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AN EXTENSION OF ITÔ'S FORMULA FOR ANTICIPATING PROCESSES

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An extension of Itô's formula for anticipating processes

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Abstract

In this paper we introduce a class of square integrable processes, denoted by $\mathbb{L}^F$, defined in the canonical probability space of the Brownian motion, which contains both the adapted processes and the processes in the Sobolev space $\mathbb{L}^{2,2}$. The processes in the class $\mathbb{L}^F$ verify that for any time t, they are twice weakly differentiable in the sense of the stochastic calculus of variations in points (r,s) such that $r \vee s \geq t$. On the other hand, processes belonging to the class $\mathbb{L}^F$ are Skorohod integrable, and the indefinite Skorohod integral verifies properties similar to those of the Itô integral. In particular we prove a change-of-variable formula that extends the classical Itô formula. Those results are generalization of similar properties proved in [7] for processes in $\mathbb{L}^{2,2}$.

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1 Introduction

A stochastic integral for processes which are not necessarily adapted to the Brownian motion was introduced by Skorohod in [10]. The Skorohod integral turns out to be a generalization of the classical Itô integral. In [3] Gaveau and Trauber proved that the Skorohod integral coincides with the adjoint of the derivative operator on the Wiener space. Starting from this fact, one can use the techniques of the stochastic calculus of variations, introduced by Malliavin in [5], in order to study the Skorohod integral. More precisely, the Sobolev space $\mathbb{L}^{1,2}$ is included into the domain of the Skorohod integral, and in this space the Skorohod integral verifies some of the usual properties of the classical Itô integral like the quadratic variation property and the local property. Results of this type were proved by Nualart and Pardoux in [7], where a change-of-variable formula was established for the Skorohod integral of processes in the space $\mathbb{L}^{2,2}$ (twice weakly differentiable). So, we know that a stochastic calculus can be developed for processes u in the Brownian filtration which belong to one of the following classes:

(i) The class of adapted processes such that $\int_0^1 u_t^2 dt < \infty$ a.s.

(ii) The class $\mathbb{L}^{2,2}_{loc}$ of processes that are locally in the Sobolev space $\mathbb{L}^{2,2}$.

The purpose of this paper is to introduce a class of processes included in the domain of the Skorohod integral, that contain both the space of square integrable adapted proceses $L^2_a(\Omega, \mathcal{F}, P)$, and the Sobolev space $\mathbb{L}^{2,2}$. This class will be denoted by $\mathbb{L}^F$. A process $u = \{u_t, t \in [0,T]\}$ in $\mathbb{L}^F$ is required to have square integrable derivatives $D_s u_t$ and $D^2_{r,s} u_t$ in the regions $\{s \geq t\}$ and $\{s \vee r \geq t\}$, respectively. We will show that the L^2-norm of the Skorohod integral of a process u in the space $\mathbb{L}^F$ is dominated by the L^2-norm of the process u and the derivatives $D_s u_t$ and $D^2_{r,s} u_t$ in the regions $\{s \geq t\}$ and $\{s \vee r \geq t\}$. Using this fact, we will establish the local property and the quadratic variation property for the Skorohod integral of a process u in $\mathbb{L}^F$. Afterwards we discuss the existence of a continuous version for the indefinite Skorohod integral of a process in $\mathbb{L}^F$ using the techniques introduced recently by Hu and Nualart in [4], and we establish a change-of-variable formula for the Skorohod integral of processes in $\mathbb{L}^F$ verifying some additional properties. These results generalize similar properties proved in [7] for processes in the spaces $\mathbb{L}^{1,2}$ or $\mathbb{L}^{2,2}$.

The paper is organized as follows. In Section 2 we present some preliminaries on the stochastic calculus of variations on the canonical Wiener space, we introduce the class $\mathbb{L}^F$ of stochastic processes, and we show the main properties of the Skorohod integral of processes in this class. Section 3 contains the change-of-variable formula for the Skorohod integral, which extends both the cassical Itô formula and the Itô formula for the Skorohod integral. Finally in Section 4 we introduce a class of processes containing the space $L^2_a([0,T] \times \Omega)$ for which the forward integral (see [9]) exists and generalizes the Itô integral.

2 A class of Skorohod integrable processes

Let $\Omega = C([0,T])$ be the space of continuous functions from $[0,T]$ into $\mathbb{R}$ equipped with the uniform topology, $\mathcal{F}$ the Borel σ-field on Ω and let P be the Wiener measure on $(\Omega, \mathcal{F})$. The canonical process $W = \{W_t, t \in [0,T]\}$ defined by $W_t(\omega) = \omega(t)$ is a standard Brownian motion. Let $\mathcal{F}_t^0 = \sigma\{W_s, 0 \leq s \leq t\}$ and set $\mathcal{F}_t = \mathcal{F}_t^0 \bigvee \mathcal{N}$, where $\mathcal{N}$ is the class of P-negligeable sets. Let H be the Hilbert space $L^2([0,T])$. For any $h \in H$ we denote by $W(h)$ the Wiener integral

$$W(h) = \int_0^T h(t) dW_t.$$

Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form:

$$F = f(W(h_1), ..., W(h_n)), \tag{2.1}$$

where $n \geq 1$, $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its derivatives are bounded), and $h_1, ..., h_n \in H$. Given a random variable F of the form (2.1), we define its derivative as the stochastic process $\{D_t F, t \in [0,T]\}$ given by

$$D_t F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(h_1), ..., W(h_n)) h_j(t), \quad t \in [0,T].$$

In this way the derivative DF is an element of $L^2([0,T] \times \Omega) \cong L^2(\Omega; H)$. More generally, we can define the iterated derivative operator on a cylindrical random variable by setting

$$D^n_{t_1,...,t_n} F = D_{t_1} \cdots D_{t_n} F.$$

The iterated derivative operator D^n is a closable unbounded operator from $L^2(\Omega)$ into $L^2([0,T]^n \times \Omega)$ for each $n \geq 1$. We denote by $\mathbb{D}^{n,2}$ the closure of

$\mathcal{S}$ with respect to the norm defined by

$$\| F \|_{n,2}^2 = \| F \|_{L^2(\Omega)}^2 + \sum_{l=1}^{n} \| D^l F \|_{L^2([0,T]^l \times \Omega)}^2 .$$

For any Borel subset A of $[0,T]$ we will denote by $\mathcal{F}_A$ the σ-field generated by the random variables $\{\int_0^T \mathbf{1}_B(s) dW_s, B \in \mathcal{B}([0,T]), B \subset A\}$. The following result is proved in [7, Lemma 2.4]:

Proposition 2.1 *Let A be a Borel subset of $[0,T]$ and consider a random variable $F \in \mathbb{D}^{1,2}$ which is $\mathcal{F}_A$-measurable. Then $D_t F = 0$ almost everywhere in $A^c \times \Omega$.*

For any smooth random variable $F \in \mathcal{S}$ and for any $h \in H$ we can define

$$D_h F = \langle DF, h \rangle_H .$$

We have that, for all $h \in H$, D_h is a closed unbounded operator from $L^2(\Omega)$ into $L^2(\Omega)$, and we will denote $\mathbb{D}_h^{1,2}$ the closure of $\mathcal{S}$ by the norm

$$\| F \|_{1,2,h}^2 = \| F \|_{L^2(\Omega)}^2 + \| D_h F \|_{L^2(\Omega)}^2 .$$

When $h = \mathbf{1}_A$, with $A \in \mathcal{B}([0,T])$ we will simply write D_A and $\mathbb{D}_A^{1,2}$. If $F \in \mathbb{D}_A^{1,2}$, the derivative process $\{D_t F, t \in A\}$ is well defined as an element of $L^2(A \times \Omega)$.

