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CONVERGENCE WITHIN NONISOTROPIC REGIONS OF HARMONIC FUNCTIONS IN B^n

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Convergence within nonisotropic regions of harmonic functions in $\mathbf{B}^n$

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Abstract

In this paper we study the boundedness in L^p of the projections over spaces of functions with spectrum contained in horizontal strips. We obtain some results concerning convergence along nonisotropic regions of harmonic extensions of functions in $L^p(\mathbf{S}^n)$, with spectrum included in these horizontal strips.

1 Introduction

This work deals with some topics related to the expansion of functions in $L^2(\mathbf{S}^n)$, $\mathbf{S}^n$ the unit sphere in $\mathbf{C}^n$, in terms of harmonic homogeneous polynomials $H(r,s)$ of bidegree (r,s). The projections K_{rs} of $L^2(\mathbf{S}^n)$ onto $H(r,s)$, extend to $L^1(\mathbf{S}^n)$ and permit to define for every $f \in L^1(\mathbf{S}^n)$ the spectrum of f, $\operatorname{spec} f = \{ (r,s) \in \mathbf{Z}_+ \times \mathbf{Z}_+ ;\ K_{rs} f \neq 0 \}$. The orthogonal projection from $\oplus_{r,s} H(r,s)$ to $\oplus_r H(r,0)$ can be identified to the Cauchy-Szegö projection and it is well known that it can be continuously extended to L^p, $p > 1$. What happens if we project to others $\oplus_{(r,s)\in\Omega} H(r,s)$? This is a very difficult problem whose answer is not known even for the Fourier expansions when $n = 1$. The first object of this work is to study the boundedness in L^p when Ω is a horizontal strip $\Omega_{0k} = \{(r,s) \in \mathbf{Z}_+ \times \mathbf{Z}_+ : 0 \leq s \leq k\}$.

It is well known that the harmonic extensions of $L^p(\mathbf{S}^n)$ to $\mathbf{B}^n$ have limit a.e. along non-tangential regions and that if the functions is in H^p, that is, its spectrum is in $\mathbf{Z}_+ \times \{0\}$, there exist convergence along admissible regions that are tangential in some directions, if $n > 1$. Is there some relation of this fact with the spectrum of the functions? The second topic of this work is to study convergence along admissible and other tangential regions of harmonic extensions of functions with spectrum in Ω_{0k}.

The paper is organized as follows: in the second section we show that, as it happens with the Cauchy-Szegö projection, the orthogonal projection $C^*_{\Omega_{0k}}$, from $L^2(\mathbf{S}^n)$ onto $L^2_{\Omega_{0k}}$, induces a bounded operator from L^p to $L^p_{\Omega_{0k}}$, $p > 1$. This will be proved by obtaining an explicit formula for the projections, from which we deduce that they are operators of order 0, in the sense of [**NaRoStWa**].

In the third section we show that the space of harmonic transforms of functions in L^p with spectrum in Ω_{0k} behaves very much alike the space of holomorphic functions in $H^p(\mathbf{B}^n)$.

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In particular, there exists for such spaces of harmonic functions, a strong L^p estimate of the admissible maximal function. We also prove that the theorem of Nagel, Rudin and Wainger, [**NaRu**] and [**NaWa**], which shows that for any function in $H^\infty(\mathbf{B}^n)$, there exists radial limits at almost every point of a transverse curve, extends to bounded harmonic functions with spectrum in Ω_{0k}.

In the fourth section we study convergence within tangential approaches to the boundary of harmonic tranforms of nonisotropic potentials of functions in $L^p_{\Omega_{0k}}$. The spaces of potentials, given by nonisotropic convolution with Riesz-type kernels, coincide, in the integer, case with nonisotropic Sobolev spaces in the unit sphere, and for the general case, when restricted to $L_{\Omega_{0k}}$, can be obtained by complex interpolation method. A direct proof of this last fact is given in the appendix.

2 Boundedness in L^p.

We begin the section with some definitions. Given $\Omega \subset \mathbf{Z}^2_+$, $\mathbf{Z}_+ = \{r \in \mathbf{Z}\,;\, r \geq 0\}$, let $\oplus_{(r,s)\in\Omega} H(r,s)$ be the algebraic sum of all spaces $H(r,s)$ with $(r,s) \in \Omega$. Here $H(r,s)$ is the space of harmonic homogeneous polynomials in $\mathbf{C}^n$ that have total degree r in the variables $z_1, \cdots, z_n$ and total degree s in the variables $\bar{z}_1, \cdots, \bar{z}_n$. If $k \leq m$ we will write $\Omega_{km} = \{(r,s) \in \mathbf{Z}^2_+\,;\, k \leq s \leq m\}$. If X is a Banach space of integrable functions on $\mathbf{S}^n$, containing the spaces $H(r,s)$, X_Ω will denote the closure of the space $\oplus_{(r,s)\in\Omega} H(r,s)$ in X. The space of harmonic extensions of functions in X_Ω will be denoted by $Q[X_\Omega]$. The Poisson kernel in the unit ball is given by

$$Q(z,\zeta) = \frac{1-|z|^2}{|z-\zeta|^{2n}}, \qquad z \in \mathbf{B}^n,\ \zeta \in \mathbf{S}^n,$$

and if $f \in L^1(\mathbf{S}^n)$, and $z \in \mathbf{B}^n$, we denote the Poisson transform by

$$Q[f](z) = \int_{\mathbf{S}^n} Q(z,\zeta)\, f(\zeta)\, d\sigma(\zeta).$$

When $X = L^2(\mathbf{S}^n)$, the identification of the Hilbert space $L^2_\Omega(\mathbf{S}^n)$ with its harmonic extension will give for each $z \in \mathbf{B}^n$ a function $C_\Omega(z,\cdot) \in L^2_\Omega(\mathbf{S}^n)$ such that for any f in $Q[L^2_\Omega]$,

$$f(z) = \int_{\mathbf{S}^n} C_\Omega(z,\zeta)\, f(\zeta)\, d\sigma(\zeta).$$

If K_{rs} is the kernel associated to the orthogonal projection from $L^2(\mathbf{S}^n)$ onto $H(r,s)$, it is shown in [**Do**] that for $\zeta \in \mathbf{S}^n$, $z \in \mathbf{B}^n$,

$$C_\Omega(z,\zeta) = \sum_{(r,s)\in\Omega} K_{rs}(z,\zeta). \tag{2.1}$$

The convergence is uniformly in $\zeta \in \mathbf{S}^n$, z in compacts of $\mathbf{B}^n$.

An explicit formula for the kernels $K_{r\,s}$ can be found in **[Al]** (see also **[Ru]**). If $z \in \mathbf{B}^n$, $\zeta \in \mathbf{S}^n$, and $n \geq 2$

$$K_{r\,s}(z,\zeta) = D(r,s,n)(z\overline{\zeta})^r(\overline{z}\zeta)^s F(-r,-s,n-1,1-\frac{|z|^2}{|z\overline{\zeta}|^2}), \tag{2.2}$$

where

$$D(r,s,n) = \binom{r+n-2}{r}\binom{s+n-2}{s}\frac{r+s+n-1}{n-1}, \tag{2.3}$$

is the dimension of $H(r,s)$, and where

$$F(a,b,c,x) = \sum_{k=0}^{+\infty}\frac{(a)_k(b)_k}{(c)_k k!}x^k$$

is the hypergeometric function (here $(a)_k = a(a+1)\dots(a+k-1)$).

It is also well known that if $z \in \mathbf{S}^n$,

$$||K_{r\,s}(z,\cdot)||_2^2 = K_{r\,s}(z,z) = D(r,s,n). \tag{2.4}$$

When $r \geq s$, it is also shown in **[Al]** that

$$K_{r\,s}(z,\zeta) = D(r,s,n)(z\overline{\zeta})^{r-s}\mathcal{P}_s^{r-s\ n-2}\left(\frac{|z\overline{\zeta}|^2}{|z|^2}\right)|z|^{2s}. \tag{2.5}$$

where

$$\mathcal{P}_s^{\alpha\ \beta}(x) = C\left\{\sum_{m=0}^{s}\binom{s+\beta}{m}\binom{s+\alpha}{s-m}(x-1)^{s-m}x^m\right\}$$

are the Jacobi polynomials orthogonal, in $L^2([0,1],x^\alpha(1-x)^\beta dx)$, to all such polynomials of degree lesser than s, normalized by the condition $\mathcal{P}_s^{\alpha\ \beta}(1) = 1$. Observe that when $\Omega = \Omega_{00}$, $C_{\Omega_{00}}(z,\zeta)$ is the Cauchy-Szegö kernel and when $\Omega = \mathbf{Z}_+^2$, $C_\Omega(z,\zeta)$ is the harmonic Poisson kernel $Q(z,\zeta)$.

First we will calculate, for any $k \in \mathbf{Z}_+$, the kernel $C_{\Omega_{0\,k}}(z,\zeta)$. We want to obtain an expression to which we could apply the theory of operators of order 0 (in the sense of **[NaRoStWa]**).

For $s \in \mathbf{Z}_+$, $z \in \mathbf{B}^n$, $\zeta \in \mathbf{S}^n$, we will simply write $C_s(z,\zeta)$ instead of $C_{\Omega_{s\,s}}(z,\zeta)$ and $C_{0\,k}$ instead of $C_{\Omega_{0\,k}}$.

Theorem 2.1 *Assume $n \geq 2$. We then have for any $z \in \mathbf{B}^n$, $\zeta \in \mathbf{S}^n$,*

(i) $C_s(z,\zeta) =$

$$\sum_{m=0}^{s}\frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!}(|z\overline{\zeta}|^2-|z|^2)^{s-m}(\overline{z}\zeta)^m\left(\frac{(n-m+s-1)!}{(1-z\overline{\zeta})^{n-m+s}}+\frac{s(n-m+s-2)!}{(1-z\overline{\zeta})^{n-m+s-1}}\right)$$

(ii) $$C_{0\,k}(z,\zeta) = \binom{n+k-1}{k}\frac{(\overline{z}\zeta-|z|^2)^k}{(1-z\overline{\zeta})^{n+k}} + \sum_{j=1}^{k}\binom{n+j-2}{j-1}\frac{(\overline{z}\zeta-|z|^2)^{j-1}(1-|z|^2)}{(1-z\overline{\zeta})^{n+j-1}}.$$

Remark: Since $|\overline{z}\zeta - |z|^2| < |1 - z\overline{\zeta}|$, the limit as $k \to +\infty$ of the first summand in (ii) is zero, whereas the second summand tends to

$$\frac{(1-|z|^2)}{(1-z\overline{\zeta})^n}\left(1 - \frac{\overline{z}\zeta - |z|^2}{1 - z\overline{\zeta}}\right)^{-n} = \frac{1-|z|^2}{|\zeta - z|^{2n}},$$

the harmonic Poisson kernel in $\mathbf{B}^n$.

Proof of theorem 2.1:
If we use (2.5), we obtain

$$\sum_{r\geq s} K_{rs}(z,\zeta) = \sum_{r\geq s}\binom{r+n-2}{r}\frac{r+s+n-1}{n-1}(z\overline{\zeta})^{r-s}\left\{\sum_{m=0}^{s}\binom{s+n-2}{m}\binom{r}{s-m}\right.$$
$$\left.\times\left(\frac{|z\overline{\zeta}|^2}{|z|^2}-1\right)^{s-m}\left(\frac{|z\overline{\zeta}|^2}{|z|^2}\right)^{m}\right\}|z|^{2s}$$
$$= \sum_{r\geq s}\binom{r+n-2}{r}\frac{r+s+n-1}{n-1}(z\overline{\zeta})^{r-s}\left\{\sum_{m=0}^{s}\binom{s+n-2}{m}\binom{r}{s-m}(|z\overline{\zeta}|^2-|z|^2)^{s-m}|z\overline{\zeta}|^{2m}\right\}$$
$$= \sum_{r\geq 0}\binom{r+s+n-2}{r+s}\frac{r+2s+n-1}{n-1}(z\overline{\zeta})^{r}\left\{\sum_{m=0}^{s}\binom{s+n-2}{m}\binom{r+s}{s-m}(|z\overline{\zeta}|^2-|z|^2)^{s-m}|z\overline{\zeta}|^{2m}\right\}$$
$$= \sum_{m=0}^{s}\frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!}\sum_{r\geq 0}(r+2s+n-1)\frac{(r+s+n-2)!}{(r+m)!}|z\overline{\zeta}|^{2m}(|z\overline{\zeta}|^2-|z|^2)^{s-m}(z\overline{\zeta})^{r}$$
$$= \sum_{m=0}^{s}\frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!}(|z\overline{\zeta}|^2-|z|^2)^{s-m}(\overline{z}\zeta)^{m}A_{sm}(z,\zeta),$$

where we have defined

$$A_{sm}(z,\zeta) = \sum_{r\geq 0}(r+2s+n-1)(r+s+n-2)(r+s+n-3)\cdots(r+m+1)(z\overline{\zeta})^{r+m}.$$

