UNIVERSITAT DE BARCELONA

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Carles Rovira and *Marta Sanz-Solé*

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CARLES ROVIRA and MARTA SANZ–SOLÉ

Facultat de Matemàtiques
Universitat de Barcelona
Gran Via, 585
08007 BARCELONA, Spain

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1. INTRODUCTION

Let $T = [0,1] \times [0,1]$ and $\{W_z, z \in T\}$ be a Brownian sheet on T. We consider the stochastic Volterra equation on the plane

$$X_z = H_z + \int_{R_z} f(z;\eta;X_\eta)dW_\eta + \int_{R_z} b(z;\eta;X_\eta)d\eta, \tag{1.1}$$

where the functions $f = f(z;\eta;x)$ and $b = b(z;\eta;x)$ are Borel functions of $(z,\eta,x) \in T \times T \times \mathbb{R}$, $H : T \to \mathbb{R}$ and if $z = (s,t) \in T$ then $R_z := [0,s] \times [0,t]$. In the one-parameter case stochastic Volterra equations have been studied for instance in [B-M 1], [B-M 2] and [P].

Consider the stochastic partial differential equation

$$LX_{s,t} = f(X_{s,t})\dot{W}_{s,t} + b(X_{s,t}), \qquad (s,t) \in T, \tag{1.2}$$

where L is some hyperbolic second order differential operator, $(\dot{W}_{s,t})$ is white noise in the plane and the value of $X_{s,t}$ on the axis is some deterministic given function. One possibility to give a meaning to the formal equation (1.2) is to use Riemann's method, that means, if $\gamma_z(\eta), 0 \le \eta \le z \le (1,1)$, is the Green function associated to L, then a solution to (1.2) is a stochastic process $\{X_{s,t}, (s,t) \in T\}$ satisfying

$$X_z = X_{0,t} + X_{s,0} - X_{0,0} + \int_{R_z} \gamma_z(\eta)[f(X_\eta)dW_\eta + b(X_\eta)d\eta]. \tag{1.3}$$

In [R-S] we have studied the problem for the operator defined by

$$Lg(s,t) = \frac{\partial^2}{\partial s \partial t} g(s,t) - a_1(s,t)\frac{\partial g}{\partial t}(s,t) - a_2(s,t)\frac{\partial g}{\partial s}(s,t). \tag{1.4}$$

As has been shown by Norris in [N] these type of equations appear in the construction of path-valued processes in Riemannian manifolds. This is one of the motivations for the analysis of stochastic differential equations like (1.1), which include as particular examples (1.3).

The existence and uniqueness of solution to (1.2) can be stated using the usual Picard's method, assuming that the coefficients f and b are Lipschitz in the variable x and have linear growth, uniformly in z and η.

Consider $E = (\{0\} \times [0,1]) \cup ([0,1] \times \{0\})$. Our aim is to obtain sufficient conditions for the existence of a smooth density for the probability law of the solution X_z to the equation (1.1), at fixed $z \in T \setminus E$. Our theorem will cover the results obtained in [R-S] and [M-S] for the solution to the nonlinear hyperbolic SPDE (1.2) with L given by (1.4).

The paper is organized as follows. In Section 2 we state the rigorous formulation of the hypotheses and we state the main theorem. In Section 3 we present the proof of the theorem. Finally, in Section 4 we include some estimates involving the Brownian motion.

Along the paper we will use the following notation: $z = (s,t), \eta = (u,v), \alpha = (r,e)$, and $\mu = (w,y)$, $z,\eta,\alpha,\mu \in T$. If $z,\eta \in T$ then $z \otimes \eta := (s,v)$.

2. FORMULATION OF THE PROBLEM

We need the following set of hypotheses.

(H1) f and b are measurable, infinitely differentiable with respect to x with uniformly bounded derivatives of any order.
(H2) There exists a constant C such that

$$|f(z;\eta;x) - f(z';\eta';x')| \leq C(|z-z'| + |\eta - \eta'| + |x - x'|),$$
$$|\partial_x f(z;\eta;x) - \partial_x f(z;\eta';x')| \leq C(|\eta - \eta'| + |x - x'|),$$

for all $z, z', \eta, \eta' \in T$, $x, x' \in \mathbb{R}$.
(H3) f and H are two times differentiable with respect to u with bounded derivatives of second order. That is, there exists a constant C such that

$$|\partial_u^2 f(z;\eta;x)| + |\partial_u^2 H_\eta| \leq C,$$

for any $z, \eta \in T$, $x \in \mathbb{R}$. Moreover

$$|\partial_u f(z;\eta;x) - \partial_u f(z;\eta;x')| \leq C|x - x'|,$$

for any $z, \eta \in T$, $x, x' \in \mathbb{R}$.
($H4_n$) f is infinitely differentiable with respect to x and n times differentiable with respect to u with bounded partial derivatives up to order n.

We introduce some conditions on the coefficients. Fix $z \in T \setminus E$,
(C1) $f(z;0\otimes z;H_{0\otimes z}) \neq 0$,
(C2) $\partial_x f(z;0\otimes z;H_{0\otimes z}) \neq 0$,
(C3) there exists $v \in (0,t)$ such that $f(0\otimes z;(0,v);H_{(0,v)}) \neq 0$,
(C4) $\int_0^t \left(\int_y^t \partial_x f(0\otimes z;(0,v);H_{(0,v)}) f((0,v);(0,y);H_{(0,y)}) dv\right)^2 dy \neq 0$,
(C5) $\partial_u f(z;0\otimes z;H_{0\otimes z}) \neq 0$,
(C6) for all $v, y \in (0,t)$, $v \geq y$, $f((0,v);(0,y);H_{(0,y)}) = 0$,
($C7_n$) for all j, l with $j + l = k$, $k \in \{0, \cdots, n-1\}$, $\partial^k_{x^j u^l} f(z;0\otimes z;H_{0\otimes z}) = 0$,
($C8_n$) $\sum_{j+l=n} \frac{1}{j!\,l!} \partial^k_{x^j u^l} f(z;0\otimes z;H_{0\otimes z}) \left(\int_0^t b(0\otimes z;(0,v);H_{(0,v)}) dv + \partial_u H_{0\otimes z}\right)^j \neq 0$.
Notice that ($C7_n$) implies not(C1) , (C6) yields not(C4) and ($C8_1$) reduces to $\partial_x f(z;0\otimes z;H_{0\otimes z}) \left(\int_0^t b(0\otimes z;(0,v);H_{(0,v)}) dv + \partial_u H_{0\otimes z}\right) + \partial_u f(z;0\otimes z;H_{0\otimes z}) \neq 0$.

Then, we will consider the following sets of hypotheses:

(h1): (H1) and (C1)
(h2): (H1), (H2) and not(C1), (C2) and (C3)
(h3): (H1), (H2) and (H3) and not(C1), (C2), not(C3) and (C4)
(h4): (H1), (H2) and (H3) and not(C1), not(C2), not(C3) and (C5)
(h5): for some $n \geq 0$, (H1), ($H4_n$) and (C6), ($C7_n$) and ($C8_n$).

The main result reads as follows.

