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ANTICIPATING STOCHASTIC VOLTERRA EQUATIONS

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Abstract

In this paper we establish the existence and uniqueness of a solution for stochastic Volterra equations assuming that the coefficients $F(t,s,x)$ and $G_i(t,s,x)$ are $\mathcal{F}_t$-measurable, for $s \le t$, where $\{\mathcal{F}_t\}$ denotes the filtration generated by the driving Brownian motion. We have to impose some differentiability assumptions on the coefficients, in the sense of the Malliavin calculus, in the time interval $[s,t]$. Some properties of the solution are discussed.

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1 Introduction

The purpose of this paper is to study stochastic integral equations in $\mathbb{R}^d$ of the form

$$X_t = x_0 + \int_0^t F(t,s,X_s)ds + \sum_{i=1}^{k} \int_0^t G_i(t,s,X_s)dW_s^i, \quad t \in [0,T], \qquad (1.1)$$

where W is a k-dimensional Brownian motion, and the coefficients $F(t,s,x)$ and $G_i(t,s,x)$ are $\mathcal{F}_t$-measurable. Although the solution X_t will be adapted to the filtration $\{\mathcal{F}_t\}$ generated by W, the integrand of the stochastic integral appearing in Equation (1.1) is not adapted because $G_i(t,s,X_s)$ is $\mathcal{F}_t$-measurable.

In this paper we will interpret the stochastic integral appearing in (1.1) in the Skorohod sense. The Skorohod integral introduced in [12] is an extension of the Itô integral which allows to integrate nonadapted processes. In [3] Gaveau and Trauber proved that the Skorohod integral coincides with the adjoint of the derivative operator on the Wiener space. Starting from this result, the techniques of the stochastic calculus of variations on the Wiener space (see [6]) have allowed to develop a stochastic calculus for the Skorohod integral (see [8]), which extends the classical Itô calculus. The Skorohod integral possesses most of the main properties of the Itô stochastic integral like the local property, and the quadratic variation.

Stochastic Volterra equations where the diffusion coefficient $G_i(t,s,x)$ is $\mathcal{F}_s$-measurable have been studied among others in [1] and [11]. Berger and Mizel considered linear stochastic Volterra equations with anticipating integrands in [2], using the notion of forward integral. In this paper the solution was obtained by means of the Wiener chaos expansion, taking into account the linearity of the coefficients. On the other hand, in [10] Pardoux and Protter considered stochastic Volterra equations where the coefficients $F(t,s,x)$ and $G_i(t,s,x)$ are $\mathcal{F}_t$-measurable, but $G_i(t,s,x)$ can be written in the form

$$G_i(t,s,x) = G_i(H_t;t,s,x),$$

where H_t is an adapted m-dimensional process and $G_i(h;t,s,x)$ is $\mathcal{F}_s$-measurable for each $h \in \mathbb{R}^m$, $t \geq s$, and $x \in \mathbb{R}^d$. This particular form of the coefficient $G_i(t,s,x)$ permits to control the L^p-norm of the Skorohod integral $\int_0^t G_i(H_t;t,s,X_s)dW_s^i$ using the substitution formula for this integral.

Our aim is to prove the existence and uniqueness of solution for stochastic Volterra equations of the form (1.1) when the coefficients $F(t,s,x)$ and

$G_i(t,s,x)$ are $\mathcal{F}_t$-measurable and the stochastic integral is interpreted in the Skorohod sense. In order to control the L^2-norm of the Skorohod integral $\int_0^t G_i(t,s,X_s)dW_s^i$ we will assume that the coefficient $G_i(t,s,x)$ is infinitely differentiable (in the sense of the stochastic calculus of variations) in the time interval $[s,t]$, and the derivatives $D^{n,i}_{s_1,\dots,s_n}(G_i(t,s,x))$, $s_1,\dots,s_n \in [s,t]$, verify a suitable Lipschitz property in the variable x. These hypotheses generalize the case where $G_i(t,s,x)$ is $\mathcal{F}_s$-measurable.

The paper is organized as follows. In Section 2 we present some preliminary technical results concerning the Skorohod integral that will be needed later. Section 3 is devoted to show the main result on the existence and uniqueness of solution to Eq. (1.1). Finally in Section 4 we discuss the continuity of the solution in time.

2 Preliminaries

Let $\Omega = C([0,T];\mathbb{R}^k)$ be the space of continuous functions from $[0,T]$ into $\mathbb{R}^k$ equipped with the uniform topology, let $\mathcal{F}$ denote the Borel σ-field on Ω and let P be the Wiener measure on $(\Omega,\mathcal{F})$. The canonical process $W = \{W_t, t\in[0,T]\}$ defined by $W_t(\omega)=\omega(t)$ will be a k-dimensional Brownian motion. Let $\mathcal{F}_t^0 = \sigma\{W_s, 0\le s\le t\}$ and set $\mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{N}$, where $\mathcal{N}$ the class of P-negligeable sets. Let H be the Hilbert space $L^2([0,T];\mathbb{R}^k)$. For any $h\in H$ we denote by $W(h)$ the Wiener integral

$$W(h) = \sum_{i=1}^{k} \int_0^T h_i(t)dW_t^i.$$

Let $\mathcal{S}$ be the set of cylindrical random variables of the form:

$$F = f(W(h_1),...,W(h_n)), \tag{2.1}$$

where $n\ge 1$, $f\in C_b^\infty(\mathbb{R}^n)$ (f and all its derivatives are bounded), and $h_1,...,h_n \in H$. Given a random variable F of the form (2.1), we define its ith derivative, $i=1,\dots,k$, as the stochastic process $\{D_t^iF, t\in[0,T]\}$ given by

$$D_t^iF = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(W(h_1),...,W(h_n))h_j^i(t), \quad t\in[0,T].$$

In this way the derivative DF is an element of $L^2([0,T]\times\Omega;\mathbb{R}^k) \cong L^2(\Omega;H)$. For each $i=1,\dots,k$, D^i is a closable unbounded operator from $L^2(\Omega)$ into

$L^2([0,T] \times \Omega)$. We denote by $\mathbb{D}_i^{1,2}$ the closure of $\mathcal{S}$ with respect to the norm defined by

$$\| F \|_{i,1,2}^2 = \| F \|_{L^2(\Omega)}^2 + \| D^i F \|_{L^2([0,T]\times\Omega)}^2 .$$

Define $\mathbb{D}^{1,2} = \cap_{i=1}^k \mathbb{D}_i^{1,2}$, and set

$$\| F \|_{1,2}^2 = \| F \|_{L^2(\Omega)}^2 + \sum_{i=1}^k \| D^i F \|_{L^2([0,T]\times\Omega)}^2 .$$

More generally, we can define the iterated derivative operator on a cylindrical random variable by setting

$$D_{t_1,\dots,t_n}^{n,i} F = D_{t_1}^i \cdots D_{t_n}^i F.$$

These operators are closable and we denote $\mathbb{D}_i^{n,2}$ the closure of $\mathcal{S}$ by the norm:

$$\| F \|_{i,n,2}^2 = \| F \|_{L^2(\Omega)}^2 + \sum_{l=1}^n \| D^{l,i} F \|_{L^2([0,T]^l\times\Omega)}^2 .$$

Set $\mathbb{D}^{n,2} = \cap_{i=1}^k \mathbb{D}_i^{n,2}$, and

$$\| F \|_{n,2}^2 = \| F \|_{L^2(\Omega)}^2 + \sum_{i=1}^k \sum_{l=1}^n \| D^{l,i} F \|_{L^2([0,T]^l\times\Omega)}^2 .$$