Next

$$A_{sm}(z,\zeta) = \sum_{r\geq m}(r+2s-m+n-1)\frac{(r+s-m+n-2)!}{r!}(z\overline{\zeta})^{r}$$
$$= \frac{(n-m+s-1)!}{(1-z\overline{\zeta})^{n-m+s}} + \frac{s(n-m+s-2)!}{(1-z\overline{\zeta})^{n-m+s-1}}$$
$$- \sum_{r\leq m-1}(r+2s-m+n-1)\frac{(r+s-m+n-2)!}{r!}(z\overline{\zeta})^{r}.$$

So we have obtained that

(2.6) $\quad C_s(z,\zeta) = \sum K_{rs}(z,\zeta)$

$$= \sum_{r\le s-1} K_{rs}(z,\zeta) - \sum_{m=0}^{s} \frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!}(|z\overline{\zeta}|^2-|z|^2)^{s-m}(z\overline{\zeta})^m$$
$$\times\left\{\sum_{r\le m-1}\frac{(r+2s-m+n-1)(r+s-m+n-2)!}{(n-1)!(s-m)!r!}(z\overline{\zeta})^r\right\}+$$
$$+\sum_{m=0}^{s}\frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!}(|z\overline{\zeta}|^2-|z|^2)^{s-m}(\overline{z}\zeta)^m\left\{\frac{(n-m+s-1)!}{(1-z\overline{\zeta})^{n-m+s}}+\frac{s(n-m+s-2)!}{(1-z\overline{\zeta})^n}\right\}.$$

Now, for each $r<s$,

$$K_{rs}(z,\zeta)=D(r,s,n)(z\overline{\zeta})^r(\overline{z}\zeta)^sF(-r,-s,n-1,\frac{1-|z|^2}{|z\overline{\zeta}|^2})$$
$$=\binom{r+n-2}{r}\binom{s+n-2}{s}\frac{r+s+n-1}{n-1}\sum_{m=0}^{r}\frac{r!s!}{m(m!)^2}(|z\overline{\zeta}|^2-|z|^2)^m(\overline{z}\zeta)^{s-r}|z\overline{\zeta}|^{2(r-m)}.$$

Using this formula, a careful calculation of the sum of the two first sumands in (2.6) shows that its value is identically zero, and consequently we obtain part (i).

The proof of the formula in part (ii) will be a consequence of the following lemma.

Lemma 2.2 *Assume $n\ge 2$ and let $s\ge 1$ be a nonnegative integer. We then have that for $z\in\mathbf{B}^n$, $\zeta\in\mathbf{S}^n$,*

$$C_s(z,\zeta)+\binom{n+s-2}{s-1}\frac{(\overline{z}\zeta-|z|^2)^{s-1}}{(1-z\overline{\zeta})^{n+s-1}}$$
$$=\binom{n+s-1}{s}\frac{(\overline{z}\zeta-|z|^2)^s}{(1-z\overline{\zeta})^{n+s}}+\binom{n+s-2}{s-1}\frac{(\overline{z}\zeta-|z|^2)^{s-1}}{(1-z\overline{\zeta})^{n+s-1}}(1-|z|^2).$$

Proof of lemma 2.2:

In the formula obtained in part (i) of theorem 2.1 we write

$$(|z\overline{\zeta}|^2-|z|^2)^{s-m}=\Big((z\overline{\zeta}-1)\overline{z}\zeta+(\overline{z}\zeta-|z|^2)\Big)^{s-m},$$

develop the above sum, and group together the terms which have the same power of $(1-z\overline{\zeta})$ in the denominator. We then obtain:

$$C_s(z,\zeta)=$$
$$\sum_{m=0}^{s}\frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!}\left\{\frac{(n-m+s-1)!}{(1-z\overline{\zeta})^{n-m+s}}(\overline{z}\zeta)^m(\overline{z}\zeta-|z|^2)^{s-m}\right.$$
$$+\sum_{j=1}^{s-m}(-1)^j\left(\frac{1}{(1-z\overline{\zeta})^{n-m+s-j}}\left((n-m+s-1)!\binom{s-m}{j}(\overline{z}\zeta)^{j+m}(\overline{z}\zeta-|z|^2)^{s-m-j}\right.\right.$$

$$- s(n-m+s-2)!\binom{s-m}{j-1}(\overline{z}\zeta)^{j-1+m}(\overline{z}\zeta-|z|^2)^{s-m-j+1}\Big)\Big)$$
$$+(-1)^{s-m}\frac{s(n-m+s-2)!}{(1-z\overline{\zeta})^{n-1}}(\overline{z}\zeta)^s\Bigg\}$$

Since

$$\sum_{m=0}^{s}\frac{\binom{s+n-2}{m}}{(s-m)!}(-1)^{s-m}(n-m+s-2)! = \frac{(s+n-2)!}{s!}\sum_{m=0}^{s}\binom{s}{m}(-1)^{s-m}=0,$$

the above formula can be rewritten as $A+B$, where

$$A=\sum_{m=0}^{s}\frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!}\Bigg\{\frac{(n-m+s-1)!}{(1-z\overline{\zeta})^{n-m+s}}(\overline{z}\zeta)^m(\overline{z}\zeta-|z|^2)^{s-m}$$
$$+\sum_{j=1}^{s-m}(-1)^j\frac{(n-m+s-1)!\binom{s-m}{j}(\overline{z}\zeta)^{j+m}(\overline{z}\zeta-|z|^2)^{s-m-j}}{(1-z\overline{\zeta})^{n-m+s-j}}\Bigg\};$$
$$B=s\sum_{m=0}^{s}\frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!}\sum_{j=1}^{s-m}(-1)^j\frac{(n-m+s-2)!\binom{s-m}{j-1}(\overline{z}\zeta)^{j-1+m}(\overline{z}\zeta-|z|^2)^{s-m-j+1}}{(1-z\overline{\zeta})^{n-m+s-j}}.$$

For each $0\le k\le s-2$, we will check that the sum of the terms in A that have as common denominator $(1-z\overline{\zeta})^{n+k}$ is zero. Indeed this sum equals to

$$(\overline{z}\zeta)^{s-k}(\overline{z}\zeta-|z|^2)^k\Bigg\{\frac{\binom{s+n-2}{s-k}(n+k-1)!}{(n-1)!k!}$$
$$+\sum_{m=0}^{s-k-1}\frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!}(-1)^{s-m-k}(n-m+s-1)!\binom{s-m}{s-k-m}\Bigg\}.$$

But

$$\frac{(s+n-2)!(n+k-1)}{(s-k)!(n-1)!k!}+\sum_{m=0}^{s-k-1}(-1)^{s-k-m}\frac{(n-m+s-1)(s+n-2)!}{(n-1)!m!(s-m-k)!k!}$$
$$=\frac{(s+n-2)!}{(n-1)!k!}\Bigg\{\frac{(n+k-1)}{(s-k)!}+\sum_{m=0}^{s-k-1}(-1)^{s-k-m}\frac{(n-m+s-1)}{m!(s-m-k)!}\Bigg\}$$
$$=\frac{(s+n-2)!}{(n-1)!k!}\Bigg\{\sum_{m=0}^{s-k}(-1)^{s-k-m}\frac{(n-m+s-1)}{m!(s-m-k)!}\Bigg\}.$$

And

$$\sum_{m=0}^{s-k}(-1)^{s-k-m}\frac{(n-m+s-1)}{m!(s-m-k)!}$$
$$=(-1)^{s-k}\Bigg\{\frac{n+s-1}{(s-k)!}\sum_{m=0}^{s-k}\binom{s-k}{m}(-1)^m-\frac{1}{(s-k)!}\sum_{m=0}^{s-k}(-1)^m m\binom{s-k}{m}\Bigg\}.$$

The fact that

$$\frac{d}{dx}(x-1)^{q-k} = \sum_{m=1}^{s-k} \binom{s-k}{m} (-1)^m x^{m-1} m.$$

easily gives that the above sum is zero, since $k \leq s-2$.

In consequence we obtain that

$$\begin{aligned}
&A + \binom{n+s-2}{s-1} \frac{(\overline{z}\zeta - |z|^2)^{s-1}}{(1-z\overline{\zeta})^{n+s-1}} = \\
&= \left\{ (n+s-2)\binom{n+s-2}{s-1} - \binom{n+s-1}{s} s + \binom{n+s-2}{s-1} \right\} \frac{(\overline{z}\zeta - |z|^2)^{s-1}}{(1-z\overline{\zeta})^{n+s-1}} \\
&+ \binom{n+s-1}{s} \frac{(\overline{z}\zeta - |z|^2)^{s}}{(1-z\overline{\zeta})^{n+s}} = \binom{n+s-2}{s-1} \frac{(\overline{z}\zeta - |z|^2)^{s-1}}{(1-z\overline{\zeta})^{n+s-1}} (1 - z\overline{\zeta}) \\
&+ \binom{n+s-1}{s} \frac{(\overline{z}\zeta - |z|^2)^{s}}{(1-z\overline{\zeta})^{n+s}}.
\end{aligned}$$

An analogous calculation shows that

$$B = \binom{n+s-2}{s-1} \frac{(\overline{z}\zeta - |z|^2)^{s}}{(1-z\overline{\zeta})^{n+s-1}}.$$

Adding both formulas we finally obtain the lemma. ∎

Now formula (ii) in theorem 2.1 is deduced from lemma 2.2 and the expression of the Cauchy kernel. ∎

We will next check that the limit as $|z| \to 1$ of the kernels $C_{\Omega_0 k}$ arises as a singular operator of order 0 in the sense of [**NaRoStWa**]. We recall some definitions.

Let T_{ij}, $1 \leq i < j \leq n$, the complex tangential vector fields

$$T_{ij} = \overline{z}_i \frac{\partial}{\partial z_j} - \overline{z}_j \frac{\partial}{\partial z_i}, \quad \overline{T}_{ij} = z_i \frac{\partial}{\partial \overline{z}_j} - z_j \frac{\partial}{\partial \overline{z}_i}.$$

We denote by $X^I = X_1 \cdots X_m$ a complex differential operator whose vector fields X_l, $1 \leq l \leq m$ are of type T_{ij} or $\overline{T}_{ij}$ for some $i < j$. The weight of X^I, $\omega(X^I) = \frac{m}{2}$ if $X^I = X_1 \cdots X_m$.

Let $Kf(\zeta) = \int_{\mathbf{S}^n} K(\zeta,\omega) f(\omega)\, d\sigma(\omega)$, for $f \in \mathcal{C}^\infty(\mathbf{S}^n)$, where $K(\zeta,\omega)$ is a distribution which is $\mathcal{C}^\infty$ outside the diagonal. The operator K is of order m if there exists a family of operators $K_\varepsilon[f](\zeta) = \int_{\mathbf{S}^n} K_\varepsilon(\zeta,\omega) f(\omega)\, d\sigma(\omega)$, such that

- (i) $K_\varepsilon(\zeta,\omega) \in \mathcal{C}^\infty(\mathbf{S}^n \times \mathbf{S}^n)$.
- (ii) $K_\varepsilon f \to Kf$ in $\mathcal{C}^\infty(\mathbf{S}^n)$, for each f in $\mathcal{C}^\infty(\mathbf{S}^n)$.
- (iii) The following conditions hold uniformly on ε :
  - $(iii) - 1$ For any X^I, X^J $|X^I_\omega X^J_\zeta K_\varepsilon(\zeta,\omega)| \leq C_{XY} |1 - \zeta\overline{\omega}|^{m-\omega(X^I)-\omega(X^J)-n}$,
  - $(iii) - 2$ For any $l \geq 0$ there exists N_l, $C_l > 0$, so that for any smooth function φ

supported in $B(\zeta_0,\delta)=\{\zeta\in\mathbf{S}^n\,;\,|1-\zeta_0\overline{\zeta}|<\delta\}$ and every X^I with $\omega(X^I)=\frac{l}{2}$,
$|X^IK_\varepsilon[\varphi](\zeta_0)|\le C_l\delta^{-\frac{l}{2}+m}\sup_{\zeta}\sum_{\omega(X^J)\le N_l}\delta^{\omega(X^J)}|X^J\varphi(\zeta)|.$

The same estimates must hold for the adjoint operator K^* with associated kernel $\overline{K(\omega,\zeta)}$.

If l is a nonnegative integer, and $L_l^p(\mathbf{S}^n)$ is the nonisotropic Sobolev space of functions with tangential derivatives up to weight l in L^p, $p>1$, and K is an operator of order m, then (see [**NaRoStWa**]) K maps continuously $L_l^p(\mathbf{S}^n)$ in $L_{l+m}^p(\mathbf{S}^n)$.