Theorem 2.1. *Suppose that (H1) is satisfied and that H_z is a Lipschitz deterministic function. Fix $z \in T \setminus E$ and assume that one of the assumptions (h1)-(h5) hold. Then, the law of X_z is absolutely continuous with respect to Lebesgue's measure on $\mathbb{R}$ and its density is infinitely differentiable.*

Remark. *The conclusion of this theorem can be obtained by exchanging the roles of t and s, v and u, and y and w, in all the hypotheses.*

3. PROOF OF THE RESULT

The proof of Theorem 2.1 relies upon a sequel of lemmas and propositions. We recall first a technical lemma from [R-S].

Lemma 3.1. *Let $\{F_n, n \geq 1\}$ be a sequence of random variables in $\mathbb{D}^{N,p}$, $N \geq 1, p \in [2, \infty)$. Assume there exists $F \in \mathbb{D}^{N-1,p}$ such that $\{D^{N-1}F_n, n \geq 1\}$ converges to $D^{N-1}F$ in $L^p(\Omega; L^2(T^{N-1}))$ as n goes to infinity and, moreover, the sequence $\{D^N F_n, n \geq 1\}$ is bounded in $L^p(\Omega; L^2(T^N))$. Then, $F \in \mathbb{D}^{N,p}$.*

Proposition 3.2. *Assume (H1). Then X_z belongs to $\mathbb{D}^\infty$ and for $0 \leq \alpha \leq z$,*

$$D_\alpha X_z = f(z;\alpha;X_\alpha) + \int_{(\alpha,z]} \partial_x f(z;\eta;X_\eta)\, D_\alpha X_\eta dW_\eta + \int_{(\alpha,z]} \partial_x b(z;\eta;X_\eta)\, D_\alpha X_\eta d\eta. \quad (3.1)$$

Proof. Consider the Picard approximations

$$X_z^0 = H_z,$$
$$X_z^{n+1} = H_z + \int_{R_z} f(z;\eta;X_\eta^n)dW_\eta + \int_{R_z} b(z;\eta;X_\eta^n)d\eta, \qquad n \geq 0. \quad (3.2)$$

In order to deal with the derivatives of X_z^n and X_z of any order we introduce some notation inspired in Leibniz derivation rule. Let $\gamma = (\gamma_1, \cdots, \gamma_N) \in T^N$; we denote by $|\gamma|$ the length of γ, that means N. Set $\widehat{\gamma}_i = (\gamma_1, \cdots, \gamma_{i-1}, \gamma_{i+1}, \cdots, \gamma_N)$, $i = 1, \cdots, N$. For a random variable $Y \in \mathbb{D}^{N,p}$, we denote by $D_\gamma^N Y$ the iterative derivative $D_{\gamma_N} D_{\gamma_{N-1}} \cdots D_{\gamma_1} Y$. Let $f \in C_b^{0,\infty}(T \times T \times \mathbb{R})$, the space of continuous functions defined on $T \times T \times \mathbb{R}$, infinitely differentiable with respect to the last variable, with bounded derivatives. Set

$$\Gamma_\gamma(f;z;\eta;X_\eta) = \sum_{m=1}^{N} \sum \partial_x^m f(z;\eta;X_\eta) \prod_{i=1}^{m} D_{p_i}^{(p_i)} X_\eta, \quad (3.3)$$

where the second sum extends to all partitions $p_1, \cdots, p_m$ of length m of γ. $\Gamma_\gamma(f;z;\eta;X_\eta^n)$ is defined in an obvious manner.

We will check, for any $p \geq 2$, by induction on N, the following set of hypotheses $(H^{(N)})$

(a) $\{X_z^n, n \geq 0\} \subset \mathbb{D}^{N,p}$,

(b) $D^{N-1}X_z^n \longrightarrow D^{N-1}X_z$ in $L^p(\Omega, L^2(T^{N-1}))$ when $n \to \infty$,

(c) $\sup_n \sup_{z \in T} \sup_{|\gamma|=N} E(|D_\gamma^N X_z^n|^p) < \infty$.

Notice that by Lemma 3.1, $(H^{(N)})$ yields $X_z \in \mathbb{D}^{N,p}$.
Let $N = 1$. It is easy to prove the convergence of $\{X_z^n, n \geq 0\}$ to X_z in $L^p(\Omega)$. The fact that for every $n \geq 0$, $X_z^n \in \mathbb{D}^{1,p}$ is checked by induction on n, using the stochastic rules for Malliavin derivatives, (3.2) and the properties of the coefficients. Moreover, the derivatives satisfy the equations

$$\begin{aligned} D_\gamma X_z^0 &= 0, \\ D_\gamma X_z^{n+1} &= f(z;\gamma;X_\gamma^n) + \int_{(\gamma,z]} D_\gamma X_\eta^n [\partial_x f(z;\eta;X_\eta^n) dW_\eta + \partial_x b(z;\eta;X_\eta^n) d\eta], \end{aligned} \tag{3.4}$$

$0 \leq \gamma \leq z$. Hence, there exists a constant C_p such that

$$\begin{aligned} E(|D_\gamma X_z^1|^p) &\leq C_p, \\ E(|D_\gamma X_z^{n+1}|^p) &\leq C_p(1 + \int_{(\gamma,z]} E(|D_\gamma X_\eta^n|^p) d\eta), \end{aligned}$$

for all $n \geq 1, z \in T$. Consequently condition (c) holds. Taking $n \to \infty$ in (3.4) we obtain (3.1).
Suppose now that $(H^{(k)})$, $1 \leq k \leq N-1$, holds. The proof of (a) in $(H^{(N)})$ can be done using induction on n. Moreover, using (3.3) one can write

$$\begin{aligned} D_\gamma^N X_z^0 &= 0, \\ D_\gamma^N X_z^{n+1} &= \sum_{i=1}^N \Gamma_{\widehat{\gamma_i}}(f;z;\gamma_i;X_{\gamma_i}^n) + \int_{(\sup\gamma,z]} [\Gamma_\gamma(f;z;\eta;X_\eta^n) dW_\eta + \Gamma_\gamma(b;z;\eta;X_\eta^n) d\eta], \end{aligned} \tag{3.5}$$

$n \geq 0$, where $\sup\gamma := \gamma_1 \vee \cdots \vee \gamma_N$.
The convergence (b) of $(H^{(N)})$ can be checked taking into account that $D^{N-1}X_z^n$, $n \geq 1$ and $D^{N-1}X_z$ satisfy equations of the same type than (3.5).
Set, for $|\gamma| = N$

$$\Delta_\gamma(f;z;\eta;X_\eta^n) := \Gamma_\gamma(f;z;\eta;X_\eta^n) - \partial_x^N f(z;\eta;X_\eta^n) D_\gamma^N X_\eta^n.$$

Then for any $n \geq 0$ we can write (3.5) as

$$\begin{aligned} D_\gamma^N X_z^{n+1} = &\sum_{i=1}^N \Gamma_{\widehat{\gamma_i}}(f;z;\gamma_i;X_{\gamma_i}^n) + \int_{(\sup\gamma,z]} [\Delta_\gamma(f;z;\eta;X_\eta^n) dW_\eta + \Delta_\gamma(b;z;\eta;X_\eta^n) d\eta] \\ &+ \int_{(\sup\gamma,z]} D_\gamma^N X_\eta^n [\partial_x^N f(z;\eta;X_\eta^n) dW_\eta + \partial_x^N(z;\eta;X_\eta^n) d\eta]. \end{aligned} \tag{3.6}$$