For any Borel subset A of $[0,T]$ we will denote by $\mathcal{F}_A$ the σ-field generated by the random variables $\{\int_0^T \mathbf{1}_B(s) dW_s, B \in \mathcal{B}(T), B \subset A\}$. The following result is proved in [8, Lemma 2.4]:

Proposition 2.1 *Let A be a Borel subset of $[0,T]$, and consider a random variable $F \in \mathbb{D}^{1,2}$ which is $\mathcal{F}_A$-measurable. Then $D_t F = 0$ almost everywhere in $A^c \times \Omega$.*

For each $i = 1, \dots, k$ we denote by δ_i the adjoint of the derivative operator D^i that will be also called the Skorohod integral with respect to the Brownian motion $\{W_t^i\}$. That is, the domain of δ_i (denoted by $\mathrm{Dom}\,\delta_i$) is the set of elements $u \in L^2([0,T] \times \Omega)$ such that there exists a constant c verifying

$$\left| E \int_0^T D_t^i F u_t dt \right| \le c \| F \|_{L^2(\Omega)},$$

for all $F \in \mathcal{S}$. If $u \in \operatorname{Dom}\delta_i$, $\delta_i(u)$ is the element in $L^2(\Omega)$ defined by the duality relationship

$$E(\delta_i(u)F) = E\int_0^T D_t^i F u_t dt, \quad F \in \mathcal{S}.$$

We will make use of the following notation: $\int_0^T u_t dW_t^i = \delta_i(u)$.

The set $L_a^2([0,T] \times \Omega)$ of square integrable and adapted processes is included into $\operatorname{Dom}\delta_i$ and the operator δ_i restricted to $L_a^2([0,T]\times\Omega)$ coincides with the Itô stochastic integral with respect to $\{W_t^i\}$. This property can be proved as a consequence of the following lemma proved in [8]. For any $h \in L^2([0,T])$ we will set $D_h^i F = \langle D^i F, h\rangle$, and we will denote by $\mathbb{D}_i^{h,2}$ the closure of $\mathcal{S}$ by the norm $(E(|F|^2) + E(|D_h^i F|^2))^{1/2}$.

Lemma 2.2 *Let F be a random variable in the space $\mathbb{D}_i^{h,2}$ for some function $h \in L^2([0,T])$. Then the process $Fh(t)$ belongs to* $\operatorname{Dom}\delta_i$*, and*

$$\delta_i(Fh) = FW^i(h) - D_h^i F.$$

Let $\mathbb{L}_i^{n,2} = L^2([0,T];\mathbb{D}_i^{n,2})$ equipped with the norm

$$\| v \|_{i,n,2}^2 = \| v \|_{L^2([0,T]\times\Omega)}^2 + \sum_{j=1}^n \| D^{j,i} v \|_{L^2([0,T]^{j+1}\times\Omega)}^2,$$

and set $\mathbb{L}^{n,2} = \cap_{i=1}^k \mathbb{L}_i^{n,2}$, $\mathbb{L}_i^{\infty,2} = \cap_{n\geq 1}\mathbb{L}_i^{n,2}$, $\mathbb{L}^{\infty,2} = \cap_{i=1}^k \cap_{n\geq 1} \mathbb{L}_i^{n,2}$.

We recall that $\mathbb{L}_i^{1,2}$ is included in the domain of δ_i, and for a process u in $\mathbb{L}_i^{1,2}$ we can compute the variance of the Skorohod integral of u as follows:

$$E(\delta_i(u)^2) = E\int_0^T u_t^2 dt + E\int_0^T\int_0^T D_s^i u_t D_t^i u_s ds dt. \tag{2.2}$$

We will make use of the following notation:

$$\Delta_n^T = \{(s_1,\ldots,s_n,s) \in [0,T]^{n+1} : s_1 \geq \cdots \geq s_n \geq s\},$$

and

$$\tilde{\Delta}_n^T = \{(s_1,\ldots,s_n,s) \in [0,T]^{n+1} : s_1 \geq s, \cdots, s_n \geq s\}.$$

Let $\mathcal{S}_T$ be the class of cylindrical $L^2([0,T])$-valued random random variables of the form

$$v = \sum_{i=1}^q F_i h_i, \quad F_i \in \mathcal{S}, \quad h_i \in L^2([0,T]).$$

We introduce the space $\mathbb{L}_i^{n,2,f}$ as the closure of $\mathcal{S}_H$ by the norm:

$$\| v \|_{i,n,2,f}^2 = \| v \|_{L^2([0,T]\times\Omega)}^2 + \sum_{j=1}^n \| D^{j,i} v \|_{L^2(\Delta_j^T \times\Omega)}^2, \tag{2.3}$$

where

$$\| D^{j,i} v \|_{L^2(\Delta_j^T\times\Omega)}^2 = E\int_{\Delta_j^T} |D_{s_1,\dots,s_j}^{j,i} v_s|^2 ds_1 \cdots ds_j ds.$$

We set

$$\begin{aligned} \mathbb{L}^{n,2,f} &= \cap_{i=1}^k \mathbb{L}_i^{n,2,f}, \\ \mathbb{L}_i^{\infty,2,f} &= \cap_{n\geq 1}\mathbb{L}_i^{n,2,f}, \\ \mathbb{L}^{\infty,2,f} &= \cap_{i=1}^k \cap_{n\geq 1} \mathbb{L}_i^{n,2,f}. \end{aligned}$$

That is, $\mathbb{L}_i^{1,2,f}$ is the class of stochastic processes $\{v_t\}$ that are differentiable with respect to the ith component of the Wiener process (in the sense of the stochastic calculus of variations) in the future. For a process v in $\mathbb{L}_i^{1,2,f}$ we can define the square integrable kernel $\{D_s^i v_t, s \geq t\}$ which belongs to $L^2(\Delta_1^T \times \Omega)$. More generally, if $v \in \mathbb{L}_i^{n,2,f}$ we can introduce the square integrable kernel $\{D_{s_1,\dots,s_n}^i v_t, s_1, \dots, s_n \geq t\}$ which is in $L^2(\tilde{\Delta}_n^T \times \Omega)$. Notice that in the definition (2.3) we could have integrated over the set $\tilde{\Delta}_n^T$ and get an equivalent norm.

Lemma 2.3 *The space $L_a^2([0,T]\times\Omega)$ is contained in $\mathbb{L}^{\infty,2,f}$. Furthermore, for all $v \in L_a^2([0,T]\times\Omega)$ we have $D_{s_1,\dots,s_n}^i v_t = 0$ for almost all $s_1, \dots, s_n \geq t$, and for all $i = 1, \dots, k$, $n \geq 1$, and, hence,*

$$\| v \|_{i,n,2,f}^2 = \| v \|_{L^2([0,T]\times\Omega)}^2 . \tag{2.4}$$

Proof: We will denote by $\mathcal{S}_T^a$ the class of elementary processes of the form

$$v_t = \sum_{j=0}^N F_j \mathbf{1}_{(t_j,t_{j+1}]}(t), \tag{2.5}$$

where $0 = t_0 < t_1 < \cdots < t_{N+1} = T$ and for all $j = 0, \dots, N$, F_j is a smooth and $\mathcal{F}_{t_j}$-measurable random variable. The set $\mathcal{S}_T^a$ is dense in $L_a^2([0,T]\times\Omega)$. On the other hand, we have $\mathcal{S}_T^a \subset \mathbb{L}^{\infty,2,f}$ and for any v of the form (2.5) we have, using Proposition 2.1, $D_{s_1,\dots,s_n}^i v_t = 0$, for almost all $s_1, \dots, s_n \geq t$, and for all $i = 1, \ldots, k$, and $n \geq 1$. This allows us to complete the proof. QED

The next three results are extensions of known resuls for the space $\mathbb{D}^{1,2}$ (see [7, Proposition 1.2.2, Exercice 1.2.13 and Proposition 1.3.7]).