It will be convenient for some computations to consider, for $1\le i,j\le n$ the generators of the tangent vector fields to $\mathbf{S}^n$

$$N_{ij}=z_i\frac{\partial}{\partial z_j}-\overline{z}_j\frac{\partial}{\partial\overline{z}_i},$$

which verifies (see [**Ge**]) that for each smooth functions $f:\mathbf{S}^n\to\mathbf{C}$, and $\Phi:\mathbf{C}\to\mathbf{C}$,

$$\int_{\mathbf{S}^n}N_{ij}\Phi(z\overline{\zeta})\,f(\zeta)\,d\sigma(\zeta)=\int_{\mathbf{S}^n}\Phi(z\overline{\zeta})\,N_{ij}f(\zeta)d\sigma(\zeta)$$

By induction it can be proved that if $X=X_1\cdots X_k$ is a differential operator with X_i, $i=1,\cdots,k$ tangent vector fields with coefficients in $\mathcal{C}^\infty(\overline{\mathbf{B}}^n)$, there exist differential operators $Y^\alpha=Y_1^\alpha\cdots Y_{|\alpha|}^\alpha$ with $|\alpha|\le k$, Y_i^α tangent, and smooth functions $\varphi_\alpha(z,\zeta)$ satisfying $|Z_z\varphi_\alpha(z,\zeta)|\le C|1-z\overline{\zeta}|^{\max(0,\omega(Y^\alpha)-\omega(X)-\omega(Z))}$ such that

$$X_z\int_{\mathbf{S}^n}\Phi(z\overline{\zeta})f(\zeta)\,d\sigma(\zeta)=\sum_{|\alpha|\le k}\int_{\mathbf{S}^n}\Phi(z\overline{\zeta})\varphi_\alpha(z,\zeta)Y^\alpha f(\zeta)\,d\sigma(\zeta).\tag{2.7}$$

As we have already commented, the operators $C_s(z,\zeta)$ when $|z|\to1$ can be realized as operators of order 0. It is easy to verify that if we define $C_s^r(\omega,\zeta)=C_s(r\omega,\zeta)$, for ζ,ω in $\mathbf{S}^n$, and $r<1$, then there exists $\lim_{r\to1}C_s^r[f](\zeta)$, for any f in $\mathcal{C}^\infty(\mathbf{S}^n)$, and it defines a function in $\mathcal{C}^\infty(\mathbf{S}^n)$. We will write $C_s^*[f]$ the value of this limit. We then have

Proposition 2.3 *For each $s\ge0$ the operators C_s^* are of order 0.*

Proof of proposition 2.3:

Since the case $s=0$ is the Cauchy-Szegö projection and the result is well known, we will assume that $s>0$. From the definition of C_s^* it is clear that conditions (i) and (ii) are satisfied. We will next check condition (iii)-1. Formula (1) in theorem 2.1 shows that for any $i<j$,

$$\begin{aligned}&T_{ij}^\omega C_s^r(\omega,\zeta)\\&=r^s\sum_{m=0}^s\frac{\binom{s+n-2}{m}}{(n-1)!(s-m)!}\Bigg\{(s-m)(\overline{\omega}_i\overline{\zeta}_j-\overline{\omega}_j\overline{\zeta}_i)(\overline{\omega}\zeta)^{m+1}(|\omega\overline{\zeta}|^2-1)^{s-m-1}\left(\frac{(n-m+s-1)!}{(1-r\omega\overline{\zeta})^{n-m+s}}\right.\\&\left.+\frac{s(n-m+s-2)!}{(1-r\omega\overline{\zeta})^{n-m+s-1}}\right)+r(\overline{\omega}\zeta)^m(|\omega\overline{\zeta}|^2-1)^{s-m}\left(\frac{(n-m+s-1)!(n-m+s)}{(1-r\omega\overline{\zeta})^{n-m+s+1}}(\overline{\omega}_i\overline{\zeta}_j-\overline{\omega}_j\overline{\zeta}_i)\right.\\&\left.+\frac{s(n-m+s-2)!(n-m+s-1)}{(1-r\omega\overline{\zeta})^{n-m+s}}(\overline{\omega}_i\overline{\zeta}_j-\overline{\omega}_j\overline{\zeta}_i)\right)\Bigg\}.\end{aligned}$$

Now the fact that $\sum_{i<j} |\omega_i\zeta_j - \omega_j\zeta_i|^2 = 1 - |\omega\overline{\zeta}|^2$ gives that

$$T_{ij}^{\omega} C_s^r(\omega,\zeta) = O\left(\frac{1}{|1-\omega\overline{\zeta}|^{n+\frac{1}{2}}}\right).$$

The same argument can be used to show a similar estimate for $\overline{T}_{ij}^{\omega} C_s^r(\omega,\zeta)$, and that for any composition of complex tangential vector fields X_ω^I, X_ζ^J

$$|X_\omega^I X_\zeta^J C_s^r(\omega,\zeta)| = O\left(\frac{1}{|1-\omega\overline{\zeta}|^{n+\omega(X^I)+\omega(X^J)}}\right)$$

uniformly on r.

In order to finish we just need to check that the kernels C_s^r satisfy condition (iii)-2 uniformly in r. We will use the following

Lemma 2.4 *let k be a nonnegative integer. There exists $C > 0$ so that for each $\varepsilon > 0$, $z \in \mathbf{B}^n$,*

$$\left|\int_{|1-z\overline{\zeta}|>\varepsilon} C_{0k}(z,\zeta)\,d\sigma(\zeta)\right| \le C.$$

Proof of lemma 2.4:

It is inmediate to verify from formula (ii) in theorem 2.1 that all the summands there satisfy that the integral over $\mathbf{S}^n$ of their modulus are bounded independently of z, except for

$$\binom{n+k-1}{k}\frac{(|z\overline{\zeta}|^2-|z|^2)^k}{(1-z\overline{\zeta})^{n+k}}.$$

So we are led to prove that for $z \in \mathbf{B}^n$,

$$\int_{|1-z\overline{\zeta}|>\varepsilon} \frac{(|z\overline{\zeta}|^2-|z|^2)^k}{(1-z\overline{\zeta})^{n+k}}\,d\sigma(\zeta) = O(1). \tag{2.8}$$

Since for any $0 < s \le k$

$$\int_{\mathbf{S}^n} \frac{(1-|z|^2)^s(1-|z\overline{\zeta}|^2))^{k-s}}{|1-z\overline{\zeta}|^{n+k}}\,d\sigma(\zeta) = O(1),$$

the estimates in (2.8) will hold once we prove that

$$\int_{|1-z\overline{\zeta}|>\varepsilon} \frac{(1-|z\overline{\zeta}|^2)^k}{(1-z\overline{\zeta})^{n+k}}\,d\sigma(\zeta) = O(1)$$

uniformly in $\frac{1}{2} < |z| < 1$, $\varepsilon > 0$. Let $\lambda = |z|$. Then a unitary change of variables gives that the above is equivalent to show that

$$\int_{|1-\lambda\overline{\zeta_1}|>\varepsilon} \frac{(1-\lambda^2|\zeta_1|^2)^k}{(1-\lambda\overline{\zeta}_1)^{n+k}}\,d\sigma(\zeta) = O(1).$$

But the above integral equals to

$$\frac{n-1}{\pi}\int\int_{\{re^{i\theta}\in\mathbf{D}\,;\,\varepsilon<|1-\lambda re^{i\theta}|\}}\frac{(1-\lambda^2r^2)^{n-2+k}}{(1-\lambda re^{-i\theta})^{n+k}}\,r dr\,d\theta$$
$$=\frac{n-1}{\pi\lambda^2}\int\int_{\{\rho e^{i\theta}\in\mathbf{D}\,;\,\rho<\lambda,\ \varepsilon<|1-\rho e^{i\theta}|\}}\frac{(1-\rho^2)^{n-2+k}}{(1-\rho e^{-i\theta})^{n+k}}\,\rho d\rho\,d\theta,$$

integral which can be easily seen that is bounded uniformly on ε, and consequentely the lemma is finished. ∎

Going back to the proof of condition (iii)-2, we have to prove that

$$|X^I C_s^r[\varphi](\zeta_0)| \preceq \delta^{-\omega(X^I)} \sup_{\omega(Y^J)\le N_X^I} \delta^{\omega(Y^J)}||Y^J\varphi||_\infty,$$

for any C^∞ function on $\mathbf{S}^n$, φ such that $\sup\varphi\subset B(\zeta_0,\delta)$ for $\zeta_0\in\mathbf{S}^n$, $\delta>0$, and any $X^I=X_1\cdots X_k$ composition of complex tangential vector fields. When $\delta\preceq 1-r$,

$$|X^I C_s^r[\varphi](\zeta_0)| \preceq \int_{B(\zeta_0,\delta)} |X^I C_s(r\zeta_0,\zeta)||\varphi(\zeta)|\,d\sigma(\zeta) \preceq (1-r)^{-n-\omega(X^I)}\delta^n||\varphi||_\infty \preceq \delta^{-\omega(X^I)}||\varphi||_\infty.$$

Thus it is enough to assume that $1-r\preceq\delta$.

Now, let $X=X_1\cdots X_k$ be a differential operator, with X_i complex tangential vector fields. By (2.7),

$$XC_s^r[\varphi](\zeta_0)=\sum_{|\alpha|\le k}\int_{\mathbf{S}^n} C_s(r\zeta_0,\zeta)\left\{\varphi_\alpha(r\zeta_0,\zeta)Y^\alpha\varphi(\zeta)-\varphi_\alpha(r\zeta_0,\zeta_0)Y^\alpha\varphi(\zeta_0)\right\}\,d\sigma(\zeta).$$

Since φ is supported in $B(\zeta_0,\delta)$, the previous lemma shows that the part of the above integral over $\mathbf{S}^n\setminus B(\zeta_0,\delta)$ is bounded by

$$\sum_{|\alpha|\le k}\delta^{\max(0,\omega(Y^\alpha)-\frac{k}{2})}||Y^\alpha\varphi||_\infty \le \sum_{|\alpha|\le k}\delta^{-\frac{k}{2}}\delta^{\omega(Y^\alpha)}||Y^\alpha\varphi||_\infty.$$

For the integral over $B(\zeta_0,\delta)$, the regularity of $Y^\alpha\varphi$ together with the properties of the functions φ_α give that (see **[BrOr]**)

$$|Y^\alpha\varphi(\zeta)-Y^\alpha\varphi(\zeta_0)| \preceq \sum_{i<j}||T_{ij}Y^\alpha\varphi||_\infty|1-\zeta\overline{\zeta}_0|^{\frac12},$$
$$|\varphi_\alpha(r\zeta_0,\zeta))-\varphi_\alpha(r\zeta_0,\zeta_0)| \preceq \delta^{\max(0,\omega(Y^\alpha)-\omega(X)-\frac12)}|1-\zeta\overline{\zeta}_0|^{\frac12},$$

which easily give that

$$|\int_{B(\zeta_0,\delta)} C_s(r\zeta_0,\zeta)\{\varphi_\alpha(r\zeta_0,\zeta)Y^\alpha\varphi(\zeta)-\varphi_\alpha(r\zeta_0,\zeta_0)Y^\alpha\varphi(\zeta_0)\}\,d\sigma(\zeta)| \preceq \delta^{-\frac{k}{2}}\sum_Y\delta^{\omega(Y)}||Y\varphi||_\infty. \quad ∎$$

As a corollary we obtain

Corollary 2.5 *For any k, l nonnegative integers, and $1 < p < +\infty$ the sigular operator $C^*_{\Omega_{0\,k}}$ maps boundedly $L^p_l(\mathbf{S}^n)$ to itself.*

Another consequence of the above corollary is that the functions in $L^p_{l\,\Omega_{0\,k}}(\mathbf{S}^n)$ can be characterized in terms of their spectrum.

Corollary 2.6 *Let $1 < p < +\infty$, and k, l nonnegative integers. We then have*

$$L^p_{l\,\Omega_{0\,k}}(\mathbf{S}^n) = \{\, f \in L^p_l(\mathbf{S}^n)\,;\, f_{rs} = K_{r\,s}[f] = 0,\ s > k\,\}.$$

Remark: Observe that for $k = 0$ this corollary can be obtained directly. Using a Bochner-Riesz summation (see for instance [**BoCl**]), every function f in $L^p(\mathbf{S}^n)$ can be approximated by polynomials in $\oplus_{r+s\leq k} H(r,s)$, whose spectrum is included in the spectrum of f. In consequence, if $\Omega \subset \mathbf{Z}^2_+$, the functions in $L^p(\mathbf{S}^n)$ with spectrum in Ω can be approximated in L^p by polynomials with spectrum in Ω.

Given a smooth function F on $\mathbf{B}^n$, we will say that $\overline{\partial}^k F = 0$ if $\frac{\partial^k}{\partial\overline{z}_{i_1}\cdots\partial\overline{z}_{i_k}} F = 0$, for any $i_1,\cdots,i_k$ in $1,\cdots,n$. Then the following characterization of the harmonic extensions of the functions in L^p with spectrum in $\Omega_{0\,k}$ holds:

Proposition 2.7 ([Do]) *If F is in $Q[L^p_l(\mathbf{S}^n)]$ and $\overline{\partial}^{k+1} F = 0$, then $F \in Q[L^p_{l\Omega_{0\,k}}(\mathbf{S}^n)]$. Conversely, any F in $Q[L^p_{l\Omega_{0\,k}}(\mathbf{S}^n)]$ satisfies $\overline{\partial}^{k+1} F = 0$.*

3 Admissible convergence of harmonic functions

In this section we will show that some problems related to admissible convergence of holomorphic functions in the Hardy spaces, still hold in the spaces of harmonic extensions of L^p functions with spectrum lying in $\Omega_{0\,k}$. We recall some definitions.