We denote by δ the adjoint of the derivative operator D that is also called the Skorohod integral with respect to the Brownian motion $\{W_t\}$. That is, the domain of δ (denoted by $\operatorname{Dom}\delta$) is the set of elements $u \in L^2([0,T] \times \Omega)$ such that there exists a constant c verifying

$$\left| E \int_0^T D_t F u_t dt \right| \le c \| F \|_2,$$

for all $F \in \mathcal{S}$. If $u \in \operatorname{Dom}\delta$, $\delta(u)$ is the element in $L^2(\Omega)$ defined by the duality relationship

$$E(\delta(u)F) = E \int_0^T D_t F u_t dt, \quad F \in \mathcal{S}.$$

We will make use of the following notation: $\int_0^T u_t dW_t = \delta(u)$.

The Skorohod integral is an extension of the Itô integral in the sense that the set $L_a^2([0,T] \times \Omega)$ of square integrable and adapted processes is included into Dom δ and the operator δ restricted to $L^2_a([0,T]\times\Omega)$ coincides with the Itô stochastic integral (see [7]).

We will make use of the following lemma that can be proved easily by duality.

Lemma 2.2 *Consider a process $u \in L^2([0,T]\times\Omega)$ such that there exists a sequence $\{u^n, n\geq 1\}\subset$ Dom δ satisfying:*

(i) $u^n \to u$ as n tends to infinity, in $L^2([0,T]\times\Omega)$.

(ii) There exists a random variable $A\in L^2(\Omega)$ such that for all $F\in\mathcal{S}$, $E[\delta(u^n)F]$ converges to $E[AF]$.

Then we have that u belongs to the domain of δ, and $A=\delta(u)$.

The next result is proved in ([7]).

Lemma 2.3 *Let $h\in L^2([0,T])$, and $F\in\mathbb{D}^{1,2}_h$. Then the process $\{Fh(t), t\in[0,T]\}$ belongs to* Dom δ *and*

$$\delta(hF) = F\delta(h) - D_hF.$$

Let $\mathbb{L}^{n,2} = L^2([0,T];\mathbb{D}^{n,2})$ equipped with the norm

$$\| v \|^2_{n,2} = \| v \|^2_{L^2([0,T]\times\Omega)} + \sum_{j=1}^{n} \| D^j v \|^2_{L^2([0,T]^{j+1}\times\Omega)}.$$

We recall that $\mathbb{L}^{1,2}$ is included in the domain of δ, and for a process u in $\mathbb{L}^{1,2}$ we can compute the variance of the Skorohod integral of u as follows:

$$E(\delta(u)^2) = E\int_0^T u_t^2 dt + E\int_0^T\int_0^T D_s u_t D_t u_s ds dt. \qquad (2.2)$$

We will make use of the following notation

$$\begin{aligned}\Delta_1^T &= \{(s,t)\in[0,T]^2 : s\geq t\},\\ \Delta_2^T &= \{(r,s,t)\in[0,T]^3 : r\vee s\geq t\}.\end{aligned}$$

Let $\mathcal{S}_T$ be the set of processes of the form $u_t=\sum_{j=1}^q F_j h_j(t)$, where $F_j\in\mathcal{S}$ and $h_j\in H$. We will denote by $\mathbb{L}^{1,2,f}$ the closure of $\mathcal{S}_T$ by the norm:

$$\| u \|^2_{1,2,f} = E\int_0^T u_t^2 dt + E\int_{\Delta_1^T}(D_s u_t)^2 ds dt, \qquad (2.3)$$

and the space $\mathbb{L}^F$ will be defined as the closure of $\mathcal{S}_T$ by the norm:

$$\| u \|_F^2 = \| u \|_{1,2,f}^2 + E \int_{\Delta_2^T} (D_r D_s u_t)^2 dr ds dt. \tag{2.4}$$

That is, $\mathbb{L}^{1,2,f}$ is the class of stochastic processes $\{v_t, t \in [0,T]\}$ that are differentiable with respect to the Wiener process (in the sense of the stochastic calculus of variations) in the future. For a process u in $\mathbb{L}^{1,2,f}$ we can define the square integrable kernel $\{D_s u_t, s \geq t\}$ which belongs to $L^2(\Delta_1^T \times \Omega)$. Similarly, $\mathbb{L}^F$ is the class of stochastic processes $\{u_t, t \in [0,T]\}$ such that for each time t, the random variable u_t is twice weakly differentiable with respect to the Wiener process in the two-dimensional future $\{(r,s), r \vee s \geq t\}$. We also observe that $\mathbb{L}^F$ coincides with the class of processes $u \in \mathbb{L}^{1,2,f}$ such that $\{D_s u_t \mathbf{1}_{[0,s]}(t), t \in [0,T]\}$ belongs to $\mathbb{L}^{1,2}$ as a process with values in the Hilbert space $L^2([0,T])$.

Remark: Notice that if $u \in \mathbb{L}^{1,2,f}$, then $\int_a^b u_s ds \in \mathbb{D}^{1,2}_{[b,T]}$ for any $0 \leq a \leq b \leq T$.

Lemma 2.4 *The space $L_a^2([0,T] \times \Omega)$ is contained in $\mathbb{L}^F$. Furthermore, for all $u \in L_a^2([0,T] \times \Omega)$ we have $D_s u_t = 0$ for almost all $s \geq t$, and, hence,*

$$\| u \|_F^2 = \| u \|_{L^2([0,T]\times\Omega)}^2 . \tag{2.5}$$

Proof: We will denote by $\mathcal{S}_T^a$ the class of elementary processes of the form

$$v_t = \sum_{j=0}^{N} F_j \mathbf{1}_{(t_j, t_{j+1}]}(t), \tag{2.6}$$

where $0 = t_0 < t_1 < \cdots < t_{N+1} = T$ and, for all $j = 0, \ldots, N$, F_j is a smooth and $\mathcal{F}_{t_j}$-measurable random variable. The set $\mathcal{S}_T^a$ is dense in $L_a^2([0,T] \times \Omega)$. On the other hand, we have $\mathcal{S}_T^a \subset \mathbb{L}^{1,2,f}$ and for any v of the form (2.6) we have, using Proposition 2.1, $D_s v_t = 0$, for any $s \geq t$. This allows us to complete the proof. QED

The next two propositions are extensions of known resuls for the space $\mathbb{D}^{1,2}$ (see [6, Proposition 1.2.2 and Proposition 1.3.7]).