Notice that in the first two terms of the right-hand side of (3.6) only derivatives of X_z^n of order less or equal to $N-1$ do appear. Hence, condition (c) in $(H^{(k)})$, $1 \leq k \leq N-1$, implies the existence of a constant C_p such that

$$\begin{aligned} E(|D_\gamma^N X_z^1|^p) &\leq C_p, \\ E(|D_\gamma^N X_z^{n+1}|^p) &\leq C_p(1 + \int_{(\sup\gamma,z]} E(|D_\gamma^N X_\eta^n|^p) d\eta), \end{aligned}$$

for all $n \geq 1, z \in T$ and $|\gamma| = N$. This proves property (c) of the set of assumptions $(H^{(N)})$. □

We introduce some new notation. Let $\{Y_z(\alpha) : 0 \leq \alpha \leq z \leq (1,1)\}$ be the solution to

$$Y_z(\alpha) = 1 + \int_{(\alpha,z]} \partial_x f(z;\eta;X_\eta) Y_\eta(\alpha) dW_\eta + \int_{(\alpha,z]} \partial_x b(z;\eta;X_\eta) Y_\eta(\alpha) d\eta. \tag{3.7}$$

Fix $\varepsilon, \beta, \delta \in (0,1)$, $z \in T$, we define the sets $C^z_{\beta,\delta}(\varepsilon) = (0, \varepsilon^\beta) \times (t - \varepsilon^\delta, t)$ and $G^z_\beta(\varepsilon) = (0, \varepsilon^\beta) \times (0, t)$.

Using Burkholder's and Hölder's inequalities and Gronwall's lemma one easily gets the following.

Lemma 3.3. *Assume (H1). Then, for any $q \geq 1$ there exists a constant C_q such that for any $z \in T$,*

$$\sup_{\alpha \leq z} E(|Y_z(\alpha)|^{2q}) + E(|X_z|^{2q}) \leq C_q, \tag{3.8}$$

$$\sup_{\alpha \in C^z_{\beta,\delta}(\varepsilon)} E(|Y_z(\alpha) - 1|^{2q}) \leq C_q \varepsilon^{\delta q}, \tag{3.9}$$

$$\sup_{\alpha \in G^z_\beta(\varepsilon)} E(|X_\alpha - H_\alpha|^{2q}) \leq C_q \varepsilon^{\beta q}. \tag{3.10}$$

Notice that, unlike for ordinary stochastic differential equations, here $Y_z(\alpha) f(z;\alpha;X_\alpha) \neq D_\alpha X_z$ in general.

The following lemma will be useful to control the difference between the two terms.

Lemma 3.4. *Assume (H1). Then, for any $q \geq 1$ there exists a constant C_q such that for any $z \in T$,*

$$\sup_{\alpha \in C^z_{\beta,\delta}(\varepsilon)} E(|D_\alpha X_z - f(z;\alpha;X_\alpha) Y_z(\alpha)|^{2q}) \leq C_q \varepsilon^{\delta q}. \tag{3.11}$$

Proof. Burkholder's and Hölder's inequalities, (3.7) and the estimate (3.8) yield

$$\begin{aligned} &E(|D_\alpha X_z - f(z;\alpha;X_\alpha) Y_z(\alpha)|^{2q}) \\ &\leq C_q \Big(\varepsilon^{\delta(q-1)} \int_{(\alpha,z]} E(|D_\alpha X_\eta - f(z;\alpha;X_\alpha) Y_\eta(\alpha)|^{2q}) d\eta \Big) \\ &\leq C_q \varepsilon^{\delta(q-1)} \Big(\varepsilon^\delta + \int_{(\alpha,z]} E(|D_\alpha X_\eta - f(\eta;\alpha;X_\alpha) Y_\eta(\alpha)|^{2q}) d\eta \Big). \end{aligned}$$

Thus, (3.11) follows from Gronwall's lemma. □

Lemma 3.5. *Assume (H1).*

a) Suppose that (C3) does not hold. Then, for any $q \geq 1$ there exists a constant C_q such that

$$\sup_{\alpha \in C^z_{\beta,\delta}(\varepsilon)} E(|X_\alpha - H_\alpha|^{2q}) \leq C_q (\varepsilon^{2\beta q} + \varepsilon^{(\beta + 2\delta) q}). \tag{3.12}$$

b) Suppose that (C6) holds. Then, for any $q \geq 1$ there exists a constant C_q such that

$$\sup_{\alpha \in G^z_\beta(\varepsilon)} E(|X_\alpha - H_\alpha|^{2q}) \leq C_q \varepsilon^{2\beta q}. \tag{3.13}$$

Proof. Introducing $\int_{R_\alpha} f(0 \otimes z; 0 \otimes \eta; H_{0\otimes\eta}) dW_\eta = 0$, from Burkholder's and Hölder's inequalities we obtain,

$$E(|X_\alpha - H_\alpha|^{2q}) \leq C_q \Big(E(| \int_{R_\alpha} (f(\alpha;\eta;X_\eta) - f(0 \otimes z; 0 \otimes \eta; H_{0\otimes\eta})) dW_\eta|^{2q})$$

$$+ E(| \int_{R_\alpha} b(\alpha;\eta;X_\eta) d\eta|^{2q}) \Big) \leq C_q \big(\varepsilon^{2\beta q} + \varepsilon^{(\beta+2\delta)q} + \varepsilon^{\beta(q-1)} \int_{R_\alpha} E(|X_\eta - H_\eta|^{2q}) d\eta \big).$$

Then, (3.10) yields (3.12). The inequality (3.13) can be checked using similar calculations.

□

Proposition 3.6. *Suppose that (H1) is satisfied and that H is a Lipschitz function. Fix $z \in T \setminus E$ and assume that one of the conditions (h1) to (h5) hold. Then, for any $p \geq 1$, $\left(\int_{R_z} |D_\alpha X_z|^2 d\alpha \right)^{-1} \in L^p$.*

Proof. It suffices to show that

$$P(\varepsilon) := P(\int_{R_z} |D_\alpha X_z|^2 d\alpha \leq \varepsilon) \leq \varepsilon^p,$$

$p \geq 1$, for any $\varepsilon \leq \varepsilon_0$, where ε_0 depends on p, z and the coefficients f and b.
Fix $\varepsilon, \beta, \delta > 0$ such that $\varepsilon^\beta < s, \varepsilon^\delta < t$. We have $P(\varepsilon) \leq P_1(\beta,\delta,\varepsilon) + P_2(\beta,\delta,\varepsilon)$, with

$$P_1(\beta,\delta,\varepsilon) := P(\int_{C^z_{\beta,\delta}(\varepsilon)} |D_\alpha X_z - f(z;\alpha \otimes z; X_\alpha)|^2 d\alpha > \varepsilon),$$

$$P_2(\beta,\delta,\varepsilon) := P(\int_{C^z_{\beta,\delta}(\varepsilon)} |f(z;\alpha \otimes z; X_\alpha)|^2 d\alpha \leq 4\varepsilon).$$

Chebychev's inequality and Lemma 3.4 yield

$$P_1(\beta,\delta,\varepsilon) \leq C_q \varepsilon^{(\beta+\delta-1)q} (\varepsilon^{\delta q} + \sup_{\alpha \in C^z_{\beta,\delta}(\varepsilon)} E(|f(z;\alpha;X_\alpha)Y_z(\alpha) - f(z;\alpha \otimes z; X_\alpha)|^{2q})).$$

Then, using the Lipschitz property of the coefficient f, (3.8) and (3.9), we obtain

$$P_1(\beta,\delta,\varepsilon) \leq C_q \varepsilon^{(\beta+2\delta-1)q}. \tag{3.14}$$

We will now study $P_2(\beta,\delta,\varepsilon)$ under the different sets of hypotheses.