Proposition 2.4 *Let $\psi : \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives. Suppose that $u = (u^1, \ldots, u^m)$ is an m-dimensional stochastic process whose components belong to the space $\mathbb{L}_i^{1,2,f}$. Then $\psi(u) \in \mathbb{L}_i^{1,2,f}$, and*

$$D_t^i(\psi(u_s)) = \sum_{j=1}^{m} \frac{\partial \psi}{\partial x_j}(u) D_t^i u_s^j,$$

for almost all $(t,s) \in \Delta_1^T$.

Proposition 2.5 *Let $u, v \in \mathbb{L}_i^{1,2,f}$ be two stochastic processes such that u_s and $\int_0^T (D_t^i u_s)^2 dt$ are bounded uniformly in s. Then $uv \in \mathbb{L}_i^{1,2,f}$ and, for almost all $(t,s) \in \Delta_1^T$, $D_t^i(u_s v_s) = u_s D_t v_s + v_s D_t u_s$.*

Proposition 2.6 *Let $u \in \mathbb{L}_i^{1,2,f}$ and $A \in \mathcal{F}$, such that $u_s(\omega) = 0$ a.e. on the product space $[0,T] \times A$. Then $D_t^i u_s(\omega) = 0$, for almost all $(t,s,\omega) \in \Delta_1^T \times A$.*

We will use the two following results for getting L^p estimates $(p \geq 2)$for the Skorohod integral of processes in the space $\mathbb{L}_i^{1,2,f}$.

Lemma 2.7 *Consider a process u in $\mathbb{L}_i^{1,2,f}$. Suppose that $D_\theta^i u \mathbf{1}_{[r,\theta]}$ belongs to the domain of δ_i for each interval $[r,\theta] \subset [0,T]$, and, moreover,*

$$E \int_r^T \left| \int_r^\theta D_\theta^i u_s dW_s^i \right|^2 d\theta < \infty. \tag{2.6}$$

Then $u \mathbf{1}_{[r,t]}$ belongs to the domain of δ_i for any $[r,t] \subset [0,T]$, and

$$E \left| \int_r^t u_s dW_s^i \right|^2 = E \int_r^t u_s^2 ds + 2E \int_r^t u_\theta \left(\int_r^\theta D_\theta^i u_s dW_s^i \right) d\theta. \tag{2.7}$$

Proof: To simplify the proof we will assume that W is a one-dimensional Wiener process. In that case we omit the index i in all the notations.

Suppose first that u has a finite Wiener chaos expansion. Then we can write:

$$\begin{aligned} E\left|\int_r^t u_s dW_s\right|^2 &= E\int_r^t u_s^2 ds + E\int_r^t\int_r^t D_s u_\theta D_\theta u_s d\theta ds \\ &= E\int_r^t u_s^2 ds + 2E\int_r^t\int_r^\theta D_s u_\theta D_\theta u_s d\theta ds \\ &= E\int_r^t u_s^2 ds + 2E\int_r^t u_\theta\left(\int_r^\theta D_\theta u_s dW_s\right) d\theta. \end{aligned}$$

Now, let us denote by u^k the sum of the k first terms in the Wiener chaos expansion of u. It holds that u^k converges to u in the norm $\|\cdot\|_{1,2,f}$, as k tends to infinity. For each k we have

$$E\left|\int_r^t u_s^k dW_s\right|^2 = E\int_r^t (u_s^k)^2 ds + 2E\int_r^t u_\theta^k (\int_r^\theta D_\theta u_s^k dW_s) d\theta. \tag{2.8}$$

It suffices to show that the right-hand side of (2.8) converges to the right-hand side of (2.7). This convergence is obvious for the first term. The convergence of the second summand follows from condition (2.6). QED

Remark 1: In the statement of Lemma 2.7 the assumptions are equivalent to saying that $u \in \mathbb{L}_i^{1,2,f}$ is such that $\{D_\theta^i u_s \mathbf{1}_{[0,\theta]}(s), s \in [0,T]\}$ belongs to the domain of δ_i as a processes with values in the Hilbert space $L^2([0,T])$.

Remark 2: Lemma 2.7 generalizes the isometry property of the Skorohod integral for processes in the spaces $L_a^2([0,T]\times\Omega)$ and $\mathbb{L}_i^{1,2}$.

Lemma 2.8 *Let $p \in (2,4)$, $\alpha = \frac{2p}{4-p}$. Consider a process u in $\mathbb{L}_i^{1,2,f} \cap L^\alpha([0,T]\times\Omega)$. Suppose also that, for each interval $[r,\theta] \subset [0,T]$, $D_\theta^i u \mathbf{1}_{[r,\theta]}$ belongs to the domain of δ_i, and, moreover,*

$$E\int_r^T \left|\int_r^\theta D_\theta^i u_s dW_s^i\right|^2 d\theta < \infty. \tag{2.9}$$

Then $\delta_i(u\mathbf{1}_{[r,t]})$ belongs to L^p for any interval $[r,t] \subset [0,T]$ and we have:

$$E\left|\int_r^t u_s dW_s^i\right|^p = C_p(t-r)^{\frac{p}{2}-1}\{E\int_r^t |u_s|^\alpha ds + E\int_r^t |\int_r^\theta D_\theta^i u_s dW_s^i|^2 d\theta\}, \tag{2.10}$$

where C_p is a constant depending only on p and T.

Proof: We deduce from Lemma 2.7 that $u\mathbf{1}_{[r,\theta]}$ belongs to the domain of δ_i. Now using Corollary 2.2 of [4] we deduce that (2.10) is true in the set $\mathcal{P}_T$ of processes u of the form:

$$u_t = \sum_{j=0}^{N} F_j \mathbf{1}_{[t_j, t_{j+1}]}(t), \tag{2.11}$$

where $0 = t_0 < \cdots < t_{N+1} = T$ and for all $j = 0, \ldots, N$, F_j are smooth random variables of the form (2.1), f being a polynomial function. We know that $\mathcal{P}_T$ is dense in $L^\alpha([0,T] \times \Omega)$. So, we can get a sequence $\{u^n, n \geq 1\}$ of processes in $\mathcal{P}_T$ such that u^n converges to u in $L^\alpha([0,T] \times \Omega)$. Moreover, if we consider the Ornstein-Uhlenbeck semigroup $\{T_t, t \geq 0\}$, we know that, for all t, $T_t u$ is also an element of $\mathcal{P}_T$, and we can easily prove that, for all $[r,t] \subset [0,T]$:

$$\begin{aligned} \lim_n \lim_k E \int_0^T |T_{\frac{1}{k}} u_s^n - u_s|^\alpha ds &= 0, \\ \lim_n \lim_k E \int_r^t | \int_r^\theta (D_\theta^i (T_{\frac{1}{k}} u_s^n) - D_\theta^i u_s) dW_s^i |^2 d\theta &= 0, \end{aligned}$$

which allows us to complete the proof. QED

Note that (2.7) implies

$$E\left|\int_r^t u_s dW_s^i\right|^2 \leq E\int_r^t u_s^2 ds + 2\sqrt{E\int_r^t u_s^2 ds E\int_r^t \left|\int_r^\theta D_\theta^i u_s dW_s^i\right|^2 d\theta}.$$

The iteration of this inequality leads to an estimation of the L^2 norm of the Skorohod integral $\delta_i(u)$ using only derivatives $D_{s_1,\ldots,s_n} u_t$ in future times $s_1, \ldots, s_n \geq t$. In order to introduce a norm that dominates $E|\delta_i(u)|^2$ we require the definition of a suitable class of positive sequences.