Proposition 2.5 *Let $\psi : \mathbb{R}^m \to \mathbb{R}$ be a continuously differenciable function with bounded partial derivatives. Suppose that $u = (u^1, \ldots, u^m)$ is an m-dimensional random process whose components belong to the space $\mathbb{L}^{1,2,f}$. Then $\psi(u) \in \mathbb{L}^{1,2,f}$, and*

$$D_s(\psi(u_t)) = \sum_{j=1}^{m} \frac{\partial \psi}{\partial x_j}(u) D_s u_t^j,$$

for all $(s,t) \in \Delta_1^T$.

Proposition 2.6 *Let* $u \in \mathbb{L}^{1,2,f}$ *and* $A \in \mathcal{F}$, *such that* $u_t(\omega) = 0$ *a.e. on the product space* $[0,T] \times A$. *Then* $D_s u_t = 0$ *for almost all* (s,t,ω) *in* $\Delta_1^T \times A$.

The following result provides an isometry property for the Skorohod integral of a process u in the space $\mathbb{L}^{1,2,f}$ satisfying an additional condition.

Lemma 2.7 *Consider a process* u *in* $\mathbb{L}^{1,2,f}$. *Suppose that for almost all* $\theta \in [0,T]$, $D_\theta u \mathbf{1}_{[0,\theta]}$ *belongs to the domain of* δ *and, moreover,*

$$E \int_0^T \left| \int_0^\theta D_\theta u_s dW_s \right|^2 d\theta < \infty. \tag{2.7}$$

Then $u\mathbf{1}_{[0,t]}$ *belongs to the domain of* δ *and*

$$E \left| \int_0^t u_s dW_s \right|^2 = E \int_0^t u_s^2 ds + 2E \int_0^t u_\theta (\int_0^\theta D_\theta u_s dW_s) d\theta. \tag{2.8}$$

Proof: Suppose first that u has a finite Wiener chaos expansion. In this case we can write:

$$\begin{aligned} E \left| \int_0^t u_s dW_s \right|^2 &= E \int_0^t u_s^2 ds + E \int_0^t \int_0^t D_s u_\theta D_\theta u_s d\theta ds \\ &= E \int_0^t u_s^2 ds + 2E \int_0^t \int_0^\theta D_s u_\theta D_\theta u_s d\theta ds \\ &= E \int_0^t u_s^2 ds + 2E \int_0^t u_\theta \left(\int_0^\theta D_\theta u_s dW_s \right) d\theta. \end{aligned}$$

Now, let us denote by u^n the sum of the n first terms in the Wiener chaos expansion of u. It holds that u^n converges to u in the norm $\|\cdot\|_{1,2,f}$, as n tends to infinity. For each n we have

$$E \left| \int_0^t u_s^n dW_s \right|^2 = E \int_0^t (u_s^n)^2 ds + 2E \int_0^t u_\theta^n (\int_0^\theta D_\theta u_s^n dW_s) d\theta. \tag{2.9}$$

It suffices to show that the right-hand side of (2.9) converges to the right-hand side of (2.8). This convergence is obvious for the first term. The convergence of the second summand follows from condition (2.7). QED

Remark 1: In the statement of Lemma 2.7 the assumptions are equivalent to saying that $u \in \mathbb{L}^{1,2,f}$ is such that $\{D_\theta u_s \mathbf{1}_{[0,\theta]}(s), s \in [0,T]\}$ belongs to the domain of δ as a processes with values in the Hilbert space $L^2([0,T])$.

Remark 2: Lemma 2.7 generalizes the isometry properties of the Skorohod integral for processes in the spaces $L^2_a([0,T]\times\Omega)$ and $\mathbb{L}^{1,2}$.

The following estimate of the L^p-norm of the Skorohod integral is a consequence of Corollary 2.2 of [4].

Lemma 2.8 *Let $p\in(2,4)$, $\alpha=\frac{2p}{4-p}$. Consider a process u in $\mathbb{L}^{1,2,f}\cap L^\alpha([0,T]\times\Omega)$. Suppose also that, for each interval $[r,\theta]\subset[0,T]$, $D_\theta u\mathbf{1}_{[r,\theta]}$ belongs to the domain of δ, and, moreover,*

$$E\int_r^T\left|\int_r^\theta D_\theta u_s dW_s\right|^2 d\theta<\infty. \tag{2.10}$$

Then $\delta(u\mathbf{1}_{[r,t]})$ belongs to L^p for any interval $[r,t]\subset[0,T]$ and we have:

$$E\left|\int_r^t u_s dW_s\right|^p\le C_p(t-r)^{\frac{p}{2}-1}\{E\int_r^t|u_s|^\alpha ds+E\int_r^t|\int_r^\theta D_\theta u_s dW_s)|^2d\theta\}, \tag{2.11}$$

where C_p is a constant depending only on p and T.

Proof: We deduce from Lemma 2.7 applied to $u\mathbf{1}_{[r,T]}$ that $u\mathbf{1}_{[r,t]}$ belongs to the domain of δ for any interval $[r,t]\subset[0,T]$. Now using Corollary 2.2 of [4] we deduce that (2.11) is true in the set $\mathcal{P}_T$ of processes u of the form:

$$u_t=\sum_{j=0}^N F_j\mathbf{1}_{[t_j,t_{j+1}]}(t), \tag{2.12}$$

where $0=t_0<\cdots<t_{N+1}=T$ and for all $j=0,\dots,N$, F_j are smooth random variables of the form (2.1), f being a polynomial function. We know that $\mathcal{P}_T$ is dense in $L^\alpha([0,T]\times\Omega)$. So, we can get a sequence $\{u^n,n\geq1\}$ of processes in $\mathcal{P}_T$ so that $u^n\to u$ in $L^\alpha([0,T]\times\Omega)$. Moreover, if we consider the Ornstein-Uhlenbeck semigroup $\{T_t,t\geq0\}$, we know that, for all t, $T_t u$ is also an element of $\mathcal{P}_T$, and we can easily prove that, for all $[r,t]\subset[0,T]$:

$$\begin{aligned}\lim_n\lim_k E\int_0^T|T_{\frac{1}{k}}u^n_s-u_s|^\alpha ds &= 0,\\ \lim_n\lim_k E\int_r^t|\int_r^\theta(D_\theta(T_{\frac{1}{k}}u^n_s)-D_\theta u_s)dW_s|^2d\theta &= 0,\end{aligned}$$

which allows us to complete the proof. QED

The next proposition provides estimates for the L^2 and L^p norms of the Skorohod integral of processes in the space $\mathbb{L}^F$.