Assume (h1). We can write $P_2(\beta,\delta,\varepsilon) \leq P^{(1)}_{21}(\beta,\delta,\varepsilon) + P^{(1)}_{22}(\beta,\delta,\varepsilon)$, with

$$P^{(1)}_{21}(\beta,\delta,\varepsilon) := P(\int_{C^z_{\beta,\delta}(\varepsilon)} |f(z;\alpha \otimes z; X_\alpha) - f(z; 0 \otimes z; H_{0\otimes z})|^2 d\alpha > 4\varepsilon),$$

$$P^{(1)}_{22}(\beta,\delta,\varepsilon) := P(|f(z;0 \otimes z; H_{0\otimes z})|^2 \leq 16\varepsilon^{(1-\beta-\delta)}).$$

From Chebychev's inequality, the properties of f and (3.10), we have that for any $q \geq 1$,

$$P_{21}^{(1)}(\beta, \delta, \varepsilon) \leq C_q(\varepsilon^{(2\beta+\delta-1)q} + \varepsilon^{(\beta+2\delta-1)q}).$$

Choose $\beta, \delta \in (0,1)$ such that $\beta+\delta < 1$, $2\beta+\delta > 1$ and $\beta+2\delta > 1$ (for instance $\beta = \frac{1}{3}$ and $\delta = \frac{1}{2}$). Then, (C1) implies $P_{22}^{(1)}(\beta, \delta, \varepsilon) = 0$ for ε small enough and the result is proved.

Assume (h2). Using a Taylor expansion, (1.1) and not(C1), one has

$$f(z; \alpha \otimes z; X_\alpha) = \sum_{i=1}^{6} T_i^{(2)}(z; \alpha),$$

with

$$T_1^{(2)}(z;\alpha) := f(z; \alpha \otimes z; H_{0\otimes\alpha}),$$
$$T_2^{(2)}(z;\alpha) := \frac{1}{2}\partial_x^2 f(z; \alpha \otimes z; H^*)(X_\alpha - H_{0\otimes\alpha})^2,$$
$$T_3^{(2)}(z;\alpha) := \partial_x f(z; \alpha \otimes z; H_{0\otimes\alpha})\Big(\int_{R_\alpha} b(\alpha;\eta;X_\eta)d\eta + H_\alpha - H_{0\otimes\alpha}\Big),$$
$$T_4^{(2)}(z;\alpha) := \big(\partial_x f(z; \alpha \otimes z; H_{0\otimes\alpha}) - \partial_x f(z; 0 \otimes z; H_{0\otimes z})\big)\int_{R_\alpha} f(\alpha;\eta;X_\eta)dW_\eta,$$
$$T_5^{(2)}(z;\alpha) := \partial_x f(z; 0 \otimes z; H_{0\otimes z})\Big(\int_{R_\alpha} f(\alpha;\eta;X_\eta)dW_\eta - \int_{R_{\alpha\otimes z}} f(0\otimes z; 0\otimes\eta; H_{0\otimes\eta})dW_\eta\Big),$$
$$T_6^{(2)}(z;\alpha) := \partial_x f(z; 0 \otimes z; H_{0\otimes z})\int_{R_{\alpha\otimes z}} f(0\otimes z; 0\otimes\eta; H_{0\otimes\eta})dW_\eta,$$

where H^* denotes a random point between X_α and $H_{0\otimes\alpha}$.
Then $P_2(\beta,\delta,\varepsilon) \leq P_{21}^{(2)}(\beta,\delta,\varepsilon) + P_{22}^{(2)}(\beta,\delta,\varepsilon)$, with

$$P_{21}^{(2)}(\beta,\delta,\varepsilon) := P\Big(\int_{C_{\beta,\delta}^z(\varepsilon)} \Big|\sum_{i=1}^{5} T_i^{(2)}(z;\alpha)\Big|^2 d\alpha > 4\varepsilon\Big),$$
$$P_{22}^{(2)}(\beta,\delta,\varepsilon) := P\Big(\int_{C_{\beta,\delta}^z(\varepsilon)} |T_6^{(2)}(z;\alpha)|^2 d\alpha \leq 16\varepsilon\Big).$$

Using (3.8), (3.10) and Chebychev's, Burkholder's and Hölder's inequalities, we prove for any $q \geq 1$,

$$P_{21}^{(2)}(\beta,\delta,\varepsilon) \leq C_q(\varepsilon^{(\beta+2\delta-1)q} + \varepsilon^{(3\beta+\delta-1)q}). \tag{3.15}$$

Set $K_1 := (\partial_x f(z; 0\otimes z; H_{0\otimes z}))^2 > 0$ and $K_2 := \int_0^t |f(0\otimes z; (0,v); H_{(0,v)})|^2 dv > 0$. Define

$$M_r^{(1)} := \int_{R_{r,t}} f(0\otimes z; 0\otimes\eta; H_{0\otimes\eta})dW_\eta.$$

Then, $\{M_r^{(1)}, r \geq 0\}$ is a martingale with $< M^{(1)} >_r = rK_2$. So, there exists a Brownian motion $Z^{(1)}$, such that $M_r^{(1)} = Z^{(1)}_{<M^{(1)}>_r}$. Thus,

$$P_{22}^{(2)}(\beta,\delta,\varepsilon) = P\Big(\int_0^{\varepsilon^\beta} |Z^{(1)}_{rK_2}|^2 dr \leq \frac{16}{K_1}\varepsilon^{1-\delta}\Big).$$

Define now $Z_r^{(2)} := K_2^{-\frac{1}{2}}\varepsilon^{-\frac{\beta}{2}} Z^{(1)}_{rK_2\varepsilon^\beta}$. Then $\{Z_r^{(2)}, r \geq 0\}$ is also a Brownian motion and

$$P_{22}^{(2)}(\beta,\delta,\varepsilon) = P\Big(\int_0^1 |Z_r^{(2)}|^2 dr \leq \frac{16}{K_1K_2}\varepsilon^{1-\delta-2\beta}\Big). \tag{3.16}$$

Then, from (3.14), (3.15) and (3.16), choosing $\delta, \beta \in (0,1)$ such that $2\beta+\delta < 1$, $3\beta+\delta > 1$ and $\beta + 2\delta > 1$ (for instance $\beta = \frac{1}{5}$ and $\delta = \frac{1}{2}$) and using Lemma 4.1 the proof is finished.