We will denote by $\mathcal{R}$ the class of positive sequences $a = \{a_n, n \geq 0\}$ such that the sequence $b(a) = \{b_n(a), n \geq 0\}$ defined by

$$\begin{aligned} b_0(a) &= a_0^2, \\ b_n(a) &= a_0^2 + 2a_0\sqrt{a_1^2 + 2a_1\sqrt{a_2^2 + 2a_2\sqrt{\cdots\sqrt{a_{n-1}^2 + 2a_{n-1}a_n}}}}, \end{aligned}$$

for $n \geq 1$, satisfies $B(a) := \lim_{n\to\infty} b_n(a) < \infty$. Notice that the sequence $b_n(a)$ is nondecreasing.

Proposition 2.9 *The class $\mathcal{R}$ coincides with the class of positive sequences $a = \{a_n, n \geq 0\}$ such that there exists a positive sequence $\epsilon = \{\epsilon_n, n \geq 0\}$ satisfying:*

$$S_\epsilon(a) := a_0^2(1 + \frac{1}{\epsilon_0}) + \sum_{k=1}^{\infty} a_k^2(\epsilon_{k-1} + \frac{\epsilon_{k-1}^2}{\epsilon_k}) < \infty, \tag{2.12}$$

and, furthermore,

$$B(a) = \min_\epsilon S_\epsilon(a). \tag{2.13}$$

Proof: a) Let us first proof that $B(a) \leq S_\epsilon(a)$ for any positive sequence ϵ. For all $R > 0$, $x, y \geq 0$ we have

$$2xy \leq Rx^2 + \frac{1}{R}y^2. \tag{2.14}$$

Using recursively (2.14) we have that, for all $R_0, R_1, \ldots, R_{n-1} > 0$,

$$b_n(a) \leq a_0^2(1 + R_0) + \frac{a_1^2(1 + R_1)}{R_0} + \frac{a_2^2(1 + R_2)}{R_0R_1} + \ldots + \frac{a_{n-1}^2(1 + R_{n-1})}{R_0R_1 \cdots R_{n-2}}$$
$$+ \frac{a_n^2}{R_0R_1 \cdots R_{n-1}},$$

and now, denoting $S_0 = R_0, S_1 = R_0R_1, \ldots, S_n = R_0R_1 \cdots R_n$ we can write

$$\begin{aligned}
b_n(a) &\leq a_0^2(1 + S_0) + \frac{a_1^2}{S_0}(1 + \frac{S_1}{S_0}) + \frac{a_2^2}{S_1}(1 + \frac{S_2}{S_1}) \\
&\quad + \ldots + \frac{a_{n-1}^2}{S_{n-2}}(1 + \frac{S_{n-1}}{S_{n-2}}) + \frac{a_n^2}{S_{n-1}} \\
&= a_0^2(1 + S_0) + \frac{a_1^2}{S_0} + \frac{a_2^2}{S_1} + \ldots + \frac{a_{n-1}^2}{S_{n-2}} + \frac{a_n^2}{S_{n-1}} \\
&\quad + a_1^2\frac{S_1}{S_0^2} + a_2^2\frac{S_2}{S_1^2} + \ldots + a_{n-1}^2\frac{S_{n-1}}{S_{n-2}^2}.
\end{aligned}$$

Finally, putting $\epsilon_i = \frac{1}{S_i}$, $i = 1, \ldots, n-1$ we obtain:

$$\begin{aligned}
b_n(a) &\leq a_0^2(1 + \frac{1}{\epsilon_0}) + \sum_{k=1}^{n-1} a_k^2(\epsilon_{k-1} + \frac{\epsilon_{k-1}^2}{\epsilon_k}) + a_n^2\epsilon_{n-1} \\
&\leq a_0^2(1 + \frac{1}{\epsilon_0}) + \sum_{k=1}^{n} a_k^2(\epsilon_{k-1} + \frac{\epsilon_{k-1}^2}{\epsilon_k}),
\end{aligned}$$

which shows that for any positive sequence ϵ we have $B(a) \leq S_\epsilon(a)$.

b) Let us now prove that if $a \in \mathcal{R}$ there exists a positive sequence $\epsilon(a)$ such that $B(a) = S_{\epsilon(a)}(a)$. Because $B(a) < \infty$ we can define, for all $n \geq 0$,

$$Q_n(a) = \lim_{N\to\infty} a_n^2 + 2a_n\sqrt{a_n^2 + 2a_n\sqrt{a_{n+1}^2 + 2a_{n+1}\sqrt{\cdots\sqrt{a_{N-1}^2 + 2a_{N-1}a_N}}}},$$

which satisfies that, for all $n \geq 0$,

(i) $B(a) = a_0^2 + 2a_0\sqrt{a_1^2 + 2a_1\sqrt{a_2^2 + 2a_2\sqrt{\cdots\sqrt{a_{n-1}^2 + 2a_{n-1}\sqrt{Q_n(a)}}}}}$,

(ii) $Q_n(a) = a_n^2 + 2a_nQ_{n+1}(a) = a_n^2(1 + R_n(a)) + \frac{1}{R_n(a)}Q_{n+1}(a)$, where $R_n(a) = \frac{\sqrt{Q_{n+1}(a)}}{a_n}$.

With these notations we can write

$$\begin{aligned} B(a) = a_0^2(1 + R_0(a)) + \frac{a_1^2(1 + R_1(a))}{R_0(a)} + \frac{a_2^2(1 + R_2(a))}{R_0(a)R_1(a)} \\ + \ldots + \frac{a_{n-1}^2(1 + R_{n-1}(a))}{R_0(a)R_1(a)\cdots R_{n-2}(a)} + \frac{Q_n(a)}{R_0(a)R_1(a)\cdots R_{n-1}(a)}, \end{aligned}$$

and taking $\epsilon_k(a) := \frac{1}{R_0(a)R_1(a)\cdots R_k(a)}$ we have that

$$\begin{aligned} B(a) &= a_0^2(1 + \frac{1}{\epsilon_0(a)}) + \sum_{k=1}^{\infty} a_k^2(\epsilon_{k-1}(a) + \frac{\epsilon_{k-1}^2(a)}{\epsilon_k(a)}) + \lim_{n\to\infty} \epsilon_n(a)Q_n(a) \\ &= S_{\epsilon(a)}(a) + \lim_{n\to\infty} \epsilon_n(a)Q_n(a), \end{aligned}$$

which shows that $S_{\epsilon(a)}(a) \leq B(a)$. The proof is now complete. QED

Remark: Obviously all the square summable sequences belong to $\mathcal{R}$. On the other hand, it is easy to find nonsquare summable sequences in $\mathcal{R}$. For example:

(i) $\{M^n, n \geq 0\}$, where M is a positive constant,

(ii) $\{n!, n \geq 0\}$,

(iii) $\{e^{n^m}, n \geq 0\}$, where m is a positive constant.

The following property is an immediate consequence of the definition of the class $\mathcal{R}$.

Lemma 2.10 *Let $a = \{a_n, n \geq 0\}$ be a positive sequence in the class $\mathcal{R}$. Suppose that $b = \{b_n, n \geq 0\}$ is another positive sequence such that $b_n \leq \rho a_n$, for all $n \geq 0$, and for some constant $\rho > 0$. Then b belongs also to the class $\mathcal{R}$, and $B(b) \leq \rho^2 B(a)$.*

We now define L_i as the class of processes u in $\mathbb{L}_i^{\infty,2,f}$ such that the sequence $d^i(u) = \{d_n^i(u), n \geq 0\}$ defined by

$$d_0^i(u) = \|u\|_{L^2([0,T]\times\Omega)},$$

and

$$d_n^i(u) = \|D^{n,i}u\|_{L^2(\Delta_n^T\times\Omega)} = \left(E\int_{\Delta_n^T} |D_{s_1,\ldots,s_n}^{n,i} u_s|^2 ds_1 \cdots ds_n ds\right)^{1/2},$$

for $n \geq 1$ belongs to $\mathcal{R}$. For $u \in L_i$ we define

$$\| u \|_{L^i}^2 := B(d^i(u)).$$

The corresponding class $L_i(\mathbb{R}^d)$ of d-dimensional processes can be defined analogously, by considering the sequence $\|D^{n,i}u\|_{L^2(\Delta_n^T\times\Omega;\mathbb{R}^d)}$.