Given $f : \mathbf{B}^n \to \mathbf{C}$, the admissible maximal function will be denoted by $M_{adm}f(\zeta) = \sup_{z\in D_\alpha(\zeta)} |f(z)|$, where $D_\alpha(\zeta)$ is the admissible region given by $D_\alpha(\zeta) = \{\, z \in \mathbf{B}^n\,;\, |1 - z\overline{\zeta}| < \alpha(1 - |z|^2)\,\}$.

Theorem 3.1 *Let $1 < p < +\infty$ and k a nonnegative integer. Then there exists $C > 0$ so that for any f in $L^p(\mathbf{S}^n)$,*

$$||M_{adm}C_{\Omega_{0\,k}}[f]||_p \leq C||f||_p.$$

Proof of theorem 3.1:

If M_{rad} is the radial maximal operator, it is a well known fact that

$$||M_{rad}Q[f]||_p \leq C||f||_p$$

for any f in $L^p(\mathbf{S}^n)$. By corollary 2.6 $C_{\Omega_{0\,k}}[f] = Q[C^*_{\Omega_{0\,k}}[f]]$, and

$$||M_{rad}Q[C^*_{\Omega_{0\,k}}[f]]||_p \leq C||C^*_{\Omega_{0\,k}}[f]||_p \leq C||f||_p.$$

Next, by proposition 2.7, any F in $Q[L_{\Omega_{0k}}]$ satisfies $\overline{\partial}^{k+1}F = 0$, and ([**AhBr**, lemma 4.4]) shows that in this case $||M_{adm}F||_p \leq C||M_{rad}F||_p$. ∎

Corollary 3.2 *Let $1 < p < +\infty$. Any harmonic function in h^p whose spectrum lies in Ω_{0k} has admissible limit at almost every $\zeta \in \mathbf{S}^n$.*

Remark: Corollary 3.2 could have been obtained from last remark, which together with proposition 2.7 and lemma 4.4 in [**AhBr**], shows that $||M_{adm}Q[f]||_p \leq C||M_{rad}Q[f]||_p \leq ||f||_p$ for any $f \in L^p_{\Omega_{0k}}$.

In general, one can not expect to obtain for an arbitrary Ω that the admissible maximal function $M_{adm}C_\Omega$ is bounded in L^p, even in L^2. This would imply the existence, almost everywhere, of admissible limits of of functions in L^2, which as it is well known (see [**Zy**]) is not true. One could ask if for some other regions, different from the strips Ω_{0k} there exist such L^p estimate. We will next see a simple result which shows that if we restrict to the case that $L^p_\Omega(\mathbf{S}^n)$ is a module over the ball algebra $A(\mathbf{S}^n)$, then the sets Ω_{0k} are the only ones for which the admissible maximal function $M_{adm}C_\Omega$ is bounded in L^p, $p > 1$. Such module condition holds ([**Do**]) if and only if for any (r,s) in Ω, (k,m) is also in Ω provided $k \geq r$ and $m \leq s$.

We show that if the sets Ω have their second projection not bounded, then the admissible maximal function is not even weakly bounded in $L^2(\mathbf{S}^n)$.

Proposition 3.3 *Assume $\Omega \subset \mathbf{Z}^2_+$ satisfies the modulus condition. If the set of $s \in \mathbf{Z}_+$ for which there exists $r \in \mathbf{Z}_+$ with $(r,s) \in \Omega$ is not bounded in $\mathbf{Z}_+$, then the admissible maximal function does not satisfy the weak L^2-type estimate, that is it does not verify that for some $C > 0$*

$$\sigma(\{\zeta \in \mathbf{S}^n\,;\, \mathcal{M}_{\mathrm{adm}}C_\Omega[f](\zeta) > \lambda\}) \leq C\frac{||f||_2^2}{\lambda^2},$$

for each $f \in L^2(\mathbf{S}^n)$, $\lambda > 0$.

Proof of proposition 3.3:

If Ω satisfies the module condition, and its second projection is not bounded in $\mathbf{Z}_+$, it can easily be constructed a non-decreasing function $\varphi : \mathbf{Z}_+ \to \mathbf{Z}_+$ so that $\Omega = \{(r,s) \in \mathbf{Z}^2_+\,;\, r \geq \varphi(s)\}$.

Assume now that the weak L^2-estimate holds, and take $\zeta_0 \in \mathbf{S}^n$, $0 < \mu < 1$. We then have that for any $\zeta \in B(\zeta_0, \varepsilon(1-\mu))$, ($\varepsilon > 0$ small enough), $\mu\zeta_0 \in D_\alpha(\zeta)$. In particular, for any $f \in L^2_+(\mathbf{S}^n)$, with $||f||_2 = 1$, and $F = C_\Omega[f]$,

$$B(\zeta_0, \varepsilon(1-\mu)) \subset \{\zeta \in \mathbf{S}^n\,;\, \mathcal{M}_{\mathrm{adm}}F(\zeta) \geq |F(\mu\zeta_0)|\}.$$

Consequently

$$(1-\mu)^n \preceq \sigma(\{\zeta \in \mathbf{S}^n\,;\, \mathcal{M}_{\mathrm{adm}}F(\zeta) \geq |F(\mu\zeta_0)|\}) \preceq \frac{||f||_2^2}{|F(\mu\zeta_0)|^2},$$

and $|F(\mu\zeta_0)| \preceq \frac{||f||_2}{(1-\mu)^{\frac{n}{2}}}$.

The above estimate gives that for any $f \in L^2(\mathbf{S}^n)$ with $||f||_2 = 1$,

$$|\int_{\mathbf{S}^n} C_\Omega(\mu\zeta_0, \zeta) f(\zeta)\, d\sigma(\zeta)| = |F(\mu\zeta_0)| \preceq \frac{1}{(1-\mu)^{\frac{n}{2}}},$$

which by duality gives that

$$||C_\Omega(\mu\zeta_0, \cdot)||_2 \preceq \frac{1}{(1-\mu)^{\frac{n}{2}}}. \tag{3.1}$$

Next, formula (2.4) together with (2.5) leads to

$$||C_\Omega(\mu\zeta_0, \cdot)||_2^2 = \sum_{(r,s)\in\Omega} ||K_{rs}(\mu\zeta_0, \cdot)||_2^2 = \sum_{(r,s)\in\Omega} \binom{r+n-2}{r}\binom{s+n-2}{s}\frac{r+s+n-1}{n-1}\mu^{2(r+s)}.$$

If $\varphi_1(s) = \max(\varphi(s), s)$, we then have

$$||C_\Omega(\mu\zeta_0, \cdot)||_2^2 \succeq \sum_s \mu^{2s} \sum_{r\geq\varphi_1(s)} (r+n-2)\cdots(r+1)r\mu^{2r} \succeq \frac{1}{(1-\mu^2)^n}\sum_s \mu^{2(s+\varphi_1(s))}.$$

Since $\varphi_1(s) \geq s$, the above gives that

$$\frac{1}{(1-\mu^2)^n}\sum_{s\geq 0} \mu^{4\varphi_1(s)} \preceq ||C_\Omega(\mu\zeta_0, \cdot)||_2^2.$$

If $g(\mu) = \sum_{s\geq 0} \mu^{\varphi_1(s)}$, let us check that $g(\mu) \to +\infty$ as $\mu \to 1$. Let $N \in \mathbf{Z}_+$ and consider $\mu < 1$ satisfying that $\mu^{\varphi_1(2N)} > \frac{1}{2}$. Since φ_1 is nondecreasing, we have that for $i \leq 2N$, $\mu^{\varphi_1(i)} \geq \mu^{\varphi_1(2N)} > \frac{1}{2}$. Hence $g(\mu) \geq \sum_{i=1}^{2N} \mu^{\varphi_1(i)} \geq N$, which together with the previous estimate contradicts (3.1). ∎

In the last part of this section we study the convergence of bounded harmonic functions with spectrum in Ω_{0k}. Nagel, Rudin and Wainger (see [**NaRu**] and [**NaWa**]) showed that every function in $H^\infty(\mathbf{B}^n)$ has radial limits at almost every point of a transverse curve in $\mathbf{S}^n$ relative to its arc-length measure. Recall that a curve γ is transverse if for each t, $\gamma'(t)$ does not lie in the complex tangential space at $\gamma(t)$. We will prove that these theorems extend to bounded harmonic functions in $\mathbf{B}^n$ with spectrum in Ω_{0k}.

We will assume that $\gamma : I \to \mathbf{S}^n$ is a simple closed transverse curve of class C^1. By making a convenient reparametrization, we may assume that $I = [-\pi, \pi]$, and that γ is 2π-periodic. For $\zeta \in \mathbf{S}^n$, a ζ-curve is a continuous map $\varphi : [0, 1) \to \mathbf{B}^n$ so that $\lim_{t\to 1}\varphi(t) = \zeta$. It is special if

$$\lim_{t\to 1} \frac{|\varphi(t) - (\varphi(t)\overline{\zeta})\zeta|^2}{1 - |\varphi(t)\overline{\zeta}|^2} = 0,$$

and restricted if it also satisfies that

$$\frac{|\varphi(t)\overline{\zeta} - 1|}{1 - |\varphi(t)\overline{\zeta}|} = O(1),$$

for $0 \leq t < 1$.

A function $f : \mathbf{B}^n \to \mathbf{C}$ is said to have restricted K-limit L at $\zeta \in \mathbf{S}^n$ if $\lim_{t \to 1} f(\varphi(t)) = L$ for any restricted ζ-curve φ.

Theorem 3.4 *Let $\gamma : [-\pi, \pi] \to \mathbf{C}$ be a simple closed transverse curve of class C^1, and let k be a nonnegative integer. If F is a bounded harmonic function with spectrum in Ω_{0k}, then F has restricted K-limit at $\gamma(t)$, for almost every $t \in [-\pi, \pi]$.*

Proof of theorem 3.4:

The hypothesis on F implies that we may assume that $F = C_{\Omega_{0k}}[f]$, with $f \in L^\infty(\mathbf{S}^n)$. Next, part (ii) in theorem 2.1 shows that

$$F(z) = \frac{(n+k-1)!}{(n-1)!k!} \int_{\mathbf{S}^n} \frac{(\bar{z}\zeta - |z|^2)^k}{(1 - z\bar{\zeta})^{n+k}} f(\zeta)\, d\sigma(\zeta) +$$
$$\sum_{j=1}^{k} \frac{(n+j-2)!}{(n-1)!(j-1)!} (1 - |z|^2) \int_{\mathbf{S}^n} \frac{(\bar{z}\zeta - |z|^2)^{j-1}}{(1 - z\bar{\zeta})^{n+j-1}} f(\zeta)\, d\sigma(\zeta).$$

Now, for each $1 \leq j \leq k$

$$\left| \int_{\mathbf{S}^n} \frac{(\bar{z}\zeta - |z|^2)^{j-1}}{(1 - z\bar{\zeta})^{n+j-1}} f(\zeta)\, d\sigma(\zeta) \right| \preceq \|f\|_\infty |\log(1 - |z|^2)|$$

and

$$\lim_{|z| \to 1} (1 - |z|^2) \int_{\mathbf{S}^n} \frac{(\bar{z}\zeta - |z|^2)^{j-1}}{(1 - z\bar{\zeta})^{n+j-1}} f(\zeta)\, d\sigma(\zeta) = 0.$$

Thus in order to finish we just need to deal with the function

$$F_1(z) = \int_{\mathbf{S}^n} \frac{(\bar{z}\zeta - |z|^2)^k}{(1 - z\bar{\zeta})^{n+k}} f(\zeta)\, d\sigma(\zeta).$$

A direct computation gives that $\dfrac{\partial F_1}{\partial \bar{z}_i} = O((1 - |z|^2)^{-\frac{1}{2}})$, $\dfrac{\partial F_1}{\partial z_i} = O((1 - |z|^2)^{-1})$, $i = 1, \cdots, n$ and for any X complex tangential operator, $|XF_1(z)| = O((1 - |z|^2)^{-\frac{1}{2}})$.

Now, the transversality of γ allows to construct (see [**Ru**, page 238]) a "quasianalytic" disc $\Phi = (\Phi_1, \cdots, \Phi_n)$ from the closed unit disc $\overline{\mathbf{D}}$ in $\mathbf{C}$, into $\mathbf{B}^n$ with the following properties:

(i) $\Phi(e^{i\theta}) = \gamma(\theta)$, $1 - |\Phi(re^{i\theta})| \simeq 1 - r$, and for any θ in $[-\pi, \pi]$, the curve $r \to \Phi(re^{i\theta})$ is a restricted $\gamma(\theta)$-curve.