Proposition 2.9 *$\mathbb{L}^F \subset \operatorname{Dom}\delta$ and we have that, for all u in $\mathbb{L}^F$,*

$$E|\delta(u)|^2 \leq 2 \parallel u \parallel_F^2 . \tag{2.13}$$

Consider $p \in (2,4)$ and $\alpha = \frac{2p}{4-p}$. If, furthermore, u belongs to the space $L^\alpha([0,T] \times \Omega)$ we have that, for all $[r,t] \subset [0,T]$, $\delta(u\mathbf{1}_{[r,t]})$ is in L^p and

$$E|\delta(u\mathbf{1}_{[r,t]})|^p \leq C_p(t-r)^{\frac{p}{2}-1}\Bigg(\int_r^t E|u_s|^\alpha ds \tag{2.14}$$

$$+E\int_r^\theta |D_\theta u_s|^2 ds + E\int_r^\theta\int_r^\theta |D_\alpha D_\theta u_s|^2 ds d\alpha\Bigg),$$

where C_p is the constant appearing in (2.11).

Proof: Because $u \in \mathbb{L}^F$ we have that $\{D_\theta u_s \mathbf{1}_{[0,\theta]}(s), s \in [0,T]\}$ belongs to $\mathbb{L}^{1,2} \subset \operatorname{Dom}\delta$ for each $\theta \in [0,T]$, and furthermore we have

$$E\int_0^T |\int_0^\theta D_\theta u_s dW_s|^2 d\theta \leq E\int_0^T\int_0^\theta (D_\theta u_s)^2 ds d\theta \tag{2.15}$$

$$+E\int_0^T\int_0^\theta\int_0^\theta (D_\sigma D_\theta u_s)^2 d\sigma ds d\theta. \tag{2.16}$$

Then, applying Lemma 2.7 we obtain that u is Skorohod integrable and

$$E|\delta(u)|^2 = E\int_0^T u_s^2 ds + 2E\int_0^T u_\theta(\int_0^\theta D_\theta u_s dW_s) d\theta.$$

Using now the Cauchy-Schwartz inequality we have

$$E|\delta(u)|^2 \leq E\int_0^T u_s^2 ds + 2\sqrt{E\int_0^T u_\theta^2 d\theta}\sqrt{E\int_0^T |\int_0^\theta D_\theta u_s dW_s|^2 d\theta},$$

and using the fact that, for all $a, b \in \mathbb{R}$, $2ab \leq a^2 + b^2$ it follows that

$$E|\delta(u)|^2 \leq 2E\int_0^T u_s^2 ds + E\int_0^T |\int_0^\theta D_\theta u_s dW_s|^2 d\theta.$$

Now applying (2.15) we obtain

$$\begin{aligned} E|\delta(u)|^2 &\leq 2E\int_0^T u_s^2 ds + E\int_0^T\int_0^\theta (D_\theta u_s)^2 ds d\theta \\ &+ E\int_0^T\int_0^\theta\int_0^\theta (D_\sigma D_\theta u_s)^2 d\sigma ds d\theta \\ &\leq 2 \parallel u \parallel_F^2, \end{aligned}$$

which proves (2.13). Similarly, using Lemma 2.8 we can prove (2.14). QED

Note that $u \in \mathbb{L}^F$ implies $u\mathbf{1}_{[r,t]} \in \mathbb{L}^F$ for any interval $[r,t] \subset [0,T]$, and, by Proposition 2.9 we have that $u\mathbf{1}_{[r,t]} \in \operatorname{Dom}\delta$.

We have the following commutative relationship between the derivative and the Skorohod integral for processes in $\mathbb{L}^F$ (see [6] , property (3) of the Skorohod integral).

Lemma 2.10 *Suppose that $u = \{u_t, t \in [0,T]\}$ belongs to $\mathbb{L}^F$. Then the process $\{\delta(u\mathbf{1}_{[0,t]}), t \in [0,T]\}$ belongs to $\mathbb{L}^{1,2,f}$ and, for all $(s,t) \in \Delta_1^T$ we have*

$$D_s(\delta(u\mathbf{1}_{[0,t]})) = \delta(D_s u\mathbf{1}_{[0,t]}). \tag{2.17}$$

The next result provides sufficient conditions for the continuity of the process

$$\delta(u\mathbf{1}_{[0,t]}) = \int_0^t u_s dW_s,$$

where $u \in \mathbb{L}^F$.

Theorem 2.11 *Let $u = \{u_t, t \in [0,T]\}$ be a process in $\mathbb{L}^F$ such that for some $\beta > 2$, $E\int_0^T |u_s|^\beta ds < \infty$. Then the process $X_t := \int_0^t u_s dW_s$ has a continuous version. More precisely, for any $0 < \gamma < \frac{\beta-2}{4\beta}$, there is a random variable C_γ such that*

$$|X_t - X_s| \le C_\gamma |t-s|^\gamma. \tag{2.18}$$

Proof: Let $p := \frac{4\beta}{\beta+2}$. We have $E\int_0^T |u_s|^{\frac{2p}{4-p}} ds < \infty$, and applying Proposition 2.9 we obtain

$$E|X_t - X_s|^p \le C_p (t-s)^{\frac{p}{2}-1} \{ \int_s^t E|u_\theta|^\beta d\theta$$
$$+ \int_s^t \int_\theta^s E|D_r u_\theta|^2 dr d\theta + \int_s^t \int_\theta^s \int_\theta^s E|D_\sigma D_r u_\theta|^2 d\sigma dr d\theta \}.$$

Therefore, there exists a nonnegative function $A : [0,T] \to \mathbb{R}_+$ such that $\int_0^T A_r dr < \infty$ and

$$E|X_t - X_s|^p \le |t-s|^{\frac{p}{2}-1} \int_s^t A_r dr. \tag{2.19}$$

For any $2 < \alpha < \frac{p+2}{2}$, applying Fubini's theorem, we obtain

$$
\begin{aligned}
E\int_0^T \int_0^T \frac{|X_t - X_s|^p}{|t-s|^\alpha} dsdt &\le 2\int_0^T \int_0^t |t-s|^{-\alpha-1+\frac{p}{2}} (\int_s^t A_r dr) dsdt \\
&= \frac{2}{(\alpha-\frac{p}{2})(1-\alpha+\frac{p}{2})} \int_0^T (r^{-\alpha+1+\frac{p}{2}} + (1-r)^{-\alpha+1+\frac{p}{2}} - 1) A_r dr.
\end{aligned}
$$

Hence the random variable

$$
\Gamma := \int_0^T \int_0^T \frac{|X_t - X_s|^p}{|t-s|^\alpha} dsdt
$$

is finite almost surely. By the Garsia-Rodemish-Rumsey lemma (see [2]), we have that for any $\gamma := \frac{(\alpha-2)(\beta+2)}{4\beta}$, there is a random constant C_γ almost surely finite such that

$$
|X_t - X_s| \le C_\gamma |t-s|^\gamma.
$$

Noting that $2 < \alpha < \frac{2+p}{2}$ is equivalent to $0 < \gamma < \frac{\beta-2}{4\beta}$, we prove the theorem. QED

We have the following local property for the operator δ:

Proposition 2.12 *We consider u a process in $\mathbb{L}^F$ and $A \in \mathcal{F}$ so that $u_t(\omega) = 0$, a.e. on the product space $[0,T] \times A$. Then $\delta(u) = 0$ a.e. on A.*