Proposition 2.11 *$L_i \subset \operatorname{Dom}\delta_i$ and we have that, for all u in L_i,*

$$E|\delta_i(u)|^2 \leq \| u \|_{L_i}^2 . \tag{2.15}$$

Consider $p \in (2,4)$ and $\alpha = \frac{2p}{4-p}$. If, furthermore, u belongs to the space $L^\alpha([0,T]\times\Omega)$ we have that, for all $[r,t] \subset [0,T]$, $\delta_i(u\mathbf{1}_{[r,t]})$ is in L^p and

$$E|\delta_i(u\mathbf{1}_{[r,t]})|^p \leq C_p(t-r)^{\frac{p}{2}-1}\{\int_r^t E|u_s|^\alpha ds + \int_r^t \| D_\theta^i u\mathbf{1}_{[r,\theta]} \|_{L_i}^2 d\theta\}, \tag{2.16}$$

where C_p is the constant appearing in (2.10).

Proof: Let us denote by u^k the sum of the first k terms of the Wiener chaos decomposition of u. Applying Lemma 2.7 and using Schwartz inequality yields

$$
\begin{aligned}
E|\delta_i(u)|^2 &\leq \| u \|^2_{L^2([0,T]\times\Omega)} + 2 \| u \|_{L^2([0,T]\times\Omega)} \sqrt{E\int_0^T \left|\int_0^\theta D^i_\theta u_s dW^i_s\right|^2 d\theta} \\
&\leq \| u \|^2_{L^2([0,T]\times\Omega)} + 2 \| u \|_{L^2([0,T]\times\Omega)} \\
&\quad \times\Bigg(\| D^{1,i} u \|^2_{L^2(\Delta^T_1\times\Omega)} + 2 \| D^{1,i} u \|_{L^2(\Delta^T_1\times\Omega)} \\
&\quad \times\sqrt{E\int_0^T\int_0^\theta \left|\int_0^\sigma D^i_\sigma D^i_\theta u_s dW^i_s\right|^2 d\sigma d\theta}\Bigg)^{1/2}.
\end{aligned}
$$

By a recursive argument and the fact that u^k has a finite Wiener chaos decomposition it follows that (2.15) holds for every u^k. Now using the fact that $\lim_k \| u^k - u \|_{L_i} = 0$ it follows that (2.15) holds for u. In order to show (2.16) we observe that, using Lemma 2.8 we have for $p \in (2,4)$ and $\alpha = \frac{2p}{4-p}$,

$$
E|\delta_i(u\mathbf{1}_{[r,t]})|^p = C_p(t-r)^{\frac{p}{2}-1}\{E\int_r^t |u_s|^\alpha ds + E\int_r^t | \int_r^\theta D^i_\theta u_s dW^i_s|^2 d\theta\},
\tag{2.17}
$$

and now applying (2.15) to the second term of the sum, the result follows. Notice that $\int_r^t \| D^i_\theta u \mathbf{1}_{[r,\theta]} \|^2_{L_i} d\theta$ is finite if u belongs to L_i because we have

$$
\|u\mathbf{1}_{[r,t]}\|^2_{L_i} = \int_r^t E|u_s|^2 ds + 2\sqrt{\int_r^t E|u_s|^2 ds \int_r^t \| D^i_\theta u \mathbf{1}_{[r,\theta]} \|^2_{L_i} d\theta}.
$$

QED

We have the following local property for the operator δ_i:

Proposition 2.12 *Consider a process u in L_i and a set $A \in \mathcal{F}$ such that $u_t(\omega) = 0$, for almost all (t,ω) in the product space $[0,T]\times A$. Then $\delta_i(u) = 0$ a.s. on A.*

Proof: Consider the sequence of processes defined by

$$
u^m_t = \sum_{j=1}^{2^m-1} T2^m(\int_{T(j-1)2^{-m}}^{Tj2^{-m}} u_s ds)\mathbf{1}_{(Tj2^{-m},T(j+1)2^{-m}]}(t).
$$

It is easy to show that for all m the mapping $u \mapsto u^m$ is a linear bounded operator on L_i with norm bounded by T. On the other hand, it is clear that for all k, $\lim_{m\to\infty} \| (u^k)^m - u^k \|_{L_i} = 0$, where u^k denotes the sum of the first k terms of the Wiener chaos decomposition of u. This allows us to deduce that $\lim_{m\to\infty} \| u^m - u \|_{L_i} = 0$. Using now Proposition 2.11 we have that $\delta_i(u^m - u)$ tends to zero in $L^2(\Omega)$ as m tends to infinity. On the other hand, Lemma 2.2 allows us to write:

$$\begin{aligned}\delta_i(u^m) &= \sum_{j=1}^{2^m-1} T2^m \Bigg((\int_{T(j-1)2^{-m}}^{Tj2^{-m}} u_s ds)(W^i_{T(j+1)2^{-m}} - W^i_{Tj2^{-m}}) \\ &- \int_{Tj2^{-m}}^{T(j+1)2^{-m}} \int_{T(j-1)2^{-m}}^{Tj2^{-m}} D^i_\theta u_s ds d\theta \Bigg),\end{aligned}$$

and by the local property of the operator D^i in the space $\mathbb{L}_i^{1,2,f}$ (Proposition 2.6) we have that this expression is zero on the set $\{\int_0^T u_s^2 ds = 0\}$, which completes the proof.

QED

We can localize the spaces $\mathbb{L}_i^{n,2,f}$, $n \geq 1$, $\mathbb{L}_i^{\infty,2,f}$ and L_i as follows. We will denote by $\mathbb{L}_{i,loc}^{1,2,f}$ the set of random processes u such that there exists a sequence $\{(\Omega_n, u^n), n \geq 1\} \subset \mathcal{F} \times \mathbb{L}_i^{1,2,f}$ with the following properties:

(i) $\Omega_n \uparrow \Omega$, a.s.

(ii) $u = u^n$, a.s. on $[0,T] \times \Omega_n$.

We then say that $\{(\Omega_n, u^n)\}$ localizes u in $\mathbb{L}_{i,loc}^{1,2,f}$. Then, by Proposition 2.6 we can define without ambiguity the derivative $D_t^i u_s$ by setting

$$D_t^i u_s|_{\Omega_n} = D_t^i u_s^n|_{\Omega_n},$$

for each $n \geq 1$, $(t,s) \in \Delta_1^T$. In a similar way we can introduce the spaces $\mathbb{L}_{i,loc}^{\infty,2,f}$ and $L_{i,loc}$. For a process u in $\mathbb{L}_{i,loc}^{\infty,2,f}$ the iterated derivatives $D^{n,i}_{s_1,\ldots,s_n} u_s$, $s_1, \ldots, s_n \geq s$, are well defined. On the other hand, if $u \in L_{i,loc}$, and $\{(\Omega_n, u^n)\}$ localizes u in $L_{i,loc}$, then by Proposition 2.12 we can define without ambiguity the Skorohod integral $\delta_i(u)$ by putting

$$\delta_i(u)|_{\Omega_n} = \delta_i(u^n)|_{\Omega_n},$$

for each $n \geq 1$.