(ii) $\left|\sum_{i=1}^{n} \dfrac{\partial \Phi_i}{\partial \bar{z}}\right| = O(1 - |z|)$.

The above properties of Φ together with the estimates on the derivatives of F_1 show that the function $U = F_1 \circ \Phi$ satisfies that $|\frac{\partial U}{\partial \bar{z}}| = O((1 - |z|^{-\frac{1}{2}})$, and consequently, (see for instance [**BrCa**]) that there exists h in $Lip_{\frac{1}{2}}(\overline{\mathbf{D}})$ so that $\frac{\partial U}{\partial \bar{z}} = \frac{\partial h}{\partial \bar{z}}$. Now, Fatou's theorem in one complex variable applied to the bounded analytic function $U - h$ gives that there exists at almost every $t \in [-\pi, \pi]$ $\lim_{r \to 1}(U(re^{it}) - h(re^{it}))$, and hence it exists $\lim_{r \to 1}(F \circ \Phi)(re^{it})$.

Since for each t the curve $\Phi(re^{it})$, $0 \leq r < 1$ is a restricted $\gamma(t)$-curve, we will finish once we check that Chirka's theorem (see **[Ch]**), concerning sufficient conditions for the existence of restricted K-limits of bounded holomorphic functions, extends to our context. In the proof of it we will follow closely the ideas in Chirka's theorem, modifying some arguments due to the fact that our functions are not holomorphic.

Proposition 3.5 *Let F be a bounded harmonic function with spectrum in Ω_{0k}, and let Ψ_0 be a special ζ-curve, ζ in $\mathbf{S}^n$. Assume that $\lim_{t\to 1} F(\Psi_0(t)) = L$. Then F has restricted K-limit L at ζ.*

Proof of proposition 3.5:

Let Ψ be a special ζ-curve, and $\psi = (\Psi\overline{\zeta})\zeta$ be the orthogonal projection onto the complex line joinning 0 and ζ. We will first show that,

$$\lim_{t\to 1} (F(\Psi(t)) - F(\psi(t))) = 0. \tag{3.2}$$

We have that $(\Psi - \psi) \perp \psi$, and if $t \in [0,1)$, then for any $|\lambda| < R = R(t)$ with $R^2 = \frac{1-|\psi|^2}{|\Psi-\psi|^2}$, the point $(1-\lambda)\psi(t) + \lambda\Psi(t)$ is in $\mathbf{B}^n$. The fact that Ψ is a special ζ-curve gives that $R(t) \to +\infty$ as $t \to 1$.

Arguing as in theorem 3.4 we just need to show that (3.2) holds with $F = C_{\Omega_{0k}}[f]$ replaced by F_1. The fact that $\dfrac{\partial F_1}{\partial \overline{z}_i} = O((1-|z|^2)^{-\frac{1}{2}})$, $i = 1, \cdots, n$, gives that the function $g(\lambda) = F_1((1-\lambda)\psi(t) + \lambda\Psi(t))$, defined in $|\lambda| < R$ satisfies that

$$\left|\frac{\partial g}{\partial \overline{\lambda}}(\lambda)\right| \preceq \frac{|\Psi - \psi|}{(1 - |(1-\lambda)\psi + \lambda\Psi|^2)^{\frac{1}{2}}}.$$

Now, $|(1-\lambda)\psi + \lambda\Psi|^2 = |\psi|^2 + |\lambda|^2|\Psi - \psi|^2$, and the above estimate leads to

$$\left|\frac{\partial g}{\partial \overline{\lambda}}(\lambda)\right| \preceq \frac{1}{(R^2 - |\lambda|^2)^{\frac{1}{2}}}, \qquad |\lambda| < R.$$

In particular, the function $h(\lambda) = g(R\lambda)$ defined in the unit disc, is in $C^1(\mathbf{D})$ and $\left|\dfrac{\partial h}{\partial \overline{\lambda}}(\lambda)\right| = O\left(\dfrac{1}{(1-|\lambda|^2)^{\frac{1}{2}}}\right)$. Hence there exists h_1 in $Lip_{\frac{1}{2}}(\overline{\mathbf{D}})$, with $||h_1||_{Lip_{\frac{1}{2}}} \preceq ||f||_\infty$, such that the function $h - h_1$ is a bounded holomorphic function in $\mathbf{D}$. Schwarz's lemma applied to $h - h_1$ gives

$$|F_1(\Psi(t)) - F_1(\psi(t))| = |g(1) - g(0)| = |h(\tfrac{1}{R}) - h(0)| \leq$$

$$|(h-h_1)(\tfrac{1}{R}) - (h-h_1)(0)| + |h_1(\tfrac{1}{R}) - h_1(0)| \preceq \frac{||h-h_1||_\infty}{R} + \frac{||h_1||_{Lip_{\frac{1}{2}}}}{R^{\frac{1}{2}}} \preceq \frac{1}{R^{\frac{1}{2}}},$$

which since $R(t) \to +\infty$, gives (3.2).

Applying (3.2) to the given special ζ-curve Ψ_0, the hypothesis on Ψ_0 gives that

$$\lim_{t\to 1} F_1(\psi_0(t)) = L, \tag{3.3}$$

with $\psi_0 = (\Psi_0\overline{\zeta})\zeta$. Next, we want to pass from this particular curve Ψ_0 to any restricted ζ-curve by applying, like in Chirka's theorem, Lindelöf's theorem. Since the function $\lambda \to F_1(\lambda\zeta)$, defined in $\mathbf{D}$ is not holomorphic, we will correct it by solving a $\overline{\partial}$-equation. An argument like the previous one shows that there exists a function f_1 in $Lip_{\frac{1}{2}}(\overline{\mathbf{D}})$ with $F_1(\cdot\zeta)-f_1$ in $H^\infty(\mathbf{D})$. And (3.3) gives that $\lim_{t\to 1}(F_1(\psi_0(t)) - f_1(\Psi_0(t)\overline{\zeta})) = L - f_1(1)$.

Finally, if Ψ is any restricted ζ-curve, the orthogonal projection ψ is nontangential, and Lindelöf's theorem (see [**Li**]), shows that $\lim_{t\to 1}(F_1(\psi(t)) - f_1(\Psi(t)\overline{\zeta})) = L - f_1(1)$ and $\lim_{t\to 1} F_1(\psi(t)) = L$. Equation (3.2) gives then that $\lim_{t\to 1} F_1(\Psi(t)) = L$. ∎

4 Tangential convergence

In this section we will show that for harmonic extensions of nonisotropic potentials with spectrum in Ω_{0k}, there exists at almost every point in $\mathbf{S}^n$, the limit along tangential approach regions. We first need some definitions. Let $\alpha \in \mathbf{C}$ so that $0 < \mathrm{Re}\alpha < n$, $z \in \overline{\mathbf{B}}^n$, $\zeta \in \mathbf{S}^n$. The nonisotropic kernel $I_\alpha(z,\zeta)$ is defined by

$$I_\alpha(z,\zeta) = \frac{C(n,\alpha)}{|1 - z\overline{\zeta}|^{n-\alpha}},$$

where $C(n,\alpha) = \frac{(\Gamma(\frac{n+\alpha}{2}))^2}{(n-1)!\Gamma(\alpha)}$ is a constant of normalization. If $1 \le p < +\infty$, and $f \in L^p(\mathbf{S}^n)$, $z \in \overline{\mathbf{B}}^n$, the nonisotropic convolution with I_α is given by

$$I_\alpha * f(z) = \int_{\mathbf{S}^n} I_\alpha(z,\zeta) f(\zeta)\, d\sigma(\zeta).$$

The space of functions $I_\alpha * f$, with $f \in L^p(\mathbf{S}^n)$, will be denoted by $I_\alpha * L^p$. We recall that if $0 < \mathrm{Re}\,\alpha < n$, $u_{rs} \in H(r,s)$, $z \in \overline{\mathbf{B}}^n$, then, see ([**AhCa**]),

$$\begin{aligned} &I_\alpha * u_{rs}(z) = \\ &\frac{\Gamma(\frac{n+\alpha}{2})^2\Gamma(\frac{n-\alpha}{2}+r)\Gamma(\frac{n-\alpha}{2}+s)}{\Gamma(\alpha)\Gamma(\frac{n-\alpha}{2})^2\Gamma(r+s+n)} F(\frac{n-\alpha}{2}+r, \frac{n-\alpha}{2}+s, r+s+n, |z|^2) u_{rs}(z), \end{aligned} \tag{4.1}$$

where $F(a,b,c,x)$ is the hypergeometric function. The fact that if $n - \mathrm{Re}\,2a > 0$,

$$F(a+r, a+s, r+s+n, 1) = \frac{\Gamma(r+s+n)\Gamma(n-2a)}{\Gamma(n+r-a)\Gamma(n+s-a)},$$

gives in particular that $I_\alpha * 1_{|\mathbf{S}^n} \equiv 1$. If $T = \sum_{i<j} \overline{T}_{ij} T_{ij}$ and $\overline{T} = \sum_{i<j} T_{ij}\overline{T}_{ij}$, then [**AhBr**],

$$T u_{rs} = -r(s+n-1)u_{rs} \qquad \overline{T} u_{rs} = -s(r+n-1)u_{rs}. \tag{4.2}$$

In [Ge], it is proved that I_1 is the fundamental solution for the sublaplacian

$$(4.3)\qquad \mathcal{L} = \frac{1}{(\frac{n-1}{2})^2}\left(-\frac{1}{2}\sum_{i<j}(T_{ij}\overline{T}_{ij} + \overline{T}_{ij}T_{ij})\right).$$

This operator can be used to show that the spaces of potentials $I_m * L^p$, m positive integer, coincide with the nonisotropic Sobolev spaces $L^p_m(\mathbf{S}^n)$. This is the result of the next proposition.

Proposition 4.1 *Let* $1 \le m \le n-1$ *and* $1 < p < +\infty$. *Then* $I_m * L^p = L^p_m(\mathbf{S}^n)$.

Proof of proposition 4.1:

The operators I_m are of order m, and hence map L^p into $L^p_m(\mathbf{S}^n)$. The other inclusion is a consequence of the following facts:

(i) $I_1 * \overset{m}{\cdots} * I_1 * L^p = L^p_m$.

(ii) $I_1 * \overset{m}{\cdots} * I_1 * L^p \subset I_m * L^p$.

The nonisotropic convolution $I_1 * \overset{m}{\cdots} * I_1$ is an operator of order m (see [**NaRoStWa**]), and consequently maps L^p into L^p_m. Of course, $\mathcal{L}^m$ applies L^p_m in L^p. Since $\mathcal{L}^m(I_1 * \overset{m}{\cdots} * I_1)$ and $(I_1 * \overset{m}{\cdots} * I_1)\mathcal{L}^m$ are the identity on regular functions by Geller's result, we obtain (i).

For (ii) consider the differential operator given by

$$X^m = \prod_{i=0}^{m-1}(\alpha_i T + \beta_i \overline{T} + \gamma_i \mathrm{Id}),$$

where

$$\alpha_i = -\frac{1}{n-1}\left(\frac{n+m}{2} - 1 - i\right),$$
$$\beta_i = -\frac{1}{n-1}\left(\frac{n-m}{2} + i\right),$$
$$\gamma_i = \left(\frac{n-m}{2} + i\right)\left(\frac{n-m}{2} + m - 1 - i\right).$$

Applying (4.2) we easily deduce that if $u_{rs} \in H(r,s)$, then

$$I_1 * \overset{m}{\cdots} * I_1 * u_{rs} = I_m X^m I_1 * \overset{m}{\cdots} * I_1 * u_{rs}.$$

By density we have that the above equality also holds for L^p. The fact that $\omega(X^m) = m$, and that $I_1 * \overset{m}{\cdots} * I_1$ is a differential operator of order m give finally that $g = X^m I_1 * \overset{m}{\cdots} * I_1 * f$ is in L^p, and consequently that $I_1 * \overset{m}{\cdots} * I_1 * f = I_m * g$ is in $I_m * L^p$. Observe that I_m gives a topological isomorphism from L^p to L^p_m. ∎

Remark: It is also easy to check that the operators I_m give a topological isomorphism from L^p_Ω to $L^p_{m\Omega}$, which in particular gives that $I_m * L^p_{\Omega_0 k} = L^p_{m\Omega_0 k}$. If α is real and noninteger, and k is a nonnegative integer, the spaces $I_\alpha * L^p_{\Omega_0 k}$ arise as the interpolated by the complex method of Sobolev spaces with spectrum in Ω_{0k}. A direct proof of this fact will be given in the appendix.

It is a well known fact that if $\alpha p > n$, the space $I_\alpha * L^p$ consists of regular functions. If f is a holomorphic function in the Hardy-Sobolev space H^p_α, or an M-harmonic extension of a potential in $I_\alpha * L^p$, $\alpha p \leq n$, then for any $\zeta \in \mathbf{S}^n$, except for an exceptional set of small suitable size, there exists the limit of $f(z)$, as z tends to ζ within certain tangential approach regions (see [**Su**] and [**CaOr**]). We will show that this tangential convergence remains true for the space of harmonic transforms of functions in $I_\alpha * L^p$ with spectrum in Ω_{0k}.