Proof: Consider the sequence of processes defined by

$$
u_t^m = \sum_{j=1}^{2^m-1} \frac{2^m}{T} (\int_{T(j-1)2^{-m}}^{Tj2^{-m}} u_s ds) \mathbf{1}_{(Tj2^{-m}, T(j+1)2^{-m}]}(t).
$$

It is easy to show that for all m , $u \to u^m$ is an linear bounded operator from $\mathbb{L}^F$ into $\mathbb{L}^F$ whose norm is bounded by $\frac{1}{T}$ and that $\lim_{m\to\infty} \| u^m - u \|_F = 0$. Using now Proposition 2.9 we have that $\delta(u^m - u)$ tends to zero in $L^2(\Omega)$ as m tends to ∞. But, on the other hand, using Lemma 2.3 we have

$$
\begin{aligned}
\delta(u^m) &= \sum_{j=1}^{2^m-1} \frac{2^m}{T} \left((\int_{T(j-1)2^{-m}}^{Tj2^{-m}} u_s ds)(W_{T(j+1)2^{-m}} - W_{Tj2^{-m}}) \right. \\
&- \left. \int_{Tj2^{-m}}^{T(j+1)2^{-m}} \int_{T(j-1)2^{-m}}^{Tj2^{-m}} D_\theta u_s ds d\theta \right),
\end{aligned}
$$

and by the local property of the operator D in the space $\mathbb{L}^{1,2,f}$ (Proposition 2.6) we have that this expression is zero on the set $\{\int_0^T u_s^2 ds = 0\}$, which completes the proof.

QED

If we have a subset $L \subset L^2([0,T] \times \Omega)$ we can localize it as follows. We will denote by L_{loc} the set of random processes u such that there exists a sequence $\{(\Omega_n, u^n), n \geq 1\} \subset \mathcal{F} \times L$ with the following properties:

(i) $\Omega_n \uparrow \Omega$, a.s.

(ii) $u = u^n$, a.e. on $[0,T] \times \Omega_n$.

We then say that $\{(\Omega_n, u^n)\}$ localizes u in L. If $u \in \mathbb{L}^F_{loc}$, by Proposition 2.12 we can define without ambiguity $\delta(u)$ by setting

$$\delta(u)|_{\Omega_n} = \delta(u^n)|_{\Omega_n},$$

for each $n \geq 1$, where $\{(\Omega_n, u^n)\}$ is a localizing sequence for u in $\mathbb{L}^F$. The following result says that the operator δ on $\mathbb{L}^F_{loc}$ is an extension of the Itô integral.

Lemma 2.13 *Let u be an adapted process verifying $\int_0^T u_s^2 ds < \infty$ a.s. Then u belongs to $\mathbb{L}^F_{loc}$ and $\delta(u)$ coincides with the Itô integral.*

Proof: For any integer $k \geq 1$ consider an infinitely differentiable function $\varphi_k : \mathbb{R} \to \mathbb{R}$ such that $\varphi_k(x) = 1$ if $|x| \leq k$, and $\varphi_k(x) = 0$ if $|x| \geq k+1$. Define

$$u_t^k = u_t \varphi_k \left(\int_0^t u_s^2 ds \right),$$

and

$$\Omega_k = \left\{ \int_0^T u_s^2 ds \leq k \right\}.$$

Then we have $\Omega_k \uparrow \Omega$ a.s., $u = u^k$ on $[0,T] \times \Omega_k$, and $u^k \in L_a^2([0,T] \times \Omega)$ because u^k is adapted and

$$\int_0^T (u_t^k)^2 dt = \int_0^T u_t^2 \varphi_k^2 \left(\int_0^t u_s^2 ds \right) dt \leq k+1.$$

QED

The next result will show the existence of a nonzero quadratic variation for the indefinite Skorohod integral.

Theorem 2.14 *Suppose that u is a process of the space $\mathbb{L}^F_{loc}$. Then*

$$\sum_{i=1}^{n-1}(\int_{t_i}^{t_{i+1}} u_s dW_s)^2 \to \int_0^t u_s^2 ds, \tag{2.20}$$

in probability, as $|\pi| \to 0$, where π runs over all finite partitions $\{0 = t_0 < t_1 < \cdots < t_n = T\}$ of $[0,T]$. Moreover, the convergence is in $L^1(\Omega)$ if u belongs to $\mathbb{L}^F$.

Proof: We will describe the details of the proof only for the case $u \in \mathbb{L}^F$. The general case would be deduced by an easy argument of localization. For any process u in $\mathbb{L}^F$ and for any partition $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ we define

$$V^\pi(u) = \sum_{i=0}^{n-1}(\int_{t_i}^{t_{i+1}} u_s dW_s)^2.$$

Suppose that u and v are two processes in $\mathbb{L}^F$. Then we have

$$\begin{aligned} E(|V^\pi(u) - V^\pi(v)|) &\le \left(\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}(u_s - v_s)dW_s\right)^2\right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}}(u_s + v_s)dW_s\right)^2\right)^{\frac{1}{2}} \\ &\le 4 \parallel u - v \parallel_F \parallel u + v \parallel_F . \end{aligned} \tag{2.21}$$

It follows from this estimate that it suffices to show the result for a class of processes u that is dense in $\mathbb{L}^F$. So we can assume that

$$u_t = \sum_{j=0}^{m-1} F_j \mathbf{1}_{(s_j, s_{j+1}]},$$

where F_j is a smooth random variable for each j, and $0 = s_0 < \cdots < s_m = T$. We can assume that the partition π contains the points $\{s_0, \ldots, s_m\}$. In this case we have

$$\begin{aligned} V^\pi(u) &= \sum_{j=0}^{m-1} \sum_{\{i: s_j < t_i \le s_{j+1}\}} \left(F_j(W(t_{i+1}) - W(t_i)) - \int_{t_i}^{t_{i+1}} D_s F_j ds\right)^2 \\ &= \sum_{j=0}^{m-1} \left[\sum_{\{i: s_j < t_i \le s_{j+1}\}} F_j^2 (W(t_{i+1}) - W(t_i))^2 \right. \\ &\quad \left. -2(W(t_{i+1}) - W(t_i)) \int_{t_i}^{t_{i+1}} D_s F_j ds + \left(\int_{t_i}^{t_{i+1}} D_s F_j ds\right)^2 \right]. \end{aligned}$$

With the properties of the quadratic variation of the Brownian motion, this converges in $L^1(\Omega)$ to

$$\sum_{j=0}^{m-1} F_j^2(s_{j+1}-s_j) = \int_0^T u_s^2 ds,$$

as $|\pi|$ tends to zero.

QED

As a consequence of this result, if $u \in \mathbb{L}_{loc}^F$ is a process such that the Skorohod integral $\int_0^t u_s dW_s$ has a continuous version with bounded variation paths, the $u \equiv 0$.