3 Existence and uniqueness of solution for anticipating Volterra equations

Consider a d-dimensional stochastic integral equation of the following type:

$$X_t = x_0 + \int_0^t F(t,s,X_s)ds + \sum_{i=1}^k \int_0^t G_i(t,s,X_s)dW_s^i. \tag{3.1}$$

We assume that the initial condition is a fixed point $x_0 \in \mathbb{R}^d$. We will make use of the following hypotheses on the coefficients. In the sequel $M = \{M_n, n \geq 0\}$ is a positive sequence such that $M^2 = \{M_n^2, n \geq 0\}$ is in $\mathcal{R}$, and $K > 0$ is a constant.

(H1) $F, G_i : \Omega \times \Delta_1^T \times \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, \ldots, k$, are measurable functions such that $F(t,s,x)$ and $G_i(t,s,x)$ are $\mathcal{F}_t$-measurable for each (t,s,x).

(H2) For all $t \in [0,T]$, $i = 1, \ldots, k$, $j = 1, \ldots, d$, and $x \in \mathbb{R}^d$ we have $G_i^j(t,\cdot,x)\mathbf{1}_{[0,t]}(\cdot) \in \mathbb{L}_i^{\infty,2,f}$. Furthermore, $G_i(t,\cdot,0)\mathbf{1}_{[0,t]}(\cdot)$ belongs to $L_i(\mathbb{R}^d)$ for all $t \in [0,T]$, $i = 1, \ldots, k$, and $\|G_i(t,\cdot,0)\mathbf{1}_{[0,t]}(\cdot)\|_{L_i(\mathbb{R}^d)} \leq K$.

(H3) *Lipschitz property:* For all $(t,s) \in \Delta_1^T$, $x, y \in \mathbb{R}^d$, and $i = 1, \ldots, k$ we have

$$\begin{aligned} |F(t,s,x) - F(t,s,y)| &\leq M_0|x-y| \\ |G_i(t,s,x) - G_i(t,s,y)| &\leq M_0|x-y|. \end{aligned}$$

(H3') *Lipschitz property for the derivatives of G_i:* For all $(t,s) \in \Delta_1^T$, $x, y \in \mathbb{R}^d$, $i = 1, \ldots, k$, and $n \geq 1$, we have

$$\int_{\{t \geq s_1 \geq \cdots, \geq s_n \geq s\}} \left|(D_{s_1,\ldots,s_n}^{n,i} G_i)(t,s,x) - (D_{s_1,\ldots,s_n}^{n,i} G_i)(t,s,y)\right|^2 ds_1 \cdots ds_n \leq M_n^2|x-y|^2.$$

(H4) *Linear growth condition:* For all $(t,s) \in \Delta_1^T$ we have $|F(t,s,0)| \leq K$.

Remark: Suppose that for all $(t,s) \in \Delta_1^T$, and $x \in \mathbb{R}^d$ the variable $G_i(t,s,x)$ is $\mathcal{F}_s$-measurable, and $E\int_0^t |G_i(t,s,0)|^2 ds \leq K^2$ for all $t \in [0,T]$, $i = 1, \ldots, k$, and the conditions (H3) and (H4) holds. Then hypotheses (H2) and (H3') are automatically true due to Lemma 2.3. Notice that the derivatives in future times are zero due to the measurability of $G_i(t,s,x)$ with respect to $\mathcal{F}_s$.

A consequence of the above hypotheses is the following chain rule which is similar to Lemma 2.3 in [9].

Lemma 3.1 *Suppose that $G_i(t,s,x)$ satisfies the above hypothesis (H1), (H2), (H3) and (H3'), and consider an adapted process $U \in L^2_a([0,T] \times \Omega; \mathbb{R}^d)$. Then, $G_i(t,\cdot,U_\cdot)\mathbf{1}_{[0,t]}(\cdot)$ belongs to $\mathbb{L}_i^{\infty,2,f}(\mathbb{R}^d)$, and*

$$D^{n,i}_{s_1,\dots,s_n}(G_i(t,s,U_s)) = (D^{n,i}_{s_1,\dots,s_n}G_i)(t,s,U_s), \tag{3.2}$$

for almost all $s_1,\dots,s_n \geq s$. Moreover, we have that $G_i(t,\cdot,U_\cdot)\mathbf{1}_{[0,t]}(\cdot)$ belongs to $L_i(\mathbb{R}^d)$.

Proof: To simplify we will assume that $G_i(t,s,x)$ is real valued. Let $\{\psi_\epsilon, \epsilon > 0\}$ be an approximation of the identity in $\mathbb{R}^d$ such that the support of ψ_ϵ is contained in the ball of center the origin and radius ϵ. Define

$$G_i^\epsilon(t,s) := \int_{\mathbb{R}^d} \psi_\epsilon(z - U_s) G_i(t,s,z)dz.$$

Then, from hypotheses (H2), (H3) and (H3') and using Propositions 2.1, 2.4, and 2.5 it follows that $G_i^\epsilon(t,\cdot)\mathbf{1}_{[0,t]}(\cdot)$ belongs to $\mathbb{L}_i^{\infty,2,f}$, $G_i^\epsilon(t,\cdot)\mathbf{1}_{[0,t]}(\cdot)$ converges in $L^2([0,T]\times\Omega)$ to $G_i(t,\cdot,U_\cdot)\mathbf{1}_{[0,t]}(\cdot)$ as ϵ tends to zero, and the derivatives

$$D^{n,i}_{s_1,\dots,s_n}(G_i^\epsilon(t,s)) = \int_{\mathbb{R}^d} \psi_\epsilon(z - U_s) D^{n,i}_{s_1,\dots,s_n}(G_i(t,s,z))dz$$

converge in $L^2(\Delta_n^t \times \Omega)$, as ϵ tends to zero, to $(D^{n,i}_{s_1,\dots,s_n}G_i)(t,s,U_s)$. This allows us to prove (3.2). On the other hand, using this equality and the hypotheses (H2) and (H3') we can write for all $n \geq 0$,

$$\begin{aligned}
\| D^{n,i}(G_i(t,\cdot,U_\cdot)) \|^2_{L^2(\Delta_n^t\times\Omega)} &= \| (D^{n,i}G_i)(t,\cdot,U_\cdot) \|^2_{L^2(\Delta_n^t\times\Omega)} \\
&\leq 2 \| D^{n,i}G_i(t,\cdot,0) \|^2_{L^2(\Delta_n^t\times\Omega)} \\
&\quad +2M_n^2 E\int_0^t |U_s|^2 ds.
\end{aligned}$$

As a consequence, the sequence

$$d^i(G,U) = \{\| (D^{n,i}G_i(t,\cdot,U_\cdot))\mathbf{1}_{[0,t]} \|^2_{L^2(\Delta_n^T\times\Omega)}, n \geq 0\}$$

is in the class $\mathcal{R}$, and, therefore, $G_i(t, \cdot, U_\cdot)\mathbf{1}_{[0,t]}(\cdot)$ belongs to L_i.

QED

Consider a d-dimensional square integrable adapted process U, namely, $U \in L^2_a([0,T] \times \Omega; \mathbb{R}^d)$. Define, for each $t \in [0,T]$

$$I_t(U) = \int_0^t F(t, s, U_s)ds. \tag{3.3}$$

Lemma 3.2 *Assume (H1), (H3) and (H4). For all $U, V \in L^2_a([0,T]\times\Omega; \mathbb{R}^d)$ we have*

$$\sup_{t\in[0,T]} E|I_t(U)|^2 < \infty \tag{3.4}$$

$$E|I_t(U) - I_t(V)|^2 \leq M_0^2 t \int_0^t E|U_s - V_s|^2 ds. \tag{3.5}$$

Proof: Using (H4) we have

$$E|I_t(U)|^2 \leq 2t^2K^2 + 2M_0^2 tE \int_0^t |U_s|^2 ds < \infty,$$

on the other hand, (3.5) follows easily from (H3).