Assume (h3). Consider the Taylor expansions

$$f(z;\alpha\otimes z;X_\alpha) = f(z;\alpha\otimes z;H_{0\otimes\alpha}) + \partial_x f(z;\alpha\otimes z;H_{0\otimes\alpha})(X_\alpha - H_{0\otimes\alpha}) + \frac{1}{2}\partial_x^2 f(z;\alpha\otimes z;H^*)(X_\alpha - H_{0\otimes\alpha})^2,$$

$$f(z;\alpha\otimes z;H_{0\otimes\alpha}) = f(z;0\otimes z;H_{0\otimes\alpha}) + r\partial_u f(z;0\otimes z;H_{0\otimes\alpha}) + \frac{1}{2}r^2\partial_u^2 f(z;\eta^*;H_{0\otimes\alpha}),$$

$$H_\alpha = H_{0\otimes\alpha} + r\partial_u H_{0\otimes\alpha} + \frac{1}{2}r^2\partial_u^2 H_{u^*},$$

with H^* some random point between X_α and $H_{0\otimes\alpha}$, $\eta^* \in (0\otimes z, \alpha\otimes z)$, u^* in the segment joining $0\otimes\alpha$ and α. Then

$$f(z;\alpha\otimes z;X_\alpha) = \sum_{i=1}^4 T_i^{(3)}(z;\alpha),$$

with

$$T_1^{(3)}(z;\alpha) := f(z;0\otimes z;H_{0\otimes\alpha}) - f(z;0\otimes z;H_{0\otimes z}),$$
$$T_2^{(3)}(z;\alpha) := \frac{1}{2}\partial_x^2 f(z;\alpha\otimes z;H^*)(X_\alpha - H_{0\otimes\alpha})^2,$$
$$T_3^{(3)}(z;\alpha) := \frac{1}{2}r^2\big(\partial_u^2 f(z;\eta^*;H_{0\otimes\alpha}) + \partial_x f(z;\alpha\otimes z;H_{0\otimes\alpha})\partial_u^2 H_{u^*}\big),$$
$$T_4^{(3)}(z;\alpha) := \partial_x f(z;\alpha\otimes z;H_{0\otimes z})\Big(\int_{R_\alpha} f(\alpha;\eta;X_\eta)dW_\eta + \int_{R_\alpha} b(\alpha;\eta;X_\eta)d\eta + r\partial_u H_{0\otimes\alpha}\Big) + r\partial_u f(z;0\otimes z;H_{0\otimes\alpha}).$$

Then $P_2(\beta,\delta,\varepsilon) \leq P_{21}^{(3)}(\beta,\delta,\varepsilon) + P_{22}^{(3)}(\beta,\delta,\varepsilon)$, with

$$P_{21}^{(3)}(\beta,\delta,\varepsilon) := P\Big(\int_{C^z_{\beta,\delta}(\varepsilon)} |\sum_{i=1}^3 T_i^{(3)}(z;\alpha)|^2 d\alpha > 4\varepsilon\Big),$$
$$P_{22}^{(3)}(\beta,\delta,\varepsilon) := P\Big(\int_{C^z_{\beta,\delta}(\varepsilon)} |T_4^{(3)}(z;\alpha)|^2 d\alpha \leq 16\varepsilon\Big).$$

Using Lemma 3.5 and similar calculations as for (3.15), we prove

$$P_{21}^{(3)}(\beta,\delta,\varepsilon) \le C_q(\varepsilon^{(\beta+3\delta-1)q} + \varepsilon^{(5\beta+\delta-1)q}), \quad q \ge 1. \tag{3.17}$$

Define

$$\begin{aligned} S_4^{(3)}(z;\alpha) &:= \partial_x f(z;0\otimes z;H_{0\otimes z})\Big(\int_{R_\alpha} f(0\otimes z;0\otimes\eta;X_\eta)dW_\eta \\ &+ \int_{R_{\alpha\otimes z}} b(0\otimes z;0\otimes\eta;H_{0\otimes\eta})d\eta + r\partial_u H_{0\otimes z}\Big) + r\partial_u f(z;0\otimes z;H_{0\otimes z}). \end{aligned}$$

We can write $P_{22}^{(3)}(\beta,\delta,\varepsilon) \le P_{221}^{(3)}(\beta,\delta,\varepsilon) + P_{222}^{(3)}(\beta,\delta,\varepsilon)$, with

$$\begin{aligned} P_{221}^{(3)}(\beta,\delta,\varepsilon) &:= P\Big(\int_{C^z_{\beta,\delta}(\varepsilon)} |T_4^{(3)}(z;\alpha) - S_4^{(3)}(z;\alpha)|^2 d\alpha > 16\varepsilon\Big), \\ P_{222}^{(3)}(\beta,\delta,\varepsilon) &:= P\Big(\int_{C^z_{\beta,\delta}(\varepsilon)} |S_4^{(3)}(z;\alpha)|^2 d\alpha \le 64\varepsilon\Big). \end{aligned}$$

It is easy to check that for any $q \ge 1$

$$P_{221}^{(3)}(\beta,\delta,\varepsilon) \le C_q(\varepsilon^{(\beta+3\delta-1)q} + \varepsilon^{(4\beta+\delta-1)q}). \tag{3.18}$$

Set $K_3 := \partial_x f(z;0\otimes z;H_{0\otimes z})\big(\int_0^t b(0\otimes z;(0,v);H_{(0,v)})dv + \partial_u H_{0\otimes z}\big) + \partial_u f(z;0\otimes z;H_{0\otimes z})$. Using again a Taylor expansion for $f(0\otimes z;0\otimes\eta;X_\eta)$ we have $S_4^{(3)}(z;\alpha) = S_{4,1}^{(3)}(z;\alpha) + S_{4,2}^{(3)}(z;\alpha)$ with

$$\begin{aligned} S_{4,1}^{(3)}(z;\alpha) &:= rK_3 + \partial_x f(z;0\otimes z;H_{0\otimes z}) \\ &\times \int_{R_\alpha} \big(\partial_x f(0\otimes z;0\otimes\eta;H_{0\otimes\eta}) \int_{R_\eta} f(\eta;\mu;X_\mu)dW_\mu\big)dW_\eta, \\ S_{4,2}^{(3)}(z;\alpha) &:= \partial_x f(z;0\otimes z;H_{0\otimes z})\Big(\int_{R_\alpha} \big[\partial_x f(0\otimes z;0\otimes\eta;H_{0\otimes\eta})\big(\int_{R_\eta} b(\eta;\mu;X_\mu)d\mu \\ &+ H_\eta - H_{0\otimes\eta}\big) + \frac{1}{2}\partial_x^2 f(0\otimes z;0\otimes\eta;H^*)(X_\eta - H_{0\otimes\eta})^2\big]dW_\eta\Big), \end{aligned}$$

where H^* is a point depending on the Taylor expansion.
Set

$$g(v,y) := \partial_x f(z;0\otimes z;H_{0\otimes z})\partial_x f(0\otimes z;(0,v);H_{(0,v)})f((0,v);(0,y);H_{(0,y)})$$

and

$$\Psi_r := rK_3 + \int_{R_{r,t}} S_{(u,v)}dW_{u,v}, \quad \text{with} \quad S_{(u,v)} := \int_{R_{u,v}} g(v,y)dW_{w,y}.$$