3 Itô formula for the Skorohod integral

Our purpose in this section is to prove a version of the change-of-variables formula for the indefinite Skorohod integral. For a process $X \in \mathbb{L}^{1,2,f}$ we will denote D^-X the element of $L^1([0,T]\times\Omega)$ defined by

$$\lim_n \int_0^T \sup_{(s-\frac{1}{n})\vee 0\le t<s} E|D_sX_t - (D^-X)_s|ds = 0, \tag{3.22}$$

provided that this limit exists. We will denote by $\mathbb{L}_{1-}^{1,2,f}$ the class of processes in $\mathbb{L}^{1,2,f}$ such that the limit (3.22) exists. It is easy to show that a process of the form

$$X_t = X_0 + \int_0^t u_s dW_s + \int_0^t v_s ds,$$

where $X_0 \in \mathbb{D}_{loc}^{1,2}$, $u \in \mathbb{L}_{loc}^F$, and $v \in \mathbb{L}_{loc}^{1,2,f}$ belongs to the class $\mathbb{L}_{1-}^{1,2,f}$ and

$$(D^-X)_t = D_tX_0 + \int_0^t D_t v_s ds + \int_0^t D_t u_s dW_s.$$

We will also denote by $\mathbb{L}_b^F$ the space of processes $u \in \mathbb{L}^F$ such that $\|\int_0^T u_s^2 ds\|_\infty < \infty$.

Theorem 3.1 *Consider a process of the form $X_t = X_0 + \int_0^t u_s dW_s + \int_0^t v_s ds$, where $X_0 \in \mathbb{D}_{loc}^{1,2}$, $u \in (\mathbb{L}_b^F)_{loc}$, and $v \in \mathbb{L}_{loc}^{1,2,f}$. We will also assume that the indefinite Skorohod integral $\int_0^t u_s dW_s$ has a continuous version. Let $F : \mathbb{R} \to$*

$\mathbb{R}$ *be a twice continuously differentiable function. Then we have*

$$F(X_t) = F(X_0) + \int_0^t F'(X_s)dX_s + \frac{1}{2}\int_0^t F''(X_s)u_s^2 ds + \int_0^t F''(X_s)(D^- X)_s u_s ds. \qquad (3.23)$$

Proof: Suppose that $(\Omega^{n,1}, X_0^n)$, $(\Omega^{n,2}, u^n)$ and $(\Omega^{n,3}, v^n)$ are localizing sequences for X_0, u, and v, respectively. For each positive integer k let ψ_k be a smooth function such that $0 \le \psi_k \le 1$, $\psi_k(x) = 0$ if $|x| \ge k+1$, and $\psi_k(x) = 1$ if $|x| \le k$. Define

$$v_t^{n,k} = v_t^n \psi_k\left(\int_0^t |v_s^n|^2 ds\right).$$

Set $X_t^{n,k} = X_0^n + \int_0^t u_s^n dW_s + \int_0^t v_s^{n,k} ds$, and consider the family of sets

$$G^{n,k} = \Omega^{n,1} \cap \Omega^{n,2} \cap \Omega^{n,3} \cap \{\sup_{t\in[0,1]} |X_t| \le k\} \cap \{\int_0^T |v_s^n|^2 ds \le k\}.$$

Define also $F^k = F\psi_k$. Then it suffices to show the result for the processes X_0^n, u^n, and $v^{n,k}$, and for the function F^k. In this way we can assume that $X_0 \in \mathbb{D}^{1,2}$, $u \in \mathbb{L}^F$, $\int_0^T u_s^2 ds \le M$, for some constant $M > 0$, $v \in \mathbb{L}^{1,2,f}$, $\int_0^T |v_s|^2 ds \le k$, and that the functions F, F' and F'' are bounded. Moreover, we can assume that the process X_t has a continuous version.

Set $t_i^n = \frac{it}{2^n}$, $0 \le i \le 2^n$. Applying Taylor development up to the second order we obtain

$$F(X_t) = F(X_0) + \sum_{i=0}^{2^n-1} F'(X(t_i^n))(X(t_{i+1}^n) - X(t_i^n)) + \sum_{i=0}^{2^n-1} \frac{1}{2} F''(\bar{X}_i)(X(t_{i+1}^n) - X(t_i^n))^2, \qquad (3.24)$$

where $\bar{X}_i$ denotes a random intermediate point between $X(t_i^n)$ and $X(t_{i+1}^n)$. Now the proof will be decomposed in several steps.

Step 1. Let us show that

$$\sum_{i=0}^{2^n-1} F''(\bar{X}_i)(X(t_{i+1}^n) - X(t_i^n))^2 \to \int_0^t F''(X_s)u_s^2 ds, \qquad (3.25)$$

in $L^1(\Omega)$, as n tends to infinity.

The increment $(X(t_{i+1}^n) - X(t_i^n))^2$ can be decomposed into

$$\left(\int_{t_i^n}^{t_{i+1}^n} u_s dW_s\right)^2 + \left(\int_{t_i^n}^{t_{i+1}^n} v_s ds\right)^2 + 2\left(\int_{t_i^n}^{t_{i+1}^n} u_s dW_s\right)\left(\int_{t_i^n}^{t_{i+1}^n} v_s ds\right).$$

Using the boundedness of $F''(\bar{X}_t)$, the property $\int_0^1 |v_s|^2 ds \le k$, and Theorem 2.14 we can show that the contribution of the last two terms to the limit (3.25) is zero. Therefore, it suffices to show that

$$\sum_{i=0}^{2^n-1} F''(\bar{X}_i)\left(\int_{t_i^n}^{t_{i+1}^n} u_s dW_s\right)^2 \to \int_0^t F''(X_s) u_s^2 ds, \tag{3.26}$$

in $L^1(\Omega)$, as n tends to infinity. Suppose that $n \ge m$, and for any $i = 1, \ldots, 2^n$ let us denote by $t_i^{(m)}$ the point of the mth partition which is closer to t_i^n from the left. Then we have

$$\begin{aligned}
&E\left|\sum_{i=0}^{2^n-1} F''(\bar{X}_i)\left(\int_{t_i^n}^{t_{i+1}^n} u_s dW_s\right)^2 - \int_0^t F''(X_s) u_s^2 ds\right| \\
&\le E\left|\sum_{i=0}^{2^n-1} [F''(\bar{X}_i) - F''(X(t_i^{(m)}))]\left(\int_{t_i^n}^{t_{i+1}^n} u_s dW_s\right)^2\right| \\
&\quad + E\left|\sum_{j=0}^{2^m-1} F''(X(t_j^m)) \sum_{i:t_i^n \in [t_j^m, t_{j+1}^m)} \left[\left(\int_{t_i^n}^{t_{i+1}^n} u_s dW_s\right)^2 - \int_{t_i^n}^{t_{i+1}^n} u_s^2 ds\right]\right| \\
&\quad + E\left|\sum_{j=0}^{2^m-1} F''(X(t_j^m)) \int_{t_j^m}^{t_{j+1}^m} u_s^2 ds - \int_0^t F''(X_s) u_s^2 ds\right| \\
&= b_1 + b_2 + b_3.
\end{aligned}$$