Going back to the L^p-boundedness of tangential maximal operators, we recall some more definitions. Let $1 < p < +\infty$, $0 < \alpha$, $m = n - \alpha p \geq 0$, and let $\zeta \in \mathbf{S}^n$, $C > 0$. If $m > 0$, and $1 \leq \tau \leq \frac{n}{m}$, we consider the tangential approximation regions given by

$$\mathcal{D}_\tau(\zeta) = \{z \in \mathbf{B}^n\,;\, |1 - z\bar{\zeta}|^\tau < C(1 - |z|)\}$$

Observe that if $\tau = 1$, $\mathcal{D}_1$ is the admissible region, whereas if $\tau = \frac{n}{m}$, is the tangential approach region considered in [**NaRuSh**]. If $m = 0$ and $\mu \geq 1$, we also consider the regions

$$\mathcal{E}_\mu(\zeta) = \{\, z \in \mathbf{B}^n\,;\, |1 - z\bar{\zeta}| < \frac{C}{(\log \frac{1}{1-|z|})^{\frac{p-1}{n}\mu}}\,\}.$$

If $f : \mathbf{B}^n \to \mathbf{C}$, the corresponding maximal functions will be denoted by $M_\tau f(\zeta) = \sup_{z \in \mathcal{D}_\tau(\zeta)} |f(\zeta)|$ and $\mathcal{M}_\mu f(\zeta) = \sup_{z \in \mathcal{E}_\mu(\zeta)} |f(z)|$ respectively.

We then have the following theorem

Theorem 4.2 *Let* $1 < p < +\infty$, $\alpha > 0$, $m = n - \alpha p \geq 0$ *and* k *a nonnegative integer.*

(i) If $m > 0$, $1 < \tau \leq \frac{n}{m}$, *and* ν *is a positive Borel measure on* $\mathbf{S}^n$ *such that for any* $\zeta \in \mathbf{S}^n$, $\delta > 0$,

$$\nu(B(\zeta, \delta)) \preceq \delta^{\tau m},$$

then there exists $C > 0$, *such that, for any* $f \in L^p(\mathbf{S}^n)$,

$$||M_\tau C_{\Omega_{0k}}[I_\alpha * f]||_{L^p(d\nu)} \leq C||f||_p.$$

(ii) If $m = 0$, $\mu \geq 1$ *and* ν *is a finite positive Borel measure on* $\mathbf{S}^n$ *such that for any* $\zeta \in \mathbf{S}^n$, $\delta > 0$,

$$\nu(B(\zeta, \delta)) \preceq \delta^{\frac{n}{\mu}},$$

then there exists $C > 0$, *so that, for any* $f \in L^p(\mathbf{S}^n)$,

$$||\mathcal{M}_\mu C_{\Omega_{0k}}[I_\alpha * f]||_{L^p(d\nu)} \leq C||f||_p.$$

Proof of theorem 4.2:

The proof is based in the following lemma

Lemma 4.3 *Let k be a nonnegative integer and $0 < \alpha < n$. Then for any $z \in \mathbf{B}^n$, $\omega \in \mathbf{S}^n$*

$$|\int_{\mathbf{S}^n} C_{\Omega_0 k}(z,\zeta) I_\alpha(\zeta,\omega)\, d\sigma(\zeta)| \preceq \frac{1}{|1-z\overline{\omega}|^{n-\alpha}}.$$

Proof of lemma 4.3:

Let $z = rz_0$, $z_0 \in \mathbf{S}^n$. Assume first that $|1-z_0\overline{\omega}| < 1-|z|$. we then have that $|1-z\overline{\omega}| \simeq 1-|z|$, and formula (ii) in theorem 2.1 gives that

$$|\int_{\mathbf{S}^n} C_{\Omega_0 k}(z,\zeta) I_\alpha(\zeta,\omega)\, d\sigma(\zeta)| \leq \int_{B(\omega,K(1-|z|))} \frac{d\sigma(\zeta)}{|1-z\overline{\zeta}|^n |1-\zeta\overline{\omega}|^{n-\alpha}}$$
$$+\int_{B^c(\omega,K(1-|z|))} \frac{d\sigma(\zeta)}{|1-z\overline{\zeta}|^n |1-\zeta\overline{\omega}|^{n-\alpha}},$$

where K is big enough. The first integral on the right is bounded by

$$\frac{1}{(1-|z|)^n}\int_{B(\omega,K(1-|z|))} \frac{d\sigma(\zeta)}{|1-\zeta\overline{\omega}|^{n-\alpha}} \preceq \frac{1}{(1-|z|)^{n-\alpha}} \simeq \frac{1}{|1-z\overline{\omega}|^{n-\alpha}}.$$

The second summand is bounded by

$$\int_{B^c(\omega,K(1-|z|))} \frac{d\sigma(\zeta)}{|1-\zeta\overline{\omega}|^{2n-\alpha}} \preceq \frac{1}{(1-|z|)^{n-\alpha}} \simeq \frac{1}{|1-z\overline{\omega}|^{n-\alpha}}.$$

If $1-|z| \leq |1-z_0\overline{\omega}|$ we then have that $|1-z\overline{\omega}| \simeq |1-z_0\overline{\omega}|$, and then the estimate is deduced from the fact that the operators $C^*_{\Omega_0 k} * I_\alpha$ are of order α, and then satisfy

$$|\int_{\mathbf{S}^n} C_{\Omega_0 k}(z,\zeta) I_\alpha(\zeta,\omega)\, d\sigma(\zeta)| \preceq \frac{1}{|1-z_0\overline{\omega}|^{n-\alpha}},$$

uniformly on $r = |z|$. ∎

Going back to the proof of the theorem, let $f \in L^p(\mathbf{S}^n)$ and consider $F = C_{\Omega_0 k}[I_\alpha * f]$. Then the previous lemma gives that

$$|F(z)| = |\int_{\mathbf{S}^n}\int_{\mathbf{S}^n} C_{\Omega_0 k}(z,\zeta) I_\alpha(\zeta,\omega) f(\omega)\, d\sigma(\omega)\, d\sigma(\zeta)| \leq$$
$$\int_{\mathbf{S}^n} |\int_{\mathbf{S}^n} C_{\Omega_0 k}(z,\zeta) I_\alpha(\zeta,\omega)\, d\sigma(\zeta)| |f(\omega)|\, d\sigma(\omega) \preceq \int_{\mathbf{S}^n} \frac{|f(\omega)|}{|1-z\overline{\omega}|^{n-\alpha}}\, d\sigma(\omega).$$

In particular, the above estimate together with [**CaOr**, lemma 2.21] gives that $|F(z)| \leq P[I_\alpha * |f|](z)$, with P the Poisson-Szegö kernel in $\mathbf{B}^n$. Now (i) and (ii) of theorem 4.2 is a consequence of [**CaOr**, thms. 2.10 and 2.17] together with remark 2.20 there (see also [**Su**]).

A straightforward argument using Frostman's theorem gives the following

Corollary 4.4 *Let* $1 < p < +\infty$, $\alpha > 0$, $m = n - \alpha p \geq 0$ *and* k *a nonnegative integer.*

(i) If $m > 0$, $1 < \tau \leq \frac{n}{m}$, *and* f *is a harmonic extension of a function in* $I_\alpha * L^p$, *with spectrum in* Ω_{0k}, *then there exists a set* $E \subset S^n$ *with* $H^{\tau m}(E) = 0$, *so that for any* $\zeta \notin E$, *there exists the limit of* $f(z)$ *as* z *approaches* ζ *within* $\mathcal{D}_\tau(\zeta)$.

(ii) If $m = 0$, $\mu \geq 1$, *and* f *is a harmonic extension of a function in* $I_\alpha * L^p$, *with spectrum in* Ω_{0k}, *then there exists a set* $E \subset S^n$ *with* $H^{\frac{n}{\mu}}(E) = 0$, *so that for any* $\zeta \notin E$, *there exists the limit of* $f(z)$ *as* z *approaches* ζ *within* $\mathcal{E}_\mu(\zeta)$.

5 Appendix

In this appendix we will give a constructive proof of the fact that the spaces of potentials $I_\alpha * L^p_{\Omega_{0k}}$ arise as the interpolated spaces, by the complex method, of the Sobolev spaces $L^p_{j\,\Omega_{0k}}$. We begin with the case $0 < \alpha < 1$, from which we deduce the general case.

Theorem 5.1 *Let* $0 < \alpha < 1$ *and let* k *be a nonnegative integer. We then have that*

$$[L^p_{\Omega_{0k}}, L^p_{1\Omega_{0k}}]_{[\alpha]} = I_\alpha * L^p_{\Omega_{0k}}.$$

Proof of theorem 5.1:

We will first prove the inclusion

$$I_\alpha * L^p_{\Omega_{0k}} \subset [L^p_{\Omega_{0k}}, L^p_{1\Omega_{0k}}]_{[\alpha]}.$$

It will be enough, given f in $L^p_{\Omega_{0k}}$, to construct a continuous vector valued function φ from the closed strip $\overline{S} = \{\lambda + it \in \mathbf{C}\,;\, 0 \leq \lambda \leq 1\}$ to $L^p_{\Omega_{0k}} + L^p_{1\Omega_{0k}} = L^p_{\Omega_{0k}}$, holomorphic on S, with $\varphi(\alpha) = I_\alpha * f$, and satisfying that for some $\gamma \in \mathbf{R}$,

$$(5.1) \qquad ||\varphi(it)||_{L^p} \leq Ce^{\gamma|t|}||f||_p, \qquad ||\varphi(1+it)||_{L^p_1} \leq Ce^{\gamma|t|}||f||_p.$$

If $0 < \lambda \leq 1$, define $\varphi(\lambda + it) = I_{\lambda+it} * f$. Let us first check that for $g \in L^p$, $||I_{\lambda+it} * g||_p \leq Ce^{\gamma|t|}||g||_p$ for some $\gamma \in \mathbf{R}$.

Formula (4.1) gives that if $u_{rs} \in H(r,s)$,

$$(5.2) \qquad I_{\lambda+it} * u_{rs}(\zeta) = C_{\lambda+it}(r,s)u_{rs}(\zeta),$$

with

$$C_{\lambda+it}(r,s) = \frac{\Gamma(\frac{n+\lambda+it}{2})^2\,\Gamma(r+\frac{n-\lambda-it}{2})\Gamma(s+\frac{n-\lambda-it}{2})}{\Gamma(\frac{n-\lambda-it}{2})^2\,\Gamma(r+\frac{n+\lambda+it}{2})\Gamma(s+\frac{n+\lambda+it}{2})}$$
$$= \frac{(r-1+\frac{n-\lambda-it}{2})}{(r-1+\frac{n+\lambda+it}{2})}\cdots\frac{(\frac{n-\lambda-it}{2})}{(\frac{n+\lambda+it}{2})}\frac{(s-1+\frac{n-\lambda-it}{2})}{(s-1+\frac{n+\lambda+it}{2})}\cdots\frac{(\frac{n-\lambda-it}{2})}{(\frac{n+\lambda+it}{2})},$$

for $r, s \geq 1$. Since for each $x \geq 1$ and $0 < \lambda \leq 1$,

$$\left|\frac{x - \frac{\lambda+it}{2}}{x + \frac{\lambda+it}{2}}\right| \leq 1,$$

$|C_{\lambda+it}(r,s)| \leq 1$. The case r or s equal to zero are treated in a simmilar way. Then we deduce that $||I_{\lambda+it}g||_2 \leq ||g||_2$, for $g \in L^2$.

Next if $z = |z|z_0 \in \mathbf{B}^n$, $\zeta \in \mathbf{S}^n$,

$$|\nabla I_{\lambda+it}(z,\zeta)| \leq C(n + |t|)\, A_{\lambda+it}\frac{1}{|1 - z_0\bar{\zeta}|^{n-\lambda+1}}.$$

where C is a constant independent of λ and t, and

$$A_{\lambda+it} = \left|\frac{\Gamma(\frac{n+\lambda+it}{2})^2}{\Gamma(\lambda + it)}\right|.$$

Stirling's formula gives easily that $A_{\lambda+it} = O(e^{\gamma|t|})$ for some $\gamma \in \mathbf{R}$. Since the estimates on the derivatives implies that the kernels $I_{\lambda+it}$ satisfy Hormander's condition **[GaRu]** with bounds $Ce^{\gamma|t|}$, we finally get that $||I_{\lambda+it} * g||_p \leq Ce^{\gamma|t|}||g||_p$, for $g \in L^p$.