Then, $P^{(3)}_{222}(\beta,\delta,\varepsilon) \le P^{(3)}_{2221}(\beta,\delta,\varepsilon) + P^{(3)}_{2222}(\beta,\delta,\varepsilon)$, with

$$P^{(3)}_{2221}(\beta,\delta,\varepsilon) := P\Big(\int_{C^z_{\beta,\delta}(\varepsilon)} |S^{(3)}_{4,2}(z,\alpha) + S^{(3)}_{4,1}(z,\alpha) - \Psi_r|^2 d\alpha > 64\varepsilon\Big),$$

$$P^{(3)}_{2222}(\beta,\delta,\varepsilon) := P\Big(\int_0^{\varepsilon^\beta} |\Psi_r|^2 dr \le 256\,\varepsilon^{1-\delta}\Big).$$

It is not difficult to check that, for any $q \ge 1$

$$P^{(3)}_{2221}(\beta,\delta,\varepsilon) \le C_q \varepsilon^{-q}\Big(E\big(\big|\int_{C^z_{\beta,\delta}(\varepsilon)} |S^{(3)}_{4,2}(z,\alpha)|^2 d\alpha\big|^q\big) + E\big(\big|\int_{C^z_{\beta,\delta}(\varepsilon)} |S^{(3)}_{4,1}(z;\alpha) - \Psi_r|^2 d\alpha\big|^q\big)\Big) \le C_q(\varepsilon^{(2\beta+2\delta-1)q} + \varepsilon^{(4\beta+\delta-1)q}). \tag{3.19}$$

Now, putting together (3.14), (3.17), (3.18) and (3.19) the proof reduces to study $P^{(3)}_{2222}(\beta,\delta,\varepsilon)$ with $\delta,\beta \in (0,1)$ such that $\beta + 2\delta > 1$ and $4\beta + \delta > 1$ (for instance, $\delta = \frac{1}{2}$ and $\beta = \frac{1}{7}$). We will follow the method presented in Norris' lemma (see [Nu]). Set

$$A(\nu) := \{\omega;\ \sup_{0\le u\le \varepsilon^\beta} \int_0^t |S_{(u,v)}|^2 dv \le \varepsilon^\nu\},$$

$$N_r := \int_{R_{r,t}} \Psi_u S_{(u,v)} dW_{u,v},$$

$$M_r := \int_{R_{r,t}} S_{(u,v)} dW_{u,v}.$$

Then, $P^{(3)}_{2222}(\beta,\delta,\varepsilon) \le Q_1(\nu) + Q_2(\nu,q_1) + Q_3(q_1,q_2) + Q_4(q_2)$ with

$$Q_1(\nu) := P(A^c),$$

$$Q_2(\nu,q_1) := P\Big(A \cap \{\int_0^{\varepsilon^\beta} |\Psi_r|^2 dr \le 256\,\varepsilon^{1-\delta}\} \cap \{\sup_{0\le r\le \varepsilon^\beta} |N_r| > \varepsilon^{q_1}\}\Big),$$

$$Q_3(q_1,q_2) := P\Big(\int_0^{\varepsilon^\beta} |\Psi_r|^2 dr \le 256\,\varepsilon^{1-\delta},\ \sup_{0\le r\le\varepsilon^\beta} |N_r| \le \varepsilon^{q_1},\ \int_0^{\varepsilon^\beta} < M >_r dr > \varepsilon^{q_2}\Big),$$

$$Q_4(q_2) := P\Big(\int_0^{\varepsilon^\beta} < M >_r dr \le \varepsilon^{q_2}\Big).$$

Set $\nu := \beta - \mu$, $q_1 := \frac{1}{2}(1-\delta+\beta-3\mu)$, $q_2 := \frac{1}{2}(1-\delta+3\beta-5\mu)$. Choose $\delta,\beta,\mu > 0$ such that $\nu > 0, q_1 > 0$ and $1-\delta > q_2 > 3\beta$ (for instance, for $\delta = \frac{1}{2}$ and $\beta = \frac{1}{7}$, take $\mu = \frac{1}{140}$). Notice that $q_1 + \beta > q_2$ and $q_2 < \frac{1}{2}(1-\delta+3\beta)$.
By Chebychev's and Burkholder's inequalities, for any $q \ge 1$

$$Q_1(\nu) \le C_q \varepsilon^{-\nu q} \int_0^t E\big(\sup_{0\le u\le\varepsilon^\beta} |S_{(u,v)}|^{2q}\big) dv \le C_q \varepsilon^{\mu q}. \tag{3.20}$$

Fix $\omega \in A$, such that $\int_0^{\varepsilon^\beta} |\Psi_r|^2 dr \leq 256\, \varepsilon^{1-\delta}$. Then, for any $r \leq \varepsilon^\beta$

$$< N >_r = \int_{R_{r,t}} |\Psi_u S_{(u,v)}|^2 du dv = \int_0^r |\Psi_u|^2 \left(\int_0^t |S_{(u,v)}|^2 dv \right) du \leq 256\, \varepsilon^{1-\delta+\nu}.$$

Then, using the exponential inequality for martingales,

$$Q_2(\nu, q_1) \leq P\Big(\sup_{0 \leq r \leq \varepsilon^\beta} |N_r| > \varepsilon^{q_1}, < N >_{\varepsilon^\beta} \leq 256\, \varepsilon^{1-\delta+\nu} \Big) \leq 2 \exp(-C \varepsilon^{-2\mu}). \tag{3.21}$$

Fix now ω such that $\int_0^{\varepsilon^\beta} |\Psi_r|^2 dr \leq 256\, \varepsilon^{1-\delta}$ and $\sup_{0 \leq r \leq \varepsilon^\beta} |N_r| \leq \varepsilon^{q_1}$. Then, from Itô's formula we have,

$$\begin{aligned}
\int_0^{\varepsilon^\beta} < M >_r dr &\leq \int_0^{\varepsilon^\beta} |\Psi_r|^2 dr + 2 \int_0^{\varepsilon^\beta} \Big| \int_0^r \Psi_u d\Psi_u \Big| dr \\
&\leq C \Big(\varepsilon^{1-\delta} + |K_3| \int_0^{\varepsilon^\beta} \Big| \int_0^r \Psi_u du \Big| dr + \varepsilon^\beta \sup_{0 \leq r \leq \varepsilon^\beta} \Big| \int_{R_{r,t}} \Psi_u S_{(u,v)} dW_{u,v} \Big| \Big) \\
&\leq C \Big(\varepsilon^{1-\delta} + |K_3| \int_0^{\varepsilon^\beta} r^{\frac{1}{2}} \Big(\int_0^r |\Psi_u|^2 du \Big)^{\frac{1}{2}} dr + \varepsilon^{q_1+\beta} \Big) \\
&\leq C \Big(\varepsilon^{1-\delta} + |K_3| \varepsilon^{\frac{1}{2}(1-\delta+3\beta)} + \varepsilon^{q_1+\beta} \Big) \leq \varepsilon^{q_2}.
\end{aligned}$$