QED

We are going to deduce a similar estimation for the Skorohod integral of $G_i(t, s, U_s)$ with respect to the Brownian motion W^i.

Proposition 3.3 *Assume (H1), (H2), (H3), (H3') and (H4). For any process $U \in L^2_a([0,T] \times \Omega; \mathbb{R}^d)$ and for all $i = 1, \dots, k$, for all $t \in [0,T]$ we have that the Skorohod integral*

$$J_t^i(U) := \int_0^t G_i(t, s, U_s)dW_s^i \tag{3.6}$$

exists. Furthermore, if $U, V \in L^2_a([0,T] \times \Omega; \mathbb{R}^d)$ then for each $i = 1, \dots, k$ we have

$$E|J_t^i(U) - J_t^i(V)|^2 \leq B(M^2)E \int_0^t |U_s - V_s|^2 ds. \tag{3.7}$$

Proof: Fix an index $i = 1, \dots, k$ and fix $t \in [0,T]$. Thanks to Proposition 2.11 and Lemma 3.1 we have that $J_t^i(U)$ exists. In order to prove (3.7) we define the d-dimensional process

$$v_s^i = (G_i(t, s, U_s) - G_i(t, s, V_s)).$$

From hypothesis (H3') and Lemma 3.1 we obtain

$$\| D^{n,i} v^i \|^2_{L^2(\Delta^t_n \times \Omega; \mathbb{R}^d)} \leq M_n^2 E \int_0^t |U_s - V_s|^2 ds.$$

Hence, by Lemma 2.10 v^i_s belongs to the space $L_i(\mathbb{R}^d)$, and

$$\| v^i \|^2_{L_i} \leq B(M^2) E \int_0^t |U_s - V_s|^2 ds.$$

Therefore, Proposition 2.11 (properly extended to d-dimensional processes) allows us to conclude the proof of the proposition.

QED

With these preliminaries we can state and prove the main result of this paper.

Theorem 3.4 *Assume the hypotheses (H1), (H2), (H3), (H3') and (H4). Then, there is a unique solution X to Equation (3.1) in the space $L^2_a([0,T] \times \Omega; \mathbb{R}^d)$.*

Proof of uniqueness: Using the notations introduced above we can write Equation (3.1) in the form

$$X_t = x_0 + I_t(X) + \sum_{i=1}^{k} J^i_t(X). \tag{3.8}$$

Consider another solution

$$Y_t = x_0 + I_t(Y) + \sum_{i=1}^{k} J^i_t(Y). \tag{3.9}$$

Applying Proposition 3.3 and Lemma 3.2 we get

$$E|X_t - Y_t|^2 \leq (2M_0^2 T + 2kB(M^2)) \int_0^t E \mid X_s - Y_s \mid^2 ds,$$

and by Gronwall's lemma we deduce that $X_t - Y_t = 0$ for each $t \in [0,T]$.

Proof of existence: Consider the sequence of Picard approximations of defined by

$$\begin{aligned} X^0_t &= x_0, \\ X^{n+1}_t &= x_0 + I_t(X^n) + \sum_{i=1}^{k} J^i_t(X^n), \qquad n \geq 0. \end{aligned}$$

Using Proposition 3.3 and Lemma 3.2 one can prove as usual that

$$E|X_t^{n+1} - X_t^n|^2 \le S^n E \int_{\{s_1 \le \cdots \le s_n \le t\}} |X_{s_1}^1 - X_{s_1}^0|^2 ds_1 \cdots ds_n$$
$$\le \frac{S^n t^n}{n!} \sup_{t \in [0,T]} E|X_t^1 - X_t^0|^2.$$

where $S = 2M_0^2 T + 2kB(M^2)$. From hypotheses (H2), (H3) and (H4) we deduce

$$E|X_t^1 - X_t^0|^2 \le 2E\left|\int_0^t F(t,s,x_0)ds\right|^2 + 2E\left|\sum_{i=1}^k \int_0^t G_i(t,s,x_0)dW_s^i\right|^2$$
$$\le 2ST|x_0|^2 + 4E\left|\int_0^t F(t,s,0)ds\right|^2 + 4E\left|\sum_{i=1}^k \int_0^t G_i(t,s,0)dW_s^i\right|^2$$
$$\le 2ST|x_0|^2 + 4T^2K^2 + 4kK^2.$$

From these estimations it follow easily that the sequence X^n converges in $L_a^2([0,T] \times \Omega; \mathbb{R}^d)$ to a process solution of (3.1). QED

Let us now discuss the existence of local solutions. We will say that a stochastic process $X = \{X_t, t \in [0,T]\}$ is a local solution of (3.1) if for all $t \in [0,T]$, $i = 1, \ldots, k$, we have

(a) $\int_0^t |F(t,s,X_s)|^2 ds < \infty$, a.e.

(b) The stochastic process $\{G_i(t,\cdot,X_\cdot), \mathbf{1}_{[0,t]}(\cdot)\}$ belongs to $L_{i,loc}(\mathbb{R}^d)$.

(c) The process X satisfies

$$X_t = x_0 + \int_0^t F(t,s,X_s)ds + \sum_{i=1}^k \int_0^t G_i(t,s,X_s)dW_s^i, \quad \text{a.s.}.$$

Theorem 3.5 *Consider measurable functions $F, G_i : \Omega \times \Delta_1^T \times \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, \ldots, k$, such that there exists a sequence $\{\Omega_n, F^n, G_1^n, \ldots, G_k^n, n \ge 1\}$ verifying*

(i) For each $n \ge 1$, $\Omega_n \in \mathcal{F}$ and $\Omega_n \uparrow \Omega$.

(ii) For each $n \ge 1$, F^n and G_i^n satisfy the hypotheses (H1), (H2), (H3), (H3') and (H4).

(iii) For each $n \geq 1$, $F = F^n, G_i = G_i^n$ on $\Omega_n \times \Delta_1^T \times \mathbb{R}^d$.

Then Eq. (3.1) has a local solution in $L_a^2([0,T] \times \Omega)$.

Proof: By Theorem 3.4 we have that for all $n \geq 1$ there exists an unique $X^n \in L_a^2([0,T] \times \Omega)$ so that for all $t \in [0,T]$, $i = 1, \ldots, k$, $I_t(X^n)$, $J_t^i(X^n)$ are well defined and we have

$$X_t^n = x_0 + \int_0^t F^n(t,s,X_s^n)ds + \sum_{i=1}^k \int_0^t G_i^n(t,s,X_s^n)dW_s^i. \tag{3.10}$$

The solution of Eq. (3.10) is obtained as the limit of Picard approximations. As a consequence, and taking into account hypothesis (iii) and the local property of the Skorohod integral in the space L_i (Proposition 2.12) we obtain that $X_t^n = X_t^{n+1}$ a.s. on Ω_n. Now we define a stochastic process X by setting $X = X^n$ on $\Omega_n \cap \Omega_{n-1}^c$. We have $X_t = X_t^n$ a.s. on Ω_n, for each $t \in [0,T]$, and we can write

$$X_t = x_0 + \int_0^t F^n(t,s,X_s^n)ds + \sum_{i=1}^k \int_0^t G_i^n(t,s,X_s^n)dW_s^i,$$

a.s. on Ω_n. The process X verifies the above conditions (a) and (b) by localization. Finally, using Lemma 3.1 and Proposition 2.12 we obtain

$$X_t = x_0 + \int_0^t F(t,s,X_s)ds + \sum_{i=1}^k \int_0^t G_i(t,s,X_s)dW_s^i,$$

a.s. on Ω_n, which implies that X satisfies condition (c). The proof is now complete.