Next observe that formula (5.2) shows that for any $t_0 \in \mathbf{R}$, and $u_{rs} \in H(r,s)$, there exists $\lim_{\lambda+it \to it_0} I_{\lambda+it} * u_{rs}$, and equals u_{rs} if $t_0 = 0$. Since $\oplus H(r,s)$ is dense in L^p, this fact, together with the uniform boundedness of $||I_{\lambda+it}||_{p,p}$, for t bounded, shows that for any $g \in L^p$, $t_0 \in \mathbf{R}$, there exists the limit in L^p of $I_{\lambda+it} * g$ as $\lambda + it$ tends to it_0. If we denote it by $I_{it_0} * g$, we also have that $||I_{it}||_{p,p} \leq Ce^{\gamma|t|}$ and $I_0 = Id$. Thus we have continuously extended φ to all S, and proved that $||\varphi(it)||_p \leq Ce^{\gamma|t|}||f||_p$. Since $\varphi(s + it)$ is in $L^p_{\Omega_0 k}$, $\varphi(it)$ is also in $L^p_{\Omega_0 k}$.

We now will show that $||\varphi(1 + it)||_{L^p_1} \leq Ce^{\gamma|t|}||f||_p$. Given $0 < \mu < 1$, we denote by $I^{\mu}_{1+it}(\zeta,\omega) = I_{1+it}(\mu\zeta,\omega)$, $\zeta,\omega \in \mathbf{S}^n$. Since the pointwise estimates of the derivatives of order two of $I^{\mu}_{1+it}(\zeta,\omega)$ are bounded uniformly in μ by $Ce^{\gamma|t|}\frac{1}{|1-\zeta\bar{\omega}|^{n+1}}$, the classical theory of singular integrals operators shows that the desired estimate in (5.1) for $||\varphi(1 + it)||_{L^p_1}$ will follow once we show that

$$||\mathcal{L}I_{1+it} * u||_2 \leq Ce^{\gamma|t|}||u||_2, \tag{5.3}$$

for every $u \in L^2(\mathbf{S}^n)$.

Now, if $u_{rs} \in H(r,s)$, we have that $I_{1+it} * u_{rs}(\zeta) = \psi_{1+it}(r,s)u_{rs}(\zeta)$, where

$$\psi_{1+it}(r,s) = \frac{\Gamma(\frac{n+1+it}{2})^2}{\Gamma(\frac{n-1-it}{2})^2}\frac{\Gamma(r + \frac{n-1-it}{2})\Gamma(s + \frac{n-1-it}{2})}{\Gamma(r + \frac{n+1+it}{2})\Gamma(s + \frac{n+1+it}{2})}.$$

Since

$$|\psi_{1+it}(r,s)| \preceq \frac{|n - 1 + it|^2}{|r + \frac{n-1+it}{2}||s + \frac{n-1+it}{2}|},$$

and $\mathcal{L}u_{rs} = (r + \frac{n-1}{2})(s + \frac{n-1}{2})u_{rs}$, we obtain (5.3).

Remark: The above shows, in particular, that for $t \in \mathbf{R}$ the operator $S_t = \mathcal{L}I_{1-it}$ verifies $||S_t[f]||_p \leq Ce^{\gamma|t|}||f||_p$ for any $f \in L^p$. Since for every $u_{rs} \in H(r,s)$, $S_t(I_1 * u_{rs}) = I_{1-it} * u_{rs}$, and $S_t \circ I_1$, I_{1-it} are bounded in L^p, we deduce that for any $h \in L^p$, $||I_{1-it} * h||_p \leq Ce^{\gamma|t|}||I_1 * h||_p$.

In $\mathbf{R}^n$, the Riesz kernels are additive with respect to the convolution. Next lemma, which will be used to finish the theorem, gives some results concerning the non isotropic convolution of the kernels I_α.

Lemma 5.2 *Let $0 \leq j \leq n-1$, $0 < \alpha < 1$, and k a nonnegative integer. We then have:*
(i) If $f \in L^p_{\Omega_0 k}$ and $\mathcal{L}I_{1-\alpha} * f \in L^p_{\Omega_0 k}$, there exists $g \in L^p_{\Omega_0 k}$ such that $\mathcal{L}^j I_{\alpha+j} * g = f$.
(ii) $I_\alpha * L^p_{\Omega_0 k} = \{f \in L^p;\ I_{1-\alpha} * f \in L^p_{1\,\Omega_0 k}\}$.
(iii) $I_j * (I_\alpha * L^p_{\Omega_0 k}) = I_{\alpha+j} * L^p_{\Omega_0 k}$.
(iv) If $f \in L^p$, $I_1 * f$ admits an expression $I_1 * f = I_\alpha * g$ with $||g||_p \preceq ||I_{1-\alpha} * f||_p$.

Proof of lemma 5.2:
Observe that (i) would follow if we could find a bounded operator in L^p such that $\mathcal{L}^j I_{\alpha+j} S \mathcal{L} I_{1-\alpha} = Id_{L^p}$. We will show that, except for some terms with good behavior, this is what happens. The proof is based in the action of these operators on the spaces $H(r,s)$ and assymptotic developments of Gamma functions.

Assume $0 \leq j \leq n-1$, $0 < \alpha < 1$ and k a nonnegative integer. It suffices to show the lemma for each $L^p_{\Omega_{s\,s}}$, $0 \leq s \leq k$. Applying (4.1), we obtain that for any $u_{r\,s}$ in $H(r,s)$,

$$I_{j+\alpha} * I_{1-\alpha} * u_{r\,s}(\zeta) = C_s \frac{\Gamma(\frac{n-j-\alpha}{2}+r)\Gamma(\frac{n-1+\alpha}{2}+r)}{\Gamma(\frac{n+j+\alpha}{2}+r)\Gamma(\frac{n+1-\alpha}{2}+r)} u_{r\,s}(\zeta),$$

where C_s is a constant depending on n, α and s.

The assymptotic development in [**TrEr**] together with Stirling's formula give that there exist $\lambda_i(\alpha,n)$, $i \in \mathbf{Z}_+$, with $\lambda_0 \neq 0$ so that for each $l > 0$,

$$b_{r\,l} = 1 - \frac{\Gamma(\frac{n-\alpha}{2}+r)}{\Gamma(\frac{n+\alpha}{2}+r)} \left(\lambda_0(r+\frac{n-1}{2})^\alpha + \cdots \lambda_{l-1}(r+\frac{n-1}{2})^{\alpha-l+1}\right),$$

verify $|b_{r\,l}| \leq \frac{2|\lambda_l|+1}{(r+1)^l}$, if r is big enough.

In particular, there exist $\lambda_i(\alpha,n)$, $\mu_i(\alpha,n)$ such that

$$\frac{\Gamma(\frac{n-j-\alpha}{2}+r)\Gamma(\frac{n-1+\alpha}{2}+r)}{\Gamma(\frac{n+j+\alpha}{2}+r)\Gamma(\frac{n+1-\alpha}{2}+r)} \left(\lambda_0(r+\frac{n-1}{2})^{j+\alpha} + \cdots \lambda_{l-1}(r+\frac{n-1}{2})^{j+\alpha-l+1}\right) \times$$
$$\left(\mu_0(r+\frac{n-1}{2})^{1-\alpha} + \cdots \mu_{l-1}(r+\frac{n-1}{2})^{1-\alpha-l+1}\right) = 1 - c_{r\,l},$$

with $|c_{r\,l}| \preceq \frac{d_l}{(r+1)^l}$. Next

$$\left(\lambda_0(r+\frac{n-1}{2})^{j+\alpha} + \cdots \lambda_{l-1}(r+\frac{n-1}{2})^{j+\alpha-l+1}\right) \times$$
$$\left(\mu_0(r+\frac{n-1}{2})^{1-\alpha} + \cdots \mu_{l-1}(r+\frac{n-1}{2})^{1-\alpha-l+1}\right) = \sum_{k=3-2l}^{1} \gamma_k(r+\frac{n-1}{2})^{j+k}.$$

Since $\mathcal{L}u_{rs} = (r+\frac{n-1}{2})(s+\frac{n-1}{2})u_{rs}$, for $u_{rs}\in H(r,s)$, and $I_1 * u_{rs} = \frac{n-1}{2}\frac{1}{(r+\frac{n-1}{2})(s+\frac{n-1}{2})}u_{rs}$, we can write

$$\sum_{k=3-2l}^{1} \gamma_k (r+\frac{n-1}{2})^{j+k} u_{rs} = \tilde{\gamma}\mathcal{L}^{j+1}u_{rs} + \sum_{k=0}^{2l-3}\tilde{\gamma}_k \mathcal{L}^j (I_1 * \overset{k}{\cdots} * I_1) * u_{rs},$$

with $\tilde{\gamma}\neq 0$.

We define the operator in $\oplus_r H(r,s)$ given by

$$T_{rs}u_{rs} = \begin{cases} u_{rs} & \text{if } r\le r_0 \\ \frac{1}{C_s}\left(\tilde{\gamma}\mathcal{L}^{j+1} + \sum_{k=0}^{2l-3}\tilde{\gamma}_k\mathcal{L}^j(I_1 * \overset{k}{\cdots} * I_1)\right) * I_{\alpha+j} * I_{1-\alpha} * u_{rs}, & \text{if } r > r_0, \end{cases}$$

where r_0 is to be chosen. Then

$$(Id - T_{rs})u_{rs} = \begin{cases} 0 & \text{if } r\le r_0 \\ c_{rl}u_{rs}, & \text{if } r > r_0. \end{cases}$$

We will check that there exists $\varepsilon < 1$ such that for any $f\in L^p_{\Omega_{s,s}}$, $||f - Tf||_p \le \varepsilon ||f||_p$.

Assume first that $p\le 2$, and take $f\in L^2(\mathbf{S}^n)$. In [**Al**, page 118], it is shown that if $\omega,\zeta\in \mathbf{S}^n$, $|K_{rs}(\omega,\zeta)|\le D(r,s,n)$. Since $D(r,s,n) = \binom{r+n-2}{r}\binom{s+n-2}{s}\frac{r+s+n-1}{n-1}$, we deduce that $|K_{rs}(\omega,\zeta)|\preceq (r+1)^{n-1}$, with constant depending on s and n. Hence

$$|f_{rs}(\omega)| = |\int_{\mathbf{S}^n} K_{rs}(z,\zeta)f(\zeta)\,d\sigma(\zeta)| \preceq (r+1)^{n-1}||f||_1.$$

This gives that if $f\in L^2_{\Omega_{s,s}}$,

$$||f-Tf||_2^2 \preceq \sum_{r\ge r_0} c_{rl}^2 ||f_{rs}||_2^2 \preceq \sum_{r>r_0}\frac{1}{(r+1)^{2l}}\int_{\mathbf{S}^n}|f_{rs}(\omega)|^2\,d\sigma(\omega) \preceq$$
$$\sum_{r>r_0}\frac{(r+1)^{2(n-1)}}{(r+1)^{2l}}||f||_1^2 \le \varepsilon ||f||_1^2,$$

provided r_0 is big enough and l satisfies that $2l - 2(n-1) > 1$.

Finally, since $1 < p\le 2$,

$$||f-Tf||_p^2 \le ||f-Tf||_2^2 \preceq \varepsilon ||f||_1^2 \preceq \varepsilon ||f||_p^2.$$

In particular, the operator T is invertible in $L^p_{\Omega_{s,s}}$ and there exists an operator $S : L^p_{\Omega_{s,s}} \to L^p_{\Omega_{s,s})}$ with $ST = TS = Id_{L^p_{\Omega_{s,s}}}$.

The case $p\ge 2$ can be deduced from the previous one by duality, since the operator T is selfadjoint.

The definition of T gives that we can write

$$T = T_1 + R_1 + R_2, \tag{5.4}$$

where

$$T_1 = \frac{\tilde{\gamma}}{C_s}\mathcal{L}^{j+1} I_{\alpha+j} * I_{1-\alpha},$$

is an operator of order zero,

$$R_1 = \frac{1}{C_s}\sum_{k=0}^{2l-3} \tilde{\gamma}_k \mathcal{L}^j (I_1 * \overset{k}{\cdots} * I_1) * I_{\alpha+j} * I_{1-\alpha}$$

is an operator of order greater or equal to 1, and where

$$R_2 = \sum_{r \leq r_0} \left(Id - \frac{1}{C_s}\left(\tilde{\gamma}\mathcal{L}^{j+1} + \sum_{k=0}^{2l-3} \tilde{\gamma}_k \mathcal{L}^j (I_1 * \overset{k}{\cdots} * I_1) \right) * I_{\alpha+j} * I_{1-\alpha} \right) K_{r\,s}$$

is an operator of finite range, and consequently, bounded in $L^p_{\Omega_{s\,s}}$. If we apply S to equation (5.4), the boundedness in L^p of the operators S, T_1, and R_j, $j = 1, 2$, together with the fact that they conmute in L^2, shows that if we take $f \in L^p_{s\,s}$ satisfying that $\mathcal{L}I_{1-\alpha} * f$ is also in $L^p_{\Omega_{s\,s}}$, then,

$$f = \frac{\tilde{\gamma}}{C_s}\mathcal{L}^j I_{\alpha+j} * S\mathcal{L}I_{1-\alpha} * f + \frac{1}{C_s}\mathcal{L}^j I_{\alpha+j} * K_1 f + \frac{1}{C_s}\mathcal{L}^j I_{\alpha+j} * K_2 f,$$

where

$$K_1 f = \frac{1}{C_s}\sum_{k=0}^{2l-3} \tilde{\gamma}_k S(I_1 * \overset{k}{\cdots} * I_1) * I_{1-\alpha} * f$$

is in L^p, and K_2 is of finite range. Thus we have obtained that if $f \in L^p_{\Omega_{s\,s}}$ and $\mathcal{L}I_{1-\alpha} * f$ is also in $L^p_{\Omega_{s\,s}}$, then $f = \mathcal{L}^j I_{j+\alpha} * g$ with g in L^p. Thus (i) is proved.