Hence, by the choice of β, δ and μ,

$$Q_3(q_1, q_2) = 0. \tag{3.22}$$

Using Hölder's inequality and Fubini's theorem, we can obtain

$$Q_4(q_2) \leq P\Big(\int_0^{\varepsilon^\beta} \int_0^r \Big[\int_{R_{u,t}} \Big(\int_y^t g(v,y) dv \Big) dW_{w,y} \Big]^2 du dr \leq t \varepsilon^{q_2} \Big).$$

Set $K_4 := \int_0^t (\int_y^t g(v,y) dv)^2 dy$. Notice that from (C2) and (C4), $K_4 > 0$. Define

$$M_u^{(2)} := \int_{R_{u,t}} \Big(\int_y^t g(v,y) dv \Big) dW_{w,y}.$$

Then, $\{M_u^{(2)}, u \geq 0\}$ is a martingale with $< M^{(2)} >_u = u K_4$. So, there exists a Brownian motion $Z^{(3)}$, such that $M_u^{(2)} = Z^{(3)}_{< M^{(2)} >_u}$. Thus,

$$Q_4(q_2) \leq P\Big(\int_0^{\varepsilon^\beta} \int_0^r |Z^{(3)}_{u K_4}|^2 du dr \leq t \varepsilon^{q_2} \Big).$$

Define $Z_u^{(4)} := K_4^{-\frac{1}{2}}\varepsilon^{-\frac{\beta}{2}}Z^{(3)}_{uK_4\varepsilon^\beta}$. Then $\{Z_u^{(4)}, u \geq 0\}$ is also a Brownian motion and by a change of variable,

$$Q_4(q_2) \leq P\Big(\int_0^1 (1-u)|Z_u^{(4)}|^2 du \leq \frac{t}{K_4}\varepsilon^{q_2-3\beta}\Big).$$

Then, from Lemma 4.1, (3.20),(3.21) and (3.22), this part of the proof is completed.

Assume (h4). We follow the proof of the previous case till the decomposition $P_{22}^{(3)}(\beta,\delta,\varepsilon) \leq P_{221}^{(3)}(\beta,\delta,\varepsilon) + P_{222}^{(3)}(\beta,\delta,\varepsilon)$. Set $K_5 := (\partial_u f(z; 0\otimes z; H_{0\otimes z}))^2$. Notice than under (h4), $K_5 > 0$ and $S_4^{(3)} = rK_5^{\frac{1}{2}}$, so

$$P_{222}^{(3)}(\beta,\delta,\varepsilon) = P(K_5 \leq C\,\varepsilon^{1-\delta-3\beta}). \tag{3.23}$$

On the other hand, $P_{221}^{(3)}(\beta,\delta,\varepsilon)$ can be studied as in (3.18). Thus, from (3.14), (3.17), (3.18) and (3.23), choosing $\delta,\beta \in (0,1)$ such that $\beta + 2\delta > 1$, $4\beta + \delta > 1$ and $3\beta + \delta < 1$ (for instance, $\delta = \frac{1}{2}$ and $\beta = \frac{1}{7}$), the proof of this case is finished.

Assume (h5). Using a multidimensional Taylor expansion, we have

$$f(z;\alpha\otimes z;X_\alpha) = \sum_{i=1}^{3} T_i^{(5)}(z;\alpha),$$

with

$$T_1^{(5)}(z;\alpha) := \sum_{k=0}^{n-1}\Big[\sum_{j+l=k}\frac{1}{j!\,l!}\partial^k_{x^j u^l} f(z;0\otimes z;H_{0\otimes z})r^l(X_\alpha - H_{0\otimes z})^j\Big],$$

$$T_2^{(5)}(z;\alpha) := \sum_{j+l=n}\frac{1}{j!\,l!}\partial^n_{x^j u^l} f(z;0\otimes z;H_{0\otimes z})r^l(X_\alpha - H_{0\otimes z})^j,$$

$$T_3^{(5)}(z;\alpha) := \sum_{j+l=n+1}\frac{1}{j!\,l!}\partial^{n+1}_{x^j u^l} f(z;\eta^*;H^*)r^l(X_\alpha - H_{0\otimes z})^j,$$

where H^*, η^* are random points depending on the Taylor expansion.
From $(C7_n)$, $T_1^{(5)}(z;\alpha) \equiv 0$. Then $P_2(\beta,\delta,\varepsilon) \leq P_{21}^{(5)}(\beta,\delta,\varepsilon) + P_{22}^{(5)}(\beta,\delta,\varepsilon)$, with

$$P_{21}^{(5)}(\beta,\delta,\varepsilon) := P\Big(\int_{C^z_{\beta,\delta}(\varepsilon)} |T_3^{(5)}(z;\alpha)|^2 d\alpha > 4\varepsilon\Big),$$

$$P_{22}^{(5)}(\beta,\delta,\varepsilon) := P\Big(\int_{C^z_{\beta,\delta}(\varepsilon)} |T_2^{(5)}(z;\alpha)|^2 d\alpha \leq 16\varepsilon\Big).$$

By Lemma 3.5 (b) it is easy to check that for any $q \geq 1$

$$\begin{aligned} P_{21}^{(5)}(\beta,\delta,\varepsilon) &\leq C_q\varepsilon^{-q}\varepsilon^{(\beta+\delta)(q-1)}\sum_{j+l=n+1}\int_{C^z_{\beta,\delta}(\varepsilon)} r^{2ql}E(|X_\alpha - H_{0\otimes z}|^{2qj})d\alpha \\ &\leq C_q(\varepsilon^{(\beta+3\delta-1)q} + \varepsilon^{((2n+3)\beta+\delta-1)q}). \end{aligned} \tag{3.24}$$

Define $K_6 := \int_0^t b(0\otimes z;(0,v);H_{(0,v)})dv + \partial_u H_{0\otimes z}$. Using again the Taylor expansion

$$H_{\alpha\otimes z} = H_{0\otimes z} + r\partial_u H_{0\otimes z} + \frac{1}{2}r^2\partial_u^2 H_{\eta^*},$$

with η^* a random point in $(0\otimes z, \alpha\otimes z)$, we can write $X_\alpha - H_{0\otimes z} = V_1(\alpha;z) + V_2(\alpha;z)$, with

$$\begin{aligned} V_1(\alpha;z) &:= \int_{R_\alpha} (f(\alpha;\eta;X_\eta) - f(0\otimes\alpha;0\otimes\eta;H_{0\otimes\eta}))dW_\eta + \int_{R_\alpha} b(\alpha;\eta;X_\eta)d\eta \\ &\quad - \int_{R_{\alpha\otimes z}} b(0\otimes z;0\otimes\eta;H_{0\otimes\eta})d\eta + (H_\alpha - H_{\alpha\otimes z}) + \frac{1}{2}r^2\partial_u^2 H_{\eta^*}, \\ V_2(\alpha;z) &:= \int_{R_{\alpha\otimes z}} b(0\otimes z;0\otimes\eta;H_{0\otimes\eta})d\eta + r\partial_u H_{0\otimes z} = rK_6. \end{aligned}$$