The term b_3 can be bounded by

$$E\left(\sup_{|s-r| \le t2^{-m}} |F''(X_s) - F''(X_r)| \int_0^t u_s^2 ds\right),$$

which converges to zero as m tends to infinity by the continuity of the process X_t. In the same way the term b_1 is bounded by

$$E\left(\sup_{|s-r| \le t2^{-m}} |F''(X_s) - F''(X_r)| \sum_{i=0}^{2^n-1}\left(\int_{t_i^n}^{t_{i+1}^n} u_s dW_s\right)^2\right),$$

which tends to zero as m tends to infinity uniformly in n. Indeed, if we write

$$A_m = \sup_{|s-r|\leq t2^{-m}} |F''(X_s) - F''(X_r)|,$$

and

$$B_n = \sum_{i=0}^{2^n-1} (\int_{t_i^n}^{t_{i+1}^n} u_s dW_s)^2,$$

then, for any $K > 0$ we can write

$$E(A_m B_n) \leq KE(A_m) + 2\|F''\|_\infty E(B_n \mathbf{1}_{\{B_n > K\}}),$$

and this implies that

$$\lim_{K\to\infty} \lim_{m\to\infty} \sup_{n\geq m} E(A_m B_n) = 0.$$

Finally, the term b_2 converges to zero as n tends to infinity, for any fixed m, due to Theorem 2.14.

Step 2. Clearly the term

$$\sum_{i=0}^{2^n-1} F'(X(t_i^n)(\int_{t_i^n}^{t_{i+1}^n} v_s ds)$$

converges a.s. and in $L^1(\Omega)$ to $\int_0^t F'(X_s)v_s ds$, as n tends to infinity.

Step 3. From Lemma 2.4 and Proposition 2.6 we deduce

$$\begin{aligned} F'(X(t_i^n)) \int_{t_i^n}^{t_{i+1}^n} u_s dW_s &= \int_{t_i^n}^{t_{i+1}^n} F'(X(t_i^n)) u_s dW_s \\ &+ \int_{t_i^n}^{t_{i+1}^n} D_s[F'(X(t_i^n))] u_s ds \\ = \int_{t_i^n}^{t_{i+1}^n} F'(X(t_i^n)) u_s dW_s \\ &+ F''(X(t_i^n)) \int_{t_i^n}^{t_{i+1}^n} D_s X(t_i^n) u_s ds. \end{aligned}$$

Notice that

$$\int_{t_i^n}^{t_{i+1}^n} D_s X(t_i^n) u_s ds = \int_{t_i^n}^{t_{i+1}^n} (D_s X_0 + \int_0^{t_i^n} D_s u_r dW_r + \int_0^{t_i^n} D_s v_r dr) u_s ds. \tag{3.27}$$

We will show that $\sum_{i=0}^{2^n-1} F''(X(t_i^n)) \int_{t_i^n}^{t_{i+1}^n} D_s X(t_i^n) u_s ds$ converges in L^1 to $\int_0^t F''(X_s)(D^- X)_s u_s ds$ as n tends to infinity. In fact, we have

$$\begin{aligned}
&\left| \sum_{i=0}^{2^n-1} F''(X(t_i^n)) \int_{t_i^n}^{t_{i+1}^n} D_s X(t_i^n) u_s ds - \int_0^t F''(X_s)(D^- X)_s u_s ds \right| \\
&\quad \leq \left| \sum_{i=0}^{2^n-1} F''(X(t_i^n)) \int_{t_i^n}^{t_{i+1}^n} [D_s X(t_i^n) - (D^- X)_s] u_s ds \right| \\
&\quad + \left| \sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} [F''(X(t_i^n)) - F''(X_s)](D^- X)_s u_s ds \right|.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
&E\left(\left| \sum_{i=0}^{2^n-1} F''(X(t_i^n)) \int_{t_i^n}^{t_{i+1}^n} D_s X(t_i^n) u_s ds - \int_0^t F''(X_s)(D^- X)_s u_s ds \right| \right) \\
&\leq \sqrt{M} \|F''\|_\infty \left\{ E \sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} |D_s X(t_i^n) - (D^- X)_s|^2 ds \right\}^{\frac{1}{2}} \\
&+ E\left(\sup_{|s-r| \leq 2^{-n}} |F''(X_s) - F''(X_r)| \int_0^1 |(D^- X)_s u_s| ds \right) = d_1 + d_2.
\end{aligned}$$

The term d_2 converges to zero as n tends to infinity, because

$$E \int_0^1 |(D^- X)_s u_s| ds < \infty.$$

For the term d_1 the convergence to zero follows from the estimates

$$\begin{aligned}
&E \sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \left| \int_0^{t_i^n} D_s u_r dW_r - \int_0^s D_s u_r dW_r \right|^2 ds \\
&= E \sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \left| \int_{t_i^n}^s D_s u_r dW_r \right|^2 ds \\
&\leq E \sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \int_{t_i^n}^s |D_s u_r|^2 dr ds + E \sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \int_{t_i^n}^s \int_{t_i^n}^s |D_\theta D_s u_r|^2 d\theta dr ds,
\end{aligned}$$

and a similar estimate for

$$E \sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} \left| \int_0^{t_i^n} D_s v_r dr - \int_0^s D_s v_r dr \right|^2 ds.$$

Step 4. Finally, we will show that for all $t \in [0,T]$, the process

$$\Phi_s = F'(X_s)u_s\mathbf{1}_{[0,t]}(s)$$

belongs to the domain of δ and that

$$\int_0^t F'(X_s)u_s dW_s = F(X_t) - F(X_0) - \int_0^t F'(X_s)v_s ds \qquad (3.28)$$
$$-\frac{1}{2}\int_0^t F''(X_s)(D^-X)_s u_s ds.$$

To get this result we will apply Lemma 2.2 to the sequence of processes

$$\Phi_s^n = u_s \sum_{i=0}^{2^n-1} F'(X(t_i^n))\mathbf{1}_{[t_i^n,t_{i+1}^n]}(s).$$

We have that Φ^n converges in $L^2([0,T] \times \Omega)$ to Φ as n tends to infinity. From Step 1, Step 2 and Step 3, we have that

$$\delta(\Phi^n) = \sum_{i=0}^{2^n-1} \int_{t_i^n}^{t_{i+1}^n} F'(X(t_i^n))u_s dW_s$$

converges in $L^1(\Omega)$ to a random variable A equals to the right-hand side of Equation (3.28). Then in order to complete the proof it suffices to show that A belongs to $L^2(\Omega)$. This follows from the fact that $u \in \mathbb{L}_b^F$, $X_0 \in \mathbb{D}^{1,2}$ and $v \in \mathbb{L}^{1,2,f}$. QED

Remarks:

1.- The assumptions of Theorem 3.1 can be slightly modified. For instance, in order to insure that the process $\int_0^t u_s dW_s$ has continuous paths we can assume that $u \in (\mathbb{L}_b^F \cap L^\beta([0,T] \times \Omega))_{loc}$ for some $\beta > 2$, due to Theorem 2.11. On the other hand, notice that the fact that $\int_0^T u_s^2 ds$ is bounded is only used to insure that the right-hand side of Equation (3.29) is square integrable. Then, instead of assuming that the process u is locally in the space $\mathbb{L}_b^F$ we can assume that the processes u, v and the initial condition X_0 verify locally the following properties:

(i) $u \in \mathbb{L}^F$ and $E(\int_0^T u_s^2 ds)^2 < \infty$.