Next, part (i) with $j = 0$, gives that

$$\{\, f \in L^p_{\Omega_{s\,s}}\,;\, I_{1-\alpha} * f \in L^p_{1\,\Omega_{s\,s}} \,\} \subset I_\alpha * L^p_{\Omega_{s\,s}}.$$

The other inclusion follows from the fact that $I_{1-\alpha} * I_\alpha$ is an operator of order 1 and, consequently, maps L^p to L^p_1.

Since $\mathcal{L}^j I^j = Id_{L^p}$ and $I^j \mathcal{L}^j = Id_{L^p_j}$, (iii) will follow once we show that

$$I_\alpha * L^p_{\Omega_{s\,s}} = \mathcal{L}^j I_{\alpha+j} * L^P_{\Omega_{s\,s}}.$$

Now, $I_{1-\alpha}\mathcal{L}^j I_{\alpha+j}$ is an operator of order 1, and this gives that

$$\mathcal{L}^j I_{\alpha+j} * L^p_{\Omega_{s\,s}} \subset \{\, f \in L^p_{\Omega_{s\,s}}\,;\, I_{1-\alpha} * f \in L^p_{1\,\Omega_{s\,s}} \,\}.$$

Part (i) gives that the inclusion in the other direction also holds, and finally (ii) gives (iii).

If we take $j = 0$ in (i), and apply S followed by I_1 in (5.4), it is then easy to check from the above calculations that (iv) is fullfilled. ∎

Let us now finish the proof of theorem 5.1. We must show that

$$[L^p_{\Omega_{0\,k}}, L^p_{1\,\Omega_{0\,k}}]_{[\alpha]} \subset I_\alpha * L^p_{\Omega_{0\,k}}.$$

By a lemma of Stafney (see [**St**]) it is enough to show that if φ_k are holomorphic on $\mathcal{S}$ and continuous up to $\overline{\mathcal{S}}$, and $h_k \in L^p_{\Omega_0 k}$, then

$$
\begin{aligned}
&\|\sum_k \varphi_k(\alpha) I_1 * h_k\|_{I_\alpha * L^p} \leq \\
(5.5)\qquad & C \max\left(\sup_t e^{\gamma|t|}\|\sum_k \varphi_k(it) I_1 * h_k\|_p, \sup_t e^{\gamma|t|}\|\sum_k \varphi_k(1-it) I_1 * h_k\|_{I_1 * L^p}\right)
\end{aligned}
$$

By (iv) of lemma 5.2,

$$
\begin{aligned}
&\|\sum_k \varphi_k(\alpha) I_1 * h_k\|_{I_\alpha * L^p} \preceq \|I_{1-\alpha} * (\sum_k \varphi_k(\cdot) h_k)\|_p = \\
(5.6)\qquad & \sup_{\|g\|_{p'} \leq 1} \left| \int_{\mathbf{S}^n} \sum_k \varphi_k(\alpha) I_{1-\alpha} * h_k(\zeta) g(\zeta)\, d\sigma(\zeta)\right|.
\end{aligned}
$$

Now the map

$$
\omega \in \overline{\mathcal{S}} \to \int_{\mathbf{S}^n} \sum_k \varphi_k(\omega) I_{1-\omega} h_k(\zeta)\, g(\zeta)\, d\sigma(\zeta),
$$

is holomorphic in $\mathcal{S}$ and continuous up to $\overline{\mathcal{S}}$. By lemma 4.3.2 in [**BeLo**], this implies that

$$
\begin{aligned}
&\left|\int_{\mathbf{S}^n} (\varphi_k(\alpha) I_{1-\alpha} h_k(\zeta))\, g(\zeta)\, d\sigma(\zeta)\right| \\
&\leq \left(\frac{1}{1-\alpha}\int_{-\infty}^{+\infty} \left|\int_{\mathbf{S}^n} (\varphi_k(it) I_{1-it} h_k(\zeta))\, g(\zeta)\, d\sigma(\zeta)\right| \mathcal{P}_0(\alpha, t) dt\right)^{1-\alpha} \\
&\times \left(\frac{1}{\alpha}\int_{-\infty}^{+\infty} \left|\int_{\mathbf{S}^n} (\varphi_k(1+it) I_{-it} h_k(\zeta))\, g(\zeta)\, d\sigma(\zeta)\right| \mathcal{P}_1(\alpha, t) dt\right)^{\alpha},
\end{aligned}
$$

where if $m = 0, 1$,

$$
\mathcal{P}_m(s+it, \tau) = \frac{e^{-\pi(\tau-t)} \sin \pi s}{\sin^2 \pi s + (\cos \pi s - e^{im\pi - \pi(\tau-t)})^2}.
$$

Taking supremum on g in the previous estimate we get

$$
\begin{aligned}
(5.7)\qquad & \|\sum_k \varphi_k(\alpha) I_1 * h_k\|_{I_\alpha * L^p_{\Omega_0 k}} \\
(5.8)\qquad & \leq \left(\frac{1}{1-\alpha}\int_{-\infty}^{+\infty} \|\sum_k \varphi_k(it) I_{1-it} * h_k\|_p \mathcal{P}_0(\alpha, t)\, dt\right)^{1-\alpha} \\
& \times \left(\frac{1}{\alpha}\int_{-\infty}^{+\infty} \|\sum_k \varphi_k(1+it) I_{-it} * h_k\|_p \mathcal{P}_1(\alpha, t)\, dt\right)^{\alpha}.
\end{aligned}
$$

But we have seen in the previous remark that $\|I_{1-it} * h\|_p \leq e^{\gamma|t|}\|I_1 * h\|_p$ and the definition of I_{-it} shows that $\|I_{-it}(h)\|_p \leq e^{\gamma|t|}\|h\|_p$. Thus (5.8) can be estimated by

$$
\left(\frac{1}{1-\alpha}\int_{-\infty}^{+\infty} e^{\gamma|t|}\|\sum_k \varphi_k(it) I_1 * h_k\|_p \mathcal{P}_0(\alpha, t)\, dt\right)^{1-\alpha} \times
$$

$$\times \left(\frac{1}{\alpha} \int_{-\infty}^{+\infty} e^{\gamma|t|} || \sum_k \varphi_k(1+it) h_k ||_p \mathcal{P}_1(\alpha, t)\, dt \right)^{\alpha}$$

$$\leq \sup_t \left(e^{\gamma|t|} || \sum_k \varphi_k(it) I_1 * h_k ||_p \right)^{1-\alpha} \times \sup_t \left(e^{\gamma|t|} || \sum_k \varphi_k(1+it) h_k ||_p \right)^{\alpha} \leq$$

$$\max \left(\sup_t e^{\gamma|t|} || \sum_k \varphi_k(it) I_1 * h_k ||_p, \sup_t e^{\gamma|t|} || \sum_k \varphi_k(1+it) h_k ||_p \right),$$

where we have used that

$$\frac{1}{1-\alpha} \int_{-\infty}^{+\infty} \mathcal{P}_0(\alpha, t)\, dt = \frac{1}{\alpha} \int_{-\infty}^{+\infty} \mathcal{P}_1(\alpha, t)\, dt = 1. \quad \blacksquare$$

Theorem 5.3 *Let $j \in \mathbf{Z}_+$, $0 \leq j \leq n-1$, $0 < \alpha < 1$ and let k be a nonnegative integer. We then have that*

$$[L^p_{j\Omega_0 k}, L^p_{j+1\Omega_0 k}]_{[\alpha]} = I_{\alpha+j} * L^p_{\Omega_0 k}.$$

Proof of theorem 5.3:

If $j \leq 1$, $I_j : L^p_{\Omega_{s,s}} \to L^p_{j\,\Omega_{s,s}}$, and $I_j : L^p_{1\,\Omega_{s,s}} \to L^p_{j+1\,\Omega_{s,s}}$, are topological isomorphisms and consequently

$$I_j : [L^p_{\Omega_{s,s}}, L^p_{1\,\Omega_{s,s}}]_{[\alpha]} \to [L^p_{j\,\Omega_{s,s}}, L^p_{j+1\,\Omega_{s,s}}]_{[\alpha]},$$

is also a topological isomorphism. But theorem 5.1 gives $[L^p_{\Omega_{s,s}}, L^p_{1\,\Omega_{s,s}}]_{[\alpha]} = I_\alpha * L^p_{\Omega_{s,s}}$. Hence $I_j * (I_\alpha * L^p_{\Omega_{s,s}}) = [L^p_{j\,\Omega_{s,s}}, L^p_{j+1\,\Omega_{s,s}}]_{[\alpha]}$. Now, part (iii) of lemma 5.2 finishes the proof. $\blacksquare$

References

[**AhBr**] P. Ahern and J. Bruna, *Maximal and area integral characterizations of Hardy-Sobolev spaces in the unit ball of* $\mathbf{C}^n$, Revista Iberoamericana **4** (1988), 123–153.

[**AhCa**] P. Ahern and C. Cascante, *Exceptional sets for Poisson integrals of potentials on the unit sphere in* $\mathbf{C}^n$, $p \leq 1$, Pacific J. of Math. **153**, (1992), 1–13.

[**Al**] A.B. Aleksandrov, "Several Complex Variables II", Encyclopaedia of Mathematical Sciences **8**, G.M. Khenkin, A.G. Vitushkin (Eds.), Springer-Verlag 1991.

[**BeLo**] J. Bergh and J. Löfstrom, "Interpolation spaces. An introduction", Springer-Verlag 1976.

[**BoCl**] A. Bonami and J-L. Clerc, *Sommes de Cesàro et multiplicateurs des développements en harmoniques sphériques*, Trans. Amer. Math. Soc. **183** (1973), 223–263.

[**BrCa**] J. Bruna and C. Cascante, *Restriction to transverse curves of some spaces of functions in the unit ball*, Michigan Math. J. **36** (1989), 387–401.

[**BrOr**] J. Bruna and J.M. Ortega, *Closed finitely generated ideals in algebras of holomorphic functions and smooth to the boundary in strictly pseudoconvex domains*, Math. Annalen **268** (1984), 137–157.

[**CaOr**] C. Cascante and J.M. Ortega, *A characterisation of tangential exceptional sets for H^p_α, $\alpha p = n$*, Proc. of the Royal Soc. of Edinburgh, **126** (1996), 625–641.

[**Ch**] E. M. Chirka. *The Lindelöf and Fatou theorems in* $\mathbf{C}^n$, Math. Sb, **92** (1973), 619–641.

[**Do**] E. Doubtsov, *Thesis*, Universite Bordeaux I, (1995).

[**GaRu**] J. Garcia-Cuerva and J. L. Rubio de Francia, "Weighted norm Inequalities and related topics", North-Holland, 1985.

[**Ge**] D. Geller, *Some results on H^p theory for the Heisenberg group*, Duke Math. J., **47** (1980), 365–390.

[**Li**] E. Lindelöf, *Sur un principe générale de l'analyse et ses applications à la théorie de la représentation conforme* , Acta Soc. Sci. Fennicae, **46** (1915), 1–35.

[**NaRu**] A. Nagel and W. Rudin. *Local boundary behavior of bounded holomorphic functions*. Canad. J. Math. **30** (1978), 583–592.

[**NaWa**] A. Nagel and S. Wainger, *Limits of bounded holomorphic functions alomg curves*, Recent Developments in Several Complex Variables (J.E. Fornaess, ed.) Princeton Univ. Press, 1981.

[**NaRoStWa**] A. Nagel, J.P. Rosay, E.M. Stein and S. Wainger, *Estimates for the Bergman and Szegö kernels in* $\mathbf{C}^2$, Annals of Math. **129** (1989), 113–149.

[**NaRuSh**] A. Nagel, W. Rudin and J.H. Shapiro, *Tangential boundary behavior of functions in Dirichlet-type spaces*, Ann. of Math. **116** (1982), 331–360.

[**Ru**] W. Rudin, "Function theory in the unit ball", Springer Verlag, 1980.

[**St**] J. D. Stafney, *The spectrum of an operator on an interpolation space* Trans. Am. Math. Soc. **144**, (1969), 333–349.

[**Su**] J. Sueiro, *Tangential boundary limits and exceptional sets for holomorphic functions in Dirichlet-type spaces*, Math. Annal. **286**, (1990), 661–678.

[**TrEr**] F. G. Tricomi and A. Erdelyi, *The assymptotic expansion of a ratio of Gamma functions* , Pac. J. of Math. **1**, (1951), 133–142.

[**Zy**] A. Zygmund, *On a theorem of Littlewood* Summa Brasil. Math. 2, (1949), 1–7.

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