Notice that from (3.13), for any $q \geq 1$

$$E(|V_1(\alpha;z)|^{2q}) \leq C_q(\varepsilon^{2\delta q} + \varepsilon^{3\beta q}). \tag{3.25}$$

Using now a Taylor formula for the function $f(x) = x^j$, for any $j \geq 0$ there exists a random variable λ_j $(0 \leq \lambda_j \leq 1)$ such that

$$\begin{aligned} (X_\alpha - H_{0\otimes z})^j =& V_2(\alpha;z)^j + j\left(\lambda_j(X_\alpha - H_{0\otimes z}) + (1-\lambda_j)V_2(\alpha;z)\right)^{j-1} \\ &\times (X_\alpha - H_{0\otimes z} - V_2(\alpha;z)) = r^jK_6^j + jV_1(\alpha;z)(\lambda_jV_1(\alpha;z) + rK_6)^{j-1}. \end{aligned}$$

Hence $T_2^{(5)}(z;\alpha) = T_{2,1}^{(5)}(z;\alpha) + T_{2,2}^{(5)}(z;\alpha)$ with

$$\begin{aligned} T_{2,1}^{(5)}(z;\alpha) &:= \sum_{j+l=n} \frac{1}{j!\,l!}\partial^n_{x^j u^l} f(z;0\otimes z;H_{0\otimes z})r^l jV_1(\alpha;z)(\lambda_jV_1(\alpha;z) + rK_6)^{j-1}, \\ T_{2,2}^{(5)}(z;\alpha) &:= r^n \sum_{j+l=n} \frac{1}{j!\,l!}\partial^n_{x^j u^l} f(z;0\otimes z;H_{0\otimes z})K_6^j. \end{aligned}$$

Then $P_{22}^{(5)}(\beta,\delta,\varepsilon) \leq P_{221}^{(5)}(\beta,\delta,\varepsilon) + P_{222}^{(5)}(\beta,\delta,\varepsilon)$, with

$$\begin{aligned} P_{221}^{(5)}(\beta,\delta,\varepsilon) &:= P\Big(\int_{C^z_{\beta,\delta}(\varepsilon)} |T_{2,1}^{(5)}(z;\alpha)|^2 d\alpha > 16\varepsilon\Big), \\ P_{222}^{(5)}(\beta,\delta,\varepsilon) &:= P\Big(\int_{C^z_{\beta,\delta}(\varepsilon)} |T_{2,2}^{(5)}(z;\alpha)|^2 d\alpha \leq 64\varepsilon\Big). \end{aligned} \tag{3.26}$$

From (3.25), by Chebychev's and Hölder's inequalities we have that for any $q \geq 1$

$$\begin{aligned} &P_{221}^{(5)}(\beta,\delta,\varepsilon) \\ &\quad\leq C_q\varepsilon^{(\beta+\delta)(q-1)-q} \sum_{j+l=n} \int_{C^z_{\beta,\delta}(\varepsilon)} r^{2ql} E\big(|V_1(\alpha;z)|^{2q}|\lambda_jV_1(\alpha;z) + rK_6|^{2q(j-1)}\big)d\alpha \\ &\quad\leq C_q(\varepsilon^{(\beta+3\delta-1)q} + \varepsilon^{((2n+2)\beta+\delta-1)q}). \end{aligned} \tag{3.27}$$

Set $K_7 := \sum_{j+l=n} \frac{1}{j!\,l!} \partial^n_{x^j u^l} f(z; 0 \otimes z; H_{0\otimes z}) K_6^j$. Notice that from ($C8_n$), $K_7 > 0$ and

$$P^{(5)}_{222}(\beta, \delta, \varepsilon) = P(K_7^2 \le 64\varepsilon^{1-\delta-(2n+1)\beta}). \tag{3.28}$$

Then, by (3.24),(3.26), (3.27) and (3.28) and choosing $\delta, \beta \in (0,1)$ such that $(2n+1)\beta + \delta < 1$, $(2n+2)\beta + \delta > 1$ and $\beta + 2\delta > 1$ (for instance $\beta = \frac{1}{4n+3}$ and $\delta = \frac{1}{2}$), we end the proof. □

Proof of Theorem 2.1. Apply Malliavin's criterion for the existence of a smooth density (see [M]) and Propositions 3.2 and 3.6. □

4. APPENDIX

Lemma 4.1. *Let $\{W_t, t \ge 0\}$ be a Brownian motion and $\lambda, K \in \mathbb{R}$. Then for any $\beta > 0$ such that $\beta < \lambda$, there exists ε_0 such that for all $\varepsilon \le \varepsilon_0$*

$$P\Big(\int_0^1 |W_t|^2 dt \le K\varepsilon^\lambda\Big) \le 2\exp\Big(-\frac{1}{2K}\varepsilon^{-\beta}\Big), \tag{4.1}$$

$$P\Big(\int_0^1 (1-t)|W_t|^2 dt \le K\varepsilon^\lambda\Big) \le 2\exp\Big(-\frac{1}{2K}\varepsilon^{-\beta}\Big). \tag{4.2}$$

Proof. Set $\nu := \frac{1}{2}(\lambda - \beta)$. It is well-known that for any t

$$|W_t|^2 = 2\int_0^t W_u dW_u + t.$$

Then,

$$\frac{1}{2} = \int_0^1 t dt \le \int_0^1 |W_t|^2 dt + 2 \sup_{0\le t\le 1} \Big|\int_0^t W_u dW_u\Big|.$$

Hence, for ε small enough, the exponential inequality for martingales implies

$$P\Big(\int_0^1 |W_t|^2 dt \le K\varepsilon^\lambda\Big) \le P\Big(\int_0^1 |W_t|^2 dt \le K\varepsilon^\lambda, \sup_{0\le t\le 1} \Big|\int_0^t W_u dW_u\Big| > \varepsilon^\nu\Big)$$
$$\le 2\exp\Big(-\frac{1}{2K}\varepsilon^{2\nu-\lambda}\Big).$$

So, (4.1) holds true. The proof of (4.2) is similar. Indeed

$$\frac{1}{6} = \int_0^1 (1-t)|W_t|^2 dt - \int_0^1 (1-t)^2 W_t dW_t.$$

Thus, for small ε,

$$P\Big(\int_0^1 (1-t)|W_t|^2 dt \le K\varepsilon^\lambda\Big) \le P\Big(\int_0^1 (1-t)|W_t|^2 dt \le K\varepsilon^\lambda, \Big|\int_0^1 (1-t)^2 W_t dW_t\Big| > \varepsilon^\nu\Big)$$
$$\le P\Big(\int_0^1 (1-t)^4 |W_t|^2 dt \le K\varepsilon^\lambda, \sup_{0\le s\le 1} \Big|\int_0^s (1-t)^2 W_t dW_t\Big| > \varepsilon^\nu\Big) \le 2\exp\Big(-\frac{1}{2K}\varepsilon^{2\nu-\lambda}\Big).$$

□

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