QED

4 Continuity of the solution

In this section we will provide additional conditions under which the solution of Eq. (3.1) is an a.s. continuous process. The main ingredient in proving the existence of a continuous version for the solution to Eq. (3.1) will be the estimations given in Proposition 2.11.

In the sequel, we will assume that $0 < \epsilon < 1$ and $\beta > \frac{1}{2}$. We will need the following additional hypotheses:

(H5) For all $(t,r,s) \in \Delta_2^T$, $x \in \mathbb{R}^d$ we have

$$|G_i(t,s,x) - G_i(r,s,x)| \le M_0|t-r|^\beta(1+|x|)^{1-\epsilon}.$$

(H5') For all $(t,r,s) \in \Delta_2^T$, $i = 1,\dots,k$, $n \ge 1$ and $x \in \mathbb{R}^d$ we have

$$\int_{\{r\ge s_1\ge\cdots\ge s_n\ge s\}} \left|(D^{n,i}_{s_1,\dots,s_n}G_i)(t,s,x) - (D^{n,i}_{s_1,\dots,s_n}G_i)(r,s,x)\right|^2 ds_1\cdots ds_n$$
$$\le M_n^2|t-r|^{2\beta}(1+|x|)^2.$$

(H6) *Sub-linear growth condition:* For all $(t,s) \in \Delta_1^T$, $x \in \mathbb{R}^d$ we have

$$|G_i(t,s,x)| \le M_0(1+|x|)^{1-\epsilon}.$$

(H7) The mapping $t \mapsto F(t,s,x)$ is continuous on $[s,T]$ for each (s,x,ω).

Then we can prove the following result:

Theorem 4.1 *Under conditions (H1), (H2), (H3), (H3'), (H4), (H5), (H5'), (H6) and (H7), the unique solution of equation (3.1) has an a.s. continuous modification.*

Proof: Using hypotheses (H4) and (H7) and the dominated convergence theorem it is easy to show that $I_t(X)$ is a continuous function of t.

Let us now prove the continuity of the processes $\{J_t^i(X), t \in [0,T]\}$, $i = 1,\dots,k$. To simplify we will assume that all processes are one-dimensional. Fix $r < t$. For each $i = 1,\dots,k$ and $p \in (2,4)$ we can write

$$\begin{aligned} E|J_t^i(X) - J_r^i(X)|^p &\le C_p\{E|\int_0^r G_i(t,s,X_s) - G_i(r,s,X_s)dW_s^i|^p \\ &+ E|\int_r^t G_i(t,s,X_s)dW_s^i|^p\} \\ &= C_p(T_1+T_2), \end{aligned}$$

where C_p is a positive constant depending only on p. Now, using hypothesis (H6) we can show that $\{G_i(t,s,X_s)\mathbf{1}_{[0,t]}(s), s \in [0,T]\}$, is a process in the space $L^{\frac{2p}{4-p}}([0,T]\times\Omega)$, provided $2 < p \le \frac{4}{2-\epsilon}$. On the other hand we have seen that this process is in $L_i(\mathbb{R}^d)$. As a consequence, applying Proposition 2.11 and setting $\alpha = \frac{2p}{4-p}$ we have:

$$
\begin{aligned}
T_1 &\le K_p\Big(E\int_0^r |G_i(t,s,X_s) - G_i(r,s,X_s)|^\alpha ds \\
&+ \int_0^r \| D_\theta^i (G_i(t,\cdot,X_\cdot) - G_i(r,\cdot,X_\cdot))\mathbf{1}_{[0,\theta]}(\cdot) \|_{L_i}^2 \, d\theta\Big),
\end{aligned}
$$

where K_p is a constant depending only on T, p. Notice that

$$
\begin{aligned}
&E\int_0^r \int_0^\theta \int_{\Delta_n^\theta} \left| D^{n+1,i}_{\theta,s_1,\ldots,s_n}(G_i(t,s,X_s) - G_i(r,s,X_s)) \right|^2 ds_1 \cdots ds_n ds d\theta \\
&\le M_{n+1}^2 |t-r|^{2\beta} E\int_0^r (1+|X_s|)^2 ds.
\end{aligned}
$$

Hence, we obtain, using that $\alpha(1-\epsilon) \le 2$,

$$
\begin{aligned}
T_1 &\le K_p M_0^\alpha |t-r|^{\alpha\beta} E\int_0^T (1+|X_s|)^2 ds \\
&+ B(\{M_1^2, M_2^2, M_3^2, \cdots\}) |t-r|^{2\beta} E\int_0^T (1+|X_s|)^2 ds.
\end{aligned}
$$

In a similar way we can deduce the following estimates for the term T_2:

$$
\begin{aligned}
T_2 &\le K_p |t-r|^{\frac{p}{2}-1} \Big(E\Big(\int_r^t |G_i(t,s,X_s)|^\alpha ds \\
&+ \int_r^t \| D_\theta^i (G_i(t,\cdot,X_\cdot))\mathbf{1}_{[0,\theta]}(\cdot) \|_{L_i}^2 \, d\theta\Big) \\
&\le K_p (M_0^\alpha + B(\{M_1^2, M_2^2, M_3^2, \cdots\})) |t-r|^{\frac{p}{2}-1} E\int_r^t (1+|X_s|)^2) ds.
\end{aligned}
$$

Note that from the proof of Theorem 3.4 we have $\sup_{0\le t\le T} E|X_t|^2 < \infty$. Hence, we can write

$$
E|J_t^i(X) - J_r^i(X)|^p \le c|t-r|^\delta, \tag{4.1}
$$

where $\delta = \min(2\beta, \frac{p}{2}) > 1$ and c is a constant. By Kolmogorov's continuity criterion, Property (4.1) implies that the process $\{J_t^i(X), t \in [0,T]\}$ possesses a continuous version, and now the proof is complete.

QED

Remark: We can obtain another type of continuity result imposing conditions over the L^p-norm of G_i and its derivatives, and working with solutions X in the space of adapted processes $L_a^p([0,T] \times \Omega)$, for $p > 2$.

References

[1] M. A. Berger and V. J. Mizel: Volterra equations with Itô integrals. *J. Integral Equations* **2** (1980) 187-245, 319–337.

[2] M. A. Berger and V. J. Mizel: An extension of the stochastic integral. *Ann. Probab.* **10** (1982) 435–450.

[3] B. Gaveau and P. Trauber: L'intégrale stochastique comme opérateur de divergence dans l'espace fonctionnel. *J. Functional Anal.* **46** (1982) 230–238.

[4] Y. Hu and D. Nualart: Continuity of some anticipating integral processes. Preprint.

[5] M. Jolis and M. Sanz-Solé: Integrator properties of the Skorohod integral. *Stochastics* **41** (1992) 163–176.

[6] P. Malliavin: Stochastic calculus of variations and hypoelliptic operators. In: *Proc. Inter. Symp. on Stoch. Diff. Equations, Kyoto* 1976, Wiley 1978, 195–263.

[7] D. Nualart: *The Malliavin Calculus and Related Topics*, Springer, 1995.

[8] D. Nualart and E. Pardoux: Stochastic calculus with anticipating integrands. *Probab. Theory Rel. Fields* **78** (1988) 535–581.

[9] D. Ocone and E. Pardoux: A generalized Itô–Ventzell formula. Application to a class of anticipating stochastic differential equations. *Ann. Inst. Henri Poincaré* **25** (1989) 39–71.

[10] E. Pardoux and Ph. Protter: Stochastic Volterra equations with anticipating coefficients. *Annals of Probability* **18** (1990) 1635–1655.

[11] Ph. Protter: Volterra equations driven by semimartingales. *Annals of Probability* **13** (1985) 519–530.

[12] A. V. Skorohod: On a generalization of a stochastic integral. *Theory Probab. Appl.* **20** (1975) 219–233.

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