(ii) $X_0 \in \mathbb{D}^{1,2}$ and $E(\int_0^T |D_s X_0|^2 ds)^2 < \infty$.

(iii) $v \in \mathbb{L}^{1,2,f}$ and $E(\int_{\{s\geq t\}} |D_s v_t|^2 ds dt)^2 < \infty$.

2.- If X_0 is a constant, and u_t and v_t are adapted processes such that $\int_0^T |u_t|^2 dt < \infty$, $\int_0^T |v_t|^2 dt < \infty$ a.s., then these processes satisfy the assumptions of the theorem because $u \in (\mathbb{L}_b^F)_{loc}$ by Lemma 2.13, $v \in \mathbb{L}_{loc}^{1,2,f}$ (this property can be proved by a localization procedure similar to that used in the proof of Lemma 2.13), and the process $X_t = X_0 + \int_0^t u_s dW_s + \int_0^t v_s ds$ has continuous paths because it is a continuous semimartingale. Furthermore, in this case $(D^- X)_s$ vanishes, and we obtain the classical Itô formula.

3.- Instead of working with processes $v \in \mathbb{L}_{loc}^{1,2,f}$ we can rewrite the proof of Theorem 3.1 assumig that v is a process verifying locally the following properties:

(i) $\int_0^T |v_s| ds < k$.

(ii) The process $V = \{\int_0^t v_s ds, t \in [0,T]\}$ belongs to the space $\mathbb{L}_{1-}^{1,2,f}$.

In this case the proof is the same, changing the term $\int_0^t D_t v_s ds$ by the term $(D^- V)_t$. Notice that if we assume this hypothesis for v, taking into account Remark 2, the change-of-variable formula for the Skorohod integral established in Theorem 3.1 is a generalization of the classical Itô's formula, in the sense that v is an adapted process satisfying $\int_0^t |v_t| dt < \infty$ a.s.

4 Forward integral

The forward integral introduced in [1] is an extension of the Itô integral to nonadapted processes which differs from the Skorohod integral. In [9] this type of integral is defined using the following approximation approach. Let $u = \{u_t, t \in [0,T]\}$ be a process in $L^2([0,T] \times \Omega)$. Then for every $m > 1$ we can introduce the term

$$F^m(u) = \frac{2^m}{T} \int_0^T u_t (W_{(t+T2^{-m})\wedge T} - W_t) dt.$$

Definition 4.1 *We say that a process $u = \{u_t, t \in [0,T]\}$ in $L^2([0,T]\times\Omega)$ is forward integrable if the sequence $F^m(u)$ converges in probability as $m \to \infty$, and in this case the limit will be denoted by $\int_0^T u_t d^- W_t$.*

The next result gives us the relationship between the forward integral and the Skorohod integral:

Theorem 4.2 *Let $u \in (\mathbb{L}^F_{1^-})_{loc}$. Then u is forward integrable and*

$$\int_0^T u_t d^- W_t = \delta(u) + \int_0^T (D^- u)_t dt. \tag{4.29}$$

Proof: By a localization argument we can assume that $u \in \mathbb{L}^F_{1^-}$. Thanks to Lemma 2.3 we have that, for every $m > 1$,

$$F^m = \frac{2^m}{T} \int_0^T \left(\int_t^{(t+T2^{-m})\wedge T} u_t dW_r \right) dt + \frac{2^m}{T} \int_0^T \left(\int_t^{(t+T2^{-m})\wedge T} D_r u_t dr \right) dt.$$

Now the proof will be descomposed into two steps.

Step 1. Let us show that

$$\frac{2^m}{T} \int_0^T \left(\int_t^{(t+T2^{-m})\wedge T} u_t dW_r \right) dt \to \int_0^T u_r dW_r \tag{4.30}$$

in $L^2(\Omega)$, as m tends to ∞. Using the Fubini theorem for the Skorohod integral (see [6], Exercice 3.2.8) we have that

$$\frac{2^m}{T} \int_0^T \left(\int_t^{(t+T2^{-m})\wedge T} u_t dW_r \right) dt = \int_0^T U_r^m dW_r,$$

where

$$U_r^m = \frac{2^m}{T} \int_{(r-T2^{-m})\vee 0}^r u_t dt.$$

It is easy to show that for all m, $u \to U^m$ is an linear bounded operator from $\mathbb{L}^F$ into $\mathbb{L}^F$ whose norm is bounded by $\frac{1}{T}$ and that $\lim_{m\to\infty} \| U^m - u \|_F = 0$. Using now Proposition 2.10 we have that $\delta(U^m)$ tends to $\delta(u)$ in $L^2(\Omega)$ as m tends to ∞.

Step 2. Let us prove that

$$E \left| \frac{2^m}{T} \int_0^T \left(\int_t^{(t+T2^{-m})\wedge T} D_r u_t dr \right) dt - \int_0^T (D^- u)_r dr \right| \tag{4.31}$$

tends to zero as m tends to ∞.
We have that

$$\begin{aligned} & E \left| \frac{2^m}{T} \int_0^T \left(\int_t^{(t+T2^{-m})\wedge T} D_r u_t dr \right) dt - \int_0^T (D^- u)_r dr \right| \\ & \leq \frac{2^m}{T} E \left| \int_0^T \left(\int_t^{(t+T2^{-m})\wedge T} D_r u_t - (D^- u)_r dr \right) dt \right| \end{aligned}$$

$$
\begin{aligned}
&+E|\frac{2^m}{T}\int_0^T\Big(\int_t^{(t+T2^{-m})\wedge T}(D^-u)_r dr\Big)dt-\int_0^T(D^-u)_r dr| \\
&\le \int_0^T \sup_{(r-T2^{-m})\vee 0<t<r} E|D_r u_r-(D^-u)_r|dr \\
&+E|\frac{2^m}{T}\int_0^T\int_{(r-T2^-m)\vee 0}^{r}(D^-u)_r dtdr-\int_0^T(D^-u)_r dr|.
\end{aligned}
$$

Now, thanks to the definition of the class $\mathbb{L}_{1-}^F$ and the dominated convergence theorem we have that these two terms tend to zero as m tends to ∞, and now the proof is complete.

QED

Remark: The forward integral is an extension of the Itô integral. In fact, notice that $L_a^2([0,T]\times\Omega)\subset\mathbb{L}_{1-}^F$ and that for a process $u\in L_a^2([0,T]\times\Omega)$ the forward integral coincides with the Itô stochastic integral.

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