ we can take $\eta = \frac{\pi^2-9}{2\pi} \simeq 0.1384$ to have the simple value $2\pi(\delta-\eta) = 9$. It remains to estimate M to have the desired upper bound. But this is irrelevant. Indeed one can use the constructive method given in [2] or simply we can compute numerically for much larger values of J_2. Numerically one can estimate M to be of the order of units, but even a relative error by a factor of 10^{100} is irrelevant because the dominant term is $\exp\left(-\frac{9}{\log\lambda}\right) \leq \exp\left(-4.5\cdot 10^5\right)$.

§ 4. Conclusion

It has been obtained that the J_2 problem for a feasible artificial Earth satellite, even being non integrable, can be considered as integrable for all practical purposes. This behaviour is shared by many other problems (for instance, the Hénon–Heiles problem for energies less then 0.04). We remark that despite the practical integrable character it is a hard task to obtain, in general, rather good analytical approximations to the solutions. Normal forms up to high order can be very useful for this purpose